IRRATIONAL NUMBERS ASSOCIATED TO SEQUENCES WITHOUT GEOMETRIC PROGRESSIONS

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Abstract

Let s and k be integers with $s \geq 2$ and $k \geq 3$. Let $g_k^{(s)}(n)$ denote the cardinality of the largest subset of the set $\{1, 2, \ldots, n\}$ that contains no geometric progression of length k whose common ratio is a power of s. Let $r_k(\ell)$ denote the cardinality of the largest subset of the set $\{0, 1, 2, \ldots, \ell - 1\}$ that contains no arithmetic progression of length k. It is proved that the limit

$$\lim_{n\to\infty} \frac{g_k^{(s)}(n)}{n} = (s-1) \sum_{m=1}^{\infty} \left(\frac{1}{s}\right)^{\min\left(r_k^{-1}(m)\right)} = \frac{(s-1)^2}{s} \sum_{\ell=1}^{\infty} \frac{r_k(\ell)}{s^{\ell}}.$$

exists and is an irrational number.

1. Maximal Subsets Without Geometric Progressions

Let $\mathbf{N}$ and $\mathbf{N}_0$ denote the sets of positive integers and nonnegative integers, respectively. For every real number x, the *integer part* of x, denoted $[x]$, is the unique integer n such that $n \leq x < n + 1$.

Let $s \geq 2$ be an integer. Every positive integer a can be written uniquely in the form

$$a = bs^v$$

where b is a positive integer not divisible by s and v is a nonnegative integer. If G is a finite geometric progression of length k whose first term is the positive integer a and whose common ratio is a positive integral power of s, say, s^d with $d \in \mathbf{N}$, then

$$G = \{a\left(s^d\right)^j : j = 0, 1, \ldots, k-1\}.$$

Writing a in the form $a = bs^v$ with b not a multiple of s, we have

$$G = \{bs^{v+dj} : j = 0, 1, \ldots, k-1\} \subseteq \{bs^i : i \in \mathbf{N}_0\} \tag{1}$$

and so the set of exponents of s in the finite geometric progression G is the finite arithmetic progression $\{v + dj : j = 0, 1, \ldots, k-1\}$. Conversely, if $a, d \in \mathbf{N}$ and $\{v_1 + dj : j = 0, 1, \ldots, k-1\}$ is a finite arithmetic progression of k nonnegative integers, then $\{as^{v_1+dj} : j = 0, 1, \ldots, k-1\}$ is a geometric progression of length k. Writing $a = bs^{v_0}$ with $v_0 \in \mathbf{N}_0$ and b not divisible by s, we obtain $\{as^{v_1+dj} : j = 0, 1, \ldots, k-1\} = \{bs^{v+dj} : j = 0, 1, \ldots, k-1\}$, where $v = v_0 + v_1$.

Let ℓ and k be positive integers with $k \geq 3$. Let $r_k(\ell)$ denote the cardinality of the largest subset of the set $\{0, 1, 2, \ldots, \ell-1\}$ that contains no arithmetic progression of length k. Because the geometric progression $(2^i)_{i=0}^{\infty}$ contains no 3-term arithmetic progression, it follows that

$$\lim_{\ell \to \infty} r_k(\ell) = \infty. \tag{2}$$

Note that $r_k(\ell) = \ell$ for $\ell = 1, \ldots, k-1$, that $r_k(k) = k-1$, and that, for every $\ell \in \mathbf{N}$, there exists $\varepsilon_\ell \in \{0, 1\}$ such that

$$r_k(\ell + 1) = r_k(\ell) + \varepsilon_\ell. \tag{3}$$

Equivalently,

$$0 \leq r_k(\ell + 1) - r_k(\ell) \leq 1. \tag{4}$$

It follows from (2) and (4) that the function $r_k : \mathbf{N} \to \mathbf{N}$ is increasing and surjective. This implies that, for every positive integer m, the set

$$r_k^{-1}(m) = \{\ell \in \mathbf{N} : r_k(\ell) = m\}$$

is a nonempty set of consecutive integers. We define

$$u_m = \min\left(r_k^{-1}(m)\right).$$

Then $(u_m)_{m=1}^{\infty}$ is a strictly increasing sequence of positive integers.

