INCLUSION-EXCLUSION POLYNOMIALS WITH LARGE COEFFICIENTS

Bartłomiej Bzdęga
Adam Mickiewicz University, Poznań, Poland
exul@amu.edu.pl

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Abstract

We prove that for every positive integer k there exists an inclusion-exclusion polynomial $Q_{\{q_1,q_2,\ldots,q_k\}}$ having height at least $d^{2^k}\prod_{j=1}^{k-2} q_j^{2^{k-j-1}-1}$, where d is a positive constant and $q_1<q_2<\ldots<q_k$ are pairwise coprime and arbitrarily large.

1. Introduction

The nth cyclotomic polynomial is the unique monic polynomial having as its simple roots all the nth primitive roots of unity. It can be shown that its coefficients are all integers.

It is well-known that if $n=p_1p_2\ldots p_k$, where $p_1,p_2,\ldots,p_k$ are distinct primes, then

$$\Phi_n(x)=\frac{(1-x^n)\cdot\prod_{1\le j_1<j_2\le k}\left(1-x^{n/(p_{j_1}p_{j_2})}\right)\cdot\ldots}{\prod_{1\le j_1\le k}(1-x^{n/p_{j_1}})\cdot\prod_{1\le j_1<j_2<j_3\le k}\left(1-x^{n/(p_{j_1}p_{j_2}p_{j_3})}\right)\cdot\ldots}. \tag{1}$$

Bachman [1] defined a slightly more general class of polynomials, called inclusion-exclusion polynomials. The case $k=3$ is considered in [2]. If we replace the primes $p_1,p_2,\ldots,p_k$ by pairwise coprime numbers $q_1,q_2,\ldots,q_k>1$ in the formula above and put $m=q_1q_2\ldots q_k$ instead of n, then we arrive at the definition of the inclusion-exclusion polynomial Q_ρ, where $\rho=\{q_1,q_2,\ldots,q_k\}$.

We can expect that properties of inclusion-exclusion polynomials and cyclotomic polynomials are similar. In particular, we may use the same methods to bound the coefficients of polynomials of these both classes, as long as we use formula (1) only and do not make use of the assumption that the numbers $p_1,p_2,\ldots,p_k$ are prime. In the note we illustrate this by an example.

Throughout the paper we set $n=p_1p_2\ldots p_k$, $m=q_1q_2\ldots q_k$, and $\rho=\{q_1,q_2\ldots,q_k\}$. We also assume that $p_1<p_2<\ldots<p_k$ and $q_1<q_2<\ldots<q_k$.

Let A_n be the largest (in absolute value) coefficient of the polynomial Φ_n. Similarly we define A_ρ for the polynomial Q_ρ. Put

$$M_n = \prod_{j=1}^{k-2} p_j^{2^{k-j-1}-1} \quad \text{and} \quad M_\rho = \prod_{j=1}^{k-2} q_j^{2^{k-j-1}-1},$$

where both products equal 1 for $k < 3$.

We define C_k to be the smallest real number for which the inequality $A_n \leq C_k M_n$ holds for all sufficiently large $p_1 = \min\{p_1, p_2, \ldots, p_k\}$. In [4] it is proved that $C_k < (c + o(1))^{2^k}$, where $c \approx 0.9541$. In the proof the assumption that $p_1, p_2, \ldots, p_k$ are primes is never used, so this estimation is true for the inclusion-exclusion polynomials as well. More precisely, the following holds.

Theorem 1. *Let D_k be the smallest real number for which the inequality $A_\rho \leq D_k M_\rho$ holds for all sufficiently large $q_1 = \min\{q_1, q_2, \ldots, q_k\}$. We have $D_k < (c + o(1))^{2^k}$ with $c \approx 0.9541$.*

The aim of this paper is to construct for every positive integer k an infinite family of inclusion-exclusion polynomials Q_ρ with A_ρ at least $d^{2^k} M_\rho$, where d is a positive constant. In this way we prove the following theorem.

