# A DENSITY CHINESE REMAINDER THEOREM

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**Abstract**

Given collections $\mathcal{A}$ and $\mathcal{B}$ of residue classes modulo $m$ and $n$, respectively, we investigate conditions on $\mathcal{A}$ and $\mathcal{B}$ that ensure that, for at least some $(a, b) \in \mathcal{A} \times \mathcal{B}$, the system: $x \equiv a \bmod m$ and $x \equiv b \bmod n$ has an integer solution, and we quantify the number of such admissible pairs $(a, b)$. The special case where $\mathcal{A}$ and $\mathcal{B}$ consist of intervals of residue classes has application to the Lonely Runner Conjecture.

## 1. Introduction

The classical Chinese Remainder Theorem provides necessary and sufficient conditions for a system of linear congruence equations to possess a solution.

**Theorem 1 (Chinese Remainder Theorem).** *Let $m_1, \ldots, m_k$ be $k$ positive integers, and let $a_1, \ldots, a_k$ be any $k$ integers. Then the system of congruences*

$$\begin{aligned} x &\equiv a_1 \bmod m_1 \\ x &\equiv a_2 \bmod m_2 \\ &\vdots \\ x &\equiv a_k \bmod m_k \end{aligned}$$

*has a solution if and only if $a_i \equiv a_j \bmod \gcd(m_i, m_j)$ for all pairs of indices $i, j$ with $1 \leq i < j \leq k$.*

*Proof.* See Theorem 7.1 of citeHua and Exercises $19 - 23$ in Chapter 2.3 of [11]. □

This theorem admits generalization in several directions. For a statement of a Chinese Remainder Theorem in the language of commutative rings and ideals, see, e.g., Hungerford [7]. Kleinert [8] considers a quite general formalism which yields the usual statement as a special case. In the sequel, we consider a density Chinese Remainder Theorem framed in the classical context of systems of two linear congruence equations.

Specifically, given collections $\mathcal{A}$ and $\mathcal{B}$ of residue classes modulo $m$ and $n$, respectively, we investigate conditions on $\mathcal{A}$ and $\mathcal{B}$ that ensure that, for at least some $(a, b) \in \mathcal{A} \times \mathcal{B}$, the system

$$x \equiv a \bmod m \quad \text{and} \quad x \equiv b \bmod n \tag{1}$$

has an integer solution, and we quantify the number of such admissible pairs $(a, b)$. For instance, if the collections $\mathcal{A}$ and $\mathcal{B}$ satisfy $|\mathcal{A}| = |\mathcal{B}| = 1$, then the Chinese Remainder Theorem (Theorem 1) provides appropriate conditions, namely, that the greatest common divisor $\gcd(m, n)$ divides the difference $a - b$.

For fixed $m$ and $n$, if the collections $\mathcal{A}$ and $\mathcal{B}$ are not large enough (in a suitable sense), then it might occur that, because the gcd condition doesn't hold, the system (1) admits no solution with every $(a, b) \in \mathcal{A} \times \mathcal{B}$. Still, one expects that requiring $\mathcal{A}$ and $\mathcal{B}$ to have large enough density in comparison to the full set of residue classes modulo $m$ and $n$, respectively, will force the gcd condition to hold, so that the classical Chinese Remainder Theorem then yields a solution (or solutions) to the system of equations.

We make this intuition precise in Theorems 2 and 3, establishing a density Chinese Remainder Theorem for the case of arbitrary collections of residue classes and for collections of intervals of residue classes, respectively. Results of this form, particularly generalizations of Corollary 2, have application to the Lonely Runner Conjecture, which we also briefly discuss.

## 2. Intersections of Sets of Arithmetic Progressions

In this section, we state our main results and some corollaries. The proofs of Theorems 2 and 3 appear in the following two sections.

