PARTITION REGULARITY OF NONLINEAR POLYNOMIALS: A NONSTANDARD APPROACH

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Abstract

In 2011, Neil Hindman proved that for all natural numbers n, m the polynomial

$$\sum_{i=1}^{n} x_i - \prod_{j=1}^{m} y_j$$

has monochromatic solutions for every finite coloration of $\mathbb{N}$. We want to generalize this result to two classes of nonlinear polynomials. The first class consists of polynomials $P(x_1, \ldots, x_n, y_1, \ldots, y_m)$ of the following kind:

$$P(x_1, \ldots, x_n, y_1, \ldots, y_m) = \sum_{i=1}^{n} a_i x_i M_i(y_1, \ldots, y_m),$$

where n, m are natural numbers, $\sum_{i=1}^{n} a_i x_i$ has monochromatic solutions for every finite coloration of $\mathbb{N}$ and the degree of each variable $y_1, \ldots, y_m$ in $M_i(y_1, \ldots, y_m)$ is at most one. An example of such a polynomial is

$$x_1 y_1 + x_2 y_1 y_2 - x_3.$$

The second class of polynomials generalizing Hindman's result is more complicated to describe; its particularity is that the degree of some of the involved variables can be greater than one. The technique that we use relies on an approach to ultrafilters based on Nonstandard Analysis. Perhaps, the most interesting aspect of this technique is that, by carefully choosing the appropriate nonstandard setting, the proof of the main results can be obtained by very simple algebraic considerations.

1. Introduction

We say that a polynomial $P(x_1, \ldots, x_n)$ is partition regular on $\mathbb{N} = \{1, 2, \ldots\}$ if whenever the natural numbers are finitely colored there is a monochromatic solution

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to the equation $P(x_1, \ldots, x_n) = 0$. The problem of determining which polynomials are partition regular has been studied since Issai Schur's work [26], and the linear case was settled by Richard Rado (see [23]):

Theorem 1.1 (Rado). *Let $P(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i$ be a linear polynomial with nonzero coefficients. The following conditions are equivalent:*

1. *$P(x_1, \ldots, x_n)$ is partition regular on $\mathbb{N}$;*
2. *there is a nonempty subset J of $\{1, \ldots, n\}$ such that $\sum_{j \in J} a_j = 0$.*

In his work Rado also characterized the partition regular finite systems of linear equations. Since then, one of the main topics in this field has been the study of infinite systems of linear equations (for a general background on many notions related to this subject see, e.g., [13]). From our point of view, one other interesting question (which has also been approached, e.g., in [5], [8]) is: which nonlinear polynomials are partition regular? To precisely formalize the problem, we recall the following definitions:

Definition 1.2. A polynomial $P(x_1, \ldots, x_n)$ is

- *partition regular* (on $\mathbb{N}$) if for every natural number r, for every partition $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there is an index $j \leq r$ and natural numbers $a_1, \ldots, a_n \in A_j$ such that $P(a_1, \ldots, a_n) = 0$;
- *injectively partition regular*[2] (on $\mathbb{N}$) if for every natural number r, for every partition $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there is an index $j \leq r$ and mutually distinct natural numbers $a_1, \ldots, a_n \in A_j$ such that $P(a_1, \ldots, a_n) = 0$.

While the linear case is settled, very little is known in the nonlinear case. One of the few exceptions is the multiplicative analogue of Rado's Theorem, that can be deduced from Theorem 1.1 by considering the map $exp(n) = 2^n$:

Theorem 1.3. *Let $n, m \geq 1$, $a_1, \ldots, a_n, b_1, \ldots, b_m > 0$ be natural numbers, and*

$$P(x_1, \ldots, x_n, y_1, \ldots, y_m) = \prod_{i=1}^{n} x_i^{a_i} - \prod_{j=1}^{m} y_j^{b_j}.$$

The following two conditions are equivalent:

1. *$P(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is partition regular;*

[2] Neil Hindman and Imre Leader proved in [15] that every linear partition regular polynomial that has an injective solution is injectively partition regular; see also Section 2.1.

2. *there are two nonempty subsets* $I_1 \subseteq \{1, \ldots, n\}$ *and* $I_2 \subseteq \{1, \ldots, m\}$ *such that* $\sum_{i \in I_1} a_i = \sum_{j \in I_2} b_j$.

As far as we know, perhaps the most interesting result regarding the partition regularity of nonlinear polynomials is the following:

Theorem 1.4 (Hindman). *For all natural numbers* $n, m \geq 1$*, with* $n + m \geq 3$*, the nonlinear polynomial*

$$\sum_{i=1}^{n} x_i - \prod_{j=1}^{m} y_j$$

is injectively partition regular.

Theorem 1.4 is a consequence of a far more general result that has been proved in [14].

The two main results in our paper are generalizations of Theorem 1.4. In Theorem 3.3 we prove that, if $P(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i$ is a linear injectively partition regular polynomial, $y_1, \ldots, y_m$ are not variables of $P(x_1, \ldots, x_n)$, and $F_1, \ldots, F_n$ are subsets of $\{1, \ldots, m\}$, the polynomial

$$R(x_1, \ldots, x_n, y_1, \ldots, y_m) = \sum_{i=1}^{n} a_i (x_i \cdot \prod_{j \in F_i} y_j)$$

(having set $\prod_{j \in F_i} y_j = 1$ if $F_i = \emptyset$) is injectively partition regular. For example, as a consequence of Theorem 3.3 we find that the polynomial

$$P(x_1, x_2, x_3, x_4, y_1, y_2, y_3) = 2x_1 + 3x_2 y_1 y_2 - 5x_3 y_1 + x_4 y_2 y_3$$

is injectively partition regular. The particularity of polynomials considered in Theorem 3.3 is that the degree of each of their variables is one. In Theorem 4.2 we prove that, by slightly modifying the hypothesis of Theorem 3.3, we can ensure the partition regularity for many polynomials having variables with degree greater than one: e.g., as a consequence of Theorem 4.2 we get that the polynomial

$$P(x, y, z, t_1, t_2, t_3, t_4, t_5, t_6) = t_1 t_2 x^2 + t_3 t_4 y^2 - t_5 t_6 z^2$$

is injectively partition regular.

The technique we use to prove our main results is based on an approach to combinatorics by means of nonstandard analysis (see [10], [19]): the idea behind this approach is that, as is well-known, problems related to partition regularity can be reformulated in terms of ultrafilters. Following an approach that has some features in common with the one used by Christian W. Puritz in his articles [21],

[22], the one used by Joram Hirschfeld in [17] and the one used by Greg Cherlin and Joram Hirschfeld in [7], it can be shown that some properties of ultrafilters can be translated and studied in terms of sets of hyperintegers. This can be obtained by associating, in particular hyperextensions ${}^*\mathbb{N}$ of $\mathbb{N}$, to every ultrafilter $\mathcal{U}$ its monad $\mu(\mathcal{U})$:

$$\mu(\mathcal{U}) = \{\alpha \in {}^*\mathbb{N} \mid \alpha \in {}^*A \text{ for every } A \in \mathcal{U}\},$$

and then proving that some of the properties of $\mathcal{U}$ can be deduced by properties of $\mu(\mathcal{U})$ (see [19], Chapter 2). In particular, we prove that a polynomial $P(x_1, \ldots, x_n)$ is injectively partition regular if and only if there is an ultrafilter $\mathcal{U}$, and mutually distinct elements $\alpha_1, \ldots, \alpha_n$ in the monad of $\mathcal{U}$, such that $P(\alpha_1, \ldots, \alpha_n) = 0$.

We will only recall the basic results regarding this nonstandard technique, since it has already been introduced in [10] and [19] .