Lemma 1. *Let $k \geq 3$ and let $u_m = \min\left(r_k^{-1}(m)\right)$ for $m \in \mathbf{N}$. Then*

$$\limsup_{m \to \infty}(u_{m+1} - u_m) = \infty.$$

Proof. We use Szemerédi's theorem, which states that $r_k(\ell) = o(\ell)$, to prove that the increasing sequence $(u_m)_{m=1}^{\infty}$ has unbounded gaps.

Note that $u_1 = 1$. If $\limsup_{m\to\infty}(u_{m+1} - u_m) < \infty$, then there is an integer $c \geq 2$ such that $u_{m+1} - u_m < c$ for all $m \in \mathbf{N}$. It follows that

$$\begin{aligned} \max\left(r_k^{-1}(m)\right) + 1 &= \min\left(r_k^{-1}(m+1)\right) \\ &= u_{m+1} \\ &= \sum_{i=1}^{m}(u_{i+1} - u_i) + u_1 \\ &< cm + 1. \end{aligned}$$

Thus, $\max\left(r_k^{-1}(m)\right) < cm$ and so $r_k(cm) > m$. Equivalently,

$$\frac{r_k(cm)}{cm} > \frac{1}{c} > 0$$

for all $m \in \mathbf{N}$. This contradicts Szemerédi's theorem, and completes the proof. □

For $k \geq 3$, let $g_k(n)$ denote the cardinality of the largest subset of the set $\{1, 2, \ldots, n\}$ that contains no geometric progression of length k. This function, introduced by Rankin [4], has been investigated by M. Beiglböck, V. Bergelson, N. Hindman, and D. Strauss [1], by Brown and Gordon [2], and by Riddell [5]. The best upper bound for the function $g_k(n)$ is due to Nathanson and O'Bryant [3].

For $s \geq 2$ and $k \geq 3$, let $g_k^{(s)}(n)$ denote the cardinality of the largest subset of the set $\{1, 2, \ldots, n\}$ that contains no geometric progression of length k whose common ratio is a power of s. The goal of this paper is to prove that the limit

$$\lim_{n\to\infty} \frac{g_k^{(s)}(n)}{n} = (s-1)\sum_{m=1}^{\infty}\left(\frac{1}{s}\right)^{\min\left(r_k^{-1}(m)\right)} = \frac{(s-1)^2}{s}\sum_{\ell=1}^{\infty}\frac{r_k(\ell)}{s^\ell} \tag{5}$$

exists and converges to an irrational number.

2. Maximal Geometric Progression-Free Sets

Lemma 2. *Let k, s, and n be positive integers with $k \geq 3$ and $s \geq 2$, and let*

$$\mathcal{B}_n = \{b \in \{1, 2, \ldots, n\} : s \text{ does not divide } b\}.$$

Then

$$g_k^{(s)}(n) = \sum_{b\in\mathcal{B}_n} r_k\left(1 + \left[\log_s\left(\frac{n}{b}\right)\right]\right).$$

Proof. Let $b \in \mathcal{B}_n$. For $i \in \mathbf{N}_0$, we have $bs^i \leq n$ if and only if $0 \leq i \leq \log_s(n/b)$. We define

$$\begin{aligned} T(b) &= \{t \in \{1, 2, \ldots, n\} : t = bs^i \text{ for some } i \in \mathbf{N}_0\} \\ &= \left\{ bs^i : i = 0, 1, \ldots, \left[\log_s\left(\frac{n}{b}\right)\right]\right\}. \end{aligned}$$

Then $b \in T(b)$ and

$$\{1, 2, \ldots, n\} = \bigcup_{b \in \mathcal{B}_n} T(b)$$

is a partition of $\{1, 2, \ldots, n\}$ into pairwise disjoint nonempty subsets.

