REPRESENTATIONS OF SQUARES BY CERTAIN TERNARY QUADRATIC FORMS

Dongxi Ye
Department of Mathematics, University of Wisconsin, Madison, Wisconsin
lawrencefrommath@gmail.com

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Abstract

For any positive integer n, we state and prove formulas for the number of solutions in integers of $x^2+y^2+5z^2=n^2$, $x^2+y^2+6z^2=n^2$ and $x^2+2y^2+3z^2=n^2$, which were first conjectured by S. Cooper and H. Y. Lam.

1. Introduction

More than one hundred years ago, Hurwitz [6] determined the number of representations of the square of a positive integer as a sum of three squares:

Theorem 1.1 (A. Hurwitz). *Let n be a positive integer with prime factorization $n=\prod_{p} p^{\lambda_p}$. Then the number of solutions in integers of $x^2+y^2+z^2=n^2$ is given by*

$$6\prod_{p\geq 3}\left(\frac{p^{\lambda_p+1}-1}{p-1}-\left(\frac{-1}{p}\right)\frac{p^{\lambda_p}-1}{p-1}\right), \tag{1.1}$$

where the values of the Legendre symbol for odd primes p are given by

$$\left(\frac{-1}{p}\right)=\begin{cases}1 & \text{if } p\equiv 1 \pmod 4,\\ -1 & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

For a complete proof of Theorem 1.1, we refer the reader to [8]. In recent work [4], Cooper and Lam established and proved several analogues of Theorem 1.1, whose quadratic forms were replaced by $x^2+y^2+2z^2$, $x^2+y^2+3z^2$, $x^2+2y^2+2z^2$ and $x^2+3y^2+3z^2$. Moreover, at the end of their work, they raised a conjecture:

Conjecture 1.2 (S. Cooper and H. Y. Lam). Let b and c have any of the values given in Table 1. Let n be a positive integer with prime factorization $n=\prod_{p} p^{\lambda_p}$.

Then the number of solutions in integers of $x^2+by^2+cz^2=n^2$ is given by a formula of the type

$$\left(\prod_{p|2bc} g(b,c,p,\lambda_p)\right)\left(\prod_{p\nmid 2bc} h(b,c,p,\lambda_p)\right),$$

where

$$h(b,c,p,\lambda_p)=\frac{p^{\lambda_p+1}-1}{p-1}-\left(\frac{-bc}{p}\right)\frac{p^{\lambda_p-1}}{p-1}$$

and $g(b,c,p,\lambda_p)$ has to be determined on an individual and case-by-case basis.

Table 1: Data for Conjecture 1.2

b	c
1	$1,2,3,4,5,6,8,9,12,21,24$
2	$2,3,4,5,6,8,10,13,16,22,40,70$
3	$3,4,5,6,9,10,12,18,21,30,45$
4	$4,6,8,12,24$
5	$5,8,10,13,25,40$
6	$6,9,16,18,24$
8	$8,10,13,16,40$
9	$9,12,21,24$
10	30
12	12
16	24
21	21
24	24

The goal of this work is to state and prove formulas for the cases $(b,c)=(1,5)$, $(1,6)$ and $(2,3)$ in Conjecture 1.2, which are summarized in Theorems 1.3–1.5, respectively.

Theorem 1.3. *Let n be a positive integer with prime factorization $n=\prod_p p^{\lambda_p}$. Then the number of solutions in integers of $x^2+y^2+5z^2=n^2$ is given by*

$$2\left(5^{\lambda_5+1}-3\right)\prod_{p\nmid 10}\left(\frac{p^{\lambda_p+1}-1}{p-1}-\left(\frac{-5}{p}\right)\frac{p^{\lambda_p}-1}{p-1}\right), \tag{1.2}$$

where the values of the Legendre symbol for primes p that are relatively prime to 10 *are given by*

$$\left(\frac{-5}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1,\ 3,\ 7 \text{ or } 9 \pmod{20}, \\ -1 & \text{if } p \equiv 11,\ 13,\ 17 \text{ or } 19 \pmod{20}. \end{cases}$$

Theorem 1.4. *Let n be a positive integer with prime factorization $n = \prod_p p^{\lambda_p}$. Then the number of solutions in integers of*

$$x^2 + y^2 + 6z^2 = n^2$$

is given by

$$4\left|2^{\lambda_2+1} - 3\right| \prod_{p\geq 5} \left(\frac{p^{\lambda_p+1}-1}{p-1} - \left(\frac{-6}{p}\right) \frac{p^{\lambda_p}-1}{p-1} \right), \tag{1.3}$$

where the values of the Legendre symbol for primes p that are relatively prime to 6 are given by

$$\left(\frac{-6}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1,\ 5,\ 7 \text{ or } 11 \pmod{24}, \\ -1 & \text{if } p \equiv 13,\ 17,\ 19 \text{ or } 23 \pmod{24}. \end{cases}$$