**Theorem 2 (Density CRT).** *Let $\mathcal{A}$ and $\mathcal{B}$ be collections of residue classes modulo $m$ and $n$, respectively. Let $g = \gcd(m, n)$, and let $h$ denote the number of solutions to the linear system*

$$x \equiv a \bmod m \quad \textit{and} \quad x \equiv b \bmod n \tag{2}$$

*modulo $mn/g$ as $(a, b)$ ranges over $\mathcal{A} \times \mathcal{B}$.*

*Write*

$$\begin{aligned} |\mathcal{A}| &= A\frac{m}{g} + r_A, \qquad 0 \le r_A < \frac{m}{g}, \\ |\mathcal{B}| &= B\frac{n}{g} + r_B, \qquad 0 \le r_B < \frac{n}{g}. \end{aligned} \tag{3}$$

*Then $h$, the number of solutions to the linear system (2) modulo $mn/g$, satisfies*

$$h \geq \begin{cases} 0, & \text{if } A+B < g-1, \\ r_A r_B, & \text{if } A+B = g-1, \\ (A+B-g)mn/g^2 + r_A n/g + r_B m/g, & \text{if } A+B > g-1. \end{cases} \tag{4}$$

**Theorem 3 (Density CRT for intervals).** *Let $\mathcal{A}$ and $\mathcal{B}$ be intervals of residue classes modulo $m$ and $n$, respectively. Let $g = \gcd(m,n)$, and let $h$ denote the number of solutions to the linear system*

$$\begin{aligned} x &\equiv a \bmod m \\ x &\equiv b \bmod n \end{aligned} \tag{5}$$

*modulo $mn/g$ as $(a,b)$ ranges over $\mathcal{A} \times \mathcal{B}$.*

*Write*

$$\begin{aligned} |\mathcal{A}| &= Ag + r_A, \qquad 0 \leq r_A < g, \\ |\mathcal{B}| &= Bg + r_B, \qquad 0 \leq r_B < g. \end{aligned} \tag{6}$$

*Then $h$, the number of solutions to the linear system (5) modulo $mn/g$, satisfies*

$$h \geq ABg + Ar_B + Br_A + \max(0, r_A + r_B - g). \tag{7}$$

**Corollary 1 ($\gcd(m,n) = 1$).** *Given the conditions of Theorem 3 with the additional assumption that $\gcd(m,n) = 1$, then the linear system (5) possesses exactly $|\mathcal{A}||\mathcal{B}|$ solutions.*

*Proof.* This follows immediately from Theorem 3.

Also, more directly, note that the condition $\gcd(m,n) = 1$ ensures that each choice $(a,b) \in \mathcal{A} \times \mathcal{B}$ of residue classes leads to a solution modulo $mn$ of the linear system by way of the classical Chinese Remainder Theorem (Proposition 1). □

**Corollary 2 (Density statement).** *Given the conditions of Theorem 3, with the additional assumptions that*

$$|\mathcal{A}| > \frac{1}{3}m \quad \text{and} \quad |\mathcal{B}| > \frac{1}{3}n, \tag{8}$$

*and that $m$ and $n$ are distinct, then the linear system (5) possesses a solution. Also, the constant $\frac{1}{3}$ can not be taken to be smaller while keeping the above conclusion of this corollary.*

*Proof.* Following Theorem 3, write

$$\begin{aligned} |\mathcal{A}| &= Ag + r_A, \qquad 0 \leq r_A < g, \\ |\mathcal{B}| &= Bg + r_B, \qquad 0 \leq r_B < g, \end{aligned} \tag{9}$$

with $g = \gcd(m, n)$. To establish the desired claim of this corollary, we must examine the terms appearing on the right side of (7). If $AB > 0$, then the corollary follows immediately, and so it remains to consider the possibility that $AB = 0$.

To that end, suppose that $A = 0$ and $B = 0$. (If $B > 0$, then the $Br_A$ term in (7) is nonzero, and again we are done.) Then $g > r_A = |\mathcal{A}| > \frac{1}{3}m$ implies that $\frac{m}{g} < 3$. This forces $\frac{m}{g}$ to be either 1 or 2.

In the first case, we have $m = g$. Since $m$ and $n$ are distinct and $g$ must divide $n$, it must be that $n = mn_0$, where the integer $n_0$ satisfies $n_0 \geq 2$. Then we have

$$|\mathcal{B}| > \frac{1}{3}n = \frac{1}{3}mn_0 \geq \frac{2}{3}m = \frac{2}{3}g. \tag{10}$$

With $r_A > \frac{1}{3}m = \frac{1}{3}g$ and $r_B = |\mathcal{B}| > \frac{2}{3}g$, this establishes that the $r_A + r_B - g$ term of (7) is nonzero, and then the linear system posseses a solution by Theorem 3.

