ON THE PRIME K-TUPLE CONJECTURE

R. Thangadurai[1]
Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad-211019, India thanga@hri.res.in

G. K. Viswanadham
Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad-211019, India viswanath@hri.res.in

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Abstract

We show that the qualitative form of the prime k-tuple conjecture implies a certain conjecture related to the least size of the interval containing exactly ℓ prime numbers. Also, we prove that the latter conjecture implies the prime k-tuple conjecture for some minimal admissible set. The proof is self-contained and elementary.

1. Introduction

Since 2 and 3 are consecutive primes, the least size of an interval containing exactly two prime numbers is one. Since these are the only primes placed consecutively, by omitting this pair, the least size of an interval containing exactly two prime numbers is two. Since the tuples $(2, 3, 5)$ and $(3, 5, 7)$ are the only prime tuples contained in an the interval of length not exceeding four, by omitting these tuples, we see that the least size of an interval having exactly three primes is six (for instance, the interval $[5, 11]$ contains exactly three primes). In general, we pose the following question.

Question 1: For a given positive integer $\ell \geq 2$, what is the least size of an interval which contains exactly ℓ prime numbers such that each of them is greater than ℓ?

A recent result in [1] motivates the above question, though, in [1], the authors deal with a folklore conjecture on the number of prime factors of the product of k consecutive integers.

To answer Question 1, we need to find an asymptotic formula for the following function. We first enumerate those primes greater than ℓ as $q_1 < q_2 < \cdots < q_n < \cdots$ where q_n denotes the n-th prime greater than ℓ. We define a function

$$\kappa_\ell := \min_j \{q_{j+\ell-1} - q_j \mid j = 1, 2, \ldots\}.$$

[1]Dedicated to Professor R. Balasubramanian

Various authors, namely, P. Erdős [7], [12], [9], [10] and [11], D. Hensley and I. Richards [5], C. Hooley [6], H. L. Montgomery and R. C. Vaughan [13] and T. Vijayaraghavan [14] were considered an interesting sequence of integers as follows.

For a given positive integer $\ell \geq 2$, we first let $A_\ell = \prod_{p \leq \ell} p$ and we enumerate all the positive integers greater than 1 which are coprime to A_ℓ as $a_1 < a_2 < \cdots < a_n < \cdots$. In [9], P. Erdős mentions that this sequence might be expected to show behavior that is somewhat similar to that of the primes. Most of the papers mentioned above deal with the maximum gap between the consecutive elements of the sequence of a_i's. Here, we consider the complementary problem and define

$$\lambda_\ell := \min_j \{a_{j+\ell-1} - a_j \mid j = 1, 2, \ldots\}.$$

Since each prime $q_j > \ell$ is co-prime to A_ℓ, it is clear that $\lambda_\ell \leq \kappa_\ell$ for all $\ell \geq 2$. By the explicit computations, one can easily see that $\lambda_\ell = \kappa_\ell$ for all $\ell \leq 10$.

In terms of λ_ℓ and κ_ℓ, the twin prime conjecture asserts that $\lambda_2 = \kappa_2$ and there are infinitely many intervals of length 2 which contain two prime numbers. Now we can generalize the twin prime conjecture as follows.

Conjecture 1: Let $\ell \geq 2$ be a given integer. Then we have $\lambda_\ell = \kappa_\ell$, and there are infinitely many intervals of length κ_ℓ having exactly ℓ prime numbers.

Remark: Conjecture 1 has a connection to the recent results on the prime differences by D. Goldston, J. Pintz and C Yildirim [3] and Y. Zhang [15]. In [3], under the Elliott - Halberstam Conjecture, the authors proved that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 16,$$

where p_n denotes the n^{th} prime. In [15], Zhang proved unconditionally that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) \leq 7 \times 10^7.$$

Conjecture 1 implies the more general case as follows: for any integer $\ell \geq 2$, we get

$$\liminf_{n\to\infty}(p_{n+\ell-1} - p_n) = \kappa_\ell.$$

The "simple-minded" Conjecture 1 is indeed connected with the prime k-tuple conjecture formulated first by Dickson (qualitative version) and then later by Hardy and Littlewood (quantitative version).

For any given integer $k \geq 2$, let $\mathcal{H}$ be a set of non-negative integers with $|\mathcal{H}| = k$. For any prime p, we let

$$\nu_{\mathcal{H}}(p) = \#\{h (\text{mod } p) : h \in \mathcal{H}\}.$$

Clearly, $\nu_{\mathcal{H}}(p)$ counts the number of distinct residue classes $h \in \mathcal{H}$ modulo p. A finite subset $\mathcal{A} \subset \mathbb{N} \cup \{0\}$ is said to be *admissible* if $\nu_{\mathcal{A}}(p) < p$ for every prime p.