Theorem 1.5. *Let n be a positive integer with prime factorization $n = \prod_p p^{\lambda_p}$. Then the number of solutions in integers of $x^2 + 2y^2 + 3z^2 = n^2$ is given by*

$$2\left(2 + (-1)^n\right)\left(3^{\lambda_3+1} - 2\right) \prod_{p\geq 5} \left(\frac{p^{\lambda_p+1}-1}{p-1} - \left(\frac{-6}{p}\right) \frac{p^{\lambda_p}-1}{p-1} \right). \tag{1.4}$$

This work is organized as follows. In Section 2, some notation and some preliminary results for theta functions will be defined. Sections 3 and 4 will be devoted to the proof of Theorem 1.3: the primes $p = 2$ and $p = 5$ are handled in Section 3 and the remaining primes p that are relatively prime to 10 are treated in Section 4. The proof of Theorem 1.4 is similar to that of Theorem 1.3 and will be given in Section 5. The proof of Theorem 1.5 is the simplest one and will be given in Section 6.

2. Notation and Background Results

The theta functions $\varphi(q)$, $\psi(q)$, $X(q)$, $P(q)$, $D(q)$ and $E(q)$ are defined by

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2}, \quad \psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}, \quad X(q) = \sum_{j=-\infty}^{\infty} q^{3j^2+2j},$$

$$P(q) = \sum_{j=-\infty}^{\infty} q^{(3j^2+j)/2}, \quad D(q) = \sum_{j=-\infty}^{\infty} q^{5j^2+2j} \quad \text{and} \quad E(q) = \sum_{j=-\infty}^{\infty} q^{5j^2+4j},$$

and for any positive integer k, we define φ_k, ψ_k, X_k, P_k, D_k and E_k by

$$\varphi_k = \varphi(q^k), \quad \psi_k = \psi(q^k), \quad X_k = X(q^k), \tag{2.1}$$

$$P_k = P(q^k), \quad D_k = D(q^k) \quad \text{and} \quad E_k = E(q^k). \tag{2.2}$$

For positive integers a, b and c, and for any nonnegative integer n, let $r_{(a,b,c)}(n)$ denote the number of solutions in integers of $ax^2 + by^2 + cz^2 = n$. Clearly,

$$\sum_{m=0}^{\infty} r_{(a,b,c)}(m)q^m = \varphi(q^a)\varphi(q^b)\varphi(q^c) = \varphi_a\varphi_b\varphi_c.$$

Lemma 2.1. *Let φ_k, ψ_k, X_k, P_k, D_k and E_k be defined by* (2.1) *and* (2.2)*. Then the following identities hold.*

$$\varphi_1 = \varphi_4 + 2q\psi_8, \tag{2.3}$$

$$\psi_1^2 = \varphi_4\psi_2 + 2q\psi_2\psi_8, \tag{2.4}$$

$$\psi_1\psi_3 = \varphi_6\psi_4 + q\varphi_2\psi_{12}, \tag{2.5}$$

$$\varphi_1^2 = \varphi_2^2 + 4q\psi_4^2, \tag{2.6}$$

$$\varphi_1 = \varphi_9 + 2qX_3, \tag{2.7}$$

$$P_1^2 = \varphi_3 P_2 + 2qX_1\psi_6, \tag{2.8}$$

$$\varphi_1 = \varphi_{25} + 2qD_5 + 2q^4E_5, \tag{2.9}$$

$$\varphi_1^2 = \varphi_5^2 + 4qD_1E_1. \tag{2.10}$$

Proof. See (i), (xiii), (xxxiii), (ii), (v), (xxx), (x) and (xi), respectively, in [3]. □

The following result is due to Hurwitz and has been summarized by Sandham [9].

Lemma 2.2. *Suppose that $a(n)$ is a function, defined for all nonnegative integers n, that satisfies the property*

$$a(pn) = a(p)a(n) - \chi(p)a(n/p) \tag{2.11}$$

for all primes p, where χ is a completely multiplicative function. Then the coefficient of q^{n^2} in

$$\left(\sum_{j=-\infty}^{\infty} q^{j^2}\right)\left(\sum_{k=0}^{\infty} a(k)q^k\right)$$

is equal to

$$\sum_{r=1}^{\infty} A(2n/r)\chi(r)\mu(r)$$

where μ is the Möbius function, $A(n)$ is defined by

$$\sum_{n=0}^{\infty} A(n)q^n = \left(\sum_{k=0}^{\infty} a(k)q^k\right)^2$$

and $A(x)$ is defined to be 0 if x is not a nonnegative integer.