Theorem 2. *We have $D_k > (d + o(1))^{2^k}$ with $d \approx 0.5496$.*

2. Proof of Theorem 2

Bateman, Pomerance and Vaughan ([3], Lemma 5, p. 188) proved that if r is a positive integer and $p_j \equiv 2r \pm 1 \pmod{4r}$ for $j = 1, 2, \ldots, k$, then $A_n \geq (4r/\pi)^{2^{k-1}}/n$, where $n = p_1 p_2 \ldots p_k$. Their proof uses only formula (1) and does not require the assumption that $p_1, p_2, \ldots, p_k$ are primes. Therefore we deduce the following inclusion-exclusion polynomial version of this lemma.

Lemma 3. *Let r be a positive integer. If $q_j \equiv 2r \pm 1 \pmod{4r}$ for $j = 1, 2, \ldots, k$, then $A_\rho \geq (4r/\pi)^{2^{k-1}}/m$, where $m = q_1 q_2 \ldots q_k$.*

Now we are ready to prove the main result of this paper.

Proof of Theorem 2. Let N be a positive integer, $r = Nk!$ and $q_j = (4j-2)r+1$ for $j = 1, 2, \ldots, k$. First we check that the numbers $q_1, q_2, \ldots, q_k$ are pairwise coprime. By the Euclidean algorithm we have $(q_i, q_j) = ((4i-2)r+1, (4j-2)r+1) = (4(i-j)r, (4j-2)r+1) = 1$, because every prime divisor of $4(i-j)r$ divides N or is not greater than k, and the number $(4j-2)r+1$ has no such prime divisors.

By Lemma 3 we have

$$(A_\rho/M_\rho)^{2^{-k}} > \left(\frac{(4r/\pi)^{2^{k-1}}/m}{\prod_{j=1}^{k-2} q_j^{2^{k-j-1}-1}}\right)^{2^{-k}} = \frac{2}{\sqrt{\pi}}\left(\frac{r^{2^{k-1}}/q_k}{\prod_{j=1}^{k-1} q_j^{2^{k-j-1}}}\right)^{2^{-k}}$$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{r}{q_k}\prod_{j=1}^{k-1}\left(\frac{r}{q_j}\right)^{2^{k-j-1}}\right)^{2^{-k}} = \frac{2}{\sqrt{\pi}}\prod_{j=1}^{\infty}(4j-2)^{-2^{-j-1}} + o(1) = d + o(1),$$

because the product is convergent. Numerical computations give $d \approx 0.5496$, which completes the proof. □

3. Discussion

Put $\overline{c} = \limsup_{k\to\infty}(C_k)^{2^{-k}}$ and $\underline{c} = \liminf_{k\to\infty}(C_k)^{2^{-k}}$ for cyclotomic polynomials, and similarly $\overline{d} = \limsup_{k\to\infty}(D_k)^{2^{-k}}$ and $\underline{d} = \liminf_{k\to\infty}(D_k)^{2^{-k}}$ for inclusion-exclusion polynomials. The following corollary is an immediate consequence of Theorems 1 and 2.

Corollary 4. *We have $d \leq \underline{d} \leq \overline{d} \leq c$ with $d \approx 0.5496$ and $c \approx 0.9541$.*

It would be much more challenging to prove the analogous results for cyclotomic polynomials. Currently we know only that $\overline{c} \leq c$, where $c \approx 0.9541$. We do not even know if $\overline{c} > 0$.

In the definition of C_k the assumption that $p_1 = \min\{p_1, p_2, \ldots, p_k\}$ is sufficiently large is important. Without it we would have (see [4], concluding remarks)

$$\overline{c} \geq \underline{c} \geq \prod_{j=1}^{\infty} p_j^{-2^{3-j}} = c_1 \approx 0.0001442,$$

where p_j denotes the jth prime number.

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