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SMITH NUMBERS WITH EXTRA DIGITAL FEATURES

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Abstract

Using a recently introduced technique, we construct a new infinite sequence of Smith numbers which are palindromic and divisible by their digital sum.

1. Introduction

We deal with natural numbers N as represented in the base-10 counting system. For such a number N, we define the digital sum $S(N)$ to be the sum of the digits in N, e.g., $S(1725) = 1 + 7 + 2 + 5 = 15$. Similarly, the p-digit sum $S_p(N)$ is the sum of the digits of all the prime factors of N, e.g., having factored $1725 = 3 \times 5^2 \times 23$, we have that $S_p(1725) = 3 + (2 \cdot 5) + (2 + 3) = 18$.

A number N is called a Smith number when N is a composite such that $S(N) = S_p(N)$. The name Smith was coined in a 1982 article by Wilansky [7]. A few years later, McDaniel [2] proved that Smith numbers are infinitely many.

Being a Smith number, then, is a digital feature that is entirely dependent upon the radix representation: a number N that is Smith in the decimal sense may lose this very property when N is represented in another base $b \neq 10$. Nevertheless, McDaniel [4, 6] also showed that there exist infinitely many base-b Smith numbers for $b \geq 8$ and $b = 2$, and then Wilson [8] completed this proposition by extending the results for all $b \geq 3$.

The abundance of Smith numbers is also reflected in the fact that it is not hard to construct all kinds of sequences of Smith numbers, even ones that enjoy additional digital features. McDaniel [5] gave an alternate construction of Smith numbers N which are divisible by the digital sum $S(N)$. A number N that is divisible by $S(N)$ is called a Niven number, as introduced in an article by Kennedy et al. [1] in 1980.

In a separate article, McDaniel [3] also constructed an infinite sequence of palindromic Smith numbers. McDaniel's definition of a palindromic number includes numbers of the form $N \times 10^e$, where N is itself a palindrome, since we argue that e leading zeros may be prefixed in order to see the number $N \times 10^e$ as a genuine palindrome. We shall go with this definition:

Definition 1. A natural number N is called a *palindrome* when the digits in N read the same left to right as they do right to left, e.g., $N = 84148$. Moreover, we say that a number M is *palindromic* if M is either a palindrome or $M = N \times 10^e$ for some palindrome N.

In this article, we aim to demonstrate the existence of an infinite sequence of Smith numbers with the extra property that each one is also a palindromic Niven number. This sequence is a special case of a known construction of Smith numbers that we have given recently [9]. In this construction, the Smith numbers are of the form $9P_{k,n}t_n \times 10^{f_k}$, where $P_{k,n} = \sum_{i=0}^{n-1} 10^{ki}$ and t_n is to be chosen from a set M_k of seven numbers with distinct p-digit sums modulo 7. We will achieve our goal by considering the case $k = 3$ with a new set M_3 consisting of seven palindromes.

2. Known Facts

We will employ the notation $D(N)$ and $\Omega(N)$ to denote the number of digits in N and the number of prime factors of N, respectively.

Definition 2. For every pair of integers $k, n \geq 1$, we define the number $P_{k,n}$ according to the formula

$$P_{k,n} = \sum_{i=0}^{n-1} 10^{ki}.$$

It is an easy observation that $S(P_{k,n}) = n$ and that $D(P_{k,n}) = kn - k + 1$ for every pair k, n. Furthermore, we have the identity

$$P_{k,n} = \frac{R_{kn}}{R_k},$$

where $R_k = (10^k - 1)/9$ and which is also known as the k-th repunit.

In the following theorem we restate previous results [9, Theorems 7, 9] upon which our new construction will be based.

Theorem 1. *Let $k \geq 2$, and let M_k be a set of seven numbers of the form $\sum_{j=1}^{k} 10^{e_j}$, such that $\{e_j \mid 1 \leq j \leq k\}$ is a complete residue system modulo k. Then $S(9P_{k,n}t) = 9kn$ for all $t \in M_k$. Moreover, if $\{S_p(t) \mid t \in M_k\}$ is a complete residue system modulo 7, then there are infinitely many values of n for which $S(9P_{k,n}t_n) - S_p(9P_{k,n}t_n) = 7f$ for some integer $f \geq 0$ and some $t_n \in M_k$.*

The number f in Theorem 1 varies with n and determines the Smith number $N_{n,k} = 9P_{k,n}t_n \times 10^f$, for in this case $S_p(10^f) = 7f$ and so

$$S_p(N_{n,k}) = S_p(9P_{k,n}t_n) + S_p(10^f) = S(9P_{k,n}t_n) = S(N_{n,k}).$$

We will also rely on the following fact [2, Theorem 1] concerning the upperbound for the p-digit sum $S_p(N)$ of an arbitrary number N.