If the set $\{1, 2, \ldots, n\}$ contains a finite geometric progression of length k whose common ratio is a power of s, then, by (1), this geometric progression is a subset of $T(b)$ for some $b \in \mathcal{B}_n$, and the set of exponents of s is a finite arithmetic progression of length k contained in the set of consecutive integers $\{0, 1, \ldots, [\log_s(n/b)]\}$. It follows that the largest cardinality of a subset of $T(b)$ that contains no k-term geometric progression whose common ratio is a power of s is equal to the largest cardinality of a subset of $\{0, 1, \ldots, [\log_s(n/b)]\}$ that contains no k-term arithmetic progression. This number is

$$r_k\left(1 + \left[\log_s\left(\frac{n}{b}\right)\right]\right).$$

If A_n is a subset of $\{1, 2, \ldots, n\}$ of maximum cardinality that contains no k-term geometric progression whose common ratio is a power of s, then

$$|A_n \cap T(b)| = r_k\left(1 + \left[\log_s\left(\frac{n}{b}\right)\right]\right).$$

Because $A = \bigcup_{b \in \mathcal{B}_n} T(b)$ is a partition of $\{1, \ldots, n\}$, it follows that

$$|A_n| = \sum_{b \in \mathcal{B}_n} |A_n \cap T(b)| = \sum_{b \in \mathcal{B}_n} r_k\left(1 + \left[\log_s\left(\frac{n}{b}\right)\right]\right).$$

This completes the proof. □

3. Construction of an Irrational Number

Lemma 3. *Let s be an integer with $s \geq 2$. Let x and y be real numbers with $x < y$. The number of integers n such that $x < n \leq y$ and s does not divide n is*

$$\left(\frac{s-1}{s}\right)(y - x) + O(1).$$

Proof. This is a straightforward calculation. □

Theorem 1. *Let k and s be integers with $k \geq 3$ and $s \geq 2$. The limit*

$$\lim_{n\to\infty} \frac{g_k^{(s)}(n)}{n} = \frac{(s-1)^2}{s} \sum_{\ell=1}^{\infty} \frac{r_k(\ell)}{s^\ell} = (s-1) \sum_{m=1}^{\infty} \left(\frac{1}{s}\right)^{\min\left(r_k^{-1}(m)\right)}$$

exists and converges to an irrational number.

Proof. For every positive integer b we have

$$1 + [\log_s(n/b)] = \ell$$

if and only if

$$\frac{n}{s^\ell} < b \leq \frac{sn}{s^\ell}.$$

By Lemma 3, the number of integers in this interval that are also in $\mathcal{B}_n$, that is, are not divisible by s, is

$$\left(\frac{s-1}{s}\right) \frac{(s-1)n}{s^\ell} + O(1) = \frac{n(s-1)^2}{s^{\ell+1}} + O(1).$$

Let $n \geq s$. Because $1 \in \mathcal{B}_n$, we have

$$L(n) = \max\{1 + [\log_s(n/b)] : b \in \mathcal{B}_n\} = 1 + [\log_s n] \leq 2\log_s n,$$

Also, if $\ell \leq L(n)$, then $r_k(\ell) \leq \ell \leq L(n)$. By Lemma 2,

$$\begin{aligned}
g_k^{(s)}(n) &= \sum_{b\in\mathcal{B}_n} r_k\left(1 + [\log_s(n/b)]\right) \\
&= \sum_{\ell=1}^{L(n)} r_k(\ell) \times |\{b \in \mathcal{B}_n : \ell = 1 + [\log_s(n/b)]\}| \\
&= \sum_{\ell=1}^{L(n)} r_k(\ell) \left(\frac{n(s-1)^2}{s^{\ell+1}} + O(1)\right) \\
&= \frac{n(s-1)^2}{s} \sum_{\ell=1}^{L(n)} \frac{r_k(\ell)}{s^\ell} + O\left(\sum_{\ell=1}^{L(n)} r_k(\ell)\right) \\
&= \frac{n(s-1)^2}{s} \sum_{\ell=1}^{L(n)} \frac{r_k(\ell)}{s^\ell} + O\left(L(n)^2\right) \\
&= n\left(\frac{(s-1)^2}{s} \sum_{\ell=1}^{L(n)} \frac{r_k(\ell)}{s^\ell} + O\left(\frac{\log_s^2 n}{n}\right)\right) \\
&= n\left(\frac{(s-1)^2}{s} \sum_{\ell=1}^{L(n)} \frac{r_k(\ell)}{s^\ell} + o(1)\right).
\end{aligned}$$