3. Proof of Theorem 1.3: Part 1

This section is devoted to establishing the parts of the formula in Theorem 1.3 that involve the primes 2 and 5.

Lemma 3.1. *Fix an odd integer j. Then for any nonnegative integer k, we have*

$$r_{(1,1,5)}(2^{2k}j^2) = r_{(1,1,5)}(j^2). \tag{3.1}$$

Proof. By (2.3), we have $\varphi_1^2\varphi_5 = (\varphi_4 + 2q\psi_8)^2(\varphi_{20} + 2q^5\psi_{40})$. Expanding, extracting the terms of the form q^{4n}, and then replacing q^4 with q we deduce

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(2^2 n)q^n = \varphi_1^2\varphi_5.$$

This leads to

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(2^{2k} n)q^n = \varphi_1^2\varphi_5 = \sum_{n=0}^{\infty} r_{(1,1,5)}(n)q^n$$

and this implies (3.1). □

Lemma 3.2. *Fix an integer j that is not divisible by 5. For any nonnegative integer k, let $f(k) = r_{(1,1,5)}(5^{2k}j^2)$. Then*

$$f(k+2) = 6f(k+1) - 5f(k), \tag{3.2}$$

$$f(1) = 11f(0). \tag{3.3}$$

Hence,

$$f(k) = \frac{5^{k+1} - 3}{2} f(0). \tag{3.4}$$

Proof. For (3.2), by (2.9), we have

$$\varphi_1^2\varphi_5 = (\varphi_{25} + 2qD_5 + 2q^4E_5)^2\varphi_5. \tag{3.5}$$

Expanding, extracting the terms of the form q^{5n}, replacing q^5 with q, and then applying (2.10), we deduce

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(5n)q^n = 2\varphi_1^3 - \varphi_1\varphi_5^2. \tag{3.6}$$

Applying (2.9) to (3.6) gives

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(5n)q^n = 2(\varphi_{25} + 2qD_5 + 2q^4E_5)^3 - (\varphi_{25} + 2qD_5 + 2q^4E_5)\varphi_5^2. \quad (3.7)$$

Applying the process of obtaining (3.6) to the terms of the form q^{5n} in (3.7), we deduce

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(5^2n)q^n = 11\varphi_1^2\varphi_5 - 10\varphi_5^3. \quad (3.8)$$

A similar process applied to the terms of the form q^{25n} in (3.8) leads to

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(5^4n)q^n = 61\varphi_1^2\varphi_5 - 60\varphi_5^3. \quad (3.9)$$

By (3.8) and (3.9), we deduce that

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(5^4n)q^n = 6\sum_{n=0}^{\infty} r_{(1,1,5)}(5^2n)q^n - 5\varphi_1^2\varphi_5$$

and this implies (3.2).

For (3.3), applying a similar process of obtaining (3.6) to the terms of the form q^{5n+1} in (3.5) and (3.8), we deduce

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(5n+1)q^n = 4\varphi_1\varphi_5D_1$$

and

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(5^2(5n+1))q^n = 44\varphi_1\varphi_5D_1,$$

respectively, and this implies

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(5^2(5n+1))q^n = 11\sum_{n=0}^{\infty} r_{(1,1,5)}(5n+1)q^n. \quad (3.10)$$

Similarly, we deduce that

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(5n+4)q^n = 4\varphi_1\varphi_5E_1,$$

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(5^2(5n+4))q^n = 44\varphi_1\varphi_5E_1,$$

and this implies

$$\sum_{n=0}^{\infty} r_{(1,1,5)}(5^2(5n+4))q^n = 11 \sum_{n=0}^{\infty} r_{(1,1,5)}(5n+4)q^n. \tag{3.11}$$

By (3.10) and (3.11), we deduce (3.3). Finally, (3.4) follows from (3.2) and (3.3). □

By Lemmas 3.1 and 3.2, we immediately deduce:

Proposition 3.3. *Let n be a positive integer with prime factorization*

$$n = \prod_p p^{\lambda_p} = 2^{\lambda_2} 5^{\lambda_5} m \quad \textit{where} \quad m = \prod_{p \nmid 10} p^{\lambda_p}.$$