Theorem 2 (McDaniel). *For any number N, $S_p(N) < 9D(N) - 0.54\Omega(N)$.*

3. The New Sequence

We begin by introducing the set M_3, given below, which will play the role of M_k in Theorem 1 for the specific value of $k = 3$.

Definition 3. Let M_3 be the set of seven numbers of distinct p-digit sums modulo 7 given by

$$M_3 = \{10^{2e} + 10^e + 1 \mid e = 1, 2, 4, 5, 7, 10, 11\}.$$

The fact that $\{S_p(t) \mid t \in M_3\}$ is a complete residue system modulo 7 is verified in Table 1, where we provide the prime factorizations of the elements $t \in M_3$ in order to compute $S_p(t)$.

e	$t = 10^{2e} + 10^e + 1$	$S_p(t)$	(mod 7)
1	3×37	13	6
2	$3 \times 7 \times 13 \times 37$	24	3
4	$3 \times 7 \times 13 \times 37 \times 9901$	43	1
5	$3 \times 31 \times 37 \times 2906161$	42	0
7	$3 \times 37 \times 43 \times 1933 \times 10838689$	79	2
10	$3 \times 7 \times 13 \times 31 \times 37 \times 211 \times 241 \times 2161 \times 2906161$	74	4
11	$3 \times 37 \times 67 \times 1344628210313298373$	96	5

Table 1: The prime factorization of $t \in M_3$ and the resulting $S_p(t)$ mod 7.

Moreover, this set M_3 satisfies the hypothesis of Theorem 1 when $k = 3$. To see this fact, for every $t \in M_3$ we have the form $t = 10^{2e} + 10^e + 10^0$, where $\{2e, e, 0\}$ is a complete residue system modulo 3 for the seven choices of $e \in \{1, 2, 4, 5, 7, 10, 11\}$.

Hence, we have $S(9P_{3,n}t) = 27n$ whenever $t \in M_3$, which allows us to set the following definition.

Definition 4. For every $n \geq 1$, we let $N_n = 9 \times P_{3,n} \times t_n$, with the unique element $t_n \in M_3$ such that $S(N_n) \equiv S_p(N_n) \pmod 7$.

We conclude that there exist infinitely many values of $n \geq 1$ for which $S(N_n) - S_p(N_n) = 7f_n$ with $f_n \geq 0$, giving the Smith number $N_n \times 10^{f_n}$. Furthermore, we claim that for each such n, the resulting Smith number is palindromic:

Theorem 3. *For all $n \geq 1$, the number N_n is a palindrome.*

Proof. Let $t = 10^{2e} + 10^e + 1 \in M_3$ with any one of the seven values of e. Then,

$$P_{3,n}t = \sum_{i=0}^{n-1} \left(10^{3i+2e} + 10^{3i+e} + 10^{3i}\right). \tag{1}$$

By the fact that $2e$, e, and 0 are all distinct modulo 3, we see that $P_{3,n}t$ is the sum of $3n$ distinct powers of ten. The largest among these powers is $10^{3n-3+2e}$.

Now let $D = D(P_{3,n}t) = 3n-2+2e$. We will claim that $P_{3,n}t$ is a palindrome by showing that the power 10^j appears among the summands if and only if $10^{(D-1)-j}$ does too. This equivalence statement is clear once we substitute $i = n-1-j$ in (1):

$$P_{3,n}t = \sum_{j=0}^{n-1} \left(10^{(D-1)-3j} + 10^{(D-1)-(3j+e)} + 10^{(D-1)-(3j+2e)}\right).$$

Lastly, since the digits in $P_{3,n}t$ are all zeros and ones, the number $9P_{3,n}t$ is as well a palindrome, which is the desired result. □

We next show the existence of an infinite subsequence of N_n in which each number N_n is divisible by $S(N_n)$.

Theorem 4. *For every $k \geq 1$, the number N_{3^k} is divisible by $S(N_{3^k})$.*

Proof. Note that $S(N_{3^k}) = 3^{k+3}$ according to Theorem 1.