Because $(L(n))_{n=1}^{\infty}$ is an increasing sequence and $\lim_{n\to\infty} L(n) = \infty$, we have

$$\lim_{n\to\infty} \frac{g_k^{(s)}(n)}{n} = \lim_{n\to\infty} \left(\frac{(s-1)^2}{s} \sum_{\ell=1}^{L(n)} \frac{r_k(\ell)}{s^\ell} + o(1) \right) = \frac{(s-1)^2}{s} \sum_{\ell=1}^{\infty} \frac{r_k(\ell)}{s^\ell}.$$

Let $M(n) = r_k(L(n))$ and let $u_m = \min\left(r_k^{-1}(m)\right)$ and $U_m = \max\left(r_k^{-1}(m)\right)$. Then $M(n) \le L(n)$ and $U_m + 1 = u_{m+1}$ for $m = 1, \ldots, M(n)$. We have

$$\begin{aligned}
\sum_{\ell=1}^{L(n)} \frac{r_k(\ell)}{s^\ell} &= \sum_{m=1}^{M(n)-1} m \sum_{\ell \in r_k^{-1}(m)} \frac{1}{s^\ell} + M(n) \sum_{\ell \in r_k^{-1}(M(n)) \cap \{1,\ldots,L(n)\}} \frac{1}{s^\ell} \\
&= \sum_{m=1}^{M(n)-1} m \sum_{\ell=u_m}^{U_m} \frac{1}{s^\ell} + M(n) \sum_{\ell=u_{M(n)}}^{L(n)} \frac{1}{s^\ell} \\
&= \frac{s}{s-1} \sum_{m=1}^{M(n)-1} m \left(\left(\frac{1}{s}\right)^{u_m} - \left(\frac{1}{s}\right)^{U_m+1} \right) \\
&\qquad + \frac{s}{s-1} M(n) \left(\left(\frac{1}{s}\right)^{u_{M(n)}} - \left(\frac{1}{s}\right)^{L(n)+1} \right) \\
&= \frac{s}{s-1} \sum_{m=1}^{M(n)-1} m \left(\left(\frac{1}{s}\right)^{u_m} - \left(\frac{1}{s}\right)^{u_{m+1}} \right) \\
&\qquad + \frac{s}{s-1} M(n) \left(\left(\frac{1}{s}\right)^{u_{M(n)}} - \left(\frac{1}{s}\right)^{L(n)+1} \right) \\
&= \frac{s}{s-1} \left(\sum_{m=1}^{M(n)} m \left(\frac{1}{s}\right)^{u_m} - \sum_{m=2}^{M(n)} (m-1) \left(\frac{1}{s}\right)^{u_m} - M(n) \left(\frac{1}{s}\right)^{L(n)+1} \right) \\
&= \frac{s}{s-1} \sum_{m=1}^{M(n)} \frac{1}{s^{u_m}} - \frac{1}{s-1} \left(\frac{M(n)}{s^{L(n)}} \right) \\
&= \frac{s}{s-1} \sum_{m=1}^{M(n)} \frac{1}{s^{u_m}} + o(1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{g_k^{(s)}(n)}{n} &= \frac{(s-1)^2}{s} \left(\frac{s}{s-1} \sum_{m=1}^{M(n)} \left(\frac{1}{s}\right)^{u_m} + o(1) \right) + o(1) \\
&= (s-1) \sum_{m=1}^{M(n)} \left(\frac{1}{s}\right)^{u_m} + o(1)
\end{aligned}$$

and so

$$\lim_{n\to\infty} \frac{g_k^{(s)}(n)}{n} = \sum_{m=1}^{\infty} \frac{s-1}{s^{u_m}}.$$

Note that $u_1 = 1$. The infinite series converges to a real number $\theta_s \in (1 - 1/s, 1)$, and the "decimal digits to base s" of θ_s are 0 or $s - 1$. The number θ_s is rational if and only if these digits are eventually periodic, but Lemma 1 implies that there are unbounded gaps between successive digits equal to 1. Therefore, θ is irrational. This completes the proof. □

4. Two Remarks

The irrationality statement in Theorem 1 is a special case of the following observation.