In the second case, that $\frac{m}{g} = 2$, we have $m = 2g$. The fact that $A = B = 0$, together with $|\mathcal{A}| > \frac{1}{3}m$ and $|\mathcal{B}| > \frac{1}{3}n$, implies that both $\frac{m}{g}$ and $\frac{n}{g}$ are either 1 or 2. Hence, if $m = 2g$, then $n = g$, and the result follows, being symmetric to the first case.

Finally, to see that the constant $\frac{1}{3}$ is optimal, let $M$ be a large integer, and set $m = 3M$, $n = 2 \cdot 3M$. Let

$$\mathcal{A} = \{0, 1, \ldots, M-1\} \tag{11}$$

and

$$\mathcal{B} = \{M, M+1, \ldots, 3M-1\}. \tag{12}$$

Then $\gcd(m, n) = \gcd(3M, 2\cdot 3M) = 3M$, with $|\mathcal{A}| = M = \frac{1}{3}m$ and $|\mathcal{B}| = 2M = \frac{1}{3}n$. If $(a, b) \in \mathcal{A} \times \mathcal{B}$, then $a - b$ is nonzero modulo $\gcd(m, n) = 3M$, so that the classical Chinese Remainder Theorem (Proposition 1) implies that the linear system has no solution. □

## 3. Arbitrary Collections

In this section, we prove Theorem 2. Given collections $\mathcal{A}$ and $\mathcal{B}$ of residue classes modulo $m$ and $n$, respectively, the classical Chinese Remainder Theorem reduces counting the number of solutions of the linear system (2) to counting the pairs $(a, b) \in \mathcal{A} \times \mathcal{B}$ with $a \equiv b \bmod \gcd(m, n)$.

To that end, let $g = \gcd(m, n)$, and, for a collection of residue classes $\mathcal{C}$ and integer $i$ with $1 \leq i \leq g$, partition the collection $\mathcal{C}$ into sets

$$\mathcal{C}_i = \{c \in \mathcal{C} : c \equiv i \bmod g\}, \tag{13}$$

and define the counting function $f$ by

$$f(i, \mathcal{C}) = |\mathcal{C}_i|. \tag{14}$$

The Chinese Remainder Theorem then yields that the number of solutions to the linear system (2) is given by the sum

$$\sum_{i=1}^{g} f(i,\mathcal{A})f(i,\mathcal{B}). \tag{15}$$

Note that we have

$$|\mathcal{A}| = \sum_{i=1}^{g} f(i,\mathcal{A}) \quad \text{and} \quad |\mathcal{B}| = \sum_{i=1}^{g} f(i,\mathcal{B}). \tag{16}$$

Moreover, for $1 \leq i \leq g$, the right-hand summands in (16) are each bounded above:

$$f(i,\mathcal{A}) \leq \frac{m}{g} \quad \text{and} \quad f(i,\mathcal{B}) \leq \frac{n}{g}. \tag{17}$$

To establish Theorem 2, we must find a lower bound on (15) under the conditions (16) and (17). We do so in three steps. First, we use an inequality on rearrangements, Lemma 1, to reduce to the case where the values $f(i,\mathcal{A})$ are increasing and the values $f(i,\mathcal{B})$ are decreasing. Next, in this case, Lemma 2 yields an extremal distribution for these values under the constraints of (16) and (17). Finally, Lemma 3 provides a case analysis for the explicit evaluation of the sum (15) in this extremal case. Theorem 2 follows directly from these three lemmas.