The paper is organized as follows: the first part, consisting of Section 2, contains an introduction that covers all the needed nonstandard results. In the second part, that consists of Sections 3 and 4, we apply the nonstandard technique to prove that there are many nonlinear injectively partition regular polynomials. In the conclusions we pose two questions that we think are quite interesting and challenging.

2. Basic Results and Definitions

2.1. Notions about Polynomials

In this work, by "polynomial" we mean any $P(x_1, \ldots, x_n) \in \mathbb{Z}[\mathbf{X}]$, where $\mathbf{X}$ is a countable set of variables, $\wp_{fin}(\mathbf{X})$ is the set of finite subsets of $\mathbf{X}$ and

$$\mathbb{Z}[\mathbf{X}] = \bigcup_{Y \in \wp_{fin}(\mathbf{X})} \mathbb{Z}[Y].$$

Given a variable $x \in \mathbf{X}$ and a polynomial $P(x_1, \ldots, x_n)$, we denote by $d_P(x)$ the degree of x in $P(x_1, \ldots, x_n)$.

Convention: When we write $P(x_1, \ldots, x_n)$ we mean that $x_1, \ldots, x_n$ are exactly the variables of $P(x_1, \ldots, x_n)$: for every variable $x \in \mathbf{X}$, $d_P(x) \geq 1$ if and only if $x \in \{x_1, \ldots, x_n\}$. The only exception is when we have a polynomial $P(x_1, \ldots, x_n)$ and we consider one of its monomials: in this case, for the sake of simplicity, we write the monomial as $M(x_1, \ldots, x_n)$ even if some of the variables $x_1, \ldots, x_n$ may not divide $M(x_1, \ldots, x_n)$.

Given the polynomial $P(x_1, \ldots, x_n)$, we call *set of variables* of $P(x_1, \ldots, x_n)$ the set $V(P) = \{x_1, \ldots, x_n\}$, and we call *partial degree* of $P(x_1, \ldots, x_n)$ the maximum degree of its variables.

We recall that a polynomial is linear if all its monomials have degree equal to one and that it is homogeneous if all its monomials have the same degree. Among the nonlinear polynomials, an important class for our purposes is the following:

Definition 2.1. A polynomial $P(x_1, \ldots, x_n)$ is *linear in each variable* (from now on abbreviated as l.e.v.) if its partial degree is equal to one.

Rado's Theorem 1.1 leads us to introduce the following definition:

Definition 2.2. A polynomial

$$P(x_1, \ldots, x_n) = \sum_{i=1}^{k} a_i M_i(x_1, \ldots, x_n),$$

where $M_1(x_1, \ldots, x_n), \ldots, M_k(x_1, \ldots, x_n)$ are its distinct monic monomials, satisfies *Rado's Condition* if there is a nonempty subset $J \subseteq \{1, \ldots, k\}$ such that $\sum_{j \in J} a_j = 0$.

We observe that Rado's Theorem talks about polynomials with the constant term equal to zero. In fact Rado, in [23], proved that when the constant term is not zero, the problem of the partition regularity of $P(x_1, \ldots, x_n)$ becomes, in a certain sense, trivial:

Theorem 2.3 (Rado). *Suppose that*

$$P(x_1, \ldots, x_n) = (\sum_{i=1}^{n} a_i x_i) + c$$

is a polynomial with a non-zero constant term c*. Then* $P(x_1, \ldots, x_n)$ *is partition regular on* $\mathbb{N}$ *if and only if either*

1. *there exists a natural number* k *such that* $P(k, k, \ldots, k) = 0$*;*
2. *there exists an integer* z *such that* $P(z, z, \ldots, z) = 0$ *and there is a nonempty subset* J *of* $\{1, \ldots, n\}$ *such that* $\sum_{j \in J} a_j = 0$.

In order to avoid similar problems, we make the following decision: all the polynomials that we consider in this paper have the constant term equal to zero.

The last fact that we will often use regards the injective partition regularity of linear polynomials. In [15] the authors proved, as a particular consequence of their Theorem 3.1, that a linear partition regular polynomial is injectively partition regular if it has at least one injective solution. Since this last condition is true for every such polynomial (except the polynomial $P(x, y) = x - y$, of course), they concluded that every linear partition regular polynomial on $\mathbb{N}$ is also injectively partition regular. We will often use this fact when studying the injective partition regularity of nonlinear polynomials.

2.2. The Nonstandard Point of View

In this section we recall the results that allow us to study the problem of partition regularity of polynomials by means of nonstandard techniques applied to ultrafilters (see also [10] and [19]). We suggest [16] as a general reference about ultrafilters, [1], [4] or [25] as introductions to nonstandard methods and [6] as a reference for the model theoretic notions that we use. We assume knowledge of the nonstandard notions and tools that we use, in particular the knowledge of superstructures, star map and enlarging properties (see, e.g., [6]). We simply recall the definition of a superstructure model of nonstandard methods, since these are the models that we use:

Definition 2.4. A *superstructure model of nonstandard methods* is a triple $\langle \mathbb{V}(X), \mathbb{V}(Y), * \rangle$ where

1. a copy of $\mathbb{N}$ is included in X and in Y;
2. $\mathbb{V}(X)$ and $\mathbb{V}(Y)$ are superstructures on the infinite sets X, Y respectively;
3. $*$ is a proper star map from $\mathbb{V}(X)$ to $\mathbb{V}(Y)$ that satisfies the transfer property.

In particular, we use single superstructure models of nonstandard methods, i.e., models where $\mathbb{V}(X) = \mathbb{V}(Y)$, whose existence is proved in [2], [3] and [9]. These models have been chosen because they allow us to iterate the star map and this, in our nonstandard technique, is needed to translate the operations between ultrafilters in a nonstandard setting.

The study of partition regular polynomials can be seen as a particular case of a more general problem:

Definition 2.5. Let $\mathcal{F}$ be a family, closed under superset, of nonempty subsets of a set S. $\mathcal{F}$ is *partition regular* if, whenever $S = A_1 \cup \cdots \cup A_n$, there exists an index $i \leq n$ such that $A_i \in \mathcal{F}$.

Given a polynomial $P(x_1, \ldots, x_n)$, we have that $P(x_1, \ldots, x_n)$ is (injectively) partition regular if and only if the family of subsets of $\mathbb{N}$ that contain a(n injective) solution to $P(x_1, \ldots, x_n)$ is partition regular. We recall that partition regular families of subsets of a set S are related to ultrafilters on S:

Theorem 2.6. *Let S be a set, and $\mathcal{F}$ a family, closed under supersets, of nonempty subsets of S. Then $\mathcal{F}$ is partition regular if and only if there exists an ultrafilter $\mathcal{U}$ on S such that $\mathcal{U} \subseteq \mathcal{F}$.*

Proof. This is just a slightly changed formulation of Theorem 3.11 in [16]. □

Theorem 2.6 leads to introduce two special classes of ultrafilters:

Definition 2.7. Let $P(x_1, \ldots, x_n)$ be a polynomial, and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. Then:

1. $\mathcal{U}$ is a σ_P*-ultrafilter* if and only if for every set $A \in \mathcal{U}$ there are $a_1, \ldots, a_n \in A$ such that $P(a_1, .., a_n) = 0$;

2. $\mathcal{U}$ is a ι_P*-ultrafilter* if and only if for every set $A \in \mathcal{U}$ there are mutually distinct elements $a_1, \ldots, a_n \in A$ such that $P(a_1, .., a_n) = 0$.

