EMBEDDABILITY PROPERTIES OF DIFFERENCE SETS

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Abstract

By using nonstandard analysis, we prove embeddability properties of differences $A - B$ of sets of integers. (A set A is "embeddable" into B if every finite configuration of A has shifted copies in B.) As corollaries of our main theorem, we obtain improvements of results by I.Z. Ruzsa about intersections of difference sets, and of Jin's theorem (as refined by V. Bergelson, H. Fürstenberg and B. Weiss), where a precise bound is given on the number of shifts of $A - B$ which are needed to cover arbitrarily large intervals.

Introduction

In several areas of combinatorics of numbers, diverse non-elementary techniques have been successfully used, including ergodic theory, Fourier analysis, (discrete) topological dynamics, and algebra on the space of ultrafilters (see *e.g.* [14, 6, 19, 34, 15] and references therein). Also *nonstandard analysis* has been applied in this context, starting from some early work that appeared in the last years of the 80s (see [20, 26]), and recently producing interesting results in density problems (see *e.g.*, [22, 23, 24]).

An important topic in combinatorics of numbers is the study of *sumsets* and of *difference sets*. In 2000, R. Jin [21] proved by nonstandard methods the following beautiful property: If A and B are sets of natural numbers with positive *upper Banach density*, then the corresponding sumset $A + B$ is *piecewise syndetic*. (A set C is piecewise syndetic if it has "bounded gaps" in arbitrarily long intervals; equivalently, if a suitable finite union of shifts $C + x_i$ covers arbitrarily long intervals. The upper Banach density is a refinement of the upper asymptotic density. See below for precise definitions.)

Jin's result raised the attention of several researchers, who tried to translate his nonstandard proof into more familiar terms, and to improve on it. In 2006, by using ergodic-theoretical tools, V. Bergelson, H. Furstenberg and B. Weiss [9] gave a new proof by showing that the set $A + B$ is in fact *piecewise Bohr*, a property stronger than piecewise syndeticity. In 2008, V. Bergelson, M. Beiglböck and A. Fish found a shorter proof of that theorem, and extended its validity to countable *amenable groups*. They showed also that a converse result holds, namely that every piecewise Bohr set includes a sumset $A + B$ for suitable sets A, B of positive density (see [1]). This result was then extended by J.T. Griesmer [18] to cases where one of the summands has zero upper Banach density. In 2010, M. Beiglböck [2] found a very short and neat *ultrafilter* proof of the afore-mentioned piecewise Bohr property.

In this paper, we work in the setting of the *hyperintegers* of nonstandard analysis, and we prove some "embeddability properties" of sets of differences. (General results on difference sets of integers $A - B$ immediately imply corresponding results on sumsets $A + B$ since B and $-B$ have the same upper Banach density.) A set A is "embeddable" into B if every finite configuration of A has shifted copies in B, so that the finite combinatorial structure of B is at least as rich as that of A. As corollaries to our main theorem, we obtain at once improvements of results by I.Z. Rusza about intersections of difference sets, and a sharpening of Jin's theorem (as refined by V. Bergelson, H. Fürstenberg and and B. Weiss). We remark that many of the results proved here for sets of integers can be generalized to amenable groups (see [13]).

The first section of this paper contains the basic notions and notation, and the statements of the main results. In the second section, characterizations of several combinatorial notions in the nonstandard setting of hyperintegers are presented, which will be used in the sequel. Section 3 is focused on *delta sets* $A - A$ and, more generally, on density-delta sets. In the fourth section, we isolate notions of *finite embeddability* for sets of integers, and show their basic properties. The main results of this paper about difference sets $A - B$, along with several corollaries, are proved in the last Section 5.

1. Preliminaries and Statement of the Main Results

If not specified otherwise, throughout the paper by "set" we shall always mean a set of *integers*. By the set $\mathbb{N}$ of natural numbers we mean the set of *positive integers*, so that $0 \notin \mathbb{N}$.

We recall the following basic definitions (see *e.g.*, [34]). The *difference set* and the *sumset* of A and B are respectively:

$$A - B \;=\; \{a - b \mid a \in A,\ b \in B\}\ ; \quad A + B \;=\; \{a + b \mid a \in A,\ b \in B\}.$$

The set of differences $\Delta(A) = A - A$ when the two sets are equal, is called the *delta set* of A. Clearly, delta sets are symmetric around 0, *i.e.*, $t \in \Delta(A) \Leftrightarrow -t \in \Delta(A)$.[1]

[1] Some authors include in $\Delta(A)$ only the positive numbers of $A - A$.

A set is *thick* if it includes arbitrarily long intervals; it is *syndetic* if it has bounded gaps, *i.e.*, if its complement is not thick; it is *piecewise syndetic* if $A = B \cap C$ where B is syndetic and C is thick. The following characterizations directly follow from the definitions: A is syndetic if and only if $A + F = \mathbb{Z}$ for a suitable finite set F; A is piecewise syndetic if and only if $A + F$ is thick for a suitable finite set F.

The *lower asymptotic density* $\underline{d}(A)$ and the *upper asymptotic density* $\overline{d}(A)$ of a set A of natural numbers are defined by putting:

$$\underline{d}(A) \;=\; \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n} \quad \text{and} \quad \overline{d}(A) \;=\; \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}\,.$$

Another notion of density for sets of natural numbers that is widely used in number theory is the *Schnirelmann density*:

$$\sigma(A) \;=\; \inf_{n\in\mathbb{N}} \frac{|A \cap [1,n]|}{n}.$$

The *upper Banach density* $\mathrm{BD}(A)$ (known also as *uniform density*) generalizes the upper density by considering arbitrary intervals in place of initial intervals:

$$\mathrm{BD}(A) \;=\; \lim_{n\to\infty} \left(\max_{x\in\mathbb{Z}} \frac{|A \cap [x+1, x+n]|}{n} \right) \;=\; \inf_{n\in\mathbb{N}} \left\{ \max_{x\in\mathbb{Z}} \frac{|A \cap [x+1, x+n]|}{n} \right\}.$$

We shall consider also the *lower Banach density*:

$$\underline{\mathrm{BD}}(A) \;=\; \lim_{n\to\infty} \left(\min_{x\in\mathbb{Z}} \frac{|A \cap [x+1, x+n]|}{n} \right) \;=\; \sup_{n\in\mathbb{N}} \left\{ \min_{x\in\mathbb{Z}} \frac{|A \cap [x+1, x+n]|}{n} \right\}.$$

(See *e.g.*, [17], for details about equivalent definitions of Banach density.) All the above densities are *shift invariant*, that is a set A has the same density of any shift $A + t$. It is readily verified that $\sigma(A) \le \underline{d}(A)$ and that

$$\underline{\mathrm{BD}}\,(A) \;\le\; \underline{d}(A) \;\le\; \overline{d}(A) \;\le\; \mathrm{BD}(A).^2$$

Notice that $\underline{d}(A^c) = 1 - \overline{d}(A)$ and $\underline{\mathrm{BD}}(A^c) = 1 - \mathrm{BD}(A)$. We remark also that a set A is thick if and only if $\mathrm{BD}(A) = 1$, and hence, a set A is syndetic if and only if $\underline{\mathrm{BD}}(A) > 0$. The following is a well-known intersection property of delta sets.

Proposition 1.1. *Assume that $BD(A) > 0$. Then $\Delta(A) \cap \Delta(B) \ne \emptyset$ for* any *infinite set B. In consequence, $\Delta(A)$ is syndetic.*

Proof. The proof essentially consists of a direct application of the *pigeonhole principle* argument. Precisely, one considers the family of shifts $\{A + b_i \mid i = 1, \ldots, n\}$ by distinct elements $b_i \in B$. As each $A + b_i$ has the same upper Banach density as A,

[2] Actually, for any choice of real numbers $0 \le r_1 \le r_2 \le r_3 \le r_4 \le 1$, it is not hard to find sets A such that $\underline{\mathrm{BD}}(A) = r_1$, $\underline{d}(A) = r_2$, $\overline{d}(A) = r_3$ and $\mathrm{BD}(A) = r_4$.

if n is sufficiently large, then those shifts cannot be pairwise disjoint, as otherwise $\mathrm{BD}(\bigcup_{i=1}^{n} A+b_i) = \sum_{i=1}^{n} \mathrm{BD}(A+b_i) = n \cdot \mathrm{BD}(A) > 1$, a contradiction. But then $(A+b_i) \cap (A+b_j) \neq \emptyset$ for suitable $i \neq j$, and hence $\Delta(A) \cap \Delta(B) \neq \emptyset$, as desired. Now assume by contradiction that the complement of $\Delta(A)$ is thick. By symmetry, its positive part $T = \Delta(A)^c \cap \mathbb{N}$ is thick as well. For any thick set $T \subseteq \mathbb{N}$, it is not hard to construct an increasing sequence $B = \{b_1 < b_2 < \ldots\}$ such that $b_j - b_i \in T$ for all $j > i$. But then $\Delta(B) \subseteq -T \cup \{0\} \cup T = \Delta(A)^c$, *i.e.*, $\Delta(B) \cap \Delta(A) = \emptyset$, a contradiction. □

The above property is just a hint of the rich combinatorial structure of sets of differences, whose investigation seems to still be far from complete (see *e.g.*, the recent papers [30, 10, 27]).