Dickson [4] formulated the following conjecture.

Conjecture 2. (Prime k-tuple conjecture): For any given integer $k \geq 2$, let $\mathcal{H}$ be an admissible set with $|\mathcal{H}| = k$. Then there exist infinitely many integers n such that $n + h$ is a prime for every $h \in \mathcal{H}$.

Conjecture 2 is wide open. When $k = 2$ and $\mathcal{H} = \{0, 2\}$, Conjecture 2 is the twin-prime conjecture. In this article, we study the relationships between the two generalizations of the twin prime conjecture, namely Conjecture 1, and the prime k- tuple conjecture.

P. Erdős and H. Riesel [12], and T. Forbes [2], computed some numbers connected with the prime k-tuple conjecture. More precisely, for a given integer $\ell \geq 2$, they computed the integer $s(\ell)$ which is defined to be the least positive integer s for which there exists an admissible set $\mathcal{H} := \{0 = b_1, b_2, \ldots, b_\ell = s\}$ with $|\mathcal{H}| = \ell$. We call such an admissible set $\mathcal{H}$ a *minimal admissible* set. The following is a weak form of Conjecture 2.

Conjecture 3. For a given integer $\ell \geq 2$, let $\mathcal{H}$ be a minimal admissible set with $|\mathcal{H}| = \ell$. Then there exist infinitely many integers n such that the set $n + \mathcal{H}$ consists of only prime numbers.

In this article, we shall prove the following results.

Theorem 1. *For any given integers $k, \ell \geq 2$, let $\mathcal{H}$ be an admissible set with $|\mathcal{H}| = k$ and $\mathcal{A}_\ell = \prod_{p \leq \ell} p$. Then there exist infinitely many integers n such that $(n + h, \mathcal{A}_\ell) = 1$ for every $h \in \mathcal{H}$.*

In answering Question 1, we prove the following results.

Theorem 2. *For every integer $\ell \geq 2$, we have $s(\ell) = \lambda_\ell$.*

Theorem 3. *Conjecture 3 implies Conjecture 1.*

Theorem 4. *If Conjecture 1 is true, then for any given $\ell \geq 2$, there exists a minimal admissible set for which Conjecture 3 is true.*

Theorem 5. *For every large integer ℓ, we have*

$$\frac{\ell \log \ell}{2} \leq \lambda_\ell \leq \left(1 + O\left(\frac{\log \log \ell}{\log^2 \ell}\right)\right) \ell \log \ell$$

and

$$\frac{\ell \log \ell}{2} \leq \kappa_\ell \leq (1 + \epsilon) \ell \log \ell$$

for any given $\epsilon > 0$.

Remarks. (1) It is clear that Conjecture 2 implies Conjecture 3 and by Theorem 3, Conjecture 3 implies Conjecture 1. (2) We believe that Conjecture 1 implies Conjecture 3, but our method is unable to prove this implication.

In the last section, we list the exact values of κ_ℓ for $\ell \leq 10$ and λ_ℓ for $\ell \leq 28$.

2. Proofs

Proof of Theorem 1. To motivate the proof of this theorem, first we shall prove the case $\ell = 2$. Let $\mathcal{H} = \{b_1, b_2, \ldots, b_k\}$ be the given admissible set. We want to prove that there are infinitely many integers n such that $n + b_i$ is coprime to $\mathcal{A}_2 = 2$. If b_1 is even (respectively, odd), then so is b_i for all i. So we choose all the natural numbers n which are of opposite parity with the b_i's. Then, clearly, the $(n + b_i)$'s are coprime to $2 = \mathcal{A}_2$.

Now we shall assume that $\ell > 2$. Let $\mathcal{H} = \{b_1, b_2, \ldots, b_k\}$ be the given admissible set. Let $2 = p_1, p_2, \ldots, p_r$ be all primes $p \leq \ell$. Since $\mathcal{H}$ is admissible, we have

$$\nu_{\mathcal{H}}(p_i) \ \leq \ p_i - 1 \ \text{ for all } \ i = 1, 2, \ldots, r.$$

Hence, for any $n \in \mathbb{N}$,

$$\nu_{\mathcal{H}}(p_i) = \#\{n + b_j (\text{mod } p_i) : j = 1, 2, \ldots, k\} \leq p_i - 1.$$

Since $\nu_{\mathcal{H}}(p_i) < p_i$ for all $i = 1, 2, \ldots, r$, for any fixed i satisfying $1 \leq i \leq r$, there exists an integer n_i such that $0 (\text{mod } p_i) \notin \{n_i + b_j (\text{mod } p_i) \ / \ j = 1, 2, \ldots, k\}$.

