ON THE INCREASES OF THE SEQUENCE $\lfloor k\sqrt{n} \rfloor$

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Abstract

We give explicit formulas for the increases of the sequence $\lfloor k\sqrt{n} \rfloor$ for any fixed positive integer k. For certain values of n mod k, we give simplified expressions for the increases. We also provide simplified upper and lower bounds for the distance between increases.

1. Introduction

In this paper we determine the increases of the sequence $\left(\lfloor k\sqrt{n} \rfloor\right)_{n=1}^{\infty}$ for any fixed positive integer k. As usual, we say a sequence $(c_n)_{n=1}^{\infty}$ has an *increase* or *ascent* at n if $c_n < c_{n+1}$.

For $k = 1$, the increases of the sequence $\left(\lfloor \sqrt{n} \rfloor\right)_{n=1}^{\infty}$ only occur right before each perfect square. This sequence appears in *The On-Line Encyclopedia of Integer Sequences* [5] as A000196. The sequence with $k = 2$, i.e., the sequence $\left(\lfloor 2\sqrt{n} \rfloor\right)_{n=1}^{\infty}$, appears in [5] as A060018. A subsequence of this sequence appears in the work of Griggs [3]. For any prime p larger than 3, the floor of $2\sqrt{p-2}$ is the maximum size of a nonspanning subset of $\mathbb{Z}_p$. This result is a solution of a problem posed by Erdős and Heilbronn [2].

The greatest integer function $\lfloor k\sqrt{n} \rfloor$ plays an important role in other applications as well. The work of Dobrić, Skyers and Stanley [1] shows that much of the fine structure of the random walk on $(0, 1)$ depends on how often the sequence $\left(\lfloor k\sqrt{n} \rfloor\right)_{n=1}^{\infty}$ increases, for a fixed positive integer k. For work on the random walk in the square lattice $\mathbb{Z}^2$, see Niederhausen [4].

2. Results

Clearly the sequence $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases right before each perfect square value of n. Since the sequence is weakly increasing, there are at most $k-1$ increases before the next perfect square. We have the following explicit formula for all increases of $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$.

Theorem 1. *For each positive integer k the sequence $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at n if and only if*

$$n = i^2 + \beta_{k,j}(i)$$

for $j = 0, 1, \ldots, k-1$ and some positive integer i, where

$$\beta_{k,j}(i) = \left\lceil \frac{j}{k^2}\left(2ki + j\right)\right\rceil - 1.$$

Moreover, for all $i \geq \left\lfloor \frac{k}{2} \right\rfloor$, we have that $i^2 + \beta_{k,j}(i)$ is the j^{th} increase after $i^2 - 1$, and $\left\lfloor k\sqrt{i^2 + \beta_{k,j}(i) + 1}\right\rfloor = ki + j$.

Remark 1. Note that $\beta_{k,0}(i) = -1$ for all k and i. This accounts for the increases right before each perfect square value of n.

Remark 2. For example, if $k = 3$, then we have three different families of increases for the sequence $\left(\lfloor 3\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ for each $j = 0, 1, 2$. In other words, the sequence $\left(\lfloor 3\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at n if and only if $n = i^2 + \beta_{3,0}(i), i^2 + \beta_{3,1}(i)$, or $i^2 + \beta_{3,2}(i)$ for some positive integer i, where

$$\begin{aligned}
\beta_{3,0}(i) &= \left\lceil \frac{0}{9}(6i+0)\right\rceil - 1 = 0 - 1 = -1 \\
\beta_{3,1}(i) &= \left\lceil \frac{1}{9}(6i+1)\right\rceil - 1 = \left\lceil \frac{2}{3}i + \frac{1}{9}\right\rceil - 1 \\
\beta_{3,2}(i) &= \left\lceil \frac{2}{9}(6i+2)\right\rceil - 1 = \left\lceil \frac{4}{3}i + \frac{4}{9}\right\rceil - 1.
\end{aligned}$$

Remark 3. To show how the sequence behaves differently at the beginning, i.e., when $i < \left\lfloor \frac{k}{2} \right\rfloor$, consider the case when $k = 5$. When $i = 1$, the values of $\beta_{5,j}(1)$ are not distinct. Indeed $\beta_{5,1}(1) = \beta_{5,2}(1) = 0$, $\beta_{5,3}(1) = 1$ and $\beta_{5,4}(1) = 2$. So the increases of $\left(\lfloor k\sqrt{n}\rfloor\right)$ immediately after the perfect square 1 occur at $n = 1, 2, 3$.

For $i \geq \left\lfloor \frac{5}{2} \right\rfloor$, the values of $\beta_{5,j}(i)$ for $j = 0, 1, 2, 3, 4$ are all distinct. They are in one-to-one correspondence with the five increases of $\left(\lfloor k\sqrt{n}\rfloor\right)$ for $i^2 - 1 \leq n < (i+1)^2 - 1$. For example, the values of $\beta_{5,j}(2)$ are $-1, 0, 1, 2, 3$, so the increases for $4-1 \leq n < 9-1$ occur at $n = 3, 4, 5, 6, 7$. And the values of $\beta_{5,j}(3)$ are $-1, 1, 2, 3, 5$, so the increases for $9-1 \le n < 16-1$ occur at $n = 8, 10, 11, 12, 14$. See the table below.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lfloor 5\sqrt{n} \rfloor$	5	7	8	10	11	12	13	14	15	15	16	17	18	18	19

Proof. To prove Theorem 1, we first show that the sequence $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ has an increase at each $n = i^2 + \beta_{k,j}(i)$. Indeed we have

$$\left\lceil \frac{j}{k^2}\left(2ki+j\right)\right\rceil - 1 < \frac{j}{k^2}\left(2ki+j\right),$$

which holds, in particular, for all positive integers k and i and all $j = 0, 1, \ldots, k-1$. This is equivalent to the following inequalities:

$$\begin{aligned}
i^2 + \left\lceil \frac{j}{k^2}\left(2ki+j\right)\right\rceil - 1 &< i^2 + \frac{j}{k^2}\left(2ki+j\right) \\
k^2\left(i^2 + \left\lceil \frac{j}{k^2}\left(2ki+j\right)\right\rceil - 1\right) &< k^2 i^2 + j\left(2ki+j\right) \\
k^2\left(i^2 + \left\lceil \frac{j}{k^2}\left(2ki+j\right)\right\rceil - 1\right) &< \left(ki+j\right)^2 \\
k\sqrt{i^2 + \left\lceil \frac{j}{k^2}\left(2ki+j\right)\right\rceil - 1} &< ki+j. \qquad (1)
\end{aligned}$$

We also have

$$\left\lceil \frac{j}{k^2}\left(2ki+j\right)\right\rceil \ge \frac{j}{k^2}\left(2ki+j\right),$$

which holds, in particular, for all positive integers k and i and all $j = 0, 1, \ldots, k-1$. This is equivalent to the following inequalities:

$$\begin{aligned}
i^2 + \left\lceil \frