NEWTON BASIS RELATIONS AND APPLICATIONS TO INTEGER-VALUED POLYNOMIALS AND q-BINOMIAL COEFFICIENTS

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Abstract

Let K be a field. Generalizing the binomial coefficient polynomials $\binom{X}{n}$, the *Newton interpolation basis polynomials* relative to an infinite sequence $\mathbf{a} = (a_i)_{i=0}^{\infty}$ of distinct elements of K are defined by $\begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}} = \prod_{i=0}^{n-1} \frac{X-a_i}{a_n-a_i} \in K[X]$. There is a complete and natural set of relations for these polynomials, namely, $\begin{bmatrix} X \\ m \end{bmatrix}_{\mathbf{a}} \begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}} = \sum_{l=\max(m,n)}^{m+n} [m,n,l]_{\mathbf{a}} \begin{bmatrix} X \\ l \end{bmatrix}_{\mathbf{a}}$, where the coefficients $[m,n,l]_{\mathbf{a}}$ are in K for all m,n,l. We derive formulas for the $[m,n,l]_{\mathbf{a}}$ that uniquely characterize the linear iterative sequences and the sequences of squares and triangular numbers and are expressed, respectively, in terms of the q-multinomial coefficients and the ordinary multinomial coefficients. We also use these relations to find a D-algebra presentation of the ring $\mathrm{Int}(S,D) = \{f \in K[X] : f(S) \subseteq D\}$ of *integer-valued polynomials on S in D*, where D is any integral domain with quotient field K and S is any subset of D such that $a_n \in S$ and $\begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}} \in \mathrm{Int}(S,D)$ for all n.

1. Introduction

Let $\mathbf{a} = (a_i)_{i=0}^{\infty}$ be an infinite sequence of distinct elements of a field K. The *nth* $\mathbf{a}$*-binomial coefficient polynomial* $\begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}}$, for any nonnegative integer n, is the (unique degree at most n) interpolating polynomial of the $n+1$ points $(a_0, 0)$, $(a_1, 0)$, ... , $(a_{n-1}, 0)$, $(a_n, 1)$ in K^2. Equivalently, one sets

$$\begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}} = \prod_{i=0}^{n-1} \frac{X-a_i}{a_n-a_i}.$$

Two examples that are important from a combinatorial perspective are as follows.

Example 1.1.

1. Let $\mathbf{n} = (0, 1, 2, 3, \ldots)$. If K is of characteristic zero, then

$$\begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{n}} = \binom{X}{n} = \frac{X(X-1)(X-2)\cdots(X-n+1)}{n!} \in \mathbb{Q}[X] \subseteq K[X]$$

is the *nth binomial coefficient polynomial.*

2. Let $\mathbf{q} = (1, q, q^2, q^3, \ldots)$ denote the sequence of powers of a fixed $q \in K$ that is neither zero nor a root of unity. Then

$$\begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{q}} = \frac{(X-1)(X-q)(X-q^2)\cdots(X-q^{n-1})}{(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1})}$$

is the *nth Gaussian binomial coefficient polynomial.*

The $\mathbf{a}$-binomial coefficient polynomials $\left[\begin{smallmatrix} X \\ n \end{smallmatrix}\right]_{\mathbf{a}}$ for $n \in \mathbb{Z}_{\geq 0}$ form a K-vectorspace basis of $K[X]$, which we call the *Newton basis of* $K[X]$ *(relative to* $\mathbf{a}$*).* Although there are explicit formulas for the coefficients of a polynomial in $K[X]$ with respect to the Newton basis of $K[X]$ relative to $\mathbf{a}$ (see Section 6), there are more efficient ways of computing the coefficients [13] [14], including the following.

Proposition 1.2 (Newton basis theorem)**.** *For any polynomial $f \in K[X]$ of degree at most n, there exist unique elements $c_k = \delta^k_{\mathbf{a}}(f)$ of K such that*

$$f = \sum_{i=0}^{n} c_i \begin{bmatrix} X \\ i \end{bmatrix}_{\mathbf{a}}.$$

Moreover, the c_k can be computed iteratively from the equations

$$c_k = b_k - \sum_{i=0}^{k-1} c_i \begin{bmatrix} a_k \\ i \end{bmatrix}_{\mathbf{a}}, \quad k = 0, 1, 2, \ldots, n,$$

where $b_k = f(a_k)$ for all k. Conversely, given any $b_0, b_1, \ldots, b_n$ in K, if the c_k are defined recursively as above, then $f = \sum_{i=0}^{n} c_i \left[\begin{smallmatrix} X \\ i \end{smallmatrix}\right]_{\mathbf{a}}$ is the interpolating polynomial of the $n+1$ points $(a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n)$ of K^2.

Proof. Substitute $a_0, a_1, \ldots, a_n$ successively for X into the given expression for f and solve for c_k in the kth resulting equation. $\square$

From the Newton basis theorem it follows that there are unique elements $[m, n, l]_{\mathbf{a}}$ of K, depending on $\mathbf{a}$, m, n, and l, such that

$$\begin{bmatrix} X \\ m \end{bmatrix}_{\mathbf{a}} \begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}} = \sum_{l=0}^{m+n} [m, n, l]_{\mathbf{a}} \begin{bmatrix} X \\ l \end{bmatrix}_{\mathbf{a}}. \tag{1.1}$$

It is easy to show that the coefficients $[m,n,l]_{\mathbf{a}}$ determine the terms of the sequence $\mathbf{a}$ uniquely up to a linear transformation (Proposition 2.7). Moreover, the coefficients $[m,n,l]_{\mathbf{a}}$ reflect various arithmetic properties of the sequence $\mathbf{a}$. For instance, their "boundary" values coincide with two $\mathbf{a}$-analogues of the binomial coefficients $\binom{n}{k}$. To define them we first define the $\mathbf{a}$-analogue of $n!$ to be the *nth* $\mathbf{a}$-*factorial* $n!_{\mathbf{a}} = \prod_{i=0}^{n-1}(a_n - a_i)$ (the reciprocal of which is the leading coefficient of $\left[{X \atop n}\right]_{\mathbf{a}}$). By [4, Lemma 16], if $\mathbf{a}$ is "simultaneously ordered" in the sense of [3, Theorem 19], or more broadly in the sense discussed later in this section, then $n!_{\mathbf{a}}$ coincides with "the 'correct' number-theoretic generalization of the factorial" discussed in [4].

Example 1.3. Let n be a nonnegative integer.

1. $n!_{\mathbf{n}} = n!$, where $\mathbf{n} = (i)_{i=0}^{\infty}$.
2. $n!_{\mathbf{q}} = (q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})$, where $\mathbf{q} = (q^i)_{i=0}^{\infty}$.
3. $n!_{\mathbf{s}} = (2n)!/2$ for all $n > 0$, where $\mathbf{s} = (i^2)_{i=0}^{\infty}$.
4. $n!_{\mathbf{t}} = \frac{(2n)!}{2^n}$ and $n!_{2\mathbf{t}} = (2n)!$, where $\mathbf{t} = ((i^2+i)/2)_{i=0}^{\infty}$ and $2\mathbf{t} = (i^2+i)_{i=0}^{\infty}$.

The two $\mathbf{a}$-analogues of $\binom{n}{k}$ are

$$\begin{bmatrix} n \\ k, n-k \end{bmatrix}_{\mathbf{a}} = \frac{n!_{\mathbf{a}}}{k!_{\mathbf{a}}(n-k)!_{\mathbf{a}}}$$

and

$$\begin{bmatrix} a_n \\ k \end{bmatrix}_{\mathbf{a}} = \prod_{i=0}^{k-1} \frac{a_n - a_i}{a_k - a_i}.$$

The first expression is symmetric with respect to k and $n-k$, but the second need not be. Moreover, the two are equal for all n,k if and only if $\mathbf{a}$ is an arithmetic sequence (Corollary 1.7).

From Eq. (1.1) it is straightforward to show that $[m,n,l]_{\mathbf{a}} = 0$ if $l < \max(m,n)$, and at the "boundaries" $l = m+n$ and $l = m$, $l = n$ one has

$$[m,n,m+n]_{\mathbf{a}} = \begin{bmatrix} m+n \\ m,n \end{bmatrix}_{\mathbf{a}}$$

and

$$[m,n,m]_{\mathbf{a}} = \begin{bmatrix} a_m \\ n \end{bmatrix}_{\mathbf{a}}, \quad [m,n,n]_{\mathbf{a}} = \begin{bmatrix} a_n \\ m \end{bmatrix}_{\mathbf{a}},$$

respectively (Proposition 2.2). In this sense the coefficients $[m,n,l]_{\mathbf{a}}$ interpolate both $\mathbf{a}$-analogues of the binomial coefficients $\binom{n}{k}$.

In Sections 4 and 5 we find simple formulas for the coefficients $[m,n,l]_{\mathbf{a}}$ uniquely characterizing particular arithmetically and combinatorially significant sequences $\mathbf{a}$. The following elementary result, which can be proved by equating coefficients in the multinomial expansions of $(1+S)^t(1+T)^t$ and $(1+S+T+ST)^t$ in $\mathbb{Z}[S,T]$, is the motivating example.

Proposition 1.4. *One has*

$$\binom{X}{m}\binom{X}{n} = \sum_{l=\max(m,n)}^{m+n} \binom{l}{l-m,\, l-n,\, m+n-l}\binom{X}{l}$$

for all $m, n \in \mathbb{Z}_{\geq 0}$. Equivalently, for any sequence $\mathbf{a}$ of distinct integers, one has $[m,n,l]_{\mathbf{a}} = \binom{l}{l-m,l-n,m+n-l}$ for all m, n, l, where one sets $\binom{l}{n_1,n_2,\ldots,n_r} = 0$ if n_i is negative for some i, if and only if $\mathbf{a}$ is an arithmetic sequence.