By the Chinese Remainder Theorem, the following set of congruences has a solution modulo A_ℓ

$$\begin{aligned} x &\equiv n_1 \pmod{p_1} \\ x &\equiv n_2 \pmod{p_2} \\ &\vdots \\ x &\equiv n_r \pmod{p_r}. \end{aligned}$$

Let the least solution be n. Then n satisfies

$$n + b_j \not\equiv 0 (\text{mod } p_i), \quad \text{for all } i = 1, \ldots, r, \quad \text{and for all } j = 1, \ldots, k.$$

Therefore, for this integer n, we have $(n + b_j, A_\ell) = 1$ for every $j = 1, \ldots, k$. Also $n + kA_\ell$ satisfies the above congruence for each $k \in \mathbb{N}$. Thus there are infinitely many natural numbers n such that $(n + b_j, A_\ell) = 1$ for every $j = 1, \ldots, k$. □

Remark: Let $N_\ell(x)$ denote the number of positive integers $n \leq x$ such that $(n + h, A_\ell) = 1$ for every $h \in \mathcal{H}$. Then, we have

$$N_\ell(x) \asymp x \prod_{p \leq \ell} \left(1 - \frac{\nu_{\mathcal{H}}(p)}{p}\right).$$

To prove this, for each prime $p \le \ell$, we choose h'_p such that $h'_p \notin \mathcal{H} \pmod p$. By the Chinese Reminder theorem, the set of simultaneous congruences

$$\begin{array}{rcll} y & \equiv & -h'_{p_1} & \pmod{p_1} \\ \vdots & \vdots & \vdots & \\ y & \equiv & -h'_{p_{\pi(\ell)}} & \pmod{p_{\pi(\ell)}} \end{array}$$

has a unique solution say, $n \pmod{A_\ell}$. Then we can see that $n \in N_\ell(x)$. Conversely, every integer $n \in N_\ell(x)$ is a solution to a system of congruences of the above type. Hence

$$N_\ell(x) \asymp \frac{x}{A_\ell} \prod_{p \le \ell} \left(p - \nu_{\mathcal{H}(p)}\right) = x \prod_{p \le \ell} \left(1 - \frac{\nu_{\mathcal{H}}(p)}{p}\right).$$

Proof of Theorem 2. Let $\mathcal{H} := \{0, h_1, h_2, \ldots, h_{\ell-1} = s(\ell)\}$ be a minimal admissible set. By Theorem 1, there exists an integer n such that all the elements of the set $\{n, n + h_1, n + h_2, \ldots, n + s(\ell)\}$ are relatively prime to A_ℓ. Therefore, we have $\lambda_\ell \le s(\ell)$.

By the definition of λ_ℓ, there exists an integer n such that the interval $[n, n+\lambda_\ell]$ contains exactly ℓ numbers, say, $n, n + x(1), \ldots, n + x(\ell - 1) = n + \lambda_\ell$ that are relatively prime to A_ℓ. We will prove that $\mathcal{H} := \{0, x(1), x(2), \ldots, x(\ell - 1)\}$ is an admissible set.

So, it is enough to prove that $\nu_{\mathcal{H}}(p) < p$ for all prime $p \le \ell$. Suppose the statement is not true. Then there is a prime $p \le \ell$ such that $\nu_{\mathcal{H}}(p) = p$. That is, the number $\#\{0, x(i) \pmod p : 1 \le i \le \ell - 1\}$ equals p, which implies that for any integer n we have $\#\{n, n+x(i) \pmod p : 1 \le i \le \ell-1\} = p$. Therefore, there exists i_0 with $1 \le i_0 \le \lambda_\ell$ such that $n+x(i_0) \equiv 0 \pmod p$. Hence, $(n+x(i_0), A_\ell) \ge p > 1$ for any integer n, which is a contradiction. Therefore, $\mathcal{H}$ is an admissible set of cardinality ℓ and the largest element of $\mathcal{H}$ is λ_ℓ. Therefore, by the definition of $s(\ell)$, we get $s(\ell) \le \lambda_\ell$, which proves the theorem. □

Proof of Theorem 3. Assume that Conjecture 3 is true. We need to prove that $\lambda_\ell = \kappa_\ell$ and there are infinitely many intervals of length κ_ℓ having exactly ℓ primes.