Suitable generalizations of delta sets are the following.

Definition 1.2. Let A be a set of integers. For $\epsilon \geq 0$, the following are called the *ϵ-density-Delta sets* (or more simply *ϵ-Delta sets*) of A:

- $\overline{\Delta}_\epsilon(A) \ = \ \{t \in \mathbb{Z} \mid \overline{d}(A \cap (A-t)) > \epsilon\}$.
- $\Delta_\epsilon(A) \ = \ \{t \in \mathbb{Z} \mid \mathrm{BD}(A \cap (A-t)) > \epsilon\}$.

Similarly to delta sets, ϵ-Delta sets also are symmetric around 0. Moreover, it is readily seen that $\overline{\Delta}_\epsilon(A) \subseteq \Delta_\epsilon(A) \subseteq \Delta(A)$ for all $\epsilon \geq 0$. We remark that if $t \in \Delta_\epsilon(C)$ (or if $t \in \overline{\Delta}_\epsilon(C)$), then t is indeed the common difference of "many" pairs of elements of A, in the sense that the set $\{x \in \mathbb{Z} \mid x, x+t \in A\}$ has upper Banach density (or upper asymptotic density, respectively) greater than ϵ.

We shall find it convenient to isolate the following notions of embeddability for sets of integers X, Y.

Definition 1.3. Let X, Y be sets of integers.

- X is *(finitely) embeddable* in Y, denoted $X \lhd Y$, if every finite configuration $F \subseteq X$ has a shifted copy $t + F \subseteq Y$.
- X is *densely embeddable* in Y, denoted $X \lhd_d Y$, if every finite configuration $F \subseteq X$ has "densely-many" shifted copies included in Y, *i.e.*, if the intersection $\bigcap_{x \in F}(Y - x) = \{t \in \mathbb{Z} \mid t + F \subseteq Y\}$ has positive upper Banach density.

Trivially $X \lhd_d Y \Rightarrow X \lhd Y$, and it is easily seen that the converse implication does not hold. Finite embeddability preserves several of the fundamental combinatorial notions that are commonly considered in combinatorics of integer numbers (see Section 4).[3]

The main results obtained in this paper are contained in the following three theorems. The first one is about the syndeticity property of ϵ-Delta sets.

Theorem I. *Let $BD(A) = \alpha > 0$ (or $\overline{d}(A) = \alpha > 0$), and let $0 \leq \epsilon < \alpha^2$. Then for every infinite $X \subseteq \mathbb{Z}$ and for every $x \in X$ there exists a finite subset $F \subset X$ such that:*

1. $x \in F$;
2. $|F| \leq \lfloor \frac{\alpha - \epsilon}{\alpha^2 - \epsilon} \rfloor = k$;
3. $X \subseteq \Delta_\epsilon(A) + F$ (or $X \subseteq \overline{\Delta}_\epsilon(A) + F$, respectively).

In consequence, the set $\Delta_\epsilon(A)$ (or $\overline{\Delta}_\epsilon(A)$, respectively) is syndetic, and its lower Banach density is not smaller than $1/k$.

The second theorem is a general property that holds for all sets of positive upper Banach density.

Theorem II. *Let $BD(A) = \alpha > 0$. Then there exists a set $E \subseteq \mathbb{N}$ such that:*

1. $\sigma(E) \geq \alpha$;
2. $E \lhd_d A$*, and hence* $\Delta(E) \subseteq \Delta_0(A)$ *and* $\Delta_\epsilon(E) \subseteq \Delta_\epsilon(A)$ *for all* $\epsilon \geq 0$.

The main result in this paper concerns an embeddability property of difference sets.

Theorem III. *Let $BD(A) = \alpha > 0$ and $BD(B) = \beta > 0$. Then there exists a set $E \subseteq \mathbb{N}$ such that:*

1. *The Schnirelmann density* $\sigma(E) \geq \alpha\beta$;
2. *For every finite $F \subset E$ there exists $\epsilon > 0$ such that for arbitrarily large intervals J one finds a suitable shift $A_J = A - t_J$ with the property that*

$$\frac{\left| \left(\bigcap_{e \in F} (A_J \cap B) - e\right) \cap J \right|}{|J|} \;\geq\; \epsilon\,;$$

[3] The notions of embeddability isolated above seem to be of interest for their own sake. *E.g.*, one can extend finite embeddability to *ultrafilters* on $\mathbb{N}$, by putting $\mathcal{U} \lhd \mathcal{V}$ when for every $B \in \mathcal{V}$ there exists $A \in \mathcal{U}$ with $A \lhd B$. The resulting relation in the space of ultrafilters $\beta\mathbb{N}$ satisfies several nice properties, which are investigated in [11].

3. *Both $E \lhd_d A$ and $E \lhd_d B$, and hence:*

   - $\Delta(E) \subseteq \Delta_0(A) \cap \Delta_0(B)$;
   - $\Delta_\epsilon(E) \subseteq \Delta_\epsilon(A) \cap \Delta_\epsilon(B)$ *for all* $\epsilon \geq 0$;
   - $\Delta(E) \lhd_d A - B$.

Several corollaries can be derived from the above theorems. The first one is a sharpening of a result about intersections of Delta sets by I.Z. Ruzsa [29], which improved on a previous theorem by C.L. Stewart and R. Tijdeman [32].

Corollary. *Assume that $A_1, \ldots, A_n \subseteq \mathbb{Z}$ have positive upper Banach densities $BD(A_i) = \alpha_i$. Then there exists a set $E \subseteq \mathbb{N}$ with $\sigma(E) \geq \prod_{i=1}^n \alpha_i$ and such that $\Delta_\epsilon(E) \subseteq \bigcap_{i=1}^n \Delta_\epsilon(A_i)$ for every $\epsilon \geq 0$.*

A second corollary is about the syndeticity of intersections of density-Delta sets.

Corollary. *Assume that $BD(A) = \alpha > 0$ and $BD(B) = \beta > 0$. Then for every $0 \leq \epsilon < \alpha^2\beta^2$, for every infinite $X \subseteq \mathbb{Z}$, and for every $x \in X$, there exists a finite subset $F \subset X$ such that*

1. $x \in F$;
2. $|F| \leq \lfloor \frac{\alpha\beta - \epsilon}{\alpha^2\beta^2 - \epsilon} \rfloor = k$;
3. $X \subseteq (\Delta_\epsilon(A) \cap \Delta_\epsilon(B)) + F$.

In consequence, the set $\Delta_\epsilon(A) \cap \Delta_\epsilon(B)$ is syndetic, and its lower Banach density is not smaller than $1/k$.

A similar result is obtained also about the syndeticity of difference sets.

Corollary. *Assume that $BD(A) = \alpha > 0$ and $BD(B) = \beta > 0$. Then for every infinite $X \subseteq \mathbb{Z}$ and for every $x \in X$, there exists a finite subset $F \subset X$ such that*

1. $x \in F$;
2. $|F| \leq \lfloor \frac{1}{\alpha\beta} \rfloor$;
3. $X \lhd_d (A - B) + F$.

If we let $X = \mathbb{Z}$, we obtain a refinement of Jin's theorem [21] where a precise bound on the number of shifts of $A - B$ which are needed to cover a thick set is given.

Corollary. *Assume that $BD(A) = \alpha > 0$ and $BD(B) = \beta > 0$. Then there exists a finite set F such that $|F| \leq \lfloor 1/\alpha\beta \rfloor$ and $A - B + F$ is thick.*

Finally, by the embedding $\Delta(E) \lhd_d A - B$ where E has a positive Schnirelmann density, we can also recover the *Bohr property* of difference sets proved by V. Bergelson, H. Fürstenberg and B. Weiss in [9].

Corollary. *Let A and B have positive upper Banach density. Then the difference set $A - B$ is piecewise Bohr.*

2. Nonstandard Characterizations of Combinatorial Properties

In the proofs of this paper, we shall use the basics of *nonstandard analysis*, including the *transfer principle* and the notion of *internal set* and of *hyperfinite set*. In particular, the reader is assumed to be familiar with the fundamental properties of the *hyperintegers* $^*\mathbb{Z}$ and of the *hyperreals* $^*\mathbb{R}$. The hyperintegers are special *elementary extensions* of the integers, namely *complete extensions*. (See §3.1 and 6.4 of [12] for the definitions.) Informally, one could say that the hyperintegers are a sort of "weakly isomorphic extension" of the integers, in the sense that they share the same "elementary" (*i.e.*, first-order) properties of $\mathbb{Z}$; in particular, $^*\mathbb{Z}$ is a discretely ordered ring whose positive part is the set $^*\mathbb{N}$ of *hypernatural* numbers. We recall that the natural numbers $\mathbb{N}$ are an initial segment of $^*\mathbb{N}$. Similarly, the hyperreal numbers $^*\mathbb{R} \supset \mathbb{R}$ have the same first-order properties as the reals, and so they are an ordered field. As a proper extension of the real line, $^*\mathbb{R}$ is necessarily non-Archimedean, and hence it contains *infinitesimal* and *infinite* numbers. We recall that a number $\xi \in {}^*\mathbb{R}$ is infinitesimal if $-1/n < \xi < 1/n$ for all $n \in \mathbb{N}$; ξ is infinite if its reciprocal $1/\xi$ is infinitesimal, *i.e.*, if $|\xi| > n$ for all $n \in \mathbb{N}$; ξ is finite if it is not infinite, *i.e.*, if $-n < \xi < n$ for some $n \in \mathbb{N}$. In one occasion (proof of Proposition 4.3), we shall apply the *overspill principle*, namely the property that if an internal set contains arbitrarily large (finite) natural numbers, then it necessarily contains also an *infinite* hypernatural number.