In Theorem 1.5 below, we express the coefficients $[m,n,l]_{\mathbf{a}}$ for the sequence $\mathbf{a} = \mathbf{q}$, where $\mathbf{q} = (q^i)_{i=0}^{\infty}$, in terms of the well-known *q-multinomial coefficients* $\binom{n}{n_1,n_2,\ldots,n_r}_q$ [9] [15]. (Their definition is recalled in Section 4.) These arise naturally in several contexts, including combinatorics, linear algebra and algebraic and projective geometry over finite fields, quantum calculus, and the theory of quantum groups [10] [11]. It is well-known, for example, that if $q \in \mathbb{Z}$ is a prime power, then $\binom{n}{k}_q$ is equal to the number of k-dimensional subspaces of an n-dimensional vectorspace over the finite field $\mathbb{F}_q$ [9, Theorem 1] [15, Proposition 1.7.2].

Theorem 1.5 (with Angela Adams, Ryan DeMoss, Margaret Freaney, and Andrew Mostowa)**.** *Let $\mathbf{q} = (q^i)_{i=0}^{\infty}$, where $q \in K$ is neither zero nor a root of unity. Then*

$$\begin{bmatrix} X \\ m \end{bmatrix}_{\mathbf{q}} \begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{q}} = \sum_{l=\max(m,n)}^{m+n} q^{(l-m)(l-n)} \binom{l}{l-m,\, l-n,\, m+n-l}_q \begin{bmatrix} X \\ l \end{bmatrix}_{\mathbf{q}}$$

for all $m, n \in \mathbb{Z}_{\geq 0}$. Equivalently, one has $[m,n,l]_{\mathbf{q}} = q^{(l-m)(l-n)}\binom{l}{l-m,l-n,m+n-l}_q$ for all $m, n, l \in \mathbb{Z}_{\geq 0}$.

Two undergraduate students, Margaret Freaney and Andrew Mostowa, conjectured the theorem in 2009 after using a computer (and Maple) to compute the coefficients $[m,n,l]_{\mathbf{q}}$ for various m, n, l and compare their values with values of various hypothetical q-multinomial versions of Proposition 1.4. In 2012, two more undergraduate students, Angela Adams and Ryan DeMoss, helped prove the conjecture by induction. (This proof, along with a proof from the *q-Vandermonde identity*, is provided in Section 4.) That same year two computer science students, Michael Kaiser and Gregory Ball, wrote programs in C and Maple to compute the coefficients $[m,n,l]_{\mathbf{a}}$ for any sequence $\mathbf{a}$, which led to the discovery of Propositions 2.2 and 2.7, as well as Theorem 1.10 below. Finally, in 2013, two math students, Lucas Mattick and Blaine Kutin, attempted to find formulas for $[m,n,l]_{\mathbf{a}}$ for further sequences $\mathbf{a}$ and proved Propositions 3.4 and 5.1.

Proposition 1.4 and Theorem 1.5 generalize as follows. We say that $\mathbf{a}$ is a *linear iterative sequence* if $\mathbf{a}$ is obtained by iterating a linear polynomial $qX + r$ in $K[X]$. Assuming $a_1 \neq a_0$, the polynomial is unique and one has $q = \frac{a_2 - a_1}{a_1 - a_0}$.

Theorem 1.6. *Let* $\mathbf{a}$ *be an infinite sequence of distinct elements of* K*, and let* $q = \frac{a_2-a_1}{a_1-a_0}$*. The following conditions are equivalent.*

1. $\mathbf{a}$ *is a linear iterative sequence.*
2. $\mathbf{a} = \alpha + \beta\mathbf{b}$ *for unique* $\alpha, \beta \in K$*, where* $\mathbf{b} = \mathbf{n}$ *if* $q = 1$ *and* $\mathbf{b} = \mathbf{q}$ *if* $q \neq 1$*.*
3. $\mathbf{a} = \alpha + \beta\mathbf{b}$ *for unique* $\alpha, \beta \in K$*, where* $\mathbf{b} = \left(\sum_{j=0}^{i-1} q^j\right)_{i=0}^{\infty}$*.*
4. $\begin{bmatrix} a_n \\ 1 \end{bmatrix}_{\mathbf{a}} = \binom{n}{1}_q$ *for all* n*.*
5. $\begin{bmatrix} a_n \\ k \end{bmatrix}_{\mathbf{a}} = \binom{n}{k}_q$ *for all* n, k *with* $k \leq n$*.*
6. $[m, n, l]_{\mathbf{a}} = q^{(l-m)(l-n)} \binom{l}{l-m, l-n, m+n-l}_q$ *for all* m, n, l*.*
7. $\begin{bmatrix} n \\ k, l \end{bmatrix}_{\mathbf{a}} = q^{kl} \begin{bmatrix} a_n \\ k \end{bmatrix}_{\mathbf{a}}$ *for all* n, k, l *with* $n = k + l$*.*
8. $\begin{bmatrix} a_n \\ k \end{bmatrix}_{\mathbf{a}} = \begin{bmatrix} a_n \\ n-k \end{bmatrix}_{\mathbf{a}}$ *for all* n, k *with* $k \leq n$*.*
9. $\begin{bmatrix} a_n \\ k \end{bmatrix}_{\mathbf{a}} = \begin{bmatrix} a_n \\ n-k \end{bmatrix}_{\mathbf{a}}$ *for* n *odd and* $k = 2, \lfloor n/2 \rfloor$ *and for* $n = 4, 5, 6$ *and* $k = 1, 2$*.*

Corollary 1.7. *Let* $\mathbf{a}$ *be an infinite sequence of distinct elements of* K*. Then* $\begin{bmatrix} a_n \\ k \end{bmatrix}_{\mathbf{a}} = \begin{bmatrix} a_n \\ n-k \end{bmatrix}_{\mathbf{a}}$ *for all* n, k *with* $k \leq n$ *if and only if* $\mathbf{a}$ *is a linear iterative sequence, and* $\begin{bmatrix} a_n \\ k \end{bmatrix}_{\mathbf{a}} = \begin{bmatrix} n \\ k, n-k \end{bmatrix}_{\mathbf{a}}$ *for all* n, k *with* $k \leq n$ *if and only if* $\mathbf{a}$ *is an arithmetic sequence.*

Another motivation for studying the coefficients $[m, n, l]_{\mathbf{a}}$ is their application to generalized integer-valued polynomials. Using the Newton basis relations (1.1), one can express the product $fg = \sum_{m,n} c_m d_n \begin{bmatrix} X \\ m \end{bmatrix}_{\mathbf{a}} \begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}}$ (and sum $f + g$) explicitly in terms of the Newton basis provided that $f = \sum_n c_n \begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}}$ and $g = \sum_n d_n \begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}}$ in $K[X]$ are so expressed. Moreover, the relations (1.1) account for all of the polynomial relations among these generators of $K[X]$. (Together with the relation $X - a_0 = (a_1 - a_0)\begin{bmatrix} X \\ 1 \end{bmatrix}_{\mathbf{a}}$ one can recover the expressions for the Newton basis polynomials and hence the sequence $\mathbf{a}$ itself.) This yields a K-algebra presentation of $K[X]$ that is particularly suited to studying subrings of $K[X]$ of generalized integer-valued polynomials, defined as follows.

Let D be an integral domain with quotient field K and S a subset of D. The set

$$\mathrm{Int}(S, D) = \{f \in K[X] : f(S) \subseteq D\}$$

is a subring of $K[X]$ called the ring of *D-integer-valued polynomials on S*. One writes $\mathrm{Int}(D) = \mathrm{Int}(D, D)$. Following [1] we say that an infinite sequence $\mathbf{a}$ of distinct elements a_i of S is a *simultaneous ordering of S (in D)* if $\begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}} \in \mathrm{Int}(S, D)$ for all n, or equivalently, if $\prod_{i=0}^{n-1}(a_n - a_i)$ divides $\prod_{i=0}^{n-1}(x - a_i)$ in D for all $x \in S$ and all n. (In [7] simultaneous orderings are called *Newtonian orderings*.) This generalizes the definition of a simultaneous ordering in terms of $\mathfrak{p}$-orderings given for Dedekind domains in [3] [4]. We say that S *admits a simultaneous ordering (in D)* if there is a sequence $\mathbf{a}$ such that $\mathbf{a}$ is a simultaneous ordering of S in D. We also say that $\mathbf{a}$ is *simultaneously ordered (in D)* if $\mathbf{a}$ is a simultaneous ordering of some subset of D, or equivalently if $\mathbf{a}$ is a simultaneous ordering of the set $\{a_0, a_1, a_2, \ldots\}$.

By the Newton basis theorem, if $\mathbf{a}$ is a simultaneous ordering of S in D, then the polynomials $\left[{X \atop n}\right]_{\mathbf{a}}$ form a free D-module basis for $\operatorname{Int}(S,D)$, and therefore $[m,n,l]_{\mathbf{a}} \in D$ for all m,n,l. Conversely, if $[m,n,l]_{\mathbf{a}} \in D$ for all m,n,l, then $\mathbf{a}$ is simultaneously ordered, since $[m,n,m]_{\mathbf{a}} = \left[{a_m \atop n}\right]_{\mathbf{a}}$. Thus the coefficients $[m,n,l]_{\mathbf{a}}$ can detect simultaneous orderings. The following result, proved in Section 3, provides a D-algebra presentation for $\operatorname{Int}(S,D)$ when S admits a simultaneous ordering.