As a consequence of Theorem 2.6, it follows that a polynomial $P(x_1, \ldots, x_n)$ is partition regular on $\mathbb{N}$ if and only if there is a σ_P-ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and it is injectively partition regular if and only if there is a ι_P-ultrafilter on $\mathbb{N}$. The idea behind the research presented in this paper is that such ultrafilters can be studied, with some important advantages, from the point of view of nonstandard analysis. The models of nonstandard analysis that we use are the single superstructure models satisfying the $\mathfrak{c}^+$-enlarging property. These models allow us to associate hypernatural numbers to ultrafilters on $\mathbb{N}$:

Proposition 2.8. (1) *Let* $^*\mathbb{N}$ *be a hyperextension of* $\mathbb{N}$*. For every hypernatural number* α *in* $^*\mathbb{N}$*, the set*

$$\mathfrak{U}_\alpha = \{A \in \mathbb{N} \mid \alpha \in {}^*A\}$$

is an ultrafilter on $\mathbb{N}$*.*

(2) *Let* $^*\mathbb{N}$ *be a hyperextension of* $\mathbb{N}$ *with the* $\mathfrak{c}^+$*-enlarging property. For every ultrafilter* $\mathcal{U}$ *on* $\mathbb{N}$ *there exists an element* α *in* $^*\mathbb{N}$ *such that* $\mathcal{U} = \mathfrak{U}_\alpha$*.*

Proof. These facts are proved, e.g., in [18] and in [20]. □

Definition 2.9. Given an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, its *set of generators* is

$$G_\mathcal{U} = \{\alpha \in {}^*\mathbb{N} \mid \mathcal{U} = \mathfrak{U}_\alpha\}.$$

For example, if $\mathcal{U} = \mathfrak{U}_n$ is the principal ultrafilter on n, then $G_\mathcal{U} = \{n\}$.

Here a disclaimer is in order: usually, the set $G_\mathcal{U}$ is called "monad of $\mathcal{U}$"; in this paper, from this moment on, the monad on $\mathcal{U}$ will be called "set of generators of $\mathcal{U}$" because, as we will show in Theorem 2.10, many combinatorial properties of $\mathcal{U}$ can be seen as actually "generated" by properties of the elements in $G_\mathcal{U}$.

The following is the result that motivates our nonstandard point of view:

Theorem 2.10 (Polynomial Bridge Theorem). *Let* $P(x_1, \ldots, x_n)$ *be a polynomial, and* $\mathcal{U}$ *an ultrafilter on* $\beta\mathbb{N}$*. The following two conditions are equivalent:*

1. $\mathcal{U}$ *is a* ι_P*-ultrafilter;*

2. *there are mutually distinct elements* $\alpha_1, \ldots, \alpha_n$ *in* $G_\mathcal{U}$ *such that* $P(\alpha_1, \ldots, \alpha_n) = 0$*.*

Proof. (1) $\Rightarrow$ (2): Given a set A in $\mathcal{U}$, we consider

$S_A = \{(a_1, \ldots, a_n) \in A^n \mid a_1, \ldots, a_n$ are mutually distinct and $P(a_1, \ldots, a_n) = 0\}$.

We observe that, by hypothesis, S_A is nonempty for every set A in $\mathcal{U}$, and that the family $\{S_A\}_{A \in \mathcal{U}}$ has the finite intersection property. In fact, if $A_1, \ldots, A_m \in \mathcal{U}$, then

$$S_{A_1} \cap \cdots \cap S_{A_m} = S_{A_1 \cap \cdots \cap A_m} \neq \emptyset.$$

By the $\mathfrak{c}^+$-enlarging property, the intersection

$$S = \bigcap_{A \in \mathcal{U}} {}^*S_A$$

is nonempty. From the way S_A is constructed it follows that

for every $(a_1, \ldots, a_n) \in S_A$ $a_1, \ldots, a_n$ are mutually distinct and $P(a_1, \ldots, a_n) = 0$,

so by transfer it follows that

$$\text{for every } (\alpha_1, \ldots, \alpha_n) \in {}^*S_A \ \alpha_1, \ldots, \alpha_n \text{ are mutually distinct and } P(\alpha_1, \ldots, \alpha_n) = 0.$$

Let $(\alpha_1, \ldots, \alpha_n)$ be an element of S. As we observed, $P(\alpha_1, \ldots, \alpha_n) = 0$, $\alpha_1, \ldots, \alpha_n$ are mutually distinct and, by construction, $\alpha_1, \ldots, \alpha_n \in G_{\mathcal{U}}$ since, for every index $i \leq n$, for every set A in $\mathcal{U}$, $\alpha_i \in {}^*A$.

To show that (2) implies (1), let $\alpha_1, \ldots, \alpha_n \in G_{\mathcal{U}}$ be mutually distinct elements such that $P(\alpha_1, \ldots, \alpha_n) = 0$, and let us suppose that $\mathcal{U}$ is not a ι_P-ultrafilter. Let A be an element of $\mathcal{U}$ such that, for every mutually distinct $a_1, \ldots, a_n$ in $A \setminus \{0\}$, $P(a_1, \ldots, a_n) \neq 0$.

Then by transfer it follows that, for every mutually distinct $\xi_1, \ldots, \xi_n$ in ${}^*A$,

$$P(\xi_1, \ldots, \xi_n) \neq 0;$$

in particular, as $G_{\mathcal{U}} \subseteq {}^*A$, for every set of mutually distinct $\xi_1, \ldots, \xi_n$ in $G_{\mathcal{U}}$ we have $P(\xi_1, \ldots, \xi_n) \neq 0$, which is absurd. Hence $\mathcal{U}$ is a ι_P-ultrafilter. □

Remarks. We obtain similar results if we require that only some of the variables take distinct values: for example, if we ask for solutions where $x_1 \neq x_2$, we have for every set A in $\mathcal{U}$ that there are $a_1, \ldots, a_n$ with $a_1 \neq a_2$ and $P(a_1, \ldots, a_n) = 0$ if and only if in $G_{\mathcal{U}}$ there are $\alpha_1, \ldots, \alpha_n$ with $\alpha_1 \neq \alpha_2$ and $P(\alpha_1, \ldots, \alpha_n) = 0$. Moreover we observe that the Polynomial Bridge Theorem is a particular case of a far more general result, that we proved in [19] (Theorem 2.2.9) and we called the Bridge Theorem. Roughly speaking, the Bridge Theorem states that, given an ultrafilter $\mathcal{U}$ and a first order open formula $\varphi(x_1, \ldots, x_n)$, for every set $A \in \mathcal{U}$ there are elements $a_1, \ldots, a_n \in A$ such that $\varphi(a_1, \ldots, a_n)$ holds if and only if there are elements $\alpha_1, \ldots, \alpha_n$ in $G_{\mathcal{U}}$ such that $\varphi(\alpha_1, \ldots, \alpha_n)$ holds. For example, every set

A in $\mathcal{U}$ contains an arithmetic progression of length 7 if and only if $G_{\mathcal{U}}$ contains an arithmetic progression of length 7.

Since, in the following, we also use operations between ultrafilters, we recall a few definitions about the space $\beta\mathbb{N}$ (for a complete study of this space, we suggest [16]):

Definition 2.11. $\beta\mathbb{N}$ is the space of ultrafilters on $\mathbb{N}$, endowed with the topology generated by the family $\langle \Theta_A \mid A \subseteq \mathbb{N} \rangle$, where

$$\Theta_A = \{\mathcal{U} \in \beta\mathbb{N} \mid A \in \mathcal{U}\}.$$

An ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ is called *principal* if there exists a natural number $n \in \mathbb{N}$ such that $\mathcal{U} = \{A \subseteq \mathbb{N} \mid n \in A\}$. Given two ultrafilters $\mathcal{U}, \mathcal{V}$, $\mathcal{U} \oplus \mathcal{V}$ is the ultrafilter such that, for every set $A \subseteq \mathbb{N}$,

$$A \in \mathcal{U} \oplus \mathcal{V} \Leftrightarrow \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid n + m \in A\} \in \mathcal{V}\} \in \mathcal{U}.$$

Similarly, $\mathcal{U} \odot \mathcal{V}$ is the ultrafilter such that, for every set $A \subseteq \mathbb{N}$,

$$A \in \mathcal{U} \odot \mathcal{V} \mid \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid n \cdot m \in A\} \in \mathcal{V}\} \in \mathcal{U}.$$

An ultrafilter $\mathcal{U}$ is an *additive idempotent* if $\mathcal{U} = \mathcal{U} \oplus \mathcal{U}$; similarly, $\mathcal{U}$ is a *multiplicative idempotent* if $\mathcal{U} = \mathcal{U} \odot \mathcal{U}$.