A semi-formal introduction to the basic ideas of nonstandard analysis can be found in the first part of the survey [3]; as for the general theory, several books can be used as references, including the classical monographies [31, 25], or the more recent textbook [16]; finally, we refer the reader to §4.4 of [12] for the logical foundations.

Let us now fix our notation. If $\xi, \zeta \in {}^*\mathbb{R}$ are hyperreal numbers, we write $\xi \approx \zeta$ when ξ and ζ are *infinitely close*, *i.e.*, when their distance $|\xi - \zeta|$ is infinitesimal. If $\xi \in {}^*\mathbb{R}$ is finite, then its *standard part* $\mathrm{st}(\xi) = \inf\{r \in \mathbb{R} \mid r > \xi\}$ is the unique real number which is infinitely close to ξ. For $x \in \mathbb{R}$, $\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leq x\}$ is the *integer part* of x; and the same notion transfers to the *hyperinteger part* of an hyperreal number $\lfloor \xi \rfloor = \max\{\nu \in {}^*\mathbb{Z} \mid \nu \leq \xi\}$. The notions of sumset $C + D$ and of difference set $C - D$ for sets of integers, transfer to internal sets $C, D \subseteq {}^*\mathbb{Z}$. If

C is a hyperfinite set, we shall abuse notation and denote by $|C| \in {}^*\mathbb{N}$ its *internal cardinality.* An infinite interval of hyperintegers is an interval $I = [\Omega+1, \Omega+N] \subset {}^*\mathbb{Z}$ whose length N is an infinite hypernatural number. Clearly, the internal cardinality $|I| = N$.

We shall use the following nonstandard characterizations (see *e.g.*, [21, 22]).

- A is *thick* $\Leftrightarrow I \subseteq {}^*A$ for some infinite interval I of hyperintegers.
- A is *syndetic* $\Leftrightarrow {}^*A$ has only finite gaps, *i.e.*, the distance of consecutive elements of ${}^*A$ is always a (finite) natural number.
- A is *piecewise syndetic* $\Leftrightarrow$ there is an infinite interval I of hyperintegers where ${}^*A$ has only finite gaps.
- $\underline{d}(A) \leq \alpha$ (or $\overline{d}(A) \geq \alpha$) $\Leftrightarrow$ there is an infinite hypernatural number N such that $\mathrm{st}(|{}^*A \cap [1,N]|/N) \leq \alpha$ (or $\mathrm{st}(|{}^*A \cap [1,N]|/N) \geq \alpha$, respectively).
- $d(A) = \alpha \Leftrightarrow |{}^*A \cap [1,N]|/N \approx \alpha$ for all infinite N.
- $\mathrm{BD}(A) \geq \alpha \Leftrightarrow$ there exists an infinite interval of hyperintegers $I \subset {}^*\mathbb{Z}$ such that $\mathrm{st}(|{}^*A \cap I|/|I|) \geq \alpha \Leftrightarrow$ for every infinite $N \in {}^*\mathbb{N}$ there exists an interval $I \subset {}^*\mathbb{Z}$ of length N such that $\mathrm{st}(|{}^*A \cap I|/|I|) \geq \alpha$.
- $\underline{\mathrm{BD}}(A) \geq \alpha \Leftrightarrow \mathrm{st}(|{}^*A \cap I|/|I|) \geq \alpha$ for every infinite interval of hyperintegers $I \subset {}^*\mathbb{Z}$.

As a warm-up for the use of the above nonstandard characterizations, let us prove a property which will be used in the sequel.

Proposition 2.1. *Let A be a set of integers and let F be a finite set with $|F| = k$.*

1. *If $A + F = \mathbb{Z}$, then $\underline{BD}(A) \geq 1/k$;*
2. *If $A + F$ is thick, then $BD(A) \geq 1/k$.*

Proof. (1). For every interval I of infinite length N, we have that:

$$I \;=\; {}^*\mathbb{Z} \cap I \;=\; {}^*\!\left(\bigcup_{x\in F}(A+x)\right) \cap I \;=\; \bigcup_{x\in F}\left(({}^*A+x)\cap I\right).$$

By the *pigeonhole principle*, there exists $\overline{x} \in F$ such that $|({}^*A+\overline{x}) \cap I| \geq |I|/k$, and hence $\mathrm{st}\left(|{}^*A \cap I|/|I|\right) = \mathrm{st}\left(|({}^*A+\overline{x}) \cap I|/|I|\right) \geq 1/k$. By the nonstandard characterization of lower Banach density, this yields the thesis $\underline{\mathrm{BD}}(A) \geq 1/k$.

(2). By the nonstandard characterization of thickness, there exists an infinite interval I with $I \subseteq {}^*(A+F) = \bigcup_{x\in F}({}^*A + x)$. Exactly as above, we can pick an element $\overline{x} \in F$ such that $|({}^*A + \overline{x}) \cap I| \geq |I|/k$, and hence $\mathrm{st}\,(|{}^*A \cap I|/|I|) \geq 1/k$. By the nonstandard characterization of Banach density, we conclude that $\mathrm{BD}(A) \geq 1/k$. □

3. Density-Delta Sets

In Section 1, we recalled the well-known property that all intersections of delta sets $\Delta(A) \cap \Delta(B)$ are non-empty, whenever A has positive upper Banach density and B is infinite (see Proposition 1.1). By the same *pigeonhole principle* argument used in the proof of that result, one also shows that:

- If $\overline{d}(A) > 0$, then $\overline{\Delta}_0(A)$ is syndetic.
- If $\mathrm{BD}(A) > 0$, then $\Delta_0(A)$ is syndetic.

This section aims at sharpening the above results by considering ϵ-Delta sets (see Definition 1.3). To this end, we shall use the following combinatorial lemma, which is proved by a straight application of the *Cauchy-Schwartz inequality*. The main point here is that this result holds in the nonstandard setting of *hyperintegers*.

Lemma 3.1. *Let $N \in {}^*\mathbb{N}$ be an infinite hypernatural number, let $\{C_i \mid i \in \Lambda\}$ be a family of internal subsets of $[1, N]$, and assume that every standard part $st(|C_i|/N) \geq \gamma$, where γ is a fixed positive real number. Then for every $0 \leq \epsilon < \gamma^2$ and for every $F \subseteq \Lambda$ with $|F| > \frac{\gamma-\epsilon}{\gamma^2-\epsilon}$, there exist distinct elements $i, j \in F$ such that $st(|C_i \cap C_j|/N) > \epsilon$.*

Proof. Assume for the sake of contradiction that there exists a finite subset $F \subseteq I$ with cardinality $k = |F| > \frac{\gamma-\epsilon}{\gamma^2-\epsilon}$ and such that $\mathrm{st}(|C_i \cap C_j|/N) \leq \epsilon$ for all distinct $i, j \in F$. To simplify matters, let us assume, without loss of generality, that all standard parts $\mathrm{st}(|C_i|/N) = \gamma$. By our hypotheses, we have the following:

- $c_i = |C_i|/N = \gamma + \eta_i$ where $\eta_i \approx 0$.
- $\sum_{i\in F} c_i = k\,\gamma + \eta$ where $\eta = \sum_{i\in F} \eta_i \approx 0$.
- $c_{ij} = |C_i \cap C_j|/N \leq \epsilon + \upsilon_{ij}$ where $\upsilon_{ij} \approx 0$.
- $\sum_{i\neq j} c_{ij} \leq \binom{k}{2} \cdot \epsilon + \upsilon$ where $\upsilon = \sum_{i\neq j} \upsilon_{ij} \approx 0$.