Then the number of solutions in integers of $x^2 + y^2 + 5z^2 = n^2$ is given by

$$r_{(1,1,5)}(n^2) = 2\left(5^{\lambda_5+1} - 3\right) r_{(1,1,5)}(m^2).$$

4. Proof of Theorem 1.3: Part 2

In this section, we work with the character modulo 20 defined on the integers by

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1,\ 3,\ 7 \text{ or } 9 \pmod{20}, \\ -1 & \text{if } n \equiv 11,\ 13,\ 17 \text{ or } 19 \pmod{20}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We note that for primes p,

$$\chi(p) = \begin{cases} \left(\frac{-5}{p}\right) & \text{if } p \neq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

The goal of this section is to prove

Proposition 4.1. *Let m be a positive integer that is relatively prime to* 10 *and has prime factorization*

$$m = \prod_{p \nmid 10} p^{\lambda_p}.$$

Then

$$r_{(1,1,5)}(m^2) = 4 \prod_{p \nmid 10} \left(\frac{p^{\lambda_p+1} - 1}{p-1} - \chi(p) \frac{p^{\lambda_p} - 1}{p-1} \right).$$

In view of (4.2), Propositions 3.3 and 4.1 immediately imply Theorem 1.3.

In order to prove Proposition 4.1, we will need some background information and several lemmas. Let $f_1 = f_1(q)$, $f_2 = f_2(q)$ be the infinite products defined by

$$f_1(q) = \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{4j})(1-q^{5j})(1-q^{10j})}{(1-q^{j})(1-q^{20j})},$$
$$f_2(q) = q\prod_{j=1}^{\infty} \frac{(1-q^{j})(1-q^{2j})(1-q^{10j})(1-q^{20j})}{(1-q^{4j})(1-q^{5j})}.$$

Let their series expansions be given by

$$f_1(q) = \sum_{n=0}^{\infty} a_1(n)q^n, \quad \text{and} \quad f_2(q) = \sum_{n=0}^{\infty} a_2(n)q^n, \tag{4.3}$$

where

$$a_1(0) = 1 \quad \text{and} \quad a_2(0) = 0.$$

Then for $j \in \{1, 2\}$, define $A_j(n)$ by

$$\sum_{n=0}^{\infty} A_j(n)q^n = \left(\sum_{k=0}^{\infty} a_j(k)q^k\right)^2.$$

Both $a_j(x)$ and $A_j(x)$ are defined to be 0 if x is not a nonnegative integer.

Lemma 4.2. *The following identity holds:*

$$\varphi(q)\varphi(q^5) = f_1(q) + f_2(q). \tag{4.4}$$

Proof. See [2, Theorem 3.1]. □

Lemma 4.3. *For any positive integer k, let*

$$L_k = 1 - 24\sum_{j=1}^{\infty} \frac{jq^{jk}}{1-q^{jk}}.$$

Then

$$f_1^2 + f_2^2 = -\frac{1}{18}L_1 + \frac{2}{9}L_4 - \frac{5}{18}L_5 + \frac{10}{9}L_{20} + \frac{2}{3}q\prod_{j=1}^{\infty}(1-q^{2j})^2(1-q^{10j})^2. \tag{4.5}$$

Proof. Squaring both sides of (4.4), we have

$$\begin{aligned}\varphi_1^2\varphi_5^2 &= f_1^2 + f_2^2 + 2f_1f_2 \\ &= f_1^2 + f_2^2 + 2q\prod_{j=1}^{\infty}(1-q^{2j})^2(1-q^{10j})^2.\end{aligned} \tag{4.6}$$

By [10, Theorem 7.1], it is known that

$$\varphi_1^2\varphi_5^2 = -\frac{1}{18}L_1 + \frac{2}{9}L_4 - \frac{5}{18}L_5 + \frac{10}{9}L_{20} + \frac{8}{3}q\prod_{j=1}^{\infty}(1-q^{2j})^2(1-q^{10j})^2. \qquad (4.7)$$

Finally, (4.5) follows from (4.7) and (4.6). □

Lemma 4.4. *Let m be a positive integer relatively prime to 10 with prime factorization*

$$m = \prod_{p\nmid 10} p^{\lambda_p}.$$

Let $c(m)$ be the coefficient of q^{2m} in $f_1^2 + f_2^2$. Then

$$c(m) = 4\sum_{d|m} d = 4\prod_{p\nmid 10}\frac{p^{\lambda_p+1}-1}{p-1}.$$

Proof. This follows immediately from Lemma 4.3, for the only term on the right hand side of (4.5) that contains terms of the form q^{20j+2}, q^{20j+6}, q^{20j+14} or q^{20j+18} is $-L_1/18$. □