To study ultrafilters from a nonstandard point of view, we need to translate the operations $\oplus, \odot$ and the notion of idempotent ultrafilter in terms of generators. These translations involve the iteration of the star map, which is possible in single superstructure models $\langle \mathbb{V}(X), \mathbb{V}(X), * \rangle$ of nonstandard methods:

Definition 2.12. For every natural number n we define the function

$$S_n : \mathbb{V}(X) \to \mathbb{V}(X)$$

by setting

$$S_1 = *$$

and, for $n \geq 1$,

$$S_{n+1} = * \circ S_n.$$

Definition 2.13. Let $\langle \mathbb{V}(X), \mathbb{V}(X), * \rangle$ be a single superstructure model of nonstandard methods. The *ω-hyperextension* of $\mathbb{N}$, which we denote by ${}^\bullet\mathbb{N}$, is the union of all the hyperextensions $S_n(\mathbb{N})$:

$${}^\bullet\mathbb{N} = \bigcup_{n \in \mathbb{N}} S_n(\mathbb{N}).$$

Observe that, as a consequence of the Elementary Chain Theorem, ${}^{\bullet}\mathbb{N}$ is a non-standard extension of $\mathbb{N}$.

To the elements of ${}^{\bullet}\mathbb{N}$ is associated a notion of "height":

Definition 2.14. Let $\alpha \in {}^{\bullet}\mathbb{N} \setminus \mathbb{N}$. The **height** of α (denoted by $h(\alpha)$) is the least natural number n such that $\alpha \in S_n(\mathbb{N})$.

By convention we set $h(\alpha) = 0$ if $\alpha \in \mathbb{N}$. We observe that, for every $\alpha \in {}^{\bullet}\mathbb{N} \setminus \mathbb{N}$ and for every natural number $n \in \mathbb{N}$, $h(S_n(\alpha)) = h(\alpha) + n$, and that, by definition of height, for every subset A of $\mathbb{N}$ and every element $\alpha \in {}^{\bullet}\mathbb{N}$, we have that $\alpha \in {}^{\bullet}A$ if and only if $\alpha \in S_{h(\alpha)}(A)$.

A fact that we will often use is that, for every polynomial $P(x_1, \ldots, x_n)$ and every ι_P-ultrafilter $\mathcal{U}$, there exists in $G_{\mathcal{U}}$ a solution $\alpha_1, \ldots, \alpha_n$ to the equation $P(x_1, \ldots, x_n) = 0$ with $h(\alpha_i) = 1$ for all $i \leq n$:

Lemma 2.15 (Reduction Lemma). *Let $P(x_1, \ldots, x_n)$ be a polynomial, and $\mathcal{U}$ a ι_P-ultrafilter. Then there are mutually distinct elements $\alpha_1, \ldots, \alpha_n \in G_{\mathcal{U}} \cap {}^{*}\mathbb{N}$ such that $P(\alpha_1, \ldots, \alpha_n) = 0$.*

Proof. It is sufficient to apply the Polynomial Bridge Theorem to ${}^{*}\mathbb{N} \subseteq {}^{\bullet}\mathbb{N}$. □

We make two observations: first of all, the analogue result holds if $\mathcal{U}$ is just a σ_P-ultrafilter; furthermore, for every natural number $m > 1$ there are mutually distinct elements of height m in $G_{\mathcal{U}}$ that form a solution to $P(x_1, \ldots, x_n)$: if $\alpha_1, \ldots, \alpha_n$ are given by the Reduction Lemma, we just have to take $S_{m-1}(\alpha_1), \ldots, S_{m-1}(\alpha_n)$.

The hyperextension ${}^{\bullet}\mathbb{N}$ provides a useful framework to translate the operations of sum and product between ultrafilters:

Proposition 2.16. *Let $\alpha, \beta \in {}^{\bullet}\mathbb{N}$, $\mathcal{U} = \mathfrak{U}_{\alpha}$ and $\mathcal{V} = \mathfrak{U}_{\beta}$, and let us suppose that $h(\alpha) = h(\beta) = 1$. Then:*

1. *for every natural number n, $\mathfrak{U}_{\alpha} = \mathfrak{U}_{S_n(\alpha)}$;*
2. *$\alpha +^{*} \beta \in G_{\mathcal{U} \oplus \mathcal{V}}$;*
3. *$\alpha \cdot^{*} \beta \in G_{\mathcal{U} \odot \mathcal{V}}$.*

Proof. These results have been proved in [10] and in [19], Chapter 2. □

Remark. In Proposition 2.16 we supposed, for the sake of simplicity, that $h(\alpha) = h(\beta) = 1$. If we drop this hypothesis, the statement in point (2) becomes

$$\alpha + S_{h(\alpha)}(\beta) \in G_{\mathcal{U} \oplus \mathcal{V}}$$

and, in point (3), the statement becomes

$$\alpha \cdot S_{h(\alpha)}(\beta) \in G_{\mathcal{U} \odot \mathcal{V}}.$$

Here a question arises: can a similar result be obtained for generic hyperextensions of $\mathbb{N}$ (with this we mean a hyperextension in which the iteration of the star map is not allowed)? There is not a clear answer to this question. In fact, the answer can be interpreted as "yes" because, as Puritz proved in ([22], Theorem 3.4), in each hyperextension that satisfies the $\mathfrak{c}^+$-enlarging property we can characterize the set of generators of the tensor product $\mathcal{U} \otimes \mathcal{V}$ in terms of $G_{\mathcal{U}}, G_{\mathcal{V}}$ for all ultrafilters $\mathcal{U}$ and $\mathcal{V}$, where $\mathcal{U} \otimes \mathcal{V}$ is the ultrafilter on $\mathbb{N}^2$ defined as follows:

$$\text{for all } A \subseteq \mathbb{N}^2, A \in \mathcal{U} \otimes \mathcal{V} \text{ if and only if } \{n \in \mathbb{N} \mid \{m \in \mathbb{N} \mid (n,m) \in A\} \in \mathcal{V}\} \in \mathcal{U}.$$

Theorem 2.17 (Puritz). *Let ${}^*\mathbb{N}$ be a hyperextension of $\mathbb{N}$ with the $\mathfrak{c}^+$-enlarging property. For all ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$,*

$$G_{\mathcal{U} \otimes \mathcal{V}} = \{(\alpha, \beta) \in {}^*\mathbb{N}^2 \mid \alpha \in G_{\mathcal{U}}, \beta \in G_{\mathcal{V}}, \alpha < er(\beta)\},$$

where

$$er(\beta) = \{{}^*f(\beta) \mid f \in \mathtt{Fun}(\mathbb{N}, \mathbb{N}), {}^*f(\beta) \in {}^*\mathbb{N} \setminus \mathbb{N}\}.$$

If we denote by $S : \mathbb{N}^2 \to \mathbb{N}$ the operation of sum on $\mathbb{N}$ and by $\hat{S} : \beta\mathbb{N}^2 \to \beta\mathbb{N}^2$ its extension to $\beta\mathbb{N}$, we see that $\mathcal{U} \oplus \mathcal{V} = \hat{S}(\mathcal{U} \otimes \mathcal{V})$. So by Puritz's Theorem it follows that

$$G_{\mathcal{U} \oplus \mathcal{V}} = \{\alpha + \beta \mid \alpha \in G_{\mathcal{U}}, \beta \in G_{\mathcal{V}}, \alpha < er(\beta)\}.$$

However, there is another way to look at the problem: the characterization given by Theorem 2.17 is, somehow, "implicit": Proposition 2.16 gives a procedure to construct, given $\alpha \in G_{\mathcal{U}}$ and $\beta \in G_{\mathcal{V}}$, an element $\gamma \in G_{\mathcal{U} \oplus \mathcal{V}}$ related to both α and β, and this fact does not hold for Theorem 2.17.