Proposition 1.8. *Let D be an integral domain and S a subset of D that admits a simultaneous ordering $\mathbf{a}$. Let*

$$\varphi : D[T_1, T_2, \ldots] \longrightarrow \operatorname{Int}(S,D)$$

denote the unique D-algebra homomorphism sending T_n to $\left[{X \atop n}\right]_{\mathbf{a}}$ for all n. Then the D-algebra homomorphism φ is surjective with kernel generated by

$$T_m T_n - \sum_{l=n}^{m+n} [m,n,l]_{\mathbf{a}} T_l$$

for all m,n with $1 \le m \le n$.

Below is a list of some important known examples of simultaneous orderings.

Example 1.9 ([1] [3] [4] [8])**.** Let $\mathbf{a}$ be sequence of distinct elements of a subset S of a domain D.

1. If $\operatorname{char} D = 0$, then the sequences $\mathbf{n} = (i)_{i=0}^{\infty}$ and $(0, 1, -1, 2, -2, \ldots)$ are simultaneous orderings of $\mathbb{Z}$ in D.

2. The sequence $\mathbf{q} = (q^i)_{i=0}^{\infty}$ of powers of any $q \in D$ that is neither zero nor a root of unity is simultaneously ordered in D.

3. If $\operatorname{char} D = 0$, then the sequence $(i^2)_{i=0}^{\infty}$ of squares and the sequence $((i^2 + i)/2)_{i=0}^{\infty}$ of triangular numbers are simultaneously ordered in D. However, the sequence $(i^k)_{i=0}^{\infty}$ does not admit a simultaneous ordering in $\mathbb{Z}$ if $k > 2$.

4. If $a_{i+1} = f(a_i)$ for all i for some polynomial $f \in \mathbb{Z}[X]$, then $\mathbf{a}$ is simultaneously ordered in $\mathbb{Z}$. For example, this holds for the sequence $(q^{r^i})_{i=0}^{\infty}$ for any integer $q \neq 0, \pm 1$ and any integer $r > 1$.

5. If $n!_{\mathbf{a}}$ divides $a_{n+1} - a_0$ for all n, then $\mathbf{a}$ is simultaneously ordered in D.

6. If $\mathbf{a}$ is a simultaneous ordering of S in D, then $\alpha + \beta\mathbf{a} = (\alpha + \beta a_i)_{i=0}^{\infty}$ is a simultaneous ordering of $\alpha + \beta S$ in D for all $\alpha, \beta \in D$ with $\beta \neq 0$.

7. $\mathbf{a}$ is a simultaneous ordering of S in D if and only if $\mathbf{a}$ is a simultaneous ordering of S in $D_{\mathfrak{p}}$ for every maximal ideal $\mathfrak{p}$ of D.

8. Suppose that D is a Dedekind domain, and let $\mathfrak{p}$ be a maximal ideal of D. Then $\mathbf{a}$ is a simultaneous ordering of S in $D_{\mathfrak{p}}$ if and only if $\mathbf{a}$ is a $\mathfrak{p}$-ordering of S in D, in the sense of [3] [4].

9. Suppose D is a local domain with principal maximal ideal $\mathfrak{m}$ and residue field of finite order q. Let t be a generator of $\mathfrak{m}$, and let $a_0 = 0, a_1, \ldots, a_{q-1} \in D$ be a complete set of representatives of $D/\mathfrak{m}$. For $n = n_0 + n_1 q + \cdots + n_k q^k$ with $0 \leq n_i < q$, let $a_n = a_{n_0} + a_{n_1} t + \cdots + a_{n_k} t^k$. Then $(a_n)_{n=0}^{\infty}$ is a simultaneous ordering of D, and in fact so is $(a_{n+k})_{n=0}^{\infty}$ for any k.

10. The simultaneous ordering of $\mathbb{F}_q[X]_{(X)}$ of Example (9) with $t = X$ and $\{a_0, a_1, \ldots, a_{q-1}\} = \mathbb{F}_q$ is also a simultaneous ordering of $\mathbb{F}_q[X]$.

It is an open problem to determine all Dedekind domains D such that D admits a simultaneous ordering [3, Question 3].

As noted above, the sequence $\mathbf{s} = (i^2)_{i=0}^{\infty}$ of squares and the sequence $\mathbf{t} = ((i^2+i)/2)_{i=0}^{\infty}$ of triangular numbers are simultaneously ordered in D if $\operatorname{char} D = 0$. In Section 5 we reveal further significance of these sequences and prove the following.

Theorem 1.10. *Suppose* $\operatorname{char} D = 0$*, and let* $m, n, l \in \mathbb{Z}_{\geq 0}$*. Then in* D *one has*

$$[m, n, l]_{\mathbf{t}} = \frac{\binom{m+n}{m,n}}{\binom{2m}{m}\binom{2n}{n}} \binom{2l}{l, l-m, l-n, m+n-l},$$

and if $m, n > 0$ *then* $[m, n, l]_{\mathbf{s}} = \frac{2l}{m+n}[m, n, l]_{\mathbf{t}}$.

Example 1.9 and the results in this paper motivate the following problems.

Problem 1.11.

1. Find a simultaneous generalization of Theorems 1.5 and 1.10.

2. Find a formula uniquely characterizing the $[m, n, l]_{\mathbf{a}}$ for (linear transformations of) the following simultaneous orderings $\mathbf{a}$.

   (a) $\mathbf{a} = (0, 1, -1, 2, -2, \ldots)$.

   (b) $\mathbf{a} = (a_0, f(a_0), f(f(a_0)), \ldots)$, where $f \in \mathbb{Z}[X]$ is of degree at least 2.

   (c) Examples (9) and (10) of Example 1.9.

3. Find a formula uniquely characterizing the $[m,n,l]_{\mathbf{a}}$ for any sequence $\mathbf{a}$ satisfying a second order linear recurrence relation, or more generally a kth order linear recurrence relations with constant coefficients for $k \geq 2$. Which of these sequences are simultaneous orderings?

4. Characterize the simultaneous orderings $\mathbf{a}$ of Example 1.9(5) in terms of the $[m,n,l]_{\mathbf{a}}$ and vice versa.

2. Basic Properties of the Newton Basis Relation Coefficients

In this section we prove some basic properties of the Newton basis relation coefficients $[m,n,l]_{\mathbf{a}}$. We often write $\left[\begin{smallmatrix} X \\ n \end{smallmatrix}\right]$ instead of $\left[\begin{smallmatrix} X \\ n \end{smallmatrix}\right]_{\mathbf{a}}$.

Lemma 2.1. *Let $f \in K[X]$, and let n be any nonnegative integer. Then $f(a_k) = 0$ for all $k < n$ if and only if $\delta_{\mathbf{a}}^k(f) = 0$ for all $k < n$.*

Proof. This is clear from the recursion provided by the Newton basis theorem (Proposition 1.2) for computing the coefficients $\delta_{\mathbf{a}}^k(f)$. $\square$

Proposition 2.2. *Let m, n be nonnegative integers. One has $[m,n,l]_{\mathbf{a}} = 0$ for any nonnegative integer l with $l < \max(m,n)$ or $l > m+n$, and therefore*

$$\begin{bmatrix} X \\ m \end{bmatrix}_{\mathbf{a}} \begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}} = \sum_{l=\max(m,n)}^{m+n} [m,n,l]_{\mathbf{a}} \begin{bmatrix} X \\ l \end{bmatrix}_{\mathbf{a}}. \tag{2.1}$$

Moreover, one has

$$[m,n,n]_{\mathbf{a}} = \begin{bmatrix} a_n \\ m \end{bmatrix}_{\mathbf{a}} \tag{2.2}$$

and

$$[m,n,m+n]_{\mathbf{a}} = \begin{bmatrix} m+n \\ m,n \end{bmatrix}_{\mathbf{a}}. \tag{2.3}$$

Proof. The polynomial $\left[\begin{smallmatrix} X \\ m \end{smallmatrix}\right]_{\mathbf{a}} \left[\begin{smallmatrix} X \\ n \end{smallmatrix}\right]_{\mathbf{a}}$ assumes the value 0 at a_i for all $i < \max(m,n)$. Equation (2.1) therefore follows from Lemma 2.1. Substituting a_n for X in (2.1) we obtain (2.2). Finally, equating the leading coefficients in (2.1) we see that $\frac{1}{m!_{\mathbf{a}}} \frac{1}{n!_{\mathbf{a}}} = [m,n,m+n]_{\mathbf{a}} \frac{1}{(m+n)!_{\mathbf{a}}}$, from which (2.3) follows. $\square$

The following corollary provides a formula for the coefficients $[1,n,l]_{\mathbf{a}}$ for any sequence $\mathbf{a}$ and any nonnegative integers n, l.