**Lemma 1 (Rearrangement inequality).** *For each pair of ordered real sequences $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$, and for each permutation $\sigma : [n] \to [n]$, we have*

$$\sum_{k=1}^{n} a_k b_{n-k+1} \leq \sum_{k=1}^{n} a_k b_{\sigma(k)} \leq \sum_{k=1}^{n} a_k b_k. \tag{18}$$

*Proof.* See Chapter 5 of [12] and Chapter X of [5]. □

**Lemma 2 (Extremal distribution).** *For each pair of non-negative ordered real sequences $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfying $A = \sum_{k=1}^{n} a_k$, $B = \sum_{k=1}^{n} b_k$, and, for all $k$ with $1 \leq k \leq n$, the bounds $a_k \leq q_A$ and $b_k \leq q_B$, then we have*

$$\sum_{k=1}^{n} a_k^* b_{n-k+1}^* \leq \sum_{k=1}^{n} a_k b_{n-k+1}, \tag{19}$$

*where*

$$a_k^* = \begin{cases} 0, & \text{if } k < n - \lfloor A/q_A \rfloor, \\ A - q_A \lfloor \frac{A}{q_A} \rfloor, & \text{if } k = n - \lfloor A/q_A \rfloor, \\ q_A, & \text{if } k > n - \lfloor A/q_A \rfloor, \end{cases} \tag{20}$$

*and*

$$b_k^* = \begin{cases} 0, & \text{if } k < n - \lfloor B/q_B \rfloor, \\ B - q_B \lfloor \frac{B}{q_B} \rfloor, & \text{if } k = n - \lfloor B/q_B \rfloor, \\ q_B, & \text{if } k > n - \lfloor B/q_B \rfloor. \end{cases} \tag{21}$$

*Proof.* Since $b_1 \le b_2 \le \cdots \le b_n$, we have

$$\sum_{k=1}^{n} a_k^* b_{n-k+1} \le \sum_{k=1}^{n} a_k b_{n-k+1}. \tag{22}$$

Then, because $a_1^* \le a_2^* \le \cdots \le a_n^*$, we have

$$\sum_{k=1}^{n} a_k^* b_{n-k+1}^* \le \sum_{k=1}^{n} a_k^* b_{n-k+1}, \tag{23}$$

and the result follows. □

**Lemma 3 (Extremal sum).** *Using the notation of Lemma 2, if*

$$\begin{aligned} s &= \lfloor A/q_A \rfloor + \lfloor B/q_B \rfloor + 1, \\ r_A &= A - q_A \lfloor A/q_A \rfloor, \\ r_B &= B - q_B \lfloor B/q_B \rfloor, \end{aligned} \tag{24}$$

*then we have*

$$\sum_{k=1}^{n} a_k^* b_{n-k+1}^* = \begin{cases} 0, & \text{if } s < n, \\ r_A r_B, & \text{if } s = n, \\ (s-n-1) q_A q_B + r_A q_B + r_B q_A, & \text{if } s > n. \end{cases} \tag{25}$$

*Proof.* First, note that (21) yields that

$$b_{n-k+1}^* = \begin{cases} q_B, & \text{if } k < \lfloor B/q_B \rfloor + 1, \\ B - q_B \lfloor \frac{B}{q_B} \rfloor, & \text{if } k = \lfloor B/q_B \rfloor + 1, \\ 0, & \text{if } k > \lfloor B/q_B \rfloor + 1. \end{cases} \tag{26}$$

Then, (20) and (26) together imply that the summands $a_k^* b_{n-k+1}^*$ in (25) vanish unless $k$ satisfies

$$n - \lfloor A/q_A \rfloor \le k \le \lfloor B/q_B \rfloor + 1. \tag{27}$$

This allows the sum in (25) to be written as

$$\sum_{k=1}^{n} a_k^* b_{n-k+1}^* = \sum_{k=n-\lfloor A/q_A \rfloor}^{\lfloor B/q_B \rfloor + 1} a_k^* b_{n-k+1}^*. \tag{28}$$

The analysis of this sum now requires examining three cases: $s < n$, $s > n$, and $s = n$. If

$$s = \lfloor A/q_A \rfloor + \lfloor B/q_B \rfloor + 1 < n, \tag{29}$$

then the sum is empty. If $s > n$, then the sum consists of a first term, last term, and $s - n - 1$ middle terms. The first and last terms contribute $r_A q_B$ and $r_B q_A$, respectively, while the remaining terms each have identical value $q_A q_B$. Finally, if $s = n$, the sum consists of a single term of value $r_A r_B$. □