An important corollary of Proposition 2.16 is that we can easily characterize the idempotent ultrafilters in the nonstandard setting:

Proposition 2.18. *Let $\mathcal{U} \in \beta\mathbb{N}$. Then:*

1. *$\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$ if and only if for all $\alpha, \beta \in G_{\mathcal{U}} \cap {}^*\mathbb{N}$, $\alpha +^* \beta \in G_{\mathcal{U}}$ if and only if there exists $\alpha, \beta \in G_{\mathcal{U}} \cap {}^*\mathbb{N}$ such that $\alpha +^* \beta \in G_{\mathcal{U}}$;*
2. *$\mathcal{U} \odot \mathcal{U} = \mathcal{U}$ if and only if for all $\alpha, \beta \in G_{\mathcal{U}} \cap {}^*\mathbb{N}$, $\alpha +^* \beta \in G_{\mathcal{U}}$ if and only if there exists $\alpha, \beta \in G_{\mathcal{U}} \cap {}^*\mathbb{N}$ such that $\alpha \cdot^* \beta \in G_{\mathcal{U}}$.*

Proof. The result follows easily by points (2) and (3) of Proposition 2.16. □

In [10] these characterizations of idempotent ultrafilters are used to prove some results in combinatorics, in particular a constructive proof of (a particular case of) Rado's Theorem. In the next two sections we show how the nonstandard approach to ultrafilters can be used to prove the partition regularity of particular nonlinear polynomials.

3. Partition Regularity for a Class of l.e.v. Polynomials

In [8], P. Csikvári, K. Gyarmati and A. Sárkzy posed the following question (that we reformulate with the terminology introduced in Section 2): is the polynomial

$$P(x_1, x_2, x_3, x_4) = x_1 + x_2 - x_3 x_4$$

injectively partition regular? This problem was solved by Neil Hindman in [14] as a particular case of Theorem 1.4. We start this section by proving Theorem 1.4 using the nonstandard approach to ultrafilters introduced in Section 2. A key result in our approach to the partition regularity of polynomials is the following:

Theorem 3.1. *If $P(x_1, \dots, x_n)$ is a homogeneous injectively partition regular polynomial then there is a nonprincipal multiplicative idempotent ι_P-ultrafilter.*

Proof. Let

$$I_P = \{\mathcal{U} \in \beta\mathbb{N} \mid \mathcal{U} \text{ is a } \iota_P\text{-ultrafilter}\}.$$

We observe that I_P is nonempty since $P(x_1, \dots, x_n)$ is partition regular. By the definition of ι_P-ultrafilter, and by Theorem 2.10, it clearly follows that every ultrafilter in $\mathcal{U}$ is nonprincipal, since $|G_{\mathcal{U}}| = 1$ for every principal ultrafilter. We claim that I_P is a closed bilateral ideal in $(\beta\mathbb{N}, \odot)$ and we observe that, if we prove this claim, the theorem follows by Ellis' Theorem (see [11]).

I_P is closed since, as is well known, for every property P the set

$$\{\mathcal{U} \in \beta\mathbb{N} \mid \forall A \in \mathcal{U} \quad A \text{ satisfies } P\}$$

is closed. Also, I_P is a bilateral ideal in $(\beta\mathbb{N}, \odot)$: let $\mathcal{U}$ be an ultrafilter in I_P, let $\alpha_1, \dots, \alpha_n$ be mutually distinct elements in $G_{\mathcal{U}} \cap {}^*\mathbb{N}$ with $P(\alpha_1, \dots, \alpha_n) = 0$ and let $\mathcal{V}$ be an ultrafilter in $\beta\mathbb{N}$. Let $\beta \in {}^*\mathbb{N}$ be a generator of $\mathcal{V}$. By Proposition 2.16 it follows that $\alpha_1 \cdot {}^*\beta, \dots, \alpha_n \cdot {}^*\beta$ are generators of $\mathcal{U} \odot \mathcal{V}$. They are mutually distinct and, since $P(x_1, \dots, x_n)$ is homogeneous, if d is the degree of $P(x_1, \dots, x_n)$ then

$$P(\alpha_1 \cdot {}^*\beta, \dots, \alpha_n \cdot {}^*\beta) = {}^*\beta^d P(\alpha_1, \dots, \alpha_n) = 0.$$

So $\mathcal{U} \odot \mathcal{V}$ is a ι_P-ultrafilter, and hence it is in I_P.

The proof for $\mathcal{V} \odot \mathcal{U}$ is completely similar: in this case, we consider the generators $\beta \cdot {}^*\alpha_1, \dots, \beta \cdot {}^*\alpha_n$, and we observe that

$$P(\beta \cdot {}^*\alpha_1, \ldots, \beta \cdot {}^*\alpha_n) = \beta^d P({}^*\alpha_1, \ldots, {}^*\alpha_n) = 0$$

since, by transfer, if $P(\alpha_1, \ldots, \alpha_n) = 0$ then $P({}^*\alpha_1, \ldots, {}^*\alpha_n) = 0$.

So I_P is a bilateral ideal, and this concludes the proof. □

Remark: Theorem 3.1 is a particular case of Theorem 3.3.5 in [19] which states that, whenever we consider a first order open formula $\varphi(x_1, \ldots, x_n)$ that is multiplicatively invariant (with this we mean that, whenever $\varphi(a_1, \ldots, a_n)$ holds, for every natural number m also $\varphi(m \cdot a_1, \ldots, m \cdot a_n)$ holds), the set

$$I_\varphi = \{\mathcal{U} \in \beta\mathbb{N} \mid \text{for all } A \in \mathcal{U} \text{ there exists } a_1, \ldots, a_n \text{ such that } \varphi(a_1, \ldots, a_n) \text{ holds}\}$$

is a bilateral ideal in $(\beta\mathbb{N}, \odot)$. This, by Ellis's Theorem, entails that I_φ contains a multiplicative idempotent ultrafilter (and we can prove that this ultrafilter can be taken to be nonprincipal). Similar results hold if $\varphi(x_1, \ldots, x_n)$ is additively invariant, and for other similar notions of invariance.

As a consequence of Theorem 3.1, we can give an alternative proof of Theorem 1.4:

Proof. If $n \geq 2$, $m = 1$, the polynomial is $\sum_{i=1}^n x_i - y$, and we can apply Rado's Theorem. If $n = 1, m \geq 2$ the polynomial is $x - \prod_{i=1}^m y_i$, and we can apply the multiplicative analogue of Rado's Theorem (Theorem 1.3).

So we suppose $n \geq 2$, $m \geq 2$ and we consider the polynomial

$$R(x_1, \ldots, x_n, y) = \sum_{i=1}^n x_i - y.$$

By Rado's Theorem, $R(x_1, \ldots, x_n, y)$ is partition regular so, as we observed in Section 2, since it is linear it is, in particular, injectively partition regular. It is also homogeneous, so there exists a multiplicative idempotent ι_R-ultrafilter $\mathcal{U}$. Let $\alpha_1, \ldots, \alpha_n, \beta$ be mutually distinct elements in $G_\mathcal{U} \cap {}^*\mathbb{N}$ with $\sum_{i=1}^n \alpha_i - \beta = 0$.