Corollary 2.3. *For all nonnegative integers n one has*

$$\begin{bmatrix} X \\ 1 \end{bmatrix}\begin{bmatrix} X \\ n \end{bmatrix} = \begin{bmatrix} a_n \\ 1 \end{bmatrix}\begin{bmatrix} X \\ n \end{bmatrix} + \begin{bmatrix} n+1 \\ 1, n \end{bmatrix}\begin{bmatrix} X \\ n+1 \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} X \\ n+1 \end{bmatrix} = \frac{(X-a_n)}{(n+1)!_{\mathbf{a}}/n!_{\mathbf{a}}}\begin{bmatrix} X \\ n \end{bmatrix} = \frac{(X-a_n)\left[\begin{smallmatrix} X \\ n \end{smallmatrix}\right]}{(a_{n+1}-a_n)\left[\begin{smallmatrix} a_{n+1} \\ n \end{smallmatrix}\right]}.$$

Next we find a simple recurrence relation for the coefficients $[m,n,l]_{\mathbf{a}}$. Let m, n be nonnegative integers. Let

$$X_{\mathbf{a}}^{\underline{n}} = (X-a_0)(X-a_1)\cdots(X-a_{n-1})$$

for all n. Let $\langle m,n,l\rangle_{\mathbf{a}}$ denote the unique constants in K such that

$$X_{\mathbf{a}}^{\underline{m}} X_{\mathbf{a}}^{\underline{n}} = \sum_{l=0}^{m+n} \langle m,n,l\rangle_{\mathbf{a}} X_{\mathbf{a}}^{\underline{l}},$$

and set $\langle m,n,l\rangle_{\mathbf{a}} = 0$ if $l > m+n$. (The $\langle m,n,l\rangle_{\mathbf{a}}$ are defined even if the a_i are not all distinct.) Also, let $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise. One has $\left[\begin{smallmatrix} X \\ n \end{smallmatrix}\right]_{\mathbf{a}} = \frac{X_{\mathbf{a}}^{\underline{n}}}{n!_{\mathbf{a}}}$ and $[m,n,l]_{\mathbf{a}} = \frac{l!_{\mathbf{a}}}{m!_{\mathbf{a}} n!_{\mathbf{a}}}\langle m,n,l\rangle_{\mathbf{a}}$ for all m,n,l.

Proposition 2.4. *One has $\langle 0,n,l\rangle_{\mathbf{a}} = \delta_{n,l}$ and $\langle m,n,0\rangle_{\mathbf{a}} = \delta_{m+n,0}$, and*

$$\langle m+1,n,l\rangle_{\mathbf{a}} = \langle m,n,l-1\rangle_{\mathbf{a}} + (a_l - a_m)\langle m,n,l\rangle_{\mathbf{a}},$$

for all m,n,l with $l > 0$. Moreover, the given recurrence relation and base cases completely determine the coefficients $\langle m,n,l\rangle_{\mathbf{a}}$ for all nonnegative integers m,n,l.

Proof. Equate the coefficients of $X_{\mathbf{a}}^{\underline{l}}$ in $X_{\mathbf{a}}^{\underline{m+1}} X_{\mathbf{a}}^{\underline{n}}$ and $(X-a_m)X_{\mathbf{a}}^{\underline{m}} X_{\mathbf{a}}^{\underline{n}}$. □

Corollary 2.5. *One has $[0,n,l]_{\mathbf{a}} = \delta_{n,l}$ and $[m,n,0]_{\mathbf{a}} = \delta_{m+n,0}$, and*

$$[m+1,n,l]_{\mathbf{a}} = \frac{m!_{\mathbf{a}}}{(m+1)!_{\mathbf{a}}}\left(\frac{l!_{\mathbf{a}}}{(l-1)!_{\mathbf{a}}}[m,n,l-1]_{\mathbf{a}} + (a_l - a_m)[m,n,l]_{\mathbf{a}}\right) \tag{2.4}$$

for all m,n,l. Moreover, the given recurrence relation and base cases completely determine the coefficients $[m,n,l]_{\mathbf{a}}$ for all nonnegative integers m,n,l.

The base cases in Proposition 2.4 and Corollary 2.5 can be replaced, respectively, with the boundary conditions

$$\langle m,n,n\rangle_{\mathbf{a}} = (a_n)_{\mathbf{a}}^{\underline{m}}, \quad \langle m,n,m+n\rangle_{\mathbf{a}} = 1$$

and

$$[m,n,n]_{\mathbf{a}} = \begin{bmatrix} a_n \\ m \end{bmatrix}_{\mathbf{a}}, \quad [m,n,m+n]_{\mathbf{a}} = \begin{bmatrix} m+n \\ m,n \end{bmatrix}_{\mathbf{a}}.$$

One defines the **a**-*multinomial coefficient*

$$\begin{bmatrix} n \\ n_1, n_2, \ldots, n_r \end{bmatrix}_{\mathbf{a}} = \frac{n!_{\mathbf{a}}}{n_1!_{\mathbf{a}} n_2!_{\mathbf{a}} \cdots n_r!_{\mathbf{a}}}$$

when $n = n_1 + n_2 + \cdots + n_r$.

Lemma 2.6. *Let $\alpha, \beta \in K$ with $\beta \neq 0$. Then*

$$n!_{\alpha+\beta\mathbf{a}} = \beta^n n!_{\mathbf{a}},$$

$$\begin{bmatrix} n \\ n_1, n_2, \ldots n_r \end{bmatrix}_{\alpha+\beta\mathbf{a}} = \begin{bmatrix} n \\ n_1, n_2, \ldots n_r \end{bmatrix}_{\mathbf{a}},$$

$$\begin{bmatrix} \alpha+\beta X \\ n \end{bmatrix}_{\alpha+\beta\mathbf{a}} = \begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}},$$

for all nonnegative integers $n, n_1, n_2, \ldots, n_r$ with $n = n_1 + n_2 + \cdots n_r$.

Proof. This is easy to verify from the definitions. □

Next we show that the coefficients $[m,n,l]_{\mathbf{a}}$ determine $\mathbf{a}$ uniquely up to a linear transformation.

Proposition 2.7. *Let $\mathbf{a} = (a_i)_{i=0}^{\infty}$ and $\mathbf{b} = (b_i)_{i=0}^{\infty}$ be infinite sequences of distinct elements of K. The following conditions are equivalent.*

1. *$[m,n,l]_{\mathbf{b}} = [m,n,l]_{\mathbf{a}}$ for all nonnegative integers m,n,l.*
2. *$[1,n,n]_{\mathbf{b}} = [1,n,n]_{\mathbf{a}}$ for all nonnegative integers n.*
3. *$\begin{bmatrix} b_n \\ 1 \end{bmatrix}_{\mathbf{b}} = \begin{bmatrix} a_n \\ 1 \end{bmatrix}_{\mathbf{a}}$ for all nonnegative integers n.*
4. *$\begin{bmatrix} b_n \\ k \end{bmatrix}_{\mathbf{b}} = \begin{bmatrix} a_n \\ k \end{bmatrix}_{\mathbf{a}}$ for all nonnegative integers n, k.*
5. *There exist $\alpha, \beta \in K$ such that $\mathbf{b} = \alpha + \beta\mathbf{a}$.*

Proof. One has (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Leftrightarrow$ (2) by Proposition 2.2. One has (3) $\Rightarrow$ (5) because the equation $\frac{b_n - b_0}{b_1 - b_0} = \begin{bmatrix} b_n \\ 1 \end{bmatrix}_{\mathbf{b}} = \begin{bmatrix} a_n \\ 1 \end{bmatrix}_{\mathbf{a}} = \frac{a_n - a_0}{a_1 - a_0}$ expresses a linear relationship between the terms b_n and a_n. Finally, one has (5) $\Rightarrow$ (1) by Lemma 2.6 and the fact that the map $K[X] \longrightarrow K[X]$ sending $f(X)$ to $f(\alpha + \beta X)$ for all $f \in K[X]$ is an automorphism of the K-algebra $K[X]$. □

3. Applications to Integer-Valued Polynomials

In this section we discuss the connections between the Newton basis and integer-valued polynomial rings and prove Proposition 1.8 of the introduction. Recall that, for any subset S of an integral domain D with quotient field K, the set

$$\mathrm{Int}(S,D)=\{f\in K[X]: f(S)\subseteq D\}$$

is a subring of $K[X]$ containing $D[X]$ called the ring of *D-integer-valued polynomials on S* [6]. The Newton basis theorem (Proposition 1.2) yields the following generalization of [3, Theorem 19].

Proposition 3.1. *Let D be an integral domain with quotient field K. Let S be a subset of D and $\mathbf{a}$ an infinite sequence of distinct elements of S. The following conditions are equivalent.*

1. *$\mathbf{a}$ is a simultaneous ordering of S in D.*
2. *The polynomials $\left[{X \atop n}\right]_{\mathbf{a}}$ for $n\in\mathbb{Z}_{\geq 0}$ form a D-module basis for $\mathrm{Int}(S,D)$.*
3. *For all $f\in K[X]$, one has $f\in\mathrm{Int}(S,D)$ if and only if $\delta_{\mathbf{a}}^k(f)\in D$ for all nonnegative integers $k\leq\deg f$.*
4. *For all $f\in K[X]$, one has $f\in\mathrm{Int}(S,D)$ if and only if $f(a_k)\in D$ for all nonnegative integers $k\leq\deg f$.*

From the above result and Proposition 2.2 we obtain the following.

Proposition 3.2. *Let D be an integral domain and $\mathbf{a}$ an infinite sequence of distinct elements of D. The following conditions are equivalent.*

1. *$\mathbf{a}$ is simultaneously ordered in D; that is, $\left[{a_n \atop m}\right]_{\mathbf{a}}\in D$ for all m,n.*
2. *$[m,n,n]_{\mathbf{a}}\in D$ for all m,n.*
3. *$[m,n,l]_{\mathbf{a}}\in D$ for all m,n,l.*

Now we prove Proposition 1.8 of the introduction.