Lemma 4.5. *Let $j \in \{1,2\}$ and let $a_j(n)$ be defined by (4.3). For any nonnegative integer n and any prime p, we have $a_j(pn) = a_j(p)a_j(n) - \chi(p)a_j(n/p)$ where χ is the completely multiplicative function defined by (4.1).*

Proof. See [7, Theorem 1]. □

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. Let $[q^k]f(q)$ denote the coefficient of q^k in the series expansion of $f(q)$. In this notation, $r_{(1,1,5)}(m^2) = [q^{m^2}](\varphi_1^2\varphi_5)$. By Lemma 4.2 and (4.3), we have

$$\begin{aligned} r_{(1,1,5)}(m^2) &= [q^{m^2}](\varphi(q)(f_1(q)+f_2(q))) \\ &= [q^{m^2}]\left(\varphi(q)\sum_{j=0}^{\infty}a_1(j)q^j\right) + [q^{m^2}]\left(\varphi(q)\sum_{j=0}^{\infty}a_2(j)q^j\right). \end{aligned}$$

By Lemmas 4.5 and 2.2, this is equivalent to

$$\begin{aligned} r_{(1,1,5)}(m^2) &= \sum_{r=1}^{\infty}A_1(2m/r)\chi(r)\mu(r) + \sum_{r=1}^{\infty}A_2(2m/r)\chi(r)\mu(r) \\ &= \sum_{r=1}^{\infty}[q^{2m/r}](f_1^2+f_2^2)\chi(r)\mu(r), \end{aligned}$$

where $\chi(r)$ is the completely multiplicative function defined by (4.1). Since $\chi(r)$ is 0 if r is even, the last sum is over odd r only. Moreover, since m is relatively prime to 10, we apply Lemma 4.4 to deduce that

$$\begin{aligned} r_{(1,1,5)}(m^2) &= \sum_{r=1}^{\infty} c(m/r)\chi(r)\mu(r) \\ &= c(m)\sum_{r|m} \frac{c(m/r)}{c(m)}\chi(r)\mu(r) \\ &= c(m)\prod_{p\nmid 10}\left(1-\chi(p)\frac{c(m/p)}{c(m)}\right) \\ &= \left(4\prod_{p\nmid 10}\frac{p^{\lambda_p+1}-1}{p-1}\right)\left(\prod_{p\nmid 10}\left(1-\left(\frac{-5}{p}\right)\frac{p^{\lambda_p}-1}{p^{\lambda_p+1}-1}\right)\right) \\ &= 4\prod_{p\nmid 10}\left(\frac{p^{\lambda_p+1}-1}{p-1}-\left(\frac{-5}{p}\right)\frac{p^{\lambda_p}-1}{p-1}\right). \end{aligned}$$

□

5. Proof of Theorem 1.4

In this section, we outline the proof of Theorem 1.4. The proof is similar to the proof of Theorem 1.3.

Lemma 5.1. *Fix an odd integer j. For any nonnegative integer k, let*

$$f(k) = r_{(1,1,6)}(2^{2k}j^2).$$

Then

$$f(k+3) = 3f(k+2) - 2f(k+1), \tag{5.1}$$

$$f(1) = f(0). \tag{5.2}$$

Hence,

$$f(k) = \left|2^{k+1}-3\right| f(0). \tag{5.3}$$

Proof. By (2.3), we have

$$\varphi_1^2\varphi_6 = (\varphi_4 + 2q\psi_8)^2(\varphi_{24} + 2q^6\psi_{48}). \tag{5.4}$$

Expanding, extracting the terms of the form q^{4n}, and then replacing q^4 with q, we deduce

$$\sum_{n=0}^{\infty} r_{(1,1,6)}(2^2n)q^n = \varphi_1^2\varphi_6 + 8q^2\psi_2^2\psi_{12}. \tag{5.5}$$

Applying (2.3) and (2.4) to (5.5) gives

$$\sum_{n=0}^{\infty} r_{(1,1,6)}(2^2n)q^n = (\varphi_4 + 2q\psi_8)^2(\varphi_{24} + 2q^6\psi_{48}) \\ + 8q^2(\varphi_8\psi_4 + 2q^2\psi_4\psi_{16})\psi_{12}. \tag{5.6}$$

Applying a similar process to the terms of the form q^{4n} in (5.6), we get

$$\sum_{n=0}^{\infty} r_{(1,1,6)}(2^4n)q^n = \varphi_1^2\varphi_6 + 8q^2\psi_2^2\psi_{12} + 16q\psi_1\psi_3\psi_4. \tag{5.7}$$