Now let us denote by $\chi_i : [1,N] \to \{0,1\}$ the characteristic function of C_i. Clearly $c_i = (1/N) \cdot \sum_{\xi=1}^{N} \chi_i(\xi)$ and $c_{ij} = (1/N) \cdot \sum_{\xi=1}^{N} \chi_i(\xi)\, \chi_j(\xi)$. By the *Cauchy-Schwartz inequality*, we obtain:

$$\begin{aligned}
k^2\gamma^2 \quad &\approx \quad (k\,\gamma+\eta)^2 \; = \; \left(\sum_{i\in F} c_i\right)^2 \; = \; \frac{1}{N^2}\cdot\left(\sum_{i\in F}\left(\sum_{\xi=1}^{N}\chi_i(\xi)\right)\right)^2 \\
&= \; \frac{1}{N^2}\cdot\left(\sum_{\xi=1}^{N} 1\cdot\left(\sum_{i\in F}\chi_i(\xi)\right)\right)^2 \; \leq \; \frac{1}{N^2}\cdot\left(\sum_{\xi=1}^{N} 1^2\right)\cdot\sum_{\xi=1}^{N}\left(\sum_{i\in F}\chi_i(\xi)\right)^2 \\
&= \; \frac{1}{N}\cdot\sum_{\xi=1}^{N}\left(\sum_{i,j\in F}\chi_i(\xi)\cdot\chi_j(\xi)\right) \; = \; \sum_{i,j\in F}\left(\frac{1}{N}\cdot\sum_{\xi=1}^{N}\chi_i(\xi)\cdot\chi_j(\xi)\right) \\
&= \; \sum_{i\in F} c_i \, + \, 2\cdot\sum_{i<j} c_{ij} \; \leq \; k\cdot\gamma+\eta \, + \, 2\binom{k}{2}\epsilon \, + \, 2\,\upsilon \\
&\approx \; k\,\gamma \, + \, k\,(k-1)\,\epsilon \; = \; k\,(\gamma+(k-1)\,\epsilon)\,,
\end{aligned}$$

and hence $k\,\gamma^2 \leq \gamma + (k-1)\,\epsilon$. This contradicts the assumption $k > \frac{\gamma-\epsilon}{\gamma^2-\epsilon}$. □

A consequence of the above lemma that is relevant to our purposes, is the following one.

Lemma 3.2. *Let $N \in {}^*\mathbb{N}$ be an infinite hypernatural number, let $C \subseteq [1,N]$ be an internal set with $st(|C|/N) = \gamma > 0$, let $0 \leq \epsilon < \gamma^2$ be a real number, and let $k = \lfloor \frac{\gamma-\epsilon}{\gamma^2-\epsilon} \rfloor$. Then for every infinite set $X \subseteq \mathbb{Z}$ and for every $x \in X$, there exists a finite subset $F \subset X$ with $x \in F$, $|F| \leq k$, and such that $X \subseteq \mathcal{D}_\epsilon(C) + F$, where*

$$\mathcal{D}_\epsilon(C) \;=\; \left\{ t \in \mathbb{Z} \,\middle|\, st\left(\frac{|C\cap(C-t)|}{N}\right) > \epsilon \right\}.$$

Proof. We proceed by induction, and define the finite subset $F = \{x_i\}_{i=1}^m \subset X$ as follows. Let $x_1 = x$. If $X \subseteq \mathcal{D}_\epsilon(C) + x_1$, then let $F = \{x_1\}$ and stop. Otherwise pick $x_2 \in X$ such that $x_2 \notin \mathcal{D}_\epsilon(C) + x_1$. Then $x_2 - x_1$ does not belong to $\mathcal{D}_\epsilon(C)$. So, $\mathrm{st}(|C \cap (C - x_2 + x_1)|/N) \leq \epsilon$, and hence also $\mathrm{st}(|(C - x_1) \cap (C - x_2)|/N) \leq \epsilon$, because $x_1/N \approx 0$. Next, if $X \subseteq \bigcup_{i=1}^2 (\mathcal{D}_\epsilon(C) + x_i)$, let $F = \{x_1, x_2\}$ and stop. Otherwise pick a witness $x_3 \in X$ such that $x_3 \notin \bigcup_{i=1}^2 \mathcal{D}_\epsilon(C) + x_i$. Then $\mathrm{st}(|C \cap (C - x_3 + x_i)|/N) \leq \epsilon$ for $i = 1, 2$, and so also $\mathrm{st}(|(C - x_i) \cap (C - x_3)|/N) \leq \epsilon$, because $x_i/N \approx 0$. We iterate this process. We now show that the procedure must stop *before* step $k+1$. If not, one could consider the family $\{C_i \mid i = 1, \ldots, k+1\}$ where $C_i = (C - x_i) \cap [1,N]$. Clearly, $\mathrm{st}(|C_i|/N) = \mathrm{st}(|C|/N) = \gamma$ for all i, and by the previous lemma one would have $\mathrm{st}(|(C - x_i) \cap (C - x_j)|/N) > \epsilon$ for suitable $i \neq j$, a contradiction. We conclude that the cardinality of $F = \{x_i\}_{i=1}^m$ has the desired bound and $X \subseteq \mathcal{D}_\epsilon(C) + F$. □

We now use the above *nonstandard* properties to prove a general result for sets of positive density.

Theorem 3.3. *Let $BD(A) = \alpha > 0$ (or $\overline{d}(A) = \alpha > 0$), and let $0 \leq \epsilon < \alpha^2$. Then for every infinite $X \subseteq \mathbb{Z}$ and for every $x \in X$ there exists a finite subset $F \subset X$ such that:*

1. *$x \in F$;*

2. *$|F| \leq \lfloor \frac{\alpha-\epsilon}{\alpha^2-\epsilon} \rfloor$;*

3. *$X \subseteq \Delta_\epsilon(A) + F$ (or $X \subseteq \overline{\Delta}_\epsilon(A) + F$, respectively).*

Proof. By the hypothesis $\mathrm{BD}(A) = \alpha$, there exists an infinite hypernatural number $N \in {}^*\mathbb{N}$ and a hyperinteger $\Omega \in {}^*\mathbb{Z}$ such that

$$\frac{|{}^*A \cap [\Omega + 1, \Omega + N]|}{N} \approx \alpha.$$

Then $C = ({}^*A - \Omega) \cap [1, N]$ is an internal subset of $[1, N]$ with $\mathrm{st}(|C|/N) = \alpha > 0$. By Lemma 3.2, there exists a finite set $F \subset X$ with $x \in F$, $|F| \leq \lfloor (\alpha - \epsilon)/(\alpha^2 - \epsilon) \rfloor$ and such that $X \subseteq \mathcal{D}_\epsilon(C) + F$. To reach the thesis, it is now enough to show that $\mathcal{D}_\epsilon(C) \subseteq \Delta_\epsilon(A)$. To see this, take an arbitrary $t \in \mathcal{D}_\epsilon(C)$. Then

$$\begin{aligned} \mathrm{BD}(A \cap (A - t)) &\geq \mathrm{st}\left(\frac{|{}^*(A \cap (A - t)) \cap [\Omega + 1, \Omega + N]|}{N}\right) = \\ &= \mathrm{st}\left(\frac{|C \cap (C - t)|}{N}\right) > \epsilon\,. \end{aligned}$$

Under the assumption that the upper asymptotic density $\overline{d}(A) = \alpha > 0$, one applies the same argument as above where $\Omega = 0$, and obtains $\mathcal{D}_\epsilon(C) \subseteq \overline{\Delta}_\epsilon(A)$. □

As the particular case when $X = \mathbb{Z}$ and $\epsilon = 0$, the above theorem gives a small improvement of a result by I.Z. Ruzsa (*cf.* [29] Theorem 2), which was a refinement of a previous result by C.L. Stewart and R. Tijdeman [32].[4]

For $h \in \mathbb{N}$, denote by:

- $h\,B = \{h\,b \mid b \in B\}$ the set of h-multiples of elements of B;

- $B/h = \{x \mid h\,x \in B\}$ the set of integers whose h-multiples belong to B.

By taking $X = h\,\mathbb{Z}$ as the set of multiples of a number h, one gets the following.

[4] The improvement here is that under the hypothesis $\mathrm{BD}(A) = \alpha > 0$, in [29] it is proved the weaker property that $\lfloor 1/\alpha \rfloor$-many shifts of $\{t \in \mathbb{Z} \mid |A \cap (A + t)| = \infty\}$ cover $\mathbb{Z}$.

Corollary 3.4. *Let $BD(A) = \alpha > 0$ (or $\overline{d}(A) = \alpha > 0$), let $0 \leq \epsilon < \alpha^2$, and let $k = \lfloor \frac{\alpha-\epsilon}{\alpha^2-\epsilon} \rfloor$. Then for every $h \in \mathbb{Z}$ there exists a finite set $|F| \leq k$ such that $\mathbb{Z} = \Delta_\epsilon(A)/h + F$ (or $\mathbb{Z} = \overline{\Delta}_\epsilon(A)/h + F$, respectively). In consequence, $\Delta_\epsilon(A)/h$ is syndetic and $\underline{BD}(\Delta_\epsilon(A)/h) \geq 1/k$ (or $\overline{\Delta}_\epsilon(A)/h$ is syndetic and $\underline{BD}(\overline{\Delta}_\epsilon(A)/h) \geq 1/k$, respectively).*

Proof. Assume first that $\mathrm{BD}(A) = \alpha > 0$. By applying the above theorem with $X = h\mathbb{Z}$, one obtains the existence of a finite set $hF \subset h\mathbb{Z}$ with $|hF| = |F| \leq k$ and such that $h\mathbb{Z} \subseteq \Delta_\epsilon(A) + hF$.[5] But then it follows that $\mathbb{Z} = \Delta_\epsilon(A)/h + F$, and thus $\Delta_\epsilon(A)/h$ is syndetic. Finally, the last property in the statement follows by Proposition 2.1. The second part of the proof where one assumes $\overline{d}(A) = \alpha > 0$ is entirely similar. □

There are potentially many examples to illustrate consequences of Theorem 3.3. For instance, assume that a set A has Banach density $\mathrm{BD}(A) = \alpha = 1/2 + \delta$ for some $\delta > 0$. Then we can conclude that $\mathrm{BD}(A \cap (A - t)) \geq \delta + 2\delta^2$ for all $t \in \mathbb{Z}$. Indeed, given $\epsilon < \delta + 2\,\delta^2$, we have that $(\alpha - \epsilon)/(\alpha^2 - \epsilon) < 2$ and so, by taking $X = \mathbb{Z}$, it follows that $\Delta_\epsilon(A) = \mathbb{Z}$. It seems worth investigating the possibility of deriving other consequences from Theorem 3.3, by means of suitable choices of the set X.