Proof of Proposition 1.8. Let $R=D[T_1,T_2,\ldots]$, and let I denote the ideal generated by the given relations. By Proposition 3.1 the homomorphism φ is surjective. Since $I\subseteq\ker\varphi$, there is an induced surjective D-algebra homomorphism $\psi: R/I\longrightarrow\mathrm{Int}(S,D)$. Let $f=f(T_1,T_2,\ldots)\in R$. By induction on $\deg f$ one can use the given relations to reduce f modulo I to a linear polynomial $g=c_0+c_1T_1+c_2T_2+\cdots+c_nT_n\in R$ in the T_i; indeed, if g is any such linear polynomial, then gT_m can be reduced modulo I to a linear polynomial for any m, since T_iT_m can be so reduced for any i. Thus, there is a linear polynomial $g\in R$ such that

$f \equiv g \pmod{I}$. If $g' \in R$ is any linear polynomial with $f \equiv g' \pmod{I}$, then $\varphi(g) = \varphi(g')$, whence $g = g'$ since the coefficients of any linear polynomial in the $\begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}}$ are unique by Proposition 3.1. Thus, for any $f \in R$ there is a unique linear polynomial $g \in R$ in the T_i such that $f \equiv g \pmod{I}$. It follows that $\psi : R/I \longrightarrow \mathrm{Int}(S, D)$ is a bijection with inverse acting by $c_0 + c_1 \begin{bmatrix} X \\ 1 \end{bmatrix}_{\mathbf{a}} + \cdots + c_n \begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}} \longmapsto c_0 + c_1 T_1 + \cdots + c_n T_n$. Therefore $I = \ker \varphi$. $\square$

Remark 3.3. Let R be a commutative ring and S a commutative R-algebra having a free R-module basis $\mathcal{B} = \{s_0, s_1, s_2, \ldots\}$. Then there exist unique $[m, n, l]_{\mathcal{B}} \in R$ such that $s_m s_n = \sum_{l=0}^{<\infty} [m, n, l]_{\mathcal{B}} s_l$ for all m, n. The proof of Proposition 1.8 easily generalizes to show that the unique R-algebra homomorphism $R[T_1, T_2, \ldots] \longrightarrow S$ sending T_i to s_i for all i is surjective with kernel generated by $T_m T_n - \sum_{l=0}^{<\infty} [m, n, l]_{\mathcal{B}} T_l$ for all m, n with $1 \le m \le n$.

The next result, whose proof is straightforward, shows the effect interchanging the first two terms of a simultaneous ordering.

Proposition 3.4 (with Lucas Mattick and Blaine Kutin)**.** *Let D be an integral domain and $\mathbf{a}$ an infinite sequence of distinct elements of D, and let $\mathbf{a}^* = (a_1, a_0, a_2, a_3, a_4, \ldots)$. Then one has the following.*

1. $\begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}^*} = \begin{bmatrix} X \\ n \end{bmatrix}_{\mathbf{a}}$ *if $n \neq 1$, and* $\begin{bmatrix} X \\ 1 \end{bmatrix}_{\mathbf{a}} + \begin{bmatrix} X \\ 1 \end{bmatrix}_{\mathbf{a}^*} = 1$.
2. *One has*

$$[m, n, l]_{\mathbf{a}^*} = \begin{cases} -[m, n, l]_{\mathbf{a}} & \text{if } m = 1 \text{ or } n = 1, \text{ and } 1 < \max(m, n) < l \\ 1 - [m, n, l]_{\mathbf{a}} & \text{if } m = 1 \text{ or } n = 1, \text{ and } 1 < \max(m, n) = l \\ [m, n, l]_{\mathbf{a}} & \text{otherwise.} \end{cases}$$

3. $\mathbf{a}^*$ *is simultaneously ordered if and only if* $\mathbf{a}$ *is simultaneously ordered.*

In the following situation, $\mathbf{a}^*$ is the only reordering of $\mathbf{a}$ that is also simultaneously ordered.

Proposition 3.5 (cf. proof of [1, Proposition 3.1])**.** *Let D be a subring of an ordered field K with $D^* = \{1, -1\}$ and $\mathbf{a}$ an increasing sequence of elements of D such that $a_i - a_j = a_1 - a_0$ implies $i = 1$ and $j = 0$. If $\mathbf{a}$ is simultaneously ordered in D, then $\mathbf{a}$ and $\mathbf{a}^*$ are the only simultaneous orderings of $\{a_0, a_1, a_2, \ldots\}$ in D. In particular, $\mathbf{a}^*$ is the only reordering of $\mathbf{a}$ that is also simultaneously ordered in D.*

Proof. Let $\mathbf{b}$ be any simultaneous ordering of $\{a_0, a_1, a_2, \ldots\}$. One has $(b_1 - b_0) \mid (a_i - b_0)$ for all i, hence $(b_1 - b_0) \mid (a_1 - a_0)$. Likewise one has $(a_1 - a_0) \mid (b_1 - b_0)$, so $(b_1 - b_0) = \pm(a_1 - a_0)$. By hypothesis, then, one has $b_0 = a_0$ and $b_1 = a_1$, or else $b_0 = a_1$ and $b_1 = a_0$. In either case one has

$$(b_2 - a_0)(b_2 - a_1) = (a_2 - a_0)(a_2 - a_1),$$

since each divides the other and both are positive since a_2 and b_2 are both greater than a_0 and a_1. If $b_2 > a_2$, then one would have $(b_2-a_0)(b_2-a_1) > (a_2-a_0)(a_2-a_1)$, which is a contradiction. Therefore one must have $b_2 = a_2$. The argument then proceeds by induction, and one concludes that $b_n = a_n$ for all $n \geq 2$. $\square$

Examples of ordered domains D to which the above result applies are $\mathbb{Z}$, the ring of hyperintegers in any ring of hyperreals, and $\mathbb{Z}[S]$ for any set S of real numbers that are algebraically independent over $\mathbb{Q}$ (and more specifically $\mathbb{Z}[\alpha]$ for any transcendental real number α).

4. Linear Iterative Sequences

In this section we prove Theorems 1.5 and 1.6 of the introduction.

In Lemma 4.1 below, the $\mathbf{q}$-factorials and $\mathbf{q}$-multinomials are expressed, respectively, in terms of the q-factorials and q-multinomials, defined as follows. Let q be an indeterminant, and let $n, k, l, n_1, n_2, \ldots, n_r$ be nonnegative integers. The *nth q-factorial* is defined by $n!_q = \prod_{i=1}^{n}(1+q+\cdots+q^{i-1}) \in \mathbb{Z}[q]$; the *q-multinomial coefficient* (which also lies in $\mathbb{Z}[q]$) is defined by $\binom{n}{n_1,n_2,\ldots,n_r}_q = \frac{n!_q}{n_1!_q n_2!_q \cdots n_r!_q} \in \mathbb{Q}(q)$ when $n = n_1 + n_2 + \cdots + n_r$; and the *q-binomial coefficient* $\binom{n}{k}_q$ is defined by $\binom{n}{k}_q = \binom{n}{k,n-k}_q$. We define the q-multinomial coefficient to be 0 if n_i is a negative integer for some i.

Lemma 4.1. *Suppose that $q \in K$ is neither zero nor a root of unity. One has the following for all nonnegative integers $n, k, l, n_1, n_2, \ldots, n_r$.*

1. $n!_{\mathbf{q}} = (q-1)^n q^{n(n-1)/2} n!_q$.
2. $\begin{bmatrix} q^n \\ k \end{bmatrix}_{\mathbf{q}} = \binom{n}{k}_q$.
3. $\begin{bmatrix} n \\ k,l \end{bmatrix}_{\mathbf{q}} = q^{kl} \begin{bmatrix} q^n \\ k \end{bmatrix}_{\mathbf{q}} = q^{kl}\binom{n}{k}_q$ *if* $n = k + l$.
4. $\begin{bmatrix} n \\ n_1,n_2,\ldots,n_r \end{bmatrix}_{\mathbf{q}} = q^{\sum_{i<j} n_i n_j} \binom{n}{n_1,n_2,\ldots,n_r}_q$ *if* $n = n_1 + n_2 + \cdots + n_r$.

Using the recurrence relation in Corollary 2.5 for the coefficients $[m,n,l]_{\mathbf{a}}$, we now prove Theorem 1.5 of the introduction.

Proof of Theorem 1.5. Since $\frac{k!_{\mathbf{q}}}{(k-1)!_{\mathbf{q}}} = q^{k-1}(q^k - 1)$ for all k, it suffices by Corollary 2.5 to show that the expression for $[m,n,l]_{\mathbf{q}}$ in the theorem satisfies the recurrence relation

$$[m+1,n,l]_{\mathbf{q}} = \frac{q^{l-1}(q^l-1)[m,n,l-1]_{\mathbf{q}} + (q^l - q^m)[m,n,l]_{\mathbf{q}}}{q^m(q^{m+1}-1)}$$

with base cases $[0,n,l]_{\mathbf{q}} = \delta_{n,l}$ and $[m,n,0]_{\mathbf{q}} = \delta_{m+n,0}$. The base cases are clear. Substituting the inductively assumed expressions for $[m,n,l]_{\mathbf{q}}$ and $[m,n,l-1]_{\mathbf{q}}$ in the right hand side of the above recurrence relation and dividing by the desired expression for $[m+1,n,l]_{\mathbf{q}}$, one is left with

$$\frac{q^{l-1}(q^l-1)q^{m-l+1}\frac{(l-1)!_q(l-n)!_q}{l!_q(l-n-1)!_q} + (q^l-q^m)q^{l-n}\frac{(l-m-1)_q(m+n-l+1)!_q}{(l-m)!_q(m+n-l)!_q}}{q^m(q^{m+1}-1)},$$

which, since $\frac{k!_q}{(k-1)!_q} = \frac{q^k-1}{q-1}$, is equal to

$$\frac{q^{l-1}(q^l-1)q^{m-l+1}\frac{q^{l-n}-1}{q^l-1} + (q^l-q^m)q^{l-n}\frac{q^{m+n-l+1}-1}{q^{l-m}-1}}{q^m(q^{m+1}-1)} = 1.$$

This completes the proof. □

We also provide an alternative proof of Theorem 1.5 based on the following.