Theorem 2. *Let $(a_n)_{n=1}^{\infty}$ be an unbounded sequence of positive integers such that $0 \leq a_n - a_{n-1} \leq s - 1$ for all $n \geq 2$. If $a_n = o(n)$, then, for every integer $s \geq 2$, the real number $\sum_{n=1}^{\infty} a_n/s^n$ is irrational.*

Proof. Let $\varepsilon_1 = a_1$ and let $\varepsilon_n = a_n - a_{n-1}$ for $n \geq 2$. Then $\varepsilon_n \geq 0$ and $a_n = \sum_{i=1}^{n} \varepsilon_i$ for all $n \geq 2$.

Suppose there exist positive integers c and n_0 such that $a_{n+c} - a_n \geq 1$ for all $n \geq n_0$. It follows that for every positive integer k we have

$$a_{n_0+kc} = a_{n_0} + \sum_{i=1}^{k} (a_{n_0+ic} - a_{n_0+(i-1)c}) \geq a_{n_0} + k$$

and so

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{a_n}{n} = \lim_{k\to\infty} \frac{a_{n_0+kc}}{n_0+kc} \\ &\geq \liminf_{k\to\infty} \frac{a_{n_0}+k}{n_0+kc} \geq \liminf_{k\to\infty} \frac{k}{n_0+kc} \\ &= \frac{1}{c} > 0 \end{aligned}$$

which is absurd. Therefore, for every positive integer c there exist infinitely many positive integers n such that

$$\sum_{i=n+1}^{n+c} \varepsilon_i = a_{n+c} - a_n = 0$$

and so $\varepsilon_i = 0$ for $i = n+1, \ldots, n+c$. Thus, the sequence $(\varepsilon_i)_{i=1}^{\infty}$ contains arbitrarily long finite sequences of zeros. Because the sequence $(a_n)_{n=1}^{\infty}$ is unbounded, it is not eventually constant, and so $(\varepsilon_i)_{i=1}^{\infty}$ is not eventually zero. It follows that $(\varepsilon_i)_{i=1}^{\infty}$ is not eventually periodic. We have

$$\sum_{n=1}^{\infty} \frac{a_n}{s^n} = \sum_{n=1}^{\infty} \frac{1}{s^n} \sum_{i=1}^{n} \varepsilon_i = \sum_{i=1}^{\infty} \varepsilon_i \sum_{n=i}^{\infty} \frac{1}{s^n} = \frac{s}{s-1} \sum_{i=1}^{\infty} \frac{\varepsilon_i}{s^i}.$$

Because $(\varepsilon_i)_{i=1}^{\infty}$ is the sequence of "decimal digits to base s" of $\sum_{i=1}^{\infty} \varepsilon_i s^{-i}$ and because $(\varepsilon_i)_{i=1}^{\infty}$ is not eventually periodic, it follows that $\sum_{n=1}^{\infty} a_n s^{-n}$ is irrational. □

We define the function $\hat{g}_k^{(s)}(n)$ as the cardinality of the largest subset of $\{1, 2, \ldots, n\}$ that contains no geometric progression of length k with common ratio s. Recall that $\mathcal{B}_n = \{b \in \{1, 2, \ldots, n\} : s \text{ does not divide } b\}$.