4. Finite Embeddability

As we already remarked in Section 1, the finite embeddability relation (see Definition 1.2) preserves the finite combinatorial structure of sets, including many familiar notions considered in combinatorics of integer numbers. A first list is given below. (All proofs follow from the definitions in a straightforward manner, and are omitted.)

Proposition 4.1.

1. *A set is $\lhd$-maximal if and only if it is $\lhd_d$-maximal if and only if it is thick;*
2. *If $X \lhd Y$ and X is piecewise syndetic, then Y also is piecewise syndetic;*
3. *If $X \lhd Y$ and X contains an arithmetic progression of length k, then Y also contains an arithmetic progression of length k;*
4. *If $X \lhd_d Y$ and if X contains an arithmetic progression of length k and common distance d, then Y contains "densely-many" such arithmetic progressions,* i.e., *$BD(\{x \in \mathbb{Z} \mid x, x+d, \ldots, x+(k-1)d \in Y\}) > 0$;*

[5] We assumed $h \neq 0$. Notice that if $h = 0$, then trivially $\Delta_\epsilon(A)/h = \mathbb{Z}$ because $0 \in \Delta_\epsilon(A)$.

5. *If $X \lhd Y$, then $BD(X) \leq BD(Y)$.*

We remark that while piecewise syndeticity is preserved under $\lhd$, the property of being syndetic is *not*. Similarly, the upper Banach density is preserved or increased under $\lhd$, but the upper asymptotic density is *not*. Another list of basic properties of embeddability that are relevant to our purposes is itemized below.

Proposition 4.2.

1. *If $X \lhd Y$ and $Y \lhd Z$, then $X \lhd Z$;*
2. *If $X \lhd Y$ and $Y \lhd_d Z$, then $X \lhd_d Z$;*
3. *If $X \lhd_d Y$ and $Y \lhd Z$, then $X \lhd_d Z$;*
4. *If $X \lhd Y$, then $\Delta(X) \subseteq \Delta(Y)$;*
5. *If $X \lhd_d Y$, then $\Delta(X) \subseteq \Delta_0(Y)$;*
6. *If $X \lhd Y$ and $X' \lhd Y'$, then $X - X' \lhd Y - Y'$;*
7. *If $X \lhd_d Y$ and $X' \lhd Y'$, then $X - X' \lhd_d Y - Y'$;*
8. *If $X \lhd Y$, then $\bigcap_{t\in G}(X-t) \lhd \bigcap_{t\in G}(Y-t)$ for every finite G;*
9. *If $X \lhd_d Y$, then $\bigcap_{t\in G}(X-t) \lhd_d \bigcap_{t\in G}(Y-t)$ for every finite G;*
10. *If $X \lhd Y$, then $\Delta_\epsilon(X) \subseteq \Delta_\epsilon(Y)$ for all $\epsilon \geq 0$.*

Proof. (1) is straightforward from the definition of $\lhd$.

(2). Given a finite $F \subseteq X$, pick t such that $t + F \subseteq Y$. As the Banach density is shift invariant, we have:

$$\mathrm{BD}\left(\bigcap_{x\in F} Z - x\right) \;=\; \mathrm{BD}\left(\bigcap_{x\in F} Z - x - t\right) \;=\; \mathrm{BD}\left(\bigcap_{s\in t+F} Z - s\right) \;>\; 0.$$

(3). Given a finite $F \subseteq X$, let $A = \bigcap_{x\in F}(Y-x)$ and let $B = \bigcap_{x\in F}(Z-x)$. By the hypothesis $X \lhd_d Y$, we know that $\mathrm{BD}(A) > 0$. If we show that $A \lhd B$, then the thesis will follow from item (5) of the previous proposition. Let $G \subseteq A$ be finite; then for all $x \in F$ and for all $\xi \in G$, we have $x + \xi \in Y$, *i.e.*, $F + G \subseteq Y$. By the hypothesis $Y \lhd Z$, we can pick t such that $t + F + G \subseteq Z$, and hence $t + G \subseteq \bigcap_{x\in F}(Z-x)$, as desired.

(4). Given $x, x' \in X$, by the hypothesis we can pick a number t such that $t + \{x, x'\} \subseteq Y$. But then $x - x' = (t+x) - (t+x') \in \Delta(Y)$.

(5). For $x, x' \in X$, we have that $\mathrm{BD}(Y \cap (Y - x + x')) = \mathrm{BD}((Y - x') \cap (Y - x)) > 0$, and so $x - x' \in \Delta_0(Y)$.

(6). Given a finite $F \subseteq X - X'$, let $G \subseteq X$ and $G' \subseteq X'$ be finite sets such that $F \subseteq G - G'$. By the hypotheses, there exist t, t' such that $t + G \subseteq Y$ and $t' + G' \subseteq Y'$. Then, $(t - t') + F \subseteq (t + G) - (t' + G') \subseteq Y - Y'$.

(7). As above, given a finite $F \subseteq X - X'$, pick finite $G \subseteq X$ and $G' \subseteq X'$ such that $F \subseteq G - G'$. By the hypothesis $X \lhd_d Y$, the set $\Gamma = \{t \mid t + G \subseteq Y\}$ has positive upper Banach density; and by the hypothesis $X' \lhd Y'$, there exists an element s such that $s + G' \subseteq Y'$. For all $t \in \Gamma$, we have that $t - s + F \subseteq t - s + (G - G') = (t + G) - (t' + G') \subseteq Y - Y'$. This shows that $\Gamma - s \subseteq \{w \mid w + F \subseteq Y - Y'\}$, and we conclude that the latter set also has positive upper Banach density, as desired.

(8). Let a finite set $F \subseteq \bigcap_{t \in G}(X - t)$ be given. Notice that $F + G \subseteq X$, so we can pick an element w such that $w + (F + G) \subseteq Y$. Then, $w + F \subseteq \bigcap_{t \in G} Y - t$.

(9). Proceed as above, by noticing that the set $\{w \mid w + F \subseteq \bigcap_{t \in G} Y - t\}$ has positive Banach density, because it is a superset of $\{w \mid w + F + G \subseteq Y\}$.

(10). By property (8), it follows that $(X \cap (X - t)) \lhd (Y \cap (Y - t))$ for every t. This implies that $\mathrm{BD}(X \cap (X - t)) \leq \mathrm{BD}(Y \cap (Y - t))$, and the desired inclusion follows. $\square$

In a nonstandard setting, the finite embeddability $X \lhd Y$ amounts to the property that a (possibly infinite) shift of X is included in the hyper-extension ${}^*Y$. This notion can be also characterized in terms of *ultrafilter-shifts*, as defined by M. Beiglböck in [2].

Proposition 4.3. *Let $X, Y \subseteq \mathbb{Z}$. Then the following are equivalent:*

1. *$X \lhd Y$;*
2. *$\mu + X \subseteq {}^*Y$ for some $\mu \in {}^*\mathbb{Z}$;*
3. *There exists an ultrafilter $\mathcal{U}$ on $\mathbb{Z}$ such that X is a subset of the "$\mathcal{U}$-shift" of Y, namely $Y - \mathcal{U} = \{t \in \mathbb{Z} \mid Y - t \in \mathcal{U}\} \supseteq X$.*

Proof. (1) $\Rightarrow$ (2). Let $X = \{x_n \mid n \in \mathbb{N}\}$. By the hypothesis $X \lhd Y$, for every $n \in \mathbb{N}$, the finite intersection $\bigcap_{i=1}^{n}(Y - x_i) \neq \emptyset$. Then, by *overspill*, there exists an infinite $N \in {}^*\mathbb{N}$ such that $\bigcap_{i=1}^{N}({}^*Y - x_i)$ is non-empty. If $\mu \in {}^*\mathbb{Z}$ is any hyperinteger in that intersection, then clearly $\mu + x_i \in {}^*Y$ for all $i \in \mathbb{N}$.

(2) $\Rightarrow$ (3). Let $\mathcal{U} = \{A \subseteq \mathbb{Z} \mid \mu \in {}^*A\}$. It is readily verified that $\mathcal{U}$ is actually an ultrafilter on $\mathbb{Z}$. For every $x \in X$, by the hypothesis, $\mu + x \in {}^*Y \Rightarrow \mu \in {}^*(Y - x)$, and hence $Y - x \in \mathcal{U}$, *i.e.*, $x \in Y - \mathcal{U}$, as desired.

