EDGE DELETION GAMES WITH PARITY RULES

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Abstract

We analyze three combinatorial games played on simple undirected graphs. A move consists of deleting a single edge; which edges can be deleted depends on the parity of each edge's endpoints. While determining the possible values of each game, we show relationships between these games and graph matchings, octal games, vertex deletion games, and graph colourings.

1. Introduction

Graphs are an abstraction that can represent any number of connected networks. As such, it is useful to think of them as an arena on which games can be played. While many games use a fixed board and allow pieces to move from place to place according to a set of rules, we can also imagine games where the board itself is changing over time while each player attempts to create a more favourable configuration.

Previous work in a similar vein has been conducted by Gallant, Gunther, Hartnell, Rall [4], Nowakowski and Ottaway [6] and Shelton [7] to name a few. In our paper, we solve an open problem of Nowakowski and Ottaway which had been discussed by numerous parties and further developed by Shelton.

Some instances of these deletion games are other games in a different guise. In particular, it has been shown that some vertex deletion games are equivalent to octal games. We expand upon this by finding an edge deletion game with similar properties. We also use a variety of constructions to show equivalence between the game values that can occur using one set of rules compared to another.

This paper assumes an understanding of impartial combinatorial games as well as

graph theory as given in Lessons in Play [1], Winning Ways [2], and Graph Theory [3]. In addition, we will make use of the definition that a graph is **equimatchable** [5] if all maximal matchings are also maximum.

1.1. The Games

The games we discuss in this paper are played on simple undirected graphs, with a move being the deletion of an edge. We abuse notation and use G to refer to both the game and the graph the game is being played on. The rules for which edges can be deleted depend on the parity of each edge's endpoints. The rules we consider are:

- BE: both endpoints have even degree;

- BO: both endpoints have odd degree;

- OE: one endpoint is even and the other is odd.

In this paper, we discuss the three impartial games on these rulesets. In anticipation of the partizan games, we label our impartial games with the notation Left's rules/Right's rules. For example, the game BO/BO is the impartial game where both players can only delete edges with both endpoints having odd degree, and the game BO/BE is the partizan game where Left can delete edges with both endpoints of odd degree and Right can deleted edges with both endpoints of even degree.

1.2. Isolated Vertices

Because all of the games discussed in this paper consider the deletion of an edge to be a move, an isolated vertex has no impact on a game. Any game on a graph with isolated vertices is equivalent to the game on the same graph with all its isolated vertices deleted. This can be seen from noticing that the game value of an isolated vertex is 0 and then considering the game sum.

1.3. Strategy for Play

We discuss BO/BO and BE/BE first, as these two games are closely related. For both games, there may be edges that cannot be deleted initially. These edges can never be deleted throughout the game, because no move can change the parity of their endpoints. Moreover, whenever an edge is deleted, its endpoints change parity so that all adjacent edges can never be deleted. This edge deletion behaviour is exactly like that of constructing a matching. So when playing the game, we consider only the edges that allow legal moves, giving us an induced subgraph based on the parity of the vertices. Once we have this induced subgraph, we ignore all rules based on parity and construct a matching.

Therefore, strategy for play on these games corresponds to constructing a maximal matching on the subgraph induced by the odd degree vertices for BO/BO, and even degree vertices for BE/BE. We specifically label these subgraphs as G_{odd} and G_{even} as shown in Figure 1.

Using the induced subgraph allows us to quickly see that if the induced subgraph is empty, neither player can make a move. Notably, in BO/BO: $P_n = 0, C_n = 0$ for $n > 2$.
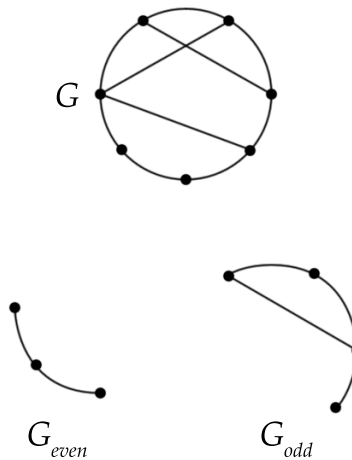

Figure 1: G_{even} and G_{odd}

2. BO/BO

Suppose G_{odd} is equimatchable, and let M be the number of edges in a maximal matching of G_{odd}. There is only one possible value for M since G_{odd} is equimatchable.