Let $\Phi_m(x)$ denote the m-th cyclotomic polynomial, and recall the familiar identity

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

We use the fact that $\Phi_{3^i}(x) = x^{2\cdot 3^{i-1}} + x^{3^{i-1}} + 1$ for all $i \geq 1$ and we obtain

$$\begin{aligned} x^{3^{k+1}} - 1 &= \Phi_1(x)\Phi_3(x)\Phi_{3^2}(x)\cdots\Phi_{3^{k+1}}(x) \\ &= (x-1)(x^2+x+1)(x^{2\cdot 3}+x^3+1)\cdots(x^{2\cdot 3^k}+x^{3^k}+1). \end{aligned}$$

Equivalently, we may write

$$\frac{x^{3^{k+1}}-1}{x^3-1} = \prod_{i=1}^{k} (x^{2\cdot 3^i} + x^{3^i} + 1).$$

Meanwhile, we also have

$$P_{3,3^k} = \frac{R_{3^{k+1}}}{R_3} = \frac{10^{3^{k+1}}-1}{10^3-1}.$$

Hence,

$$P_{3,3^k} = \prod_{i=1}^{k} (10^{2\cdot 3^i} + 10^{3^i} + 1). \tag{2}$$

Since the sum $10^{2\cdot 3^i} + 10^{3^i} + 1$ in (2) is a multiple of 3, it follows that 3^k divides $P_{3,3^k}$. And with the fact that every $t \in M_3$ is also a multiple of 3, we then conclude that 3^{k+3} divides N_{3^k} as claimed. □

Finally, we now arrive at our main result.

Theorem 5. *For all sufficiently large k, we can find $f_k > 0$ such that the number $N_{3^k} \times 10^{f_k}$ is a palindromic Smith number that is divisible by its digital sum, i.e., $S(N_{3^k} \times 10^{f_k}) = 3^{k+3}$.*

Proof. We simply let

$$f_k = \frac{S(N_{3^k}) - S_p(N_{3^k})}{7}.$$

Hence, it suffices to show that $S(N_{3^k}) > S_p(N_{3^k})$ for all k sufficiently large. Recalling that $D(9P_{3,3^k}) = 3^{k+1} - 2$ and, by Equation (2), $\Omega(P_{3,3^k}) \geq 2k$, we apply Theorem 2:

$$\begin{aligned} S_p(N_{3^k}) = S_p(t_{3^k}) + S_p(9P_{3,3^k}) &< 96 + 9(3^{k+1} - 2) - 0.54(2k + 2) \\ &= 3^{k+3} + 76.92 - 1.08k. \end{aligned}$$

Since $3^{k+3} = S(N_{3^k})$, we see that $S_p(N_{3^k}) < S(N_{3^k})$ for all sufficiently large values of k. □

4. Computational Notes

The proof of Theorem 5 implicitly implies that $S_p(N_{3^k}) < S(N_{3^k})$ for all $k \geq 72$. However, computational results suggested that the quantity $S_p(N_{3^k})$ is significantly smaller than 3^{k+3} even for small values of k, and more so as k increases. For $k \leq 5$, we recorded in Table 2 the corresponding values of f_k that yield the Smith numbers $N_{3^k} \times 10^{f_k}$. In this range, only $k = 1$ fails to generate a Smith number due to the fact that $f_1 < 0$.

k	$S(N_{3^k})$	$S_p(P_{3,3^k})$	e_k	$S_p(N_{3^k})$	f_k
1	81	31	7	116	n/a
2	243	117	4	166	11
3	729	359	5	407	46
4	2187	1093	2	1123	152
5	6561	3296	10	3376	455

Table 2: Smith numbers $N_{3^k} \times 10^{f_k}$, where $N_{3^k} = 9P_{3,3^k}(10^{2e_k} + 10^{e_k} + 1)$

These numerical observations led us to a stronger version of Theorem 5, as follows.

Theorem 6. *For every $k \geq 2$, the number $N_{3^k} \times 10^{f_k}$ is a palindromic Smith number that is also a Niven number, where f_k is positive and determined by*

$$f_k = \frac{S(N_{3^k}) - S_p(N_{3^k})}{7}.$$

Proof. We will assume that $k \geq 3$, since the claim has been verified for $k = 2$ by direct computation, and show that $3^{k+3} > S_p(N_{3^k})$. Now let

$$P' = \prod_{i=3}^{k} (10^{2\cdot 3^i} + 10^{3^i} + 1),$$

so that we have $P_{3,3^k} = P_{3,3^2} \times P'$. This identity allows us to closely estimate the number of digits in P'. In particular,

$$D(P') \leq D(P_{3,3^k}) - D(P_{3,9}) + 1 = (3^{k+1} - 2) - 25 + 1 = 3^{k+1} - 26.$$

Moreover, since P' is the product of $k-2$ composites, we note that $\Omega(P') \geq (k-2)2 = 2k-4$. Then with $S_p(P_{3,9}) = 117$, we employ Theorem 2 again to arrive at the desired result:

$$\begin{aligned} S_p(N_{3^k}) &= S_p(9) + S_p(P_{3,9}) + S_p(P') + S_p(t_{3^k}) \\ &\leq 6 + 117 + 9(3^{k+1} - 26) - 0.54(2k-4) + 96 \\ &= 3^{k+3} - 12.84 - 1.08k \\ &< 3^{k+3}. \end{aligned}$$

□

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