(3) $\Rightarrow$ (1). Given a finite $F \subseteq X$, the set $\bigcap_{x \in F}(Y - x)$ is nonempty, because it is a finite intersection of elements of $\mathcal{U}$. If $t \in \mathbb{Z}$ is any element in that intersection, then $t + F \subseteq Y$. $\square$

An interesting nonstandard property discovered by R. Jin [22] is the fact that if an internal set of hypernatural numbers $C \subseteq [1, N] \subset {}^*\mathbb{N}$ has a non-infinitesimal relative density $\mathrm{st}(|C|/N) = \gamma > 0$, then C must include a translated copy of a set $E \subseteq \mathbb{N}$ whose *Schnirelmann density* is at least γ. Below, we prove the related property that one can find a set $E \subseteq \mathbb{N}$ with Schnirelmann density at least γ, and such that "many" translated copies of its initial segments $E \cap [1, n]$ are exactly found in C.[6]

Lemma 4.4. *Let $N \in {}^*\mathbb{N}$ be an infinite hypernatural number, and let $C \subseteq [1, N]$ be an internal set with $st(|C|/N) = \gamma > 0$. Then, there exists a set $E \subseteq \mathbb{N}$ such that*

1. *The Schnirelmann density $\sigma(E) \geq \gamma$;*
2. *Every internal set $\Theta_n = \{\theta \in [1, N] \mid (C - \theta) \cap [1, n] = E \cap [1, n]\}$ is such that $st(|\Theta_n|/N) > 0$.*

Proof. For every $n \in \mathbb{N}$, let

$$\Gamma_n \;=\; \left\{\theta \in [1, N] \,\Big|\, \min_{1 \leq i \leq n} \frac{|C \cap [\theta + 1, \theta + i]|}{i} \geq \gamma\right\},$$

and let $\Lambda_n = [1, N] \setminus \Gamma_n$ be its complement. Notice that

$$\Lambda_n \;=\; \left\{\theta \in [1, N] \,\Big|\, \min_{1 \leq i \leq n} \frac{|C \cap [\theta + 1, \theta + i]|}{i} \leq \gamma_n\right\},$$

where $\gamma_n < \gamma$ is the rational number $\gamma_n = \max\{\frac{j}{i} < \gamma \mid 1 \leq i \leq n,\ 0 \leq j \leq i\}$. We define the internal map F on $[1, N]$ by putting:

$$F(\theta) \;=\; \begin{cases} 1 & \text{if } \theta \in \Gamma_n \\ s & \text{if } \theta \in \Lambda_n \text{ and } s = \min\left\{1 \leq i \leq n \,\big|\, \frac{|C \cap [\theta+1, \theta+i]|}{i} \leq \gamma_n\right\}. \end{cases}$$

By internal induction, we define a hyperfinite sequence by letting $\theta_0 = 1$, and $\theta_{m+1} = \theta_m + F(\theta_m)$ as long as $\theta_{l+1} \leq N + 1$. Notice that, since $F(\theta) \leq n$ for all θ, the set $[1, N] \setminus [\theta_0, \theta_{l+1})$ contains less than n-many elements. Then we have:

$$\begin{aligned} |C| \;&<\; |C \cap [\theta_0, \theta_{l+1})| + n \;=\; \sum_{i=0}^{l} |C \cap [\theta_i, \theta_{i+1})| + n \\ &=\; \sum_{\substack{0 \leq i \leq l \\ \theta_i \in \Gamma_n}} |C \cap [\theta_i, \theta_{i+1})| \;+\; \sum_{\substack{0 \leq i \leq l \\ \theta_i \in \Lambda_n}} |C \cap [\theta_i, \theta_{i+1})| \;+ n \\ &\leq\; \sum_{\substack{0 \leq i \leq l \\ \theta_i \in \Gamma_n}} |C \cap \{\theta_i\}| \;+\; \sum_{\substack{0 \leq i \leq l \\ \theta_i \in \Lambda_n}} |C \cap [\theta_i, \theta_{i+1})| \;+ n. \end{aligned}$$

[6] The argument used in this proof is essentially due to C.L. Stewart and R. Tijdeman (see Theorem 1 of [32]).

Now, let $X = \{\theta_i \mid i = 0, \ldots, l\}$. In the last line above, the first term equals $|C \cap X \cap \Gamma_n| \leq |X \cap \Gamma_n|$, and the second term:

$$\begin{aligned}
\sum_{\substack{0 \leq i \leq l \\ \theta_i \in \Lambda_n}} |C \cap [\theta_i, \theta_{i+1})| &\leq \sum_{\substack{0 \leq i \leq l \\ \theta_i \in \Lambda_n}} F(\theta_i) \cdot \gamma_n \\
&= \gamma_n \cdot \left(\sum_{0 \leq i \leq l} F(\theta_i) - \sum_{\substack{0 \leq i \leq l \\ \theta_i \in \Gamma_n}} 1 \right) \\
&= \gamma_n \cdot (\theta_{l+1} - 1 - |X \cap \Gamma_n|) \;\leq\; \gamma_n \cdot (N - |X \cap \Gamma_n|).
\end{aligned}$$

So, we have the inequality $|C| < M_n + \gamma_n(N - M_n) + n$ where $M_n = |X \cap \Gamma_n|$, and we obtain that:

$$\frac{|\Gamma_n|}{N} \;\geq\; \frac{M_n}{N} \;>\; \frac{|C|/N - \gamma_n - n/N}{1 - \gamma_n}.$$

Notice that the last quantity has a positive standard part. As there are 2^n-many subsets of $[1, n]$, by the *pigeonhole principle* there exists a subset $\Gamma'_n \subseteq \Gamma_n$ with $|\Gamma'_n| \geq |\Gamma_n|/2^n$, and a set $B_n \subseteq [1, n]$ with the property that $(C - \theta) \cap [1, n] = B_n$ for all $\theta \in \Gamma'_n$.

Now fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and define the set $E \subseteq \mathbb{N}$ by putting

$$n \in E \;\Leftrightarrow\; \mathcal{B}_n = \{k \geq n \mid n \in B_k\} \in \mathcal{U}.$$

We claim that E is the desired set. Given n, the following set belongs to $\mathcal{U}$, because it is a finite intersection of elements of $\mathcal{U}$:

$$\bigcap_{i \in E \cap [1,n]} \mathcal{B}_i \quad \cap \quad \bigcap_{i \in [1,n] \setminus E} \mathcal{B}_i^c \;\in\; \mathcal{U}.$$

(Notice that, since $\gamma > 0$, we have $1 \in B_k$ for all k, and so $1 \in E \cap [1, n] \neq \emptyset$.) If k is any number in the above intersection, then $B_k \cap [1, n] = E \cap [1, n]$. Moreover, for every $\theta \in \Gamma'_k$,

$$\frac{|E \cap [1, n]|}{n} \;=\; \frac{|B_k \cap [1, n]|}{n} \;\geq\; \min_{1 \leq i \leq k} \frac{|B_k \cap [1, i]|}{i} \;=\; \min_{1 \leq i \leq k} \frac{|C \cap [\theta + 1, \theta + i]|}{i} \;\geq\; \gamma.$$

This proves that $\sigma(E) \geq \gamma$. Moreover, $\theta \in \Gamma'_k \Rightarrow (C - \theta) \cap [1, k] = B_k \Rightarrow (C - \theta) \cap [1, n] = E \cap [1, n]$, and hence $\theta \in \Theta_n$. Therefore, we conclude that

$$\frac{|\Theta_n|}{N} \;\geq\; \frac{|\Gamma'_k|}{N} \;>\; \frac{|C|/N - \gamma_k - k/N}{2^k(1 - \gamma_k)},$$

where the standard part of the last quantity is $\frac{\gamma - \gamma_k}{2^k(1-\gamma_k)} > 0$. □

In consequence of the previous *nonstandard* lemma, we obtain an embeddability property that holds for all sets of positive density. It is a small refinement of a result by V. Bergelson [4], which improved on a previous result by C.L. Stewart and R. Tijdeman [33].[7]

Theorem 4.5 (*Cf.* [4] Theorem 2.2; [33] Theorem 1). *Let $BD(A) = \alpha > 0$. Then there exists a set $E \subseteq \mathbb{N}$ such that:*

1. $\sigma(E) \geq \alpha$.
2. $E \lhd_d A$*, and hence* $\Delta(E) \subseteq \Delta_0(A)$ *and* $\Delta_\epsilon(E) \subseteq \Delta_\epsilon(A)$ *for all* $\epsilon \geq 0$.