Then by (2.3)–(2.5), we have

$$\sum_{n=0}^{\infty} r_{(1,1,6)}(2^4n)q^n = (\varphi_4 + 2q\psi_8)^2(\varphi_{24} + 2q^6\psi_{48}) \\ + 8q^2(\varphi_8\psi_4 + 2q^2\psi_4\psi_{16})\psi_{12} \\ + 16q((\varphi_{24} + 2q^6\psi_{48})\psi_4 + q(\varphi_8 + 2q^2\psi_{16})\psi_{12})\psi_4. \tag{5.8}$$

Applying a similar process to the terms of the form q^{4n} in (5.8), we get

$$\sum_{n=0}^{\infty} r_{(1,1,6)}(2^6n)q^n = \varphi_1^2\varphi_6 + 8q^2\psi_2^2\psi_{12} + 48q\psi_1\psi_3\psi_4. \tag{5.9}$$

Then (5.5), (5.7) and (5.9) imply (5.1).

Now, expanding (5.4), extracting the terms of the form q^{4n+1}, dividing both sides by q, and then replacing q^4 with q, we deduce

$$\sum_{n=0}^{\infty} r_{(1,1,6)}(4n+1)q^n = 4\varphi_1\varphi_6\psi_2. \tag{5.10}$$

Applying a similar process to the terms of the form q^{4n+1} in (5.6) gives

$$\sum_{n=0}^{\infty} r_{(1,1,6)}(2^2(4n+1))q^n = 4\varphi_1\varphi_6\psi_2. \tag{5.11}$$

Then (5.10) and (5.11) imply (5.2). Finally, (5.3) follows from (5.1) and (5.2). □

Lemma 5.2. *Fix an integer j that is not divisible by 3. Then for any nonnegative integer k, we have*

$$r_{(1,1,6)}(3^{2k}j^2) = r_{(1,1,6)}(j^2). \tag{5.12}$$

Proof. By (2.7), we deduce

$$\sum_{n=0}^{\infty} r_{(1,1,6)}(n)q^n = (\varphi_9 + 2qX_3)^2\varphi_6.$$

Expanding, extracting the terms of the form q^{3n}, and then replacing q^3 with q, we get

$$\sum_{n=0}^{\infty} r_{(1,1,6)}(3n)q^n = \varphi_3^2\varphi_2. \tag{5.13}$$

Applying (2.7) to (5.13) gives

$$\sum_{n=0}^{\infty} r_{(1,1,6)}(3n)q^n = \varphi_3^2(\varphi_{18} + 2q^2X_6). \tag{5.14}$$

Applying a similar process to the terms of the form q^{3n} in (5.14), we deduce

$$\sum_{n=0}^{\infty} r_{(1,1,6)}(3^2n)q^n = \varphi_1^2\varphi_6. \tag{5.15}$$

This implies (5.12). □

Lemmas 5.1 and 5.2 immediately imply:

Proposition 5.3. *Let n be a positive integer with prime factorization*

$$n = \prod_p p^{\lambda_p} = 2^{\lambda_2}3^{\lambda_3}m \quad \textit{where} \quad m = \prod_{p\geq 5} p^{\lambda_p}.$$

Then the number of solutions in integers of $x^2 + y^2 + 6z^2 = n^2$ is given by

$$r_{(1,1,6)}(n^2) = 4\left|2^{\lambda_2} - 3\right| r_{(1,1,6)}(m^2).$$

It remains to determine $r_{(1,1,6)}(m^2)$ in the case that $(m, 6) = 1$. For any integer n let $\chi(n)$ denote the character modulo 24 by

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1,\ 5,\ 7 \text{ or } 11 \pmod{24}, \\ -1 & \text{if } n \equiv 13,\ 17,\ 19 \text{ or } 23 \pmod{24}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.16}$$

We note that for primes p

$$\chi(p) = \left(\frac{-6}{p}\right). \tag{5.17}$$

Proposition 5.4. *Let m be a positive integer that is relatively prime to 6 and has prime factorization*

$$m = \prod_{p\geq 5} p^{\lambda_p}.$$

Then

$$r_{(1,1,6)}(m^2) = 4\prod_{p\geq 5}\left(\frac{p^{\lambda_p+1} - 1}{p - 1} - \chi(p)\frac{p^{\lambda_p} - 1}{p - 1}\right).$$

In view of (5.17), Propositions (5.3) and (5.4) immediately imply Theorem 1.4.