Theorem 2.1. *If M is even, $G = 0$. If M is odd, $G = *$.*

Proof. Play continues until a maximal matching is reached in G_{odd}. When M is even, the game lasts an even number of moves, so the second player wins and $G = 0$. For M odd, both players' moves are to an even matching, so $G = \{0|0\} = *$.   □

Corollary 2.2.
$$K_n = \begin{cases} * & \text{if } n \equiv 2 \mod 4 \\ 0 & \text{otherwise} \end{cases}$$

Proof. G_{odd} is equimatchable since it is the empty graph for odd n or K_n for even n. In either case, $M = \lfloor \frac{n}{2} \rfloor$. Then the parity of M is:

$$M : \begin{cases} even & \text{if } n \equiv 0 \text{ or } 1 \mod 4 \\ odd & \text{otherwise} \end{cases}$$

When n is odd, G_{odd} is empty and $K_n = 0$.
When n is even, $G_{odd} = K_n$ and we have two cases: M even or odd.
If M is even, $G = 0$ by Theorem 2.1. So K_n only equals $*$ when n is even and M is odd. $\qquad\qquad\square$

Corollary 2.3.
$$K_{m,n} = \begin{cases} * & if \ m, n \ both \ odd \\ 0 & otherwise \end{cases}$$

Proof. If at least one of m or n is even, G_{odd} is empty because at least one endpoint of every edge will have even degree. Both players require odd degree endpoints, so neither can move and $G = 0$. If m and n are both odd, $K_{m,n} = *$ by Theorem 2.1. $\qquad\qquad\square$

   The nature of graphs is that they become computationally complex very quickly. The canonical forms of complete graphs on more than ten vertices become unreasonable to compute very quickly for all of the games discussed in this paper. Theorem 2.1 and its corollaries give us more than just values of the complete graphs. Being able to relate strategy for play to constructing matchings gives a strong tie from game theory concepts to graph theory structure, suggesting that we may use graph theory principles to understand these games. However, most work on matchings uses arguments based on augmenting paths, which do not apply within the context of these games. We instead must describe a way to force the maximal matching to be of even or odd size. Such a strategy eludes us for now, and this problem is added to our future work section.

3. BE/BE

Suppose G_{even} is equimatchable, and let M be the number of edges in a maximal matching of G_{even}. Then Theorem 2.1 holds for BE/BE.

Corollary 3.1.
$$K_n = \begin{cases} * & if \ n \equiv 3 \mod 4 \\ 0 & otherwise \end{cases}$$

Proof. This proof is similar to Corollary 2.2 and uses the argument from Theorem 2.1. K_n is equal to zero except for when n is odd (G_{even} is non-empty) and M is odd. $\square$

Corollary 3.2.

$$K_{m,n} = 0$$

Proof. If m or n is odd, G_{even} is empty so neither player has any move. If they are both even, M is even, and so the game is in P by Theorem 2.1. $\square$

We notice that BE/BE and BO/BO behave very similarly on the complete and complete bipartite graphs. This insight leads to the construction of homomorphisms between the two games in Lemma 3.5.

Lemma 3.3. *A game played on a path is equivalent to the octal game* 0.4.

Proof. We notice a couple of points about playing BE/BE on paths: A path of length one cannot have its only edge deleted, so a path cannot be made empty. The two edges on the ends of a path also may not be deleted. Finally, as always, only a single edge at a time may be deleted. So, if we consider P_n as a heap of $n-1$ tokens, play consists of removing one token (edge) and leaving two non-empty heaps (paths). Specifically, removing an edge from P_n leaves P_k and $P_{n-k}, 2 \le k \le n-2$.

These are the criteria for the octal game 0.4, which has period 34 with a maximum nim value of 7. $\square$

The octal games are defined in [2]. We will use the octal characteristics of the paths in an attempt to construct all possible game values for BE/BE and BO/BO in Conjecture 3.6.

Corollary 3.4. *We have*

$$C_n = \begin{cases} * & \text{if } P_n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. When $P_n = 0$, both player have a move to zero, so the game is $\{0|0\} = *$. Otherwise, both players have a move to something other than zero, which is in P by definition. $\square$

Lemma 3.5. *Any value that exists in BE/BE or BO/BO exists in the other.*

Proof. We construct two homomorphisms between BO/BO and BE/BE to show that BO/BO and BE/BE have the same set of game values. We attempt to find all nimbers in BE/BE to acquire all game values for both games.

`Homomorphism from BO/BO to BE/BE`: for all $v \in G$, add an isolated vertex and connect v to it. This flips the parity of every vertex, so edges that had both

endpoints odd now have both endpoints even. Let this new graph be H. All the added edges are incident to a vertex of degree one, so they cannot be deleted in BE/BE. Thus, the subgraph induced by the even degree vertices of H is exactly the subgraph induced by the odd degree vertices of G.