Proof. Pick an infinite interval $[\Omega+1, \Omega+N]$ such that $|{}^*A \cap [\Omega+1, \Omega+N]|/N \approx \alpha$. By applying the above theorem where $C = ({}^*A - \Omega) \cap [1, N]$, one gets the existence of a set $E \subseteq \mathbb{N}$ such that $\sigma(E) \geq \alpha$ and $\mathrm{st}(|\Theta_n|/N) > 0$ for all n, where

$$\Theta_n \;=\; \{\theta \in [1, N] \mid ({}^*A - \Omega - \theta) \cap [1, n] = E \cap [1, n]\}\,.$$

Now, given a finite $F \subseteq E \cap [1, n]$ and given an element $e \in F$, for every $\theta \in \Theta_n$ we have $\Omega + \theta + e \in {}^*A$. This shows that $\Omega + \Theta_n \subseteq \bigcap_{e \in F} {}^*(A - e) \cap [\Omega+1, \Omega+N]$. But then

$$\begin{aligned} \mathrm{BD}\left(\bigcap_{e \in F} (A - e)\right) \;&\geq\; \mathrm{st}\left(\frac{|\bigcap_{e \in F} {}^*(A-e) \cap [\Omega+1, \Omega+N]|}{N}\right) \\ &\geq\; \mathrm{st}\left(\frac{|\Omega + \Theta_n|}{N}\right) \;=\; \mathrm{st}\left(\frac{|\Theta_n|}{N}\right) \;>\; 0. \end{aligned}$$

□

5. Difference Sets $A - B$

In this final section, we generalize the results of Section 3 by considering sets of differences $A - B$ where $A \neq B$. We remark that, while $\Delta(A) = A - A$ is syndetic whenever A has a positive upper Banach density, the same property does not extend to the case of difference sets $A - B$ where $A \neq B$. (*E.g.*, it is not hard to construct thick sets A, B, C such that their complements A^c, B^c, C^c are thick as well, and $A - B \subset C$.)

We shall use the following elementary inequality.

Lemma 5.1. *Let $C \subseteq [1, N]$ and $D \subseteq [1, \nu]$ be sets of natural numbers. Then there exists $1 \leq \overline{x} \leq N$ such that*

$$\frac{|(C - \overline{x}) \cap D|}{\nu} \;\geq\; \frac{|C|}{N} \cdot \frac{|D|}{\nu} \;-\; \frac{|D|}{N}.$$

[7] The improvement here is that we have $\sigma(E) \geq \alpha$ instead of $\overline{d}(E) \geq \alpha$.

Proof. Let $\chi_C : [1, N] \to \{0, 1\}$ be the characteristic function of C. For every $d \in D$, we have

$$\frac{1}{N} \cdot \sum_{x=1}^{N} \chi_C(x+d) \;=\; \frac{|C \cap [1+d, N+d]|}{N} \;=\; \frac{|C|}{N} + \frac{e(d)}{N}$$

where $|e(d)| \leq d$. Then:

$$\frac{1}{N} \cdot \sum_{x=1}^{N} \left(\frac{1}{\nu} \cdot \sum_{d \in D} \chi_C(x+d) \right) \;=\; \frac{1}{\nu} \cdot \sum_{d \in D} \left(\frac{1}{N} \cdot \sum_{x=1}^{N} \chi_C(x+d) \right) \;=$$

$$= \; \frac{1}{\nu} \cdot \sum_{d \in D} \frac{|C|}{N} \;+\; \frac{1}{N \cdot \nu} \cdot \sum_{d \in D} e(d) \;=\; \frac{|C|}{N} \cdot \frac{|D|}{\nu} \;+\; e$$

where

$$|e| \;=\; \left| \frac{1}{N \cdot \nu} \sum_{d \in D} e(d) \right| \;\leq\; \frac{1}{N \cdot \nu} \sum_{d \in D} |e(d)| \;\leq\; \frac{1}{N \cdot \nu} \cdot \sum_{d \in D} d \;\leq\; \frac{1}{N \cdot \nu} \sum_{d \in D} \nu \;=\; \frac{|D|}{N}.$$

By the *pigeonhole principle*, there must exist at least one number $1 \leq \overline{x} \leq N$ such that

$$\frac{1}{\nu} \cdot \sum_{d \in D} \chi_C(\overline{x}+d) \;\geq\; \frac{|C|}{N} \cdot \frac{|D|}{\nu} \;-\; \frac{|D|}{N}.$$

The thesis is reached by noticing that

$$\frac{1}{\nu} \cdot \sum_{d \in D} \chi_C(\overline{x}+d) \;=\; \frac{|(D+\overline{x}) \cap C|}{\nu} \;=\; \frac{|(C-\overline{x}) \cap D|}{\nu}.$$

□

We are now ready to prove the main result of this paper.

Theorem 5.2. *Let $BD(A) = \alpha > 0$ and $BD(B) = \beta > 0$. Then there exists a set $E \subseteq \mathbb{N}$ such that:*

1. *The Schnirelmann density $\sigma(E) \geq \alpha\beta$;*
2. *For every finite $F \subset E$ there exists $\epsilon > 0$ such that for arbitrarily large intervals J one finds a suitable shift $A_J = A - t_J$ with the property that*

$$\frac{\left| \left(\bigcap_{e \in F} (A_J \cap B) - e \right) \cap J \right|}{|J|} \;\geq\; \epsilon;$$

3. *Both $E \lhd_d A$ and $E \lhd_d B$, and hence:*

- $\Delta(E) \subseteq \Delta_0(A) \cap \Delta_0(B)$*;*
- $\Delta_\epsilon(E) \subseteq \Delta_\epsilon(A) \cap \Delta_\epsilon(B)$ *for all* $\epsilon \geq 0$*;*
- $\Delta(E) \lhd_d A - B$.

Proof. (1). Fix $\nu, N \in {}^*\mathbb{N}$ infinite numbers with $\nu/N \approx 0$, and pick intervals $[\Omega+1, \Omega+N]$ and $[\Xi+1, \Xi+\nu]$ of length N and ν respectively, such that

$$\frac{|{}^*A \cap [\Omega+1, \Omega+N]}{N} \approx \alpha \quad \text{and} \quad \frac{|{}^*B \cap [\Xi+1, \Xi+\nu]|}{\nu} \approx \beta.$$

Then consider the internal sets

- $C = ({}^*A - \Omega) \cap [1, N]$;
- $D = ({}^*B - \Xi) \cap [1, \nu]$.

Clearly, $|C|/N \approx \alpha$ and $|D|/\nu \approx \beta$. The property of Lemma 5.1 *transfers* to the internal sets $C \subseteq [1, N]$ and $D \subseteq [1, \nu]$, and so we can pick a hyperinteger element $1 \leq \zeta \leq N$ such that

$$\frac{|(C-\zeta) \cap D|}{\nu} \geq \frac{|C|}{N} \cdot \frac{|D|}{\nu} - \frac{|D|}{N}.$$

Now let $W = (C - \zeta) \cap D \subseteq [1, \nu]$. Since $|D|/N \leq \nu/N \approx 0$, we have that

$$\gamma = \mathrm{st}\left(\frac{|W|}{\nu}\right) \geq \mathrm{st}\left(\frac{|C|}{N} \cdot \frac{|D|}{\nu}\right) = \mathrm{st}\left(\frac{|C|}{N}\right) \cdot \mathrm{st}\left(\frac{|D|}{\nu}\right) = \alpha \cdot \beta.$$

By applying Theorem 4.5 to the internal set $W \subseteq [1, \nu]$, one gets the existence of a set $E \subseteq \mathbb{N}$ that satisfies the following properties:

- $\sigma(E) \geq \gamma \geq \alpha\beta$;
- For every n, the internal set $\Theta_n = \{\theta \in [1, \nu] \mid (W - \theta) \cap [1, n] = E \cap [1, n]\}$ is such that $\mathrm{st}(|\Theta_n|/\nu) > 0$.

(2). Given a finite set $F = \{e_1 < \ldots < e_k\} \subseteq E \cap [1, n]$, for every $\theta \in \Theta_n$ and for every i, we have that $\theta + e_i \in W = ({}^*A - \Omega - \zeta) \cap ({}^*B - \Xi) \cap [1, \nu]$, and so

$$\Xi + \Theta_n \subseteq \bigcap_{i=1}^{k} [(({}^*A - \mu) \cap {}^*B) - e_i] \cap I$$

where $\mu = \Omega + \zeta - \Xi$ and $I = [\Xi + 1, \Xi + \nu]$. Then,

$$(\star) \qquad \mathrm{st}\left(\frac{|\bigcap_{i=1}^{k} [(({}^*A - \mu) \cap {}^*B) - e_i] \cap I|}{|I|}\right) \geq \mathrm{st}(|\Theta_n|/\nu) = \epsilon > 0.$$

We now want to extract a standard property out of the above nonstandard inequality ($\star$). Notice that, since $|I| = \nu$ is infinite, the following is true for every fixed $m \in \mathbb{N}$:

$$\exists I \subset {}^*\mathbb{Z} \text{ interval s.t. } |I| > m \ \& \ \exists \mu \in {}^*\mathbb{Z} \ \text{ s.t. } \frac{|\bigcap_{i=1}^k [(({}^*A - \mu) \cap {}^*B) - e_i] \cap I|}{|I|} \ \geq \ \epsilon.$$

By *transfer*, we obtain the existence of an interval $J \subset \mathbb{Z}$ of length $|J| > m$ and of an element $t_J \in \mathbb{Z}$ that satisfies:

$$\frac{|\bigcap_{i=1}^k [((A - t_J) \cap B) - e_i] \cap J|}{|J|} \ \geq \ \epsilon.$$