To prove Proposition 5.4, we need some relevant lemmas. Let

$$g_1(q) = \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{3j})(1-q^{8j})(1-q^{12j})}{(1-q^j)(1-q^{24j})},$$
$$g_2(q) = q\prod_{j=1}^{\infty} \frac{(1-q^{j})(1-q^{4j})(1-q^{6j})(1-q^{24j})}{(1-q^{3j})(1-q^{8j})}.$$

Let their series expansions be given by

$$g_1(q) = \sum_{n=0}^{\infty} b_1(n)q^n, \quad \text{and} \quad g_2(q) = \sum_{n=0}^{\infty} b_2(n)q^n, \tag{5.18}$$

where $b_1(0) = 1$ and $b_2(0) = 0$. Then for $j \in \{1, 2\}$, define $B_j(n)$ by

$$\sum_{n=0}^{\infty} B_j(n)q^n = \left(\sum_{k=0}^{\infty} b_j(k)q^k\right)^2.$$

Both $b_j(x)$ and $B_j(x)$ are defined to be 0 if x is not an integer.

Lemma 5.5. *We have*

$$\varphi(q)\varphi(q^6) = g_1(q) + g_2(q). \tag{5.19}$$

Proof. See [2, Theorem 4.1]. □

Lemma 5.6. *For any positive integer k, let*

$$L_k = 1 - 24\sum_{j=1}^{\infty} \frac{jq^{jk}}{1-q^{jk}}.$$

Then

$$g_1^2 + g_2^2 = -\frac{1}{12}L_1 + \frac{1}{12}L_2 + \frac{1}{4}L_3 + \frac{1}{6}L_4 - \frac{1}{4}L_6 - \frac{2}{3}L_8 - \frac{1}{2}L_{12} + 2L_{24}. \tag{5.20}$$

Proof. Squaring both sides of (5.19) gives

$$\begin{aligned} \varphi_1^2\varphi_6^2 &= g_1^2 + g_2^2 + 2g_1g_2 \qquad (5.21)\\ &= g_1^2 + g_2^2 + 2q\prod_{j=1}^{\infty}(1-q^{2j})(1-q^{4j})(1-q^{6j})(1-q^{12j}). \end{aligned}$$

By [1, Theorem 1.12], it is known that

$$\varphi_1^2\varphi_6^2 = -\frac{1}{12}L_1 + \frac{1}{12}L_2 + \frac{1}{4}L_3 + \frac{1}{6}L_4 - \frac{1}{4}L_6 - \frac{2}{3}L_8 - \frac{1}{2}L_{12} + 2L_{24} \tag{5.22}$$

$$+ 2q\prod_{j=1}^{\infty}(1-q^{2j})(1-q^{4j})(1-q^{6j})(1-q^{12j}).$$

Finally, (5.20) follows from (5.21) and (5.22). □

Lemma 5.7. *Let m be a positive integer relatively prime to 6 with prime factorization*

$$m = \prod_{p\geq 5} p^{\lambda_p}.$$

Let $c(m)$ be the coefficient of q^{2m} in $g_1^2 + g_2^2$. Then

$$c(m) = 4\sum_{d|m} d = 4\prod_{p\geq 5}\frac{p^{\lambda_p+1}-1}{p-1}.$$

Proof. This follows immediately from Lemma 5.6, for the only term on the right hand side of (5.20) that contains terms of the form q^{12j+2} and q^{12j+10} is $-(L_1 - L_2)/12$. □

Lemma 5.8. *Let $j \in \{1,2\}$ and let $b_j(n)$ be defined by* (5.18)*. For any nonnegative integer n and any prime p, we have $b_j(pn) = b_j(p)b_j(n) - \chi(p)b_j(n/p)$ where χ is the completely multiplicative function defined by* (5.16)*.*

Proof. See [7, Theorem 1]. □

Now we are ready for

Proof of Proposition 5.4. Let $[q^k]f(q)$ denote the coefficient of q^k in the series expansion of $f(q)$. In this notation,

$$r_{(1,1,6)}(m^2) = [q^{m^2}](\varphi_1^2\varphi_6).$$

By Lemma 5.5 and (5.18), we have

$$\begin{aligned} r_{(1,1,6)}(m^2) &= [q^{m^2}](\varphi(q)(g_1(q)+g_2(q))) \\ &= [q^{m^2}]\left(\varphi(q)\sum_{j=0}^{\infty}b_1(j)q^j\right) + [q^{m^2}]\left(\varphi(q)\sum_{j=0}^{\infty}b_2(j)q^j\right). \end{aligned}$$