Theorem 3. *Let $s \geq 2$ and $k \geq 3$ be integers, and let Y^* be the set of all integers in $\{1, \ldots, n\}$ of the form bs^{qk-1} with $b \in \mathcal{B}_n$ and $q \in \mathbf{N}$. Then $X^* = \{1, \ldots, n\} \setminus Y^*$ contains no k-term geometric progression with common ratio s, and*

$$\hat{g}_k^{(s)}(n) = |X^*|. \tag{6}$$

Moreover,

$$\lim_{n\to\infty} \frac{\hat{g}_k^{(s)}(n)}{n} = \frac{s^k - s}{s^k - 1}.$$

Proof. Every k-term geometric progression of positive integers with common ratio s is of the form $\{bs^{v+i} : i = 0, 1, \ldots, k-1\}$ for some $b \in \mathcal{B}_n$ and $v \in \mathbf{N}_0$, and so the set of exponents of s is a complete system of residues modulo k, hence contains an integer congruent to $k-1$ modulo k. It follows that the set

$$X^* = \left\{ bs^{qk-i} \in \{1, \ldots, n\} : b \in \mathcal{B}_n, q \in \mathbf{N}, \text{ and } i \in \{2, 3, \ldots, k\} \right\} \tag{7}$$

contains no k-term geometric progression with common ratio s.

Let X_0 be any subset of $\{1, \ldots, n\}$ that contains no k-term geometric progression with common ratio s. We shall prove that $|X_0| \leq |X^*|$. For each $b \in \mathcal{B}_n$, the set X_0 does not contain the geometric progression $\{bs^{k-i} : i = 1, 2, \ldots, k\}$. It follows that the set

$$X_1 = \left(X_0 \cup \{bs^{k-i} : i = 2, 3, \ldots, k\} \right) \setminus \{bs^{k-1} : b \in \mathcal{B}_n\}$$

also contains no k-term geometric progression with common ratio s. Moreover, $|X_1| \geq |X_0|$.

For $q \in \mathbf{N}$, let X_q be a subset of $\{1, \ldots, n\}$ such that

(i) X_q contains no k-term geometric progression with common ratio s,

(ii)
$$\bigcup_{q'=1}^{q} \left\{ bs^{q'k-i} : i = 2, 3, \ldots, k \right\} \subseteq X_q$$

(iii)
$$\left\{ bs^{q'k-1} : q' = 1, \ldots, q \right\} \cap X_q = \emptyset.$$

If $bs^{(q+1)k-1} > n$, let

$$X_{q+1} = \left(X_q \cup \left\{ bs^{(q+1)k-i} : i = 2, 3, \ldots, k \right\} \right) \cap \{1, \ldots, n\}.$$

If $bs^{(q+1)k-1} \leq n$, let

$$X_{q+1} = \left(X_q \cup \left\{ bs^{(q+1)k-i} : i = 2, 3, \ldots, k \right\} \right) \setminus \{bs^{(q+1)k-1}\}.$$

In both cases, the set X_{q+1} contains no k-term geometric progression with common ratio s, and $|X_{q+1}| \geq |X_q|$. Continuing inductively, we obtain the set $X^*(b) \subseteq \{1, 2, \ldots, n\}$ such that

(i) $X^*(b)$ contains no k-term geometric progression with common ratio s,

(ii) $n' \in X^*(b)$ if $n' \in \{1, 2, \ldots, n\}$ and $n' = bs^{qk-i}$ for some $q \in \mathbf{N}$ and $i \in \{2, 3, \ldots, k\}$,

(iii) $bs^{qk-1} \notin X^*(b)$ for all $q \in \mathbf{N}$,

(iv) $|X_0| \leq |X^*(b)|$.

Iterating this construction with $X_0 = X^*(b)$ and $b' \in \mathcal{B}_n \setminus \{b\}$, we obtain the set X^* defined by (7). The inequality $|X_0| \leq |X^*|$ implies (6).