## 4. Collections of Intervals

In this section, we prove Theorem 3. As in the previous section, given collections $\mathcal{A}$ and $\mathcal{B}$ of residue classes modulo $m$ and $n$, respectively, the classical Chinese Remainder Theorem reduces counting the number of solutions of the linear system (5) to counting the pairs $(a, b) \in \mathcal{A} \times \mathcal{B}$ with $a \equiv b \bmod \gcd(m, n)$. Unlike in the previous section, the fact that the collections of residue classes in Theorem 3 consist of intervals of classes, rather than arbitrary sets of classes, substantially simplifies the analysis of this counting problem.

*Proof of Theorem 3.* With $g = \gcd(m, n)$, we have

$$\begin{aligned} |\mathcal{A}| &= Ag + r_A, \qquad 0 \le r_A < g, \\ |\mathcal{B}| &= Bg + r_B, \qquad 0 \le r_B < g. \end{aligned} \tag{30}$$

It remains to count the pairs $(a, b) \in \mathcal{A} \times \mathcal{B}$ with $a \equiv b \bmod g$.

Since $\mathcal{A}$ consists of an interval of residue classes, it follows that $\mathcal{A}$ can be partitioned into $A$ sub-intervals of length $g$, each sub-interval containing $g$ distinct elements modulo $g$, along with a smaller sub-interval of length $r_A$, containing $r_A < g$ distinct elements modulo $g$. Similarly, $\mathcal{B}$ can be partitioned into $B$ sub-intervals of length $g$, each sub-interval containing $g$ distinct elements modulo $g$, along with a smaller sub-interval of length $r_B$, containing $r_B < g$ distinct elements modulo $g$.

Each of the $Ag$ elements from the $A$ sub-intervals of length $g$ from the collection $\mathcal{A}$ will agree modulo $g$ with exactly one element in each of the $B$ sub-intervals of length $g$ from the collection $\mathcal{B}$. These pairings contribute $ABg$ solutions to (5).

Each of the $r_A$ elements from the smaller sub-interval of length $r_A$ will agree modulo $g$ with exactly one element in each of the $B$ sub-intervals of length $g$ from the collection $\mathcal{B}$. Similarly, each of the $r_B$ elements from the smaller sub-interval of length $r_B$ will agree modulo $g$ with exactly one element in each of the $A$ sub-intervals of length $g$ from the collection $\mathcal{A}$. Together, these pairings contribute $r_A B + r_B A$ solutions to (5).

Finally, it might be that the two small sub-intervals of length $r_A < g$ and $r_B < g$ have no elements that agree modulo $g$. Still, by the pigeonhole principle, there must be at least $r_A + r_B - g$ matches modulo $g$. These pairings contribute $\min(0, r_A + r_B - g)$ solutions to (5).

This yields that, in total, (5) possesses at least

$$ABg + Ar_B + Br_A + \min(0, r_A + r_B - g) \tag{31}$$

solutions, completing the proof of Theorem 3. □

## 5. The Lonely Runner Conjecture

The Lonely Runner Conjecture, having its origins in view-obstruction problems and in diophantine approximation, seems to be due independently to Wills [13] and Cusick [3]. A problem in $n$-dimensional geometry view obstruction motivated Cusick's statement of the problem, while Wills viewed the question from the perspective of Diophantine approximation.

Let $k \geq 2$ be an integer, and let $m_1, \ldots, m_k$ be distinct positive integers. For $x \in \mathbb{R}$, let $\|x\|$ denote the distance from $x$ to the integer nearest, i.e.,

$$\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}. \tag{32}$$

Montgomery includes the Wills version as Problem 44 in the Diophantine Approximation section of the appendix of unsolved problems in [10].

**Conjecture 1.** (Wills) If $1 \leq m_1 < m_2 < \ldots < m_k$ then

$$\max_{\alpha \in \mathbb{R}} \min_{1 \leq k \leq K} \|m_k \alpha\| \geq \frac{1}{k+1}. \tag{33}$$

Goddyn, one of the authors of [2], gave the problem a memorable name and interpretation concerning runners on a circular track.