Homomorphism from BE/BE to BO/BO: for all $v \in G$, add a copy of P_2 and connect v to one endpoint of it. This flips the parity of every vertex, so edges that had both endpoints even now have both endpoints odd. Let this new graph be H. The end vertices of degree one in the copies of P_2 are all adjacent to the vertex of degree two that is incident to a vertex in G. In the subgraph induced by the odd vertices of H, these end vertices become isolated vertices, which are discarded. Thus, the subgraph induced by the odd degree vertices of H is exactly the subgraph induced by the even degree vertices of G.

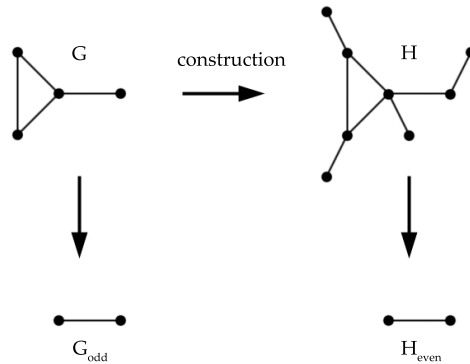These two operations are demonstrated in figures 2 and 3.

Figure 2: Homomorphism from BO/BO to BE/BE

$\square$

Conjecture 3.6. *For all n, there exists a graph G with $G = *n$.*

We have found nimbers in BE/BE up to $*15$ through the following construction:

Note that if we identify (merge) one end vertex each from an odd number of paths, their shared end vertex will have odd degree, and the paths behave exactly as if we considered their disjoint sum. Let c be the merged end vertex and assume an even number of paths are joined. If we delete any edge incident to c, one path becomes disconnected and has its length reduced by one, which is normally not an allowed move. The rest of the paths remain joined to c and behave like their disjoint sum because c now has odd degree.

We choose eight paths such that when each is disconnected, its nim value changes by some amount between 0 and 7. By ensuring each change occurs exactly once,
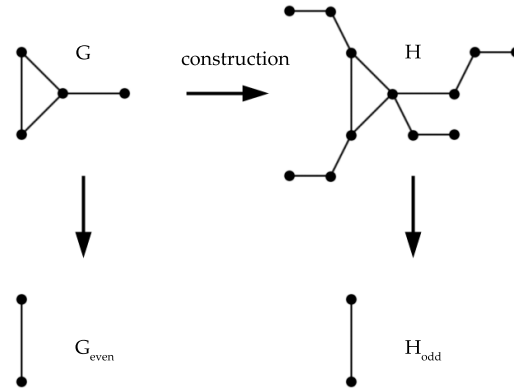
Figure 3: Homomorphism from BE/BE to BO/BO

this gives a permutation of the nim-sum of the other paths. This gives us the values $0, *, *2, *3, *4, *5, *6, *7$ in some order. Thus, the original graph of the joined paths had value at least $*8$. In this case, the paths that give the required changes are: $P_4, P_5, P_6, P_7, P_{16}, P_{17}, P_{18}, P_{32}$. Once we have $*8$, we use nim-sums to obtain all values up to $*15$.

This construction requires paths, but yields a tree. So, we cannot use our tree that has value $*8$ in a similar fashion to obtain $*16$, and so on. Ultimately, a new construction will be needed to find higher values, because we know that the maximum nim value of a path is $*7$.

4. OE/OE

Lemma 4.1. *There is a graph for every nimber value.*

Proof. We inductively construct a tree G_n having value $*n$. First, we establish that $G_0 = P_1 = 0$, $G_1 = P_3 = *$. We construct G_n from $G_0, G_1, \ldots, G_{n-1}$, which inductively have values $0, *, \ldots, *(n-1)$. This construction hinges on the fact that every G_n has exactly one even degree vertex:

- Start with an isolated vertex v;

- Add one copy of each G_i, for $i < n$;

- If n is odd, add an extra copy of G_0;

- Connect v to the even degree vertex of each G_i.

With this construction, v becomes the only even degree vertex of G_n because the previously even vertex of each G_i is now also adjacent to v. We ensure v is even by adding an extra copy of G_0 when the number of G_i graphs adjacent to v is odd. Since v is the only vertex of even degree and all its neighbours are odd, all options for both players are precisely the edges incident with v. Removing one of these edges disconnects the graph into G_k for some $0 \leq k \leq n-1$ and a component with only odd degree vertices, which has value zero. Recalling that G_k has value $*k$, we see that the edges incident v represent moves to $0, *, *2, *3, \ldots, *(n-1)$. Therefore, G_n has value $*n$.
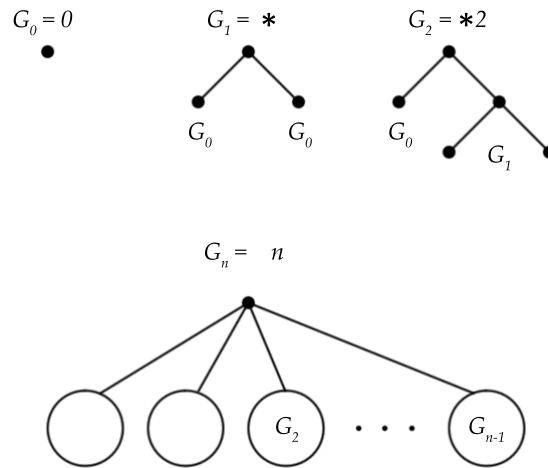