Now let

$$\eta = \prod_{j=1}^m S_j(\beta).$$

For $i = 1, \ldots, n$ we set

$$\lambda_i = \alpha_i \cdot \eta$$

and, for $j = 1, \ldots, m$, we set

$$\mu_j = S_j(\beta).$$

Now, for $i \leq n, j \leq m$ we set $x_i = \lambda_i$ and $y_j = \mu_j$. Since $\mathcal{U}$ is a multiplicative idempotent, all these elements are in $G_\mathcal{U}$. Also,

$$\sum_{i=1}^{n}\lambda_i-\prod_{j=1}^{m}\mu_j=\eta(\sum_{i=1}^{n}\alpha_i-\beta)=0,$$

and this shows that $\mathcal{U}$ is a ι_P-ultrafilter. In particular $P(x_1,\dots,x_n,y_1,\dots,y_m)$ is injectively partition regular. □

These ideas can be slightly modified to prove a more general result:

Definition 3.2. Let m be a positive natural number and let $\{y_1,\dots,y_m\}$ be a set of mutually distinct variables. For all finite set $F\subseteq\{1,..,m\}$ we denote by $Q_F(y_1,\dots,y_m)$ the monomial

$$Q_F(y_1,\dots,y_m)=\begin{cases}\prod_{j\in F}y_j, & \text{if } F\neq\emptyset;\\ 1, & \text{if } F=\emptyset.\end{cases}$$

Theorem 3.3. *Let $n\geq 2$ be a natural number, let $R(x_1,\dots,x_n)=\sum_{i=1}^{n}a_ix_i$ be a partition regular polynomial, and let m be a positive natural number. Then, for all $F_1,\dots,F_n\subseteq\{1,..,m\}$ (with the requirement that, when $n=2$, $F_1\cup F_2\neq\emptyset$), the polynomial*

$$P(x_1,\dots,x_n,y_1,\dots,y_m)=\sum_{i=1}^{n}a_ix_iQ_{F_i}(y_1,\dots,y_m)$$

is injectively partition regular.

Proof. If $n=2$, since in this case we supposed that at least one of the monomials has degree greater than one, we are in a particular case of the multiplicative analogue of Rado's Theorem with at least three variables, and this ensures that the polynomial is injectively partition regular. Hence we can suppose, from now on, $n\geq 3$.

Since $R(x_1,\dots,x_n)$ is linear (so, in particular, it is homogeneous) and partition regular, by Theorem 3.1 it follows that there exists a nonprincipal multiplicative idempotent ι_R-ultrafilter $\mathcal{U}$. Let $\alpha_1,\dots,\alpha_n\in{}^*\mathbb{N}$ be mutually distinct generators of $\mathcal{U}$ such that $R(\alpha_1,\dots,\alpha_n)=0$, and let $\beta\in{}^*\mathbb{N}$ be any generator of $\mathcal{U}$. For every index $j\leq m$, we set

$$\beta_j=S_j(\beta)\in G_{\mathcal{U}}.$$

We observe that, for every index $j\leq m$, $\beta_j\in G_{\mathcal{U}}$. We set, for every index $i\leq n$,

$$\eta_i=\alpha_i\cdot(\prod_{j\notin F_i}\beta_j).$$

Since $\mathcal{U}$ is a multiplicative idempotent, $\eta_i\in G_{\mathcal{U}}$ for every index $i\leq n$. We claim that $P(\eta_1,\dots,\eta_n,\beta_1,\dots,\beta_m)=0$. In fact,

$$P(\eta_1,\ldots,\eta_n,\beta_1,\ldots,\beta_m)=\sum_{i=1}^{n}a_i\eta_iQ_{F_i}(\beta_1,\ldots,\beta_m)=$$
$$\sum_{i=1}^{n}a_i\alpha_i(\prod_{j\notin F_i}\beta_j)(\prod_{j\in F_i}\beta_j)=\sum_{i=1}^{n}a_i\alpha_i(\prod_{j=1}^{m}\beta_j)=(\prod_{j=1}^{m}\beta_j)\sum_{i=1}^{n}a_i\alpha_i=0.$$

This shows that, if we set $x_i=\eta_i$ for $i=1,\ldots,n$ and $y_j=\beta_j$ for $j=1,\ldots,m$, we have an injective solution to the equation $P(x_1,\ldots,x_n,y_1,\ldots,y_m)=0$ in $G_{\mathcal{U}}$. □

We make three observations:

1. as a consequence of the argument used to prove the theorem, the ultrafilter $\mathcal{U}$ considered in the proof is both a ι_P-ultrafilter and a ι_R-ultrafilter;
2. some of the variables $y_1,\ldots,y_m$ may appear in more than a monomial: for example, the polynomial

   $$P(x_1,x_2,x_3,x_4,x_5,y_1,y_2,y_3)=x_1y_1y_2+4x_2y_1y_2y_3-3x_3y_3-2x_4y_1+x_5$$

   satisfies the hypothesis of Theorem 3.3, so it is injectively partition regular;
3. Theorem 1.4 is a particular case of Theorem 3.3.

Theorem 3.3 can be reformulated in a way that leads to the generalization given by Theorem 4.2:

Definition 3.4. Let

$$P(x_1,\ldots,x_n)=\sum_{i=1}^{k}a_iM_i(x_1,\ldots,x_n)$$

be a polynomial and let $M_1(x_1,\ldots,x_n),\ldots,M_k(x_1,\ldots,x_n)$ be the distinct monic monomials of $P(x_1,\ldots,x_n)$. We say that $\{v_1,\ldots,v_k\}\subseteq V(P)$ is a *set of exclusive variables* for $P(x_1,\ldots,x_n)$ if, for every $i,j\leq k$, $d_{M_i}(v_j)\geq 1\Leftrightarrow i=j$. In this case we say that the variable v_i is *exclusive* for the monomial $M_i(x_1,\ldots,x_n)$ in $P(x_1,\ldots,x_n)$.

For example, the polynomial $P(x,y,z,t,w):xyz+yt-w$ admits $\{x,t,w\}$ or $\{z,t,w\}$ as sets of exlusive variables, while the polynomial $P(x,y,z):xy+yz-xz$ does not have any exclusive variable.

Definition 3.5. Let

$$P(x_1,\ldots,x_n)=\sum_{i=1}^{k}a_iM_i(x_1,\ldots,x_n)$$

be a polynomial, and let $M_1(x_1,\ldots,x_n),\ldots,M_k(x_1,\ldots,x_n)$ be the distinct monic monomials of $P(x_1,\ldots,x_n)$. We call *reduced of* P (notation $\text{Red}(P)$) the polynomial:

$$\text{Red}(P)(y_1, \ldots, y_k) = \sum_{i=1}^{k} a_i y_i.$$

For example, if $P(x, y, z, t, w)$ is the polynomial $x_1x_2 + 4x_2x_3 - 2x_4 + x_2x_5$, then

$$\text{Red}(P)(y_1, y_2, y_3, y_4) = y_1 + 4y_2 - 2y_3 + y_4.$$

As a consequence of Rado's Theorem, we see that $P(x_1, \ldots, x_n)$ satisfies Rado's condition if and only if $Red(P)$ is partition regular. As a consequence of Theorem 3.3, we obtain the following result:

Corollary 3.6. *Let $n \geq 3$, $k \geq n$ be natural numbers and let*

$$P(x_1, \ldots, x_n) = \sum_{i=1}^{k} a_i M_i(x_1, \ldots, x_n)$$

be an l.e.v. polynomial. We suppose that $P(x_1, \ldots, x_n)$ admits a set of exclusive variables and that it satisfies Rado's Condition. Then $P(x_1, \ldots, x_n)$ is an injectively partition regular polynomial.