**Conjecture 2 (Lonely Runner Conjecture).** Suppose that $k$ runners having positive integer speeds run laps on a unit-length circular track. Then there is a time at which all $k$ runners are at distance at least $1/(k+1)$ from their common starting point.

The formulation of this problem by Wills and Cusick led to a considerable body of work on the Lonely Runner Conjecture and its various incarnations and applications in Diophantine approximation (see [1]), view-obstruction problems (see [3] and [4]), nowhere zero flows in regular matroids (see [2]), and certain graph coloring questions (e.g., [14], [9]).

At time $t$, runner $i$ has position at distance $\|m_i t\|$ from the starting point. Call a runner *distant* at time $t$ if $\|m_i t\| \geq \frac{1}{k+1}$. Given an instance of the lonely runner problem, we would like to know if there is a time at which all runners are distant.

To connect this problem to the density Chinese Remainder Theorem, we proceed by first moving from its formulation over $\mathbb{R}$ to one over $\mathbb{Q}$, and then we examine rationals with a suitable fixed denominator (a function of the number of runners $k$ and their speeds $m_1, \ldots, m_k$).

Towards that end, consider a single runner with speed $m$. We seek to investigate the set

$$T(m) = \{t \in \mathbb{R} : \|mt\| \geq \frac{1}{k+1}\}, \tag{34}$$

the times $t$ at which this runner is distant. To understand this set of times, for an integer $Q \geq 1$, let

$$T_Q(m) = \{t \in T(m) : t = a/Q, 0 \leq a \leq Q-1\}. \tag{35}$$

Imposing the condition that $(k+1)m$ divides $Q$ ensures that this set captures the extremal times where equality holds in (34). For such $Q$, the numerators of the times in $T_Q(m)$ correspond to residue classes modulo $Q$. These residue classes themselves are induced by an interval of residue classes modulo $Q/m$.

Hence, for $k$ runners with speeds $m_1, \ldots, m_k$, we select an appropriate choice of $Q$, and we seek to find at least one time $t$ that belongs to each of $T_Q(m_1), \ldots, T_Q(m_k)$. By the above remarks, the problem reduces to finding a common solution to a system of linear congruence equations.

For example, suppose that $k = 2$, with runner speeds $m$ and $n$. If we set $Q = 3mn$, then the times $T_Q(m)$ are induced by an interval $\mathcal{A}$ of residue classes modulo $3n$, and the times $T_Q(n)$ are induced by an interval $\mathcal{B}$ of residue classes modulo $3m$. Specifically,

$$T_Q(m) = \{\frac{a}{3n} : \|\frac{a}{3n}\| \geq \frac{1}{3}, 0 \leq a \leq 3mn-1\}, \tag{36}$$

and $\mathcal{A}$ consists of $n + 1 > \frac{1}{3} \cdot 3n$ residue classes, the interval of residue classes

$$\{a \bmod 3n : n \leq a \leq 2n\}, \tag{37}$$

with analogous statements holding for $T_Q(n)$ and $\mathcal{B}$. Thus, Corollary 2 applies, yielding a solution to the linear system of congruences and hence the existence of a time belonging to both $T_Q(m)$ and $T_Q(n)$.

## 6. Further Work

First, as with the classical Chinese Remainder Theorem, results for systems containing more than two linear congruence equations would be both interesting and useful. In particular, such a generalization in the case of collections of intervals can be applied to the Lonely Runner Conjecture. For instance, the constant $\frac{1}{3}$ appearing in Corollary 2 is the analogue of the same constant appearing in the Lonely

Runner Conjecture with $k = 2$ runners. One might hope for such similarities to continue for larger values of $k$.

Other possibilities for the collections $\mathcal{A}$ and $\mathcal{B}$, for example, random collections (for suitable notions of random), collections nicely distributed among residue classes, and collections with other arithmetic structure might be amenable to analysis.

Finally, regardless of the number of linear congruence equations or the types of collections involved, structural information beyond mere existence and quantity for the admissible classes and the resulting solutions could be useful in iterated applications of this, and other, density Chinese Remainder Theorems.

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