We shall estimate the cardinality of the set

$$Y^* = \{1, 2, \ldots, n\} \setminus X^* = \left\{ bs^{qk-1} \in \{1, \ldots, n\} : b \in \mathcal{B}_n \text{ and } q \in \mathbf{N} \right\}.$$

If $bs^{qk-1} \in Y^*$ and $n \geq s$, then

$$q \leq \frac{1}{k} \left(\log_s n - \log_s b + 1 \right) \leq \log_s n.$$

For $q \leq \log_s n$, we have $bs^{qk-1} \in \{1, \ldots, n\}$ if s does not divide b and

$$0 < b \leq \frac{n}{s^{qk-1}}.$$

By Lemma 3, the number of such b is

$$\left(\frac{s-1}{s}\right)\frac{n}{s^{qk-1}}+O(1)=\frac{(s-1)n}{s^{qk}}+O(1)$$

and so

$$\begin{aligned}|Y^*| &= \sum_{q=1}^{[\log_s n]}\frac{(s-1)n}{s^{qk}}+O(\log_s n)\\ &= n(s-1)\left(\sum_{q=1}^{\infty}\frac{1}{s^{qk}}-\sum_{q=[\log_s n]+1}^{\infty}\frac{1}{s^{qk}}\right)+O(\log_s n)\\ &= \frac{n(s-1)}{s^k-1}\left(1-\frac{1}{s^{[\log_s n]k}}\right)+O(\log_s n).\end{aligned}$$

Therefore

$$\frac{|Y^*|}{n}=\frac{s-1}{s^k-1}+O\left(\frac{1}{n^k}\right)+O\left(\frac{\log_s n}{n}\right)$$

and

$$\begin{aligned}\frac{\hat{g}_k^{(s)}(n)}{n} &= 1-\frac{|Y^*|}{n}\\ &= 1-\frac{s-1}{s^k-1}+O\left(\frac{1}{n^k}\right)+O\left(\frac{\log_s n}{n}\right)\\ &= \frac{s^k-s}{s^k-1}+o(n).\end{aligned}$$

This completes the proof. □

5. Open Problems

1. Let k and s be integers with $k \geq 3$ and $s \geq 2$. Is the number

$$\lim_{n\to\infty}\frac{g_k^{(s)}(n)}{n}=(s-1)\sum_{m=1}^{\infty}\left(\frac{1}{s}\right)^{\min\left(r_k^{-1}(m)\right)}$$

transcendental?

2. Let $u_m = \min\left(r_k^{-1}(m)\right)$ for $m \in \mathbf{N}$. Prove that the sequence $(u_m)_{m=1}^{\infty}$ is not eventually periodic without using Szemerédi's theorem.

3. Let s and s' be integers with $2 \leq s < s'$. Is it true that $g_k^{(s')}(n) \leq g_k^{(s)}(n)$ for all $n \in \mathbf{N}$ and that $g_k^{(s')}(n) < g_k^{(s)}(n)$ for all sufficiently large integers n?

4. Let $\mathcal{S}$ be a finite set of integers such that $s \geq 2$ for all $s \in \mathcal{S}$. For $k \geq 3$, let $g_k^{(\mathcal{S})}(n)$ denote the cardinality of the largest subset of the set $\{1, 2, \ldots, n\}$ that contains no geometric progression of length k whose common ratio is a power of s for some $s \in \mathcal{S}$. Does

$$\lim_{n\to\infty} \frac{g_k^{(\mathcal{S})}(n)}{n}$$

exist? If so, can this limit be expressed by an infinite series analogous to (5)?

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References

[1] M. Beiglböck, V. Bergelson, N. Hindman, and D. Strauss, *Multiplicative structures in additively large sets*, J. Combin. Theory Ser. A **113** (2006), no. 7, 1219–1242.

[2] B. E. Brown and D. M. Gordon, *On sequences without geometric progressions*, Math. Comp. **65** (1996), no. 216, 1749–1754.

[3] M. B. Nathanson and K. O'Bryant, *On sequences without geometric progressions*, Integers **13** (2013), #A73, 1–5.

[4] R. A. Rankin, *Sets of integers containing not more than a given number of terms in arithmetical progression*, Proc. Roy. Soc. Edinburgh Sect. A **65** (1960/1961), 332–344.

[5] J. Riddell, *Sets of integers containing no n terms in geometric progression*, Glasgow Math. J. **10** (1969), 137–146.