Proof. If $n = k$, the polynomial is linear and the result follows from Theorem 1.1. So we can suppose that $k > n$. By reordering, if necessary, we can suppose that, for $j = 1, \ldots, k$, the variable x_j is exclusive for the monomial $M_j(x_1, \ldots, x_n)$. Then, by Rado's condition, the polynomial

$$\sum_{i=1}^{k} a_i x_i$$

is partition regular. If $F = \{1, \ldots, n - k\}$, for $i \leq k$ we set

$$F_i = \{j \in F \mid x_{j+k} \text{ divides } M_i(x_1, \ldots, x_n)\}.$$

Then if we set, for $j \leq n - k$, $y_j = x_{i+k}$, the polynomial $P(x_1, \ldots, x_n)$ is, by renaming the variables, equal to

$$\sum_{i=1}^{k} a_i x_i Q_{F_i}(y_1, \ldots, y_{n-k}).$$

By Theorem 3.3 the above polynomial is injectively partition regular, so we have the result. □

Corollary 3.7 talks about l.e.v. polynomials; in section 4 we show that there are also non-l.e.v. polynomials that are partition regular, provided that they have enough exclusive variables in each monomial.

4. Partition Regularity for a Class of Nonlinear Polynomials

In this section we want to extend Theorem 3.3 to a particular class of nonlinear polynomials. To introduce our main result, we need the following notations:

Definition 4.1. Let $P(x_1, \ldots, x_n) = \sum_{i=1}^{k} a_i M_i(x_1, \ldots, x_n)$ be a polynomial, and let $M_1(x_1, \ldots, x_n), \ldots, M_k(x_1, \ldots, x_n)$ be the monic monomials of $P(x_1, \ldots, x_n)$. Then

- *NL(P)*=$\{x \in V(P) \mid d(x) \geq 2\}$ is the set of nonlinear variables of $P(x_1, \ldots, x_n)$;
- for every $i \leq k$, $l_i = \max\{d(x) - d_i(x) \mid x \in NL(P)\}$.

Theorem 4.2. *Let*

$$P(x_1, \ldots, x_n) = \sum_{i=1}^{k} a_i M_i(x_1, \ldots, x_n)$$

be a polynomial, and let $M_1(x_1, \ldots, x_n)$,...,$M_k(x_1, \ldots, x_n)$ *be the monic monomials of* $P(x_1, \ldots, x_n)$. *We suppose that* $k \geq 3$, *that* $P(x_1, \ldots, x_n)$ *satisfies Rado's Condition and that, for every index* $i \leq k$, *in the monomial* $M_i(x_1, \ldots, x_n)$ *there are at least* $m_i = \max\{1, l_i\}$ *exclusive variables with degree equal to 1.*

Then $P(x_1, \ldots, x_n)$ *is injectively partition regular.*

Proof. We rename the variables in $V(P)$ in the following way: for $i \leq k$ let $x_{i,1}, \ldots, x_{i,m_i}$ be m_i exclusive variables for $M_i(x_1, \ldots, x_n)$ with degree equal to 1. We set

$$E = \{x_{i,j} \mid i \leq k, j \leq m_i\}$$

and $NL(P) = \{y_1, \ldots, y_h\}$. Finally, we set $\{z_1, \ldots, z_r\} = V(P) \setminus (E \cup NL(P))$.

We suppose that the variables are ordered so as to have

$$P(x_1, \ldots, x_n) = P(x_{1,1}, \ldots, x_{1,m_1}, x_{2,1}, \ldots, x_{k,m_k}, z_1, \ldots, z_r, y_1, \ldots, y_h).$$

We set

$$\widetilde{P}(x_{1,1}, \ldots, z_r) = P(x_{1,1}, \ldots, z_r, 1, 1, \ldots, 1).$$

By construction, and by hypothesis, $\widetilde{P}(x_{1,1}, \ldots, z_r)$ is an l.e.v. polynomial with at least three monomials, it satisfies Rado's Condition and it has at least one exclusive variable for each monomial. So, by Theorem 3.3, it is injectively partition regular. Let $\mathcal{U}$ be a multiplicative idempotent ultrafilter such that in $G_{\mathcal{U}}$ there is an injective solution $(\alpha_{1,1}, \ldots, \alpha_{k,m_k}, \beta_1, \ldots, \beta_r)$ to the equation $\widetilde{P}(x_{1,1}, \ldots, z_r) = 0$. Let γ be an element in $G_{\mathcal{U}} \setminus \{\alpha_{1,1}, \ldots, \alpha_{k,m_k}, \beta_1, \ldots, \beta_r\}$. We consider

$$\eta = \prod_{i=1}^{h} S_i(\gamma)^{d(y_i)}.$$

For $i = 1, \ldots, k$ we set $M_i^{NL} = \prod_{j=1}^{h} S_j(\gamma)^{d_i(y_j)}$ and

$$\eta_i = \frac{\eta}{M_i^{NL}} = \prod_{j=1}^{h} S_j(\gamma)^{d(y_j)-d_i(y_j)}.$$

We observe that the maximum degree of an element $S_j(\gamma)$ in η_i is, by construction, l_i. Finally, for $1 \leq j \leq m_i$, we set $I_{i,j} = \{s \leq h \mid d(y_s) - d_i(y_s) \geq j\}$ and

$$\gamma_{i,j} = \prod_{s \in I_{i,j}} S_s(\gamma).$$

With these choices, we have

$$\prod_{j=1}^{m_i} \gamma_{i,j} = \eta_i$$

and, by construction, $\{\gamma_{i,j} \mid i \leq k, j \leq m_i\} \subseteq G_{\mathcal{U}}$ since $\mathcal{U}$ is a multiplicative idempotent.

We also observe that, for every $i \leq k$, $\left(\prod_{j=1}^{m_i} \gamma_{i,j}\right) \cdot M_i^{NL} = \eta$. Now, if we set, for $i \leq k$ and $j \leq m_i$:

$$x_{i,j} = \begin{cases} \alpha_{i,j} \cdot \gamma_{i,j} & \text{if } l_i \geq 1; \\ \\ \alpha_{i,j} & \text{if } l_i = 0; \end{cases}$$

and

- $y_i = S_i(\gamma)$ for $i \leq h$;
- $z_i = \beta_i$ for $i \leq r$

then

$$P(x_{1,1}, \ldots, x_{k,m_k}, z_1, \ldots, z_r, y_1, \ldots, y_h) =$$
$$\eta \cdot \widetilde{P}(\alpha_{1,1}, \ldots, \alpha_{k,m_k}, \beta_1, \ldots, \beta_r, 1, \ldots, 1) = 0,$$

so $P(x_{1,1}, \ldots, y_h)$ is injectively partition regular. □

In order to understand the requirement $k \geq 3$, we observe that one of the crucial points in the proof is that, when we set $y = 1$ for every $y \in NL(P)$, the polynomial $\widetilde{P}(x_{1,1}, \ldots, z_r)$ that we obtain is injectively partition regular. Now, let us suppose that $k = 2$, and let $M_1(x_1, \ldots, x_n)$ and $M_2(x_1, \ldots, x_n)$ be the two monic monomials of $P(x_1,\dots,x_n)$. If $D(x_1,\dots,x_n)$ is the greatest common divisor of $M_1(x_1,\dots,x_n)$, $M_2(x_1,\dots,x_n)$, we set

$$Q_i(x_1,\dots,x_n)=\frac{M_i(x_1,\dots,x_n)}{D(x_1,\dots,x_n)}$$

for $i=1,2$. We have

$$P(x_1,\dots,x_n)=D(x_1,\dots,x_n)(Q_1(x_1,\dots,x_n)-Q_2(x_1,\dots,x_n)),$$

and it holds that $P(x_1,\dots,x_n)$ is injectively partition regular if and only if

$$R(x_1,\dots,x_n)=Q_1(x_1,\dots,x_n)-Q_2(x_1,\dots,x_n)$$

is, since $D(x_1,\dots,x_n)$ is a nonzero monomial. Now there are two possibilities:

1. $NL(R)\neq\emptyset$, in which case, since every $y\in NL(R)$ divides $Q_1(x_1,\dots,x_n)$ if and only if it does not divide $Q_2(x_1,\dots,x_n)$ (this property holds because, by construction, $Q_1(x_1,\dots,x_n)$ and $Q_2(x_1,\dots,x_n)$ are relatively prime), in at least one of the monomials there are at least two exclusive variables. So that the polynomial $\widetilde{R}(x_{1,1},\dots,z_r)$ is injectively partition regular follows from Theorem 3.3;

2. $NL(R)=\emptyset$, in which case $R(x_1,\dots,x_n)$ is an l.e.v. polynomial with only two monomials, so it is injectively partition regular if and only if $n\geq 3$.