By Lemmas 5.8 and 2.2, this is equivalent to

$$\begin{aligned} r_{(1,1,6)}(m^2) &= \sum_{r=1}^{\infty}B_1(2m/r)\chi(r)\mu(r) + \sum_{r=1}^{\infty}B_2(2m/r)\chi(r)\mu(r) \\ &= \sum_{r=1}^{\infty}[q^{2m/r}](g_1^2+g_2^2)\chi(r)\mu(r), \end{aligned}$$

where $\chi(r)$ is the completely multiplicative function defined by (5.16). Since $\chi(r)$ is 0 if r is divisible by 2 or 3, the last sum is over the r that is relatively prime to 6 only. Moreover, since m is relatively prime to 6, we apply Lemma 5.7 to deduce that

$$\begin{aligned}
r_{(1,1,6)}(m^2) &= \sum_{r=1}^{\infty} c(m/r)\chi(r)\mu(r) \\
&= c(m)\sum_{r|m} \frac{c(m/r)}{c(m)}\chi(r)\mu(r) \\
&= c(m)\prod_{p\geq 5}\left(1-\chi(p)\frac{c(m/p)}{c(m)}\right) \\
&= \left(4\prod_{p\geq 5}\frac{p^{\lambda_p+1}-1}{p-1}\right)\left(\prod_{p\geq 5}\left(1-\left(\frac{-6}{p}\right)\frac{p^{\lambda_p}-1}{p^{\lambda_p+1}-1}\right)\right) \\
&= 4\prod_{p\geq 5}\left(\frac{p^{\lambda_p+1}-1}{p-1}-\left(\frac{-6}{p}\right)\frac{p^{\lambda_p}-1}{p-1}\right).
\end{aligned}$$

□

6. Proof of Theorem 1.5

This section is devoted to the proof of Theorem 1.5. It is much simpler than the other two because Theorem 1.5 can be deduced fairly easily from Theorem 1.4.

Proof of Theorem 1.5. By (2.3), we have

$$\sum_{n=0}^{\infty} r_{(1,2,3)}(n)q^n = (\varphi_4+2q\psi_8)(\varphi_8+2q^2\psi_{16})(\varphi_{12}+2q^3\psi_{24}). \tag{6.1}$$

Expanding, extracting the terms of the form q^{4n+1}, dividing both sides by q, and then replacing q^4 with q, we get

$$\sum_{n=0}^{\infty} r_{(1,2,3)}(4n+1)q^n = 2\varphi_2\varphi_3\psi_2+4q\varphi_1\psi_4\psi_6. \tag{6.2}$$

Applying a similar process to the terms of the form q^{4n} gives

$$\sum_{n=0}^{\infty} r_{(1,2,3)}(2^2n)q^n = 2\varphi_1\varphi_2\varphi_3+4q\varphi_2\psi_2\psi_6.$$

By (2.3) and (2.5), we have

$$\sum_{n=0}^{\infty} r_{(1,2,3)}(4n)q^n = 2(\varphi_4 + 2q\psi_8)(\varphi_8 + 2q^2\psi_{16})(\varphi_{12} + 2q^3\psi_{24}) + 4q(\varphi_8 + 2q^2\psi_{16})(\varphi_{12}\psi_8 + q^2\varphi_4\psi_{24}).$$

Expanding, extracting the terms of the form q^{4n+1}, dividing both sides by q, and then replacing q^4 with q, we get

$$\sum_{n=0}^{\infty} r_{(1,2,3)}(2^2(4n+1))q^n = 6\varphi_2\varphi_3\psi_2 + 12q\varphi_1\psi_4\psi_6. \tag{6.3}$$

Then (6.2) and (6.3) imply that for a fixed odd j, $r_{(1,2,3)}(2^{2k}j^2) = 3r_{(1,2,3)}(j^2)$ holds for all positive integer k.

Next, for a fixed integer m that is relatively prime to 6, let $f(k) = r_{(1,2,3)}(3^{2k}m^2)$. By the same methods of deriving (5.12), we may deduce that

$$f(k+2) = 4f(k+1) - 3f(k),$$
$$f(1) = 7f(0)$$

and it follows that

$$f(k) = (3^{k+1} - 2)f(0). \tag{6.4}$$

It remains to determine $f(0)$, that is, $r_{(1,2,3)}(m^2)$. By the same methods, we may verify that

$$\sum_{n=0}^{\infty} r_{(1,2,3)}(12n+1)q^n = \frac{1}{2}\sum_{n=0}^{\infty} r_{(1,1,6)}(12n+1)q^n,$$

and it follows that

$$r_{(1,2,3)}(m^2) = \frac{1}{2}r_{(1,1,6)}(m^2).$$

Therefore, by Proposition 5.4, we deduce

$$r_{(1,2,3)}(m^2) = 2\prod_{p\geq 5}\left(\frac{p^{\lambda_p+1}-1}{p-1} - \left(\frac{-6}{p}\right)\frac{p^{\lambda_p}-1}{p-1}\right). \tag{6.5}$$

□

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