Lemma 4.2 (q-Vandermonde identity)**.** *For all $t,m,n \in \mathbb{Z}_{\geq 0}$ one has*

$$\binom{t}{m}_q = \sum_{l=n}^{m+n} q^{(l-m)(l-n)} \binom{n}{m+n-l}_q \binom{t-n}{l-n}_q.$$

Proof. An equivalent identity is proved in [12, p. 237–238] and [2, Eq. (3.2)]. □

Proposition 4.3. *For all $t \in \mathbb{Z}_{\geq 0}$ one has*

$$\binom{t}{m}_q \binom{t}{n}_q = \sum_{l=\max(m,n)}^{m+n} q^{(l-m)(l-n)} \binom{l}{l-m,\, l-n,\, m+n-l}_q \binom{t}{l}_q.$$

Proof. We have

$$\begin{aligned}\binom{n}{m+n-l}_q \binom{t-n}{l-n}_q \binom{t}{n}_q &= \frac{t!_q}{(l-m)!_q(l-n)!_q(m+n-l)!_q(t-l)!_q} \\ &= \binom{l}{l-m,\, l-n,\, m+n-l}_q \binom{t}{l}_q.\end{aligned}$$

The proposition therefore follows from Lemma 4.2. □

Note that Proposition 1.4 follows from Proposition 4.3 by setting $q=1$.

Alternative proof of Theorem 1.5. Let $t \in \mathbb{Z}_{\geq 0}$. Since $\binom{t}{k}_q = \left[\begin{smallmatrix} q^t \\ k \end{smallmatrix}\right]_{\mathbf{q}}$ for all k, Proposition 4.3 implies that $\left[\begin{smallmatrix} X \\ m \end{smallmatrix}\right]_{\mathbf{q}} \left[\begin{smallmatrix} X \\ n \end{smallmatrix}\right]_{\mathbf{q}}$ and $\sum_{l=\max(m,n)}^{m+n} q^{(l-m)(l-n)} \binom{l}{l-m,l-n,m+n-l}_q \left[\begin{smallmatrix} X \\ l \end{smallmatrix}\right]_{\mathbf{q}}$ agree at q^t for all nonnegative integers t and are therefore equal. □

The following corollary, whose proof is straigtforward, expresses $[m,n,l]_{\mathbf{a}}$ for any linear iterative sequence $\mathbf{a}$ in terms of the sequence $\mathbf{a}$ itself, without reference to q.

Corollary 4.4. *Let* $\mathbf{a}$ *be a linear iterative sequence of distinct elements of* K*. Then one has*

$$[m,n,l]_{\mathbf{a}} = \begin{bmatrix} 2l-m-n \\ l-m, l-n \end{bmatrix} \begin{bmatrix} a_l \\ m+n-l \end{bmatrix}$$

and

$$[m,n,l]_{\mathbf{a}} = \frac{l!_{\mathbf{a}}(m+n-l)!_{\mathbf{a}}}{m!_{\mathbf{a}}n!_{\mathbf{a}}} \begin{bmatrix} a_m \\ m+n-l \end{bmatrix} \begin{bmatrix} a_n \\ m+n-l \end{bmatrix}$$

for all $m,n,l \in \mathbb{Z}_{\geq 0}$ *with* $\max(m,n) \leq l \leq m+n$*. In particular, setting* $r = m+n-l$*, one has*

$$\begin{bmatrix} a_m \\ r \end{bmatrix} \begin{bmatrix} a_n \\ r \end{bmatrix} = \frac{m!_{\mathbf{a}}n!_{\mathbf{a}}}{r!_{\mathbf{a}}(m+n-r)!_{\mathbf{a}}} \begin{bmatrix} m+n-2r \\ m-r, n-r \end{bmatrix} \begin{bmatrix} a_{m+n-r} \\ r \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} m+n \\ m,n \end{bmatrix} \begin{bmatrix} a_m \\ r \end{bmatrix} \begin{bmatrix} a_n \\ r \end{bmatrix} = \begin{bmatrix} m+n \\ r, m+n-r \end{bmatrix} \begin{bmatrix} m+n-2r \\ m-r, n-r \end{bmatrix} \begin{bmatrix} a_{m+n-r} \\ r \end{bmatrix},$$

for all $0 \leq r \leq \min(m,n)$*.*

The above corollary motivates the following problem.

Problem 4.5. The four equations in Corollary 4.4 are invariant under arbitrary linear transformations of $\mathbf{a}$. For each of these four equations find necessary and sufficient conditions on $\mathbf{a}$ so that the given equation holds. In particular, which if any of them uniquely characterize the linear iterative sequences?

Finally we prove Theorem 1.6.

Proof of Theorem 1.6. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are straightforward, and one has (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) by Proposition 2.7 and Theorem 1.5. By Lemma 4.1(3), statement (7) holds for the sequences $\mathbf{a} = \mathbf{n}$ and $\mathbf{a} = \mathbf{q}$. Therefore (2) $\Rightarrow$ (7) by Lemma 2.6, and clearly (7) $\Rightarrow$ (8) $\Rightarrow$ (9).

Thus it remains only to show that (9) $\Rightarrow$ (3). By applying a linear transformation to $\mathbf{a}$ we may assume without loss of generality that $a_0 = 0$ and $a_1 = 1$, so $q = a_2 - 1$. Set $c_n = \sum_{i=0}^{n-1} q^i$ for all n, so $a_n = c_n$ for $n = 0,1,2$. We show that $a_n = c_n$ for all n. First, the equation $\left[\begin{smallmatrix} X \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} X \\ 2 \end{smallmatrix}\right]$ is linear in X (after cancellation of like factors) and has both a_3 and c_3 as solutions, and therefore $a_3 = c_3$. Similarly, the equation $\left[\begin{smallmatrix} X \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} X \\ 3 \end{smallmatrix}\right]$ is linear in X and has both a_5 and c_5 as solutions, and therefore $a_5 = c_5$. Assuming for the moment that $a_4 = c_4$, we show that $a_n = c_n$ successively for

$$n = 7, 9; 6; 11, 13; 8; 15, 17; \ldots; 2k; 4k-1, 4k+1; \ldots.$$

First, for any $k \geq 2$, the cases $n = 4k-1$ and $n = 4k+1$ follow from the cases $n = 0, 1, \ldots, 2k$ because the linear equation $\begin{bmatrix} X \\ 2k-1 \end{bmatrix} = \begin{bmatrix} X \\ 2k \end{bmatrix}$ has both a_{4k-1} and c_{4k-1} as solutions and the linear equation $\begin{bmatrix} X \\ 2k \end{bmatrix} = \begin{bmatrix} X \\ 2k+1 \end{bmatrix}$ has both a_{4k+1} and c_{4k+1} as solutions. Moreover, for any $k \geq 3$, the case $n = 2k$ follows from the cases $n = 0, 1, 2, \ldots, 2k-1$ and $n = 2k+1, 2k+3$ because the equation $\begin{bmatrix} a_{2k+3} \\ 2 \end{bmatrix} = \begin{bmatrix} a_{2k+3} \\ 2k+1 \end{bmatrix}$ is linear in a_{2k} and involves only $a_0, a_1, \ldots, a_{2k+1}$ and a_{2k+3}. This proves, then, that $a_n = c_n$ for all n provided that $a_4 = c_4$.

Now, the equation $\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ 3 \end{bmatrix}$ is quadratic in X and has c_4 as a solution. Consideration of the linear term shows that the other solution is $c_1 + c_2 - c_4 = 1 - q^2 - q^3$. Therefore either $a_4 = c_4$ or $a_4 = 1 - q^2 - q^3$. Suppose to obtain a contradiction that $a_4 \neq c_4$. Then one has

$$c_5 = \begin{bmatrix} c_5 \\ 1 \end{bmatrix} = \begin{bmatrix} c_5 \\ 4 \end{bmatrix} = \frac{c_5(q^4+q^3+q^2+q)(q^4+q^3+q^2)(q^4+q^3)}{(q^3+q^2-1)(q^3+q^2)(q^3+q^2+q)(q^3+2q^2+q)},$$

which simplifies to the equation $(q-1)(2q^2+2q+1) = 0$. Thus either $q = 1$ or $2q^3 + 2q^2 + q = 0$; equivalently, either $q = 1$ or $c_4 = 1+q+q^2+q^3 = 1-q^2-q^3 = a_4$. Thus one has $q = 1$ and $a_4 = 1-q^2-q^3 = -1$. Then, since $\mathbf{a} = (0, 1, 2, 3, -1, 5, a_6, a_7, \ldots)$, one sees from the equation $\begin{bmatrix} a_6 \\ 2 \end{bmatrix} = \begin{bmatrix} a_6 \\ 4 \end{bmatrix}$ that $(a_6-2)(a_6-3) = 4!_{\mathbf{a}}/2!_{\mathbf{a}} = 24/2 = 12$ and therefore $a_6 = 6, -1$. However, $a_6 = -1 = a_4$ is not possible, so $a_6 = 6$. Therefore $6 = \begin{bmatrix} a_6 \\ 1 \end{bmatrix} = \begin{bmatrix} a_6 \\ 5 \end{bmatrix} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 7}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 6} = \frac{7}{2}$, which is our desired contradiction. $\square$

5. Quadratic Sequences

In this section we prove Theorem 1.10 of the introduction, thus computing the coefficients $[m, n, l]_{\mathbf{a}}$ for the sequences $\mathbf{s} = (i^2)_{i=0}^{\infty}$ and $\mathbf{t} = ((i^2+i)/2)_{i=0}^{\infty}$ of squares and triangular numbers, respectively. The following result, which is a reinterpretation of [1, Theorem 14], reveals the significance of these two sequences.