We always have $\lambda_\ell \le \kappa_\ell$ for all integers $\ell \ge 2$. To prove $\kappa_\ell \le \lambda_\ell$, we need to prove the existance of ℓ prime numbers $p_1, p_2, \ldots, p_\ell$ such that $p_\ell - p_1 \le \lambda_\ell$.

By the definition of λ_ℓ, there exists a positive integer j such that $\lambda_\ell = a_{j+\ell-1} - a_j$. That is, the interval $[a_j, a_j + \lambda_\ell] = [a_j, a_{j+\ell-1}]$ contains exactly ℓ numbers, namely, $a_j, a_{j+1}, \ldots, a_{j+l-1}$ which are coprime to A_ℓ. We shall rewrite these consecutive integers as $a, a+1, \ldots, a+\lambda_\ell$ with $a = a_j$. Also, for each $i = 0, 1, 2, \ldots, \ell-1$, let $a+x(i) = a_{j+i}$. Note that $x(0) = 0$. Clearly, by letting $\mathcal{H} := \{0, x(1), \ldots, x(\ell-1)\} \subset \{1, 2, \ldots, \lambda_\ell\}$, we see that each $x(i)$ is an even integer for all $i = 1, 2, \ldots, \ell - 1$.

If we can find an integer $n > \ell$ such that $n, n+x(1), n+x(2), \ldots, n+x(\ell-1)$ are all prime numbers, then, as $n + x(\ell - 1) - n = x(\ell - 1) = \lambda_\ell$, we get $\kappa_\ell \le \lambda_\ell$.

To do this, we shall apply Conjecture 3 for $\mathcal{H}$. For this, we need to prove $\mathcal{H}$ is a minimal admissible set.

Obviously, for all prime $p > \ell$, we have $\nu_{\mathcal{H}}(p) < p$ as $|\mathcal{H}| = \ell$. Hence, let p be a prime such that $p \leq \ell$ and $\nu_{\mathcal{H}}(p) = p$. Then, $\#\{0, x(i) \pmod p \ / \ 1 \leq i \leq \ell - 1\} = p$ which implies that $\#\{a, a + x(i) \pmod p \ / \ 1 \leq i \leq \ell - 1\} = p$. That is, there exists i_0 such that $a + x(i_0) \equiv 0 \pmod p$ and hence, $(a + x(i_0), A_\ell) \geq p > 1$, a contradiction. Hence $\nu_{\mathcal{H}}(p) < p$ for every prime p. Since $x(\ell - 1) = \lambda_\ell = s(\ell)$ (by Theorem 2), we conclude that $\mathcal{H}$ is a minimal admissible set. Thus, by Conjecture 3, there exists a positive integer $n > \ell$ such that $n, n + x(1), \ldots, n + x(\ell - 1)$ are all prime numbers. Hence, we get $\lambda_\ell = \kappa_\ell$.

Let I be an interval of length κ_ℓ having exactly ℓ primes, say, $p, p + y(1), p + y(2), \ldots, p + y(\ell - 1)$. By arguing as above, we can show that the set $\mathcal{H} := \{0, y(1), \ldots, y(\ell - 1)\}$ is a minimal admissible set. Therefore, by Conjecture 3, there exist infinitely many translates of I (which are intervals) having ℓ primes. Since $\lambda_\ell = \kappa_\ell$, no interval can contain more than ℓ primes. Hence Conjecture 1 is true. □

Proof of Theorem 4. Assume Conjecture 1 is true. Let I_ℓ be an interval of length κ_ℓ which contains exactly ℓ prime numbers. By the definition of κ_ℓ, the interval I_ℓ contain exactly ℓ prime numbers and they must be of the form

$$p, p + x(1), p + x(2), \ldots, p + x(\ell - 1) = p + \kappa_\ell \text{ for some prime } p.$$