(3). With the same notation as above, by ($\star$) one directly gets that

$$\text{st}\left(\frac{|\bigcap_{i=1}^k ({}^*A - e_i) \cap I'|}{|I'|}\right) \ \geq \ \epsilon \quad \text{and} \quad \text{st}\left(\frac{|\bigcap_{i=1}^k ({}^*B - e_i) \cap I|}{|I|}\right) \ \geq \epsilon,$$

where $I' = \mu + I = [\Omega + \zeta + 1, \Omega + \zeta + \nu]$. Since ${}^*\left(\bigcap_{i=1}^k (A - e_i)\right) = \bigcap_{i=1}^k ({}^*A - e_i)$, by the nonstandard characterization of the upper Banach density, we obtain the thesis $\mathrm{BD}\left(\bigcap_{e \in F} (A - e)\right) > 0$. The other inequality $\mathrm{BD}\left(\bigcap_{e \in F} (B - e)\right) > 0$ is proved in the same way. □

By a recent result obtained by M. Beiglböck, V. Bergelson and A. Fish in the general context of countable amenable groups (see [1] Proposition 4.1.), one gets the existence of a set E of positive upper Banach density with the property that $\Delta(E) \lhd A - B$. Afterwards, M. Beiglböck found a short ultrafilter proof of that property, with the refinement that one can take $\mathrm{BD}(E) \geq \alpha\beta$. Our improvement here is that one can assume also the Schnirelmann density $\sigma(E) \geq \alpha\beta$, and that there are *dense* embeddings $E \lhd_d A$ and $E \lhd_d B$ (and hence, a dense embedding $\Delta(E) \lhd_d A - B$).

As a first corollary to our main theorem, we obtain a sharpening of a result by I.Z. Rusza [29] about intersections of difference sets, which improved on a previous result by C.L. Stewart and R. Tijedman [32].

Corollary 5.3 (*cf.* [29] Theorem 1; [33] Theorem 4). *Assume $A_1, \ldots, A_n \subseteq \mathbb{Z}$ have positive upper Banach densities $BD(A_i) = \alpha_i$. Then there exists a set $E \subseteq \mathbb{N}$ with $\sigma(E) \geq \prod_{i=1}^n \alpha_i$ and such that $\Delta_\epsilon(E) \subseteq \bigcap_{i=1}^n \Delta_\epsilon(A_i)$ for every $\epsilon \geq 0$.*

Proof. We proceed by induction on n. The basis $n = 1$ is given by Theorem 4.5. At step $n + 1$, by the inductive hypothesis we can pick a set $E' \subseteq \mathbb{N}$ such that $\sigma(E') \geq \prod_{i=1}^n \alpha_i$ and $\Delta_\epsilon(E') \subseteq \bigcap_{i=1}^n \Delta_\epsilon(A_i)$. Now apply the above theorem to the sets E' and A_{n+1}, and obtain the existence of a set $E \subseteq \mathbb{N}$ whose Schnirelmann density $\sigma(E) \geq \mathrm{BD}(E') \cdot \mathrm{BD}(A_{n+1}) \geq \prod_{i=1}^{n+1} \alpha_i$, and such that $\Delta_\epsilon(E) \subseteq \Delta_\epsilon(E') \cap \Delta_\epsilon(A_{n+1}) \subseteq \bigcap_{i=1}^{n+1} \Delta_\epsilon(A_i)$, as desired. □

Two more corollaries are obtained by combining Theorem 5.2 with Theorem 3.3.

Corollary 5.4. *Assume that $BD(A) = \alpha > 0$ and $BD(B) = \beta > 0$. Then for every $0 \leq \epsilon < \alpha^2\beta^2$, for every infinite $X \subseteq \mathbb{Z}$, and for every $x \in X$, there exists a finite subset $F \subset X$ such that*

1. $x \in F$;
2. $|F| \leq \lfloor \frac{\alpha\beta-\epsilon}{\alpha^2\beta^2-\epsilon} \rfloor = k$;
3. $X \subseteq (\Delta_\epsilon(A) \cap \Delta_\epsilon(B)) + F$.

In consequence, for every h, the intersection $\Delta_\epsilon(A)/h \cap \Delta_\epsilon(B)/h$ is syndetic and has a lower Banach density not smaller than $1/k$.

Proof. Pick a set $E \subseteq \mathbb{N}$ as given by Theorem 5.2. As $\overline{d}(E) \geq \sigma(E) \geq \alpha\beta$, we can apply Theorem 3.3 and obtain the existence of a finite $F \subset X$ such that $x \in F$, $|F| \leq k$, and $X \subseteq \Delta_\epsilon(E) + F$ (in fact, $X \subseteq \overline{\Delta}_\epsilon(E) + F$). As $E \lhd A$ and $E \lhd B$ (in fact, $E \lhd_d A$ and $E \lhd_d B$), we have the inclusion $\Delta_\epsilon(E) \subseteq \Delta_\epsilon(A) \cap \Delta_\epsilon(B)$, and the thesis follows. Finally, by taking as $X = h\,\mathbb{Z}$ the set of h-multiples, one obtains that $\mathbb{Z} = (\Delta_\epsilon(A)/h \cap \Delta_\epsilon(B)/h) + G$ for a suitable $|G| \leq k$ and so, by Proposition 2.1, the last statement in the corollary is also proved. ☐

Corollary 5.5. *Assume that $BD(A) = \alpha > 0$ and $BD(B) = \beta > 0$. Then for every infinite $X \subseteq \mathbb{Z}$ and for every $x \in X$, there exists a finite subset $F \subset X$ such that*

1. $x \in F$;
2. $|F| \leq \lfloor \frac{1}{\alpha\beta} \rfloor$;
3. $X \lhd_d (A - B) + F$.

Proof. With the same notation as in the proof of the previous corollary, let $\epsilon = 0$, and pick a set $E \subseteq \mathbb{N}$ such that $\sigma(E) \geq \alpha\beta$, and a set F such that $|F| \leq \lfloor 1/\alpha\beta \rfloor$ and $X \subseteq \Delta_0(E) + F$. Now, $E \lhd_d A$ and $E \lhd_d B$ imply that $\Delta(E) \lhd_d A - B$, which in turn implies that $\Delta(E) + F \lhd_d (A - B) + F$. As $\Delta_0(E) \subseteq \Delta(E)$, we can conclude that $X \subseteq \Delta(E) + F \lhd_d (A - B) + F$. ☐

By the above result where $X = \mathbb{Z}$, one can improve Jin's theorem about the piecewise syndeticity of a difference set, by giving a precise bound on the number of shifts of $A - B$ which are needed to cover a thick set.

Corollary 5.6 (*cf.* [21] Corollary 3). *Assume that $BD(A) = \alpha > 0$ and $BD(B) = \beta > 0$, and let $k = \lfloor 1/\alpha\beta \rfloor$. Then there exists a finite set $|F| \leq k$ such that $A - B + F$ is thick, and hence $A - B$ is piecewise syndetic.*

Proof. Apply the above Corollary with $X = \mathbb{Z}$, and recall that $\mathbb{Z} \lhd_d Y$ if and only if Y is $\lhd_d$-maximal if and only if Y is thick. $\square$

We remark that the above corollary implies the same property for sumsets $A+B$ with $A, B \subseteq \mathbb{N}$.

Bohr sets are commonly used in applications of *Fourier analysis* in combinatorial number theory. We recall that $A \subseteq \mathbb{Z}$ is called a *Bohr set* if it contains a non-empty open set of the topology induced by the embedding into the *Bohr compactification* of the discrete topological group $(\mathbb{Z}, +)$. The following characterization holds: A is a Bohr set if and only if there exist $r_1, \ldots, r_k \in [0, 1)$ and a positive $\epsilon > 0$ such that a shift of $\{x \in \mathbb{Z} \mid \|r_1 \cdot x\|, \ldots, \|r_k \cdot x\| < \epsilon\}$ is included in A, where $\|z\|$ denotes the distance of z from the nearest integer. A set is *piecewise Bohr* if it is the intersection of a Bohr set with a thick set. We remark that Bohr sets are syndetic, and hence piecewise Bohr sets are piecewise syndetic, but there are syndetic sets that are not piecewise Bohr. (For a proof of this fact, and for more information about Bohr sets, we refer the reader to [9] and references therein.)

As a consequence of Theorem 5.2, one can recover also the following theorem by V. Bergelson, H. Fürstenberg and B. Weiss about the Bohr property of difference sets.

Corollary 5.7 (*cf.* [9] Theorem I). *Let A and B have positive upper Banach density. Then the difference set $A - B$ is piecewise Bohr.*

Proof. By Theorem 5.2, we can pick a set $E \subseteq \mathbb{N}$ with $\sigma(E) \geq \alpha\beta > 0$ and such that $\Delta(E) \lhd_d A - B$. Then apply Proposition 4.1 of [1], where it was shown that if $\Delta(E) \lhd A - B$ for some set E of positive upper Banach density, then $A - B$ is piecewise Bohr. $\square$

As a final remark, we point out that the nonstandard methods used in this paper for sets of integers, work also in more abstract settings. Indeed, many of the results presented here can be extended to the general framework of *amenable groups* (see [13]).

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