Proposition 5.1 (with Lucas Mattick and Blaine Kutin). *Let $f \in \mathrm{Int}(\mathbb{Z})$ be a polynomial of degree 2, and let $S = \{n^2 : n \in \mathbb{Z}_{\geq 0}\}$ and $T = \{(n^2+n)/2 : n \in \mathbb{Z}_{\geq 0}\}$. Then the following are equivalent.*

1. *$f(\mathbb{Z}_{\geq 0})$ admits a simultaneous ordering.*
2. *f is of the form $\alpha + \beta X(X - 2\lambda)$ or $\alpha + \beta X(X - 2\lambda - 1)/2$, where $\alpha, \beta, \lambda \in \mathbb{Z}$ and $\lambda \geq 0$.*
3. *$f(\mathbb{Z}_{\geq 0})$ is equal to $\alpha + \beta S$ or $\alpha + \beta T$ for some $\alpha, \beta \in \mathbb{Z}$.*
4. *$f(\mathbb{Z}_{\geq 0})$ admits a simultaneous ordering of the form $\alpha + \beta\mathbf{s}$ or $\alpha + \beta\mathbf{t}$, where $\alpha, \beta \in \mathbb{Z}$.*

Moreover, if the above conditions hold, then any simultaneous ordering of $f(\mathbb{Z}_{\geq 0})$ is of the form $\alpha + \beta\mathbf{s}$, $\alpha + \beta\mathbf{s}^*$, $\alpha + \beta\mathbf{t}$, or $\alpha + \beta\mathbf{t}^*$, where $\alpha, \beta \in \mathbb{Z}$.

Proof. The equivalence of (1) and (2) is [1, Theorem 14]. Statement (2) is easily seen to imply (3). Clearly (3) implies (4) and (4) implies (1), so the four conditions are equivalent. Finally, the last two statements follow from Proposition 3.5. □

Corollary 5.2. *The sequences $\alpha + \beta\mathbf{s}$ and $\alpha + \beta\mathbf{t}$ for $\alpha, \beta \in \mathbb{Z}$ and $\beta \neq 0$ are the only quadratic integer sequences that are simultaneously ordered.*

A similar proposition, based on [1, Theorem 17], holds for $f(\mathbb{Z})$ instead of $f(\mathbb{Z}_{\geq 0})$. Next, we pose the following conjecture.

Conjecture 5.3. Let $f \in \mathrm{Int}(\mathbb{Z})$ be a polynomial of degree greater than 2. Then $f(\mathbb{Z}_{\geq 0})$ and $f(\mathbb{Z})$ do not admit simultaneous orderings. In particular, the only simultaneously ordered polynomial integer sequences are $\alpha+\beta\mathbf{n}$, $\alpha+\beta\mathbf{s}$, and $\alpha+\beta\mathbf{t}$ for $\alpha, \beta \in \mathbb{Z}$ and $\beta \neq 0$.

We now prove Theorem 1.10, computing $[m, n, l]_{\mathbf{a}}$ for $\mathbf{a} = \mathbf{s}, \mathbf{t}$. By Proposition 5.1, this will determine the coefficients for all simultaneously ordered linear and quadratic (and perhaps even polynomial) sequences.

Lemma 5.4. *Let n, k, l be nonnegative integers with $n = k + l$. One has*

$$\begin{bmatrix} n \\ k, l \end{bmatrix}_{\mathbf{t}} = \binom{2n}{2k}, \quad \begin{bmatrix} a_n \\ k \end{bmatrix}_{\mathbf{t}} = \binom{n+k}{n-k}$$

Proof. Clear. □

Next, consider the following identities:

$$[m, n, l]_{\mathbf{s}} = \frac{2l}{m+n}[m, n, l]_{\mathbf{t}} \quad \text{if } m, n > 0, \tag{5.1}$$

$$[m, n, l-1]_{\mathbf{t}} = \frac{(l-m)(l-n)}{2(2l-1)(m+n-l)}[m, n, l]_{\mathbf{t}}, \tag{5.2}$$

$$[m, n, l]_{\mathbf{t}} = \frac{\binom{m+n}{m,n}}{\binom{2m}{m}\binom{2n}{n}} \binom{2l}{l, l-m, l-n, m+n-l}. \tag{5.3}$$

We discovered (5.1) experimentally by comparing data on the coefficients $[m, n, l]_{\mathbf{s}}$ and $[m, n, l]_{\mathbf{t}}$. Although (5.1) is simpler than the other two identites, we were unable to prove it directly; its simplicty begs for a direct proof and generalization. Instead we prove (5.1) ⇒ (5.2) ⇒ (5.3) ⇒ (5.1) and then verify (5.3) by induction.

Lemma 5.5. *One has (5.1) ⇒ (5.2) ⇒ (5.3) ⇒ (5.1).*

Proof. Note that $[m,n,l]_{\mathbf{t}} = [m,n,l]_{\mathbf{u}}$, where $\mathbf{u} = 2\mathbf{t}$. By Corollary 2.5 one has

$$[m+1,n,l]_{\mathbf{s}} = \frac{2l(2l-1)[m,n,l-1]_{\mathbf{s}} + (l^2-m^2)[m,n,l]_{\mathbf{s}}}{(2m+2)(2m+1)}. \tag{5.4}$$

Suppose that (5.1) holds. Then (5.4) implies that

$$\frac{m+n}{m+n+1}[m+1,n,l]_{\mathbf{u}} = \frac{(2l-1)(2l-2)[m,n,l-1]_{\mathbf{u}} + (l^2-m^2)[m,n,l]_{\mathbf{u}}}{(2m+2)(2m+1)}.$$

By Corollary 2.5 one also has

$$[m+1,n,l]_{\mathbf{u}} = \frac{2l(2l-1)[m,n,l-1]_{\mathbf{u}} + (l^2+l-m^2-m)[m,n,l]_{\mathbf{u}}}{(2m+2)(2m+1)}.$$

Using the previous two equations we can eliminate $[m+1,n,l]_{\mathbf{u}}$, obtaining

$$[m,n,l-1]_{\mathbf{u}} = \frac{(l-m)(l-n)}{2(2l-1)(m+n-l)}[m,n,l]_{\mathbf{u}}.$$

Thus (5.1) implies (5.2).

Now, (5.2) is a first order recurrence relation for the coefficients $[m,n,l]_{\mathbf{u}} = [m,n,l]_{\mathbf{t}}$ with base case $[m,n,m+n]_{\mathbf{u}} = \left[{m+n \atop m,n}\right]_{\mathbf{u}} = \binom{2m+2n}{2m,2n}$. One easily shows that (5.3) is the unique solution to that recurrence relation. Thus (5.2) implies (5.3). Finally, if (5.3) holds, then, carrying out the above arguments in reverse, one sees that $[m,n,l]_{\mathbf{s}}$ and $\frac{2l}{m+n}[m,n,l]_{\mathbf{u}}$ satisfy the same recurrence relation (5.4) with identical base cases thus and are equal. Therefore (5.3) implies (5.1). □

As with Theorem 1.5, Theorem 1.10 of the introduction has a proof based on the recurrence relation of Corollary 2.5.

Proof of Theorem 1.10. By Lemma 5.5, we need only prove (5.3), which is equivalent to

$$[m,n,l]_{\mathbf{t}} = \frac{m!\,n!\,(m+n)!\,(2l)!}{(2m)!\,(2n)!\,l!\,(l-m)!\,(l-n)!\,(m+n-l)!}. \tag{5.5}$$

Note that $[m,n,l]_{\mathbf{t}} = [m,n,l]_{\mathbf{u}}$, where $\mathbf{u} = 2\mathbf{t}$. Thus, by Corollary 2.5, to prove (5.5) it suffices to show that the given expression for $[m,n,l]_{\mathbf{t}} = [m,n,l]_{\mathbf{u}}$ in (5.5) satisfies the recurrence relation

$$[m+1,n,l]_{\mathbf{u}} = \frac{2l(2l-1)[m,n,l-1]_{\mathbf{u}} + (l-m)(l+m+1)[m,n,l]_{\mathbf{u}}}{(2m+2)(2m+1)}$$

and boundary conditions (2.2) and (2.3). The boundary conditions follow easily from Lemma 5.4. Substituting the inductively assumed expressions for $[m,n,l]_{\mathbf{u}}$

and $[m,n,l-1]_{\mathbf{u}}$ in the right hand side of the above recurrence relation, dividing by the desired formula for $[m+1,n,l]_{\mathbf{u}}$, and simplifying by using the identity $\frac{k!}{(k-1)!}=k$ several times, all of the factorials can be eliminated and one is left with

$$\frac{l(l-n)+(l+m+1)(m+n-l+1)}{(m+n+1)(m+1)}=1.$$

This proves (5.3) and completes the proof. □

The numbers

$$C_{m,n}=\frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{m,n}}=\frac{(2m)!(2n)!}{m!(m+n)!n!}$$

have been known to be integers since at least Catalan [5]. One has $C_{m,1}=2C_m$, where C_m is the mth Catalan number. By Theorem 1.10, we have the following.