As in the proof of Theorem 2, we can show that $\mathcal{H} := \{0, x(1), x(2), \ldots, x(\ell - 1) = \kappa_\ell\}$ is an admissible set of cardinality ℓ. Since $\lambda_\ell = \kappa_\ell$, by Theorem 2, we have $\kappa_\ell = s(\ell)$. Thus, by the definition, $\mathcal{H}$ is a minimal admissible set of cardinality ℓ for which there is an integer n, namely, p such that $n, n + x(1), \ldots, n + \kappa_\ell$ are primes. By the assumption, there are infinitely many intervals of length κ_ℓ that contain exactly ℓ prime numbers. That is, there are infinitely many primes p such that

$$p, p + x_p(1), p + x_p(2), \ldots, p + x_p(\ell - 1) = p + \kappa_\ell$$

are all primes. By the definition of κ_ℓ, we see that $I_{p,\ell} = \{0, x_p(1), \ldots, x_p(\ell - 1)\}$ is a minimal admissible set for every such prime p. Since there are only finitely many minimal admissible sets of length ℓ, and the number of $I_{p,\ell}$'s is infinite, by pigeonhole principle, there exists a sequence of primes $\{p_i\}$ for which $I_{p_i,\ell} := I_\ell = \{0, x(1), \ldots, x(\ell - 1)\}$ are the same. For this minimal admissible set, there are infinitely many integers, namely, the sequence of primes $\{p_i\}$, for which $\{p_i, p_i + x(1), \ldots, p + x(\ell - 1)\}$ are all primes. Hence Conjecture 3 is true. □

Proof of Theorem 5. In [12], Erdős and Riesel proved that

$$\frac{\ell \log \ell}{2} \leq s(\ell) \leq \ell \log \ell \left(1 + O\left(\frac{\log \log \ell}{\log^2 \ell}\right)\right).$$

By Theorem 2, the bounds for λ_ℓ follow from the above result.

Since $\lambda_\ell \leq \kappa_\ell$, the lower bound follows. Now we shall prove the upper bound. Let $\epsilon > 0$ be given. We will produce an interval of length $(1+\epsilon)\ell\log\ell$ that contains at least ℓ prime numbers. That is, we need to find an $x \gg 1$ such that $\pi(x) - \pi(\ell) > \ell$.

Since $\dfrac{x}{\log x} < \pi(x) < \dfrac{3}{2}\dfrac{x}{\log x}$, it is enough to find an $x \gg 1$ satisfying

$$\pi(x) - \pi(\ell) > \frac{x}{\log x} - \frac{3}{2}\frac{\ell}{\log\ell} > \ell.$$

It is sufficient to find $x \gg 1$ satisfying

$$\frac{x}{\log x} > \ell\left(1 + \frac{3}{2\log\ell}\right).$$

Now, choose $x = \ell + (1+\epsilon)\ell\log\ell$. We need to check that

$$\frac{x}{\log x} = \frac{\ell(1 + (1+\epsilon)\log\ell)}{\log\ell + \log(1 + (1+\epsilon)\log\ell))} \geq \ell\left(\frac{2\log\ell + 3}{2\log\ell}\right).$$

That is, we need to check the following inequality:

$$2\epsilon(\log\ell) \geq 1 + \left(2 + \frac{3}{\log\ell}\right)\log(1 + (1+\epsilon)\log\ell).$$

However, the above inequality is true for all $\ell \gg 1$ and the upper bound follows. □

In [2], Forbes computed $s(\ell)$ for all $\ell \leq 20$. In [1], λ_ℓ has been computed for all $\ell \leq 28$. Putting these together, we have the following table:

ℓ	$\kappa_\ell, s(\ell), \lambda_\ell$	$(a_{j+\ell-1}, a_j)$	ℓ	$s(\ell), \lambda_\ell$	ℓ	$s(\ell), \lambda_\ell$
2	2	(a_2, a_1)	11	36	21	84
3	6	(a_3, a_1)	12	42	22	90
4	8	(a_4, a_1)	13	48	23	94
5	12	(a_5, a_1)	14	50	24	100
6	16	(a_6, a_1)	15	56	25	110
7	20	(a_7, a_1)	16	60	26	114
8	26	(a_8, a_1)	17	66	27	120
9	30	(a_9, a_1)	18	70	28	126
10	32	(a_{10}, a_1)	19	76		
			20	80		

In the above table, the column $(a_{j+\ell-1}, a_j)$ denotes the least positive integer j for which $\lambda_\ell = a_{j+\ell-1} - a_j$ holds. Also, by explicit computations, we have checked that $\kappa_\ell = \lambda_\ell$ for every $\ell \leq 10$. However, using the tables in [1] and [2], we have listed the values of λ_ℓ and $s(\ell)$ for all $11 \leq \ell \leq 28$. Note that whenever $j = 1$ in the above, then it is clear that $a_{j+\ell-1} = a_\ell = q_\ell$ and $a_j = a_1 = q_1$.

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