By the previous discussion (and using the same notations) it follows that, when $k=2$, if the other hypothesis of Theorem 4.2 holds then the polynomial $P(x_1,\dots,x_n)$ is injectively partition regular if and only if there do not exist two variables $x_i,x_j\in V(P)$ such that $R(x_1,\dots,x_n)=x_i-x_j$.

We conclude this section by showing with an example how the proof of Theorem 4.2 works. Consider the polynomial

$$P(x_{1,1},x_{2,1},x_{2,2},x_{3,1},x_{3,2},x_{4,1},x_{4,2},z_1,z_2,y_1,y_2)=$$
$$x_{1,1}y_1^2y_2^2+x_{2,1}x_{2,2}z_1y_2^2-2x_{3,1}x_{3,2}z_2y_1+x_{4,1}x_{4,2},$$

where we have chosen the names of the variables following the notations introduced in the proof of Theorem 4.2. We set

$$\widetilde{P}(x_{1,1},\dots,x_{4,2},z_1,z_2)=x_{1,1}+x_{2,1}x_{2,2}z_1-2x_{3,1}x_{3,2}z_2+x_{4,1}x_{4,2}.$$

Let $\mathcal{U}$ be a multiplicative idempotent $\iota_{\widetilde{P}}$-ultrafilter and let $\alpha_{1,1},\dots,\alpha_{4,2},\beta_1,\beta_2\in{}^*\mathbb{N}$ be mutually distinct elements in $G_{\mathcal{U}}$ such that

$$\alpha_{1,1}+\alpha_{2,1}\alpha_{2,2}\beta_1-2\alpha_{3,1}\alpha_{3,2}\beta_2+\alpha_{4,1}\alpha_{4,2}.$$

We take $\gamma \in G_{\mathcal{U}} \setminus \{\alpha_{1,1}, \ldots, \alpha_{4,2}, \beta_1, \beta_2\}$ and we set:

$$\gamma_{2,1} = \gamma_{2,2} =^{*}\gamma,\ \gamma_{3,1} =^{*}\gamma^{**}\gamma,\ \gamma_{3,2} =^{**}\gamma,\ \gamma_{4,1} = \gamma_{4,2} =^{*}\gamma^{**}\gamma.$$

Finally, we set:

- $x_{1,1} = \alpha_{1,1}$;
- $x_{2,1} = \alpha_{2,1} \cdot \gamma_{2,1}$;
- $x_{2,2} = \alpha_{2,2} \cdot \gamma_{2,2}$;
- $x_{3,1} = \alpha_{3,1} \cdot \gamma_{3,1}$;
- $x_{3,2} = \alpha_{3,2} \cdot \gamma_{3,2}$;
- $x_{4,1} = \alpha_{4,1} \cdot \gamma_{4,1}$;
- $x_{4,2} = \alpha_{4,2} \cdot \gamma_{4,2}$;
- $z_1 = \beta_1$;
- $z_2 = \beta_2$;
- $y_1 =^{*}\gamma$;
- $y_2 =^{**}\gamma$.

With these choices, we have:

$$P(x_{1,1}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{4,1}, x_{4,2}, z_1, z_2, y_1, y_2) =$$
$$\alpha_{1,1} \cdot {}^{*}\gamma^2 \cdot {}^{**}\gamma^2 + \alpha_{2,1}\alpha_{2,2}\beta_1{}^{*}\gamma^{2**}\gamma^2 - 2\alpha_{3,1}\alpha_{3,2}\beta_2{}^{*}\gamma^{2**}\gamma^2 + \alpha_{4,1}\alpha_{4,2}{}^{*}\gamma^{2**}\gamma^2 =$$
$${}^{*}\gamma^{2**}\gamma^2 \widetilde{P}(\alpha_{1,1}, \alpha_{2,1}, \alpha_{2,2}, \alpha_{3,1}, \alpha_{3,2}, \alpha_{4,1}, \alpha_{4,2}, \beta_1, \beta_2) = 0,$$

so $P(x_{1,1}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{4,1}, x_{4,2}, z_1, z_2, y_1, y_2)$ has an injective solution in $G_{\mathcal{U}}$, and this proves that $P(x_{1,1}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{4,1}, x_{4,2}, z_1, z_2, y_1, y_2)$ is injectively partition regular.

5. Conclusions

A natural question is the following: can the implications in Theorem 3.3 and Theorem 4.2 be reversed? The hypothesis on the existence of exclusive variables is not necessary: in [8] it is proved that the polynomial

$$P(x, y, z) = xy + xz - yz$$

is partition regular (it can be proved that it is injectively partition regular), and it does not admit a set of exclusive variables. The hypothesis regarding Rado's Condition is more complicated: by slightly modifying the original arguments of Richard Rado (that can be found, for example, in [12]) we can prove that this hypothesis is necessary for every homogeneous partition regular polynomial, but it seems to be not necessary in general. For sure, it is not necessary if we ask for the partition regularity of polynomials on $\mathbb{Z}$: in fact, e.g., the polynomial

$$P(x_1, x_2, x_3, y_1, y_2) = x_1y_1 + x_2y_2 + x_3$$

is injectively partition regular on $\mathbb{Z}$ even if it does not satisfy Rado's Condition. This can be easily proved in the following way: the polynomial

$$R(x_1, x_2, x_3, y_1, y_2) = x_1y_1 + x_2y_2 - x_3$$

is, by Theorem 3.3, injectively partition regular on $\mathbb{N}$ and, if $\mathfrak{U}_\alpha$ is a ι_R-ultrafilter, then $\mathfrak{U}_{-\alpha}$ is a ι_P-ultrafilter; in fact, if $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ are elements in $G_{\mathfrak{U}_\alpha}$ such that $R(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2) = 0$, then $-\alpha_1, -\alpha_2, -\alpha_3, -\beta_1, -\beta_2$ are elements in $G_{\mathfrak{U}_{-\alpha}}$ and, by construction,

$$P(-\alpha_1, -\alpha_2, -\alpha_3, -\beta_1, -\beta_2) = 0.$$

Furthermore, the previous example also shows that, while in the homogeneous case every polynomial which is partition regular on $\mathbb{Z}$ is also partition regular on $\mathbb{N}$, in the non-homogeneous case this is false because $P(x_1, x_2, x_3, y_1, y_2)$, having only positive coefficients, cannot be partition regular on $\mathbb{N}$ (it does not even have any solution in $\mathbb{N}$). Finally, Rado's Condition alone is not sufficient to ensure the partition regularity of a nonlinear polynomial: in [8] the authors proved that the polynomial

$$x + y - z^2$$

is not partition regular on $\mathbb{N}$, even if it satisfies Rado's Condition.

We conclude the paper summarizing the previous observations in two questions:

Question 1. Is there a characterization of nonlinear partition regular polynomials on $\mathbb{N}$ in "Rado's style", i.e., that allows us to determine in a finite time if a given polynomial $P(x_1, \ldots, x_n)$ is, or is not, partition regular?

Question 1 seems particularly challenging; an easier question, that would still be interesting to answer, is the following:

Question 2. Is there a characterization of homogeneous partition regular polynomials (in the same sense of Question 1)?

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