Corollary 5.6. *Let $m,n,l\in\mathbb{Z}_{\geq 0}$. Then the integer $C_{m,n}=\frac{(2m)!(2n)!}{m!(m+n)!n!}$ divides $\binom{2l}{l,l-m,l-n,m+n-l}$, and $(m+n)C_{m,n}$ divides $2l\binom{2l}{l,l-m,l-n,m+n-l}$.*

6. Newton Basis Coefficients via Laurent Series

We exhibit in this section two generic formulas for the coefficients $[m,n,l]_{\mathbf{a}}$, to be contrasted with the specific formulas uniquely characterizing the examples in this paper and those sought in Problem 1.11.

Let $k\in\mathbb{Z}_{>0}$. Let

$$e_k(X_1,X_2,\ldots,X_n)=\sum_{1\leq i_1<i_2<\cdots<i_k\leq n}X_{i_1}X_{i_2}\cdots X_{i_k}\in\mathbb{Z}[X_1,X_2,\ldots,X_n]$$

denote the *kth elementary symmetric polynomial*, and let

$$h_k(X_1,X_2,\ldots,X_n)=\sum_{1\leq i_1\leq i_2\leq\cdots\leq i_k\leq n}X_{i_1}X_{i_2}\cdots X_{i_k}\in\mathbb{Z}[X_1,X_2,\ldots,X_n]$$

denote the *kth complete homogeneous symmetric polynomial*. We also let $X^{\underline{n}}$ denote $X^{\underline{n}}_{\mathbf{a}}=(X-a_0)(X-a_1)\cdots(X-a_{n-1})$.

Now we may write $f=\sum_{i=0}^{\deg f}f_iX^i$ and

$$f=\sum_{n=0}^{\deg f}d^n_{\mathbf{a}}(f)X^{\underline{n}},\tag{6.1}$$

where the $d^n_{\mathbf{a}}(f)\in K$ are unique (and defined even if the a_i are not distinct), and one has $\delta^n_{\mathbf{a}}(f)=n!_{\mathbf{a}}d^n_{\mathbf{a}}(f)$ for all n. From (6.1) we see that

$$f_i=\sum_{n=i}^{\deg f}(-1)^{n-i}e_{n-i}(a_0,a_1,\ldots,a_{n-1})d^n_{\mathbf{a}}(f)$$

if $0 \leq i \leq \deg f$. In Theorem 6.2 below we use formal Laurent series to invert this formula, solving for the $d^n_{\mathbf{a}}(f)$ in terms of the coefficients f_i of f.

Recall that the quotient field $K((X)) = (K[[X]])[X^{-1}]$ of the formal power series ring $K[[X]]$ is called the field of *formal Laurent series* with coefficients in K.

Lemma 6.1. *The constant coefficient of* $\frac{(1/X)(1/X)^{\underline{k}}}{(1/X)^{\underline{l+1}}} \in K((X))$ *is equal to* $\delta_{k,l}$ *for all* $k, l \in \mathbb{Z}_{\geq 0}$.

Proof. The proof can be divided into the three cases $k = l$, $k < l$, and $k > l$, and is straightforward. ☐

Theorem 6.2. *Let* $f = \sum_{i=0}^N f_i X^i \in K[X]$, *where* $N = \deg f$. *Then* $d^n_{\mathbf{a}}(f)$ *is equal to the constant coefficient of* $\frac{(1/X)f(1/X)}{(1/X)^{\underline{n+1}}} \in K((X))$ *for any* $n \in \mathbb{Z}_{\geq 0}$. *Equivalently,* $d^n_{\mathbf{a}}(f)$ *is equal to the coefficient of* X^{N-n} *in* $\frac{X^N f(1/X)}{\prod_{i=0}^n (1-a_i X)} \in K[[X]]$. *In particular, one has*

$$d^n_{\mathbf{a}}(f) = \sum_{i=N-n}^{N} h_{i-N+n}(a_0, a_1, \ldots, a_n) f_i.$$

Proof. By (6.1) one has $\frac{(1/X)f(1/X)}{(1/X)^{\underline{n+1}}} = \sum_{k=0}^N d^k_{\mathbf{a}}(f) \frac{(1/X)(1/X)^{\underline{k}}}{(1/X)^{\underline{n+1}}}$, and therefore by Lemma 6.1 the constant coefficient of the above formal Laurent series is equal to $\sum_{k=0}^N d^k_{\mathbf{a}}(f)\delta_{k,n} = d^n_{\mathbf{a}}(f)$. ☐

Corollary 6.3. *Let* $f = \sum_{i=0}^N f_i X^i \in K[X]$, *where* $N = \deg f$. *Then* $\delta^n_{\mathbf{a}}(f)$ *is equal to the constant coefficient of* $\frac{f(1/X)}{1-a_n X} \begin{bmatrix} 1/X \\ n \end{bmatrix}_{\mathbf{a}}^{-1} \in K((X))$ *for any* $n \in \mathbb{Z}_{\geq 0}$. *Equivalently,* $\delta^n_{\mathbf{a}}(f)$ *is equal to the coefficient of* X^{N-n} *in* $\frac{n!_{\mathbf{a}} X^N f(1/X)}{\prod_{i=0}^n (1-a_i X)} \in K[[X]]$. *In particular, one has*

$$\delta^n_{\mathbf{a}}(f) = n!_{\mathbf{a}} \sum_{i=N-n}^{N} h_{i-N+n}(a_0, a_1, \ldots, a_n) f_i.$$

The corollary above yields the following.

Corollary 6.4. *Let* $m, n, l \in \mathbb{Z}_{\geq 0}$. *Then* $[m, n, l]_{\mathbf{a}}$ *is equal to the constant coefficient of* $\frac{1}{1-a_l X} \begin{bmatrix} 1/X \\ m \end{bmatrix}_{\mathbf{a}} \begin{bmatrix} 1/X \\ n \end{bmatrix}_{\mathbf{a}} \begin{bmatrix} 1/X \\ l \end{bmatrix}_{\mathbf{a}}^{-1} \in K((X))$ *Equivalently,* $[m, n, l]_{\mathbf{a}}$ *is the coefficient of* X^{m+n-l} *in* $\frac{l!_{\mathbf{a}}}{m!_{\mathbf{a}} n!_{\mathbf{a}}} \frac{\prod_{i=0}^{m-1}(1-a_i X)}{\prod_{i=n}^{l}(1-a_i X)} \in K[[X]]$. *In particular, if* $m > 0$ *and* $n \leq l \leq m+n$, *then one has*

$$[m, n, l]_{\mathbf{a}} = \frac{l!_{\mathbf{a}}}{m!_{\mathbf{a}} n!_{\mathbf{a}}} \sum_{i+j=m+n-l} (-1)^i e_i(a_0, a_1, \ldots, a_{m-1}) h_j(a_n, a_{n+1}, \ldots, a_l).$$

Example 6.5. By Corollary 6.4, $[m,n,n]_{\mathbf{a}}$ is equal to

$$\frac{1}{m!_{\mathbf{a}}}\sum_{i+j=m}(-1)^i e_i(a_0,a_1,\ldots,a_{m-1})a_n^j = \frac{(a_n)^{\underline{m}}}{m!_{\mathbf{a}}} = \begin{bmatrix} a_n \\ m \end{bmatrix}_{\mathbf{a}}.$$

Similarly, $[m,n,m+n]_{\mathbf{a}}$ is the constant coefficient of $\frac{(m+n)!_{\mathbf{a}}}{m!_{\mathbf{a}}n!_{\mathbf{a}}}\frac{\prod_{i=0}^{m-1}(1-a_iX)}{\prod_{i=n}^{m+n}(1-a_iX)}$, which is equal to $\begin{bmatrix} m+n \\ m,n \end{bmatrix}_{\mathbf{a}}$. These computations are consistent with Proposition 2.2.

Finally, we note the following formula for the coefficients $\delta^n_{\mathbf{a}}(f)$ in terms of the values of f at $a_0,a_1,\ldots,a_n$.

Theorem 6.6. *Let $f\in K[X]$. One has*

$$\delta^n_{\mathbf{a}}(f) = \sum_{0\le i_0<\ldots<i_s=n}(-1)^s f(a_{i_0})\prod_{r=1}^{s}\begin{bmatrix} a_{i_r} \\ i_{r-1} \end{bmatrix}$$

for all $n\in\mathbb{Z}_{\ge 0}$.

Corollary 6.7. *One has*

$$[m,n,l]_{\mathbf{a}} = \sum_{n\le i_0<\ldots<i_s=l}(-1)^s \begin{bmatrix} a_{i_0} \\ m \end{bmatrix}\begin{bmatrix} a_{i_0} \\ n \end{bmatrix}\prod_{r=1}^{s}\begin{bmatrix} a_{i_r} \\ i_{r-1} \end{bmatrix}$$

for all $m,n,l\in\mathbb{Z}_{\ge 0}$ with $m\le n\le l$.

The theorem can be proved by expressing the first n terms of the Newton basis in terms of the well-known *Lagrange basis* of the K-vectorspace of polynomials in $K[X]$ of degree at most n and inverting the change of basis matrix.

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