Figure 4: $*n$ construction

$\square$

This shows that all possible game values can be constructed from trees, which is a very specific subset of simple graphs.

An open problem from [6] asks if there is a graph G with $G = *n$ for the game on *vertex deletion* where both players can only delete odd vertices.

Corollary 4.2. *For the vertex deletion game where players may only delete odd degree vertices, there exists a graph G with value $*n$.*

Proof. For any nimber $*n$, let G_n be the graph with that value from the construction in Lemma 4.1. Let H be the linegraph of G_n. The edges with one even degree endpoint and one odd degree endpoint in G_n correspond exactly to the vertices of odd degree in H. Also, deleting an edge in G_n changes the parity of all its neighbours, just as deleting a vertex in H removes all incident edges. Therefore, play in H behaves exactly like its parent graph in G_n, so H has the same value as

G_n. Since $*n$ exists for all n in the edge deletion game, the same is true for the vertex deletion game. □

5. Future Work

5.1. Strategy for Play

We would like to know an exact winning strategy for the games discussed in this paper. For BE/BE and BO/BO, we have a good way of describing a winning strategy: force an odd sized matching as the first player, force an even sized matching as the second player. But we do not have a precise way of describing how to play to force a favourably sized matching. Much of the work done in matchings is to do with augmenting paths, which do not apply to our matching constructions. As for OE/OE, we do not have any strategy for playing correctly.

5.2. Partizan Games

Our naming of the games discussed in this paper as "Left's Rules/Right's Rules" is in anticipation of the partizan edge deletion games. We have begun work on BE/BO, OE/BO, and OE/BO. These three games all appear to have very restricted possible games values. For example, BE/BO has only integer game values. The partizan games also translate in a more meaningful way to edge colourings, inspiring questions about the connections between edge deletion games and edge colourings.

5.3. Colourings

When playing edge deletion games, it is often easier to visualize allowed moves by using coloured edges. We use the game BE/BO as an example. Let red edges be removable by Right, blue edges by Left, and black edges by neither player. Then the parity-based rules of BE/BO can be replaced by the following colouring rules:

- If a red edge is deleted, adjacent red edges become black, and adjacent black edges become blue

- If a blue edge is deleted, adjacent blue edges become black, and adjacent black edges become red

However, the games OE/BE and OE/BO can not have their parity rules replaced by colouring rules. This leads to the questions "which games have rules replaceable by colouring rules?" and "what are the properties of edge deletion games for arbitrary colourings?"

5.4. Rule Modification

In this paper, we have only considered a move to be the deletion of a single edge. A natural extension to be considered is the effect on the game if a player can delete any number of edges with the appropriate properties. Another question which arises is due to the recent results in misere games. How do game values and strategies change when a player wins (rather than loses) by having no legal move on their turn? While this question is typically harder than those addressed in this paper, perhaps there are structures present that make it easier to answer.

References

[1] M. H. Albert, R. J. Nowakowsi, and D. Wolfe, *Lessons in Play,* Combinatorial Game Theory, A K Peters, Ltd. 2007.

[2] E. Berlekamp, J. Conway, and R. Guy, *Winning Ways for Your Mathematical Plays,* New York: Academic Press, 1982.

[3] J. A. Bondy, U.S.R. Murty, *Graph Theory,* Graphs, Springer Graduate Texts in Mathematics, 2008.

[4] R. Gallant, G. Gunther, B. Hartnell, and D.F. Rall, *A Game of Edge Removal on Graphs,* J. Comb. Math. Combin. Comput, 2006.

[5] Lesk, Plummer, and Pulleyblank, *Equi-matchable graphs,* Graph Theory and Combinatorics, London, Academic Press, 1983.

[6] R. J. Nowakowsi, P. Ottaway, *Vertex Deletion Games with Parity Rules,* Integers **5** (2005), A15.

[7] K. Shelton, *The Game of Take Turn,* Integers **7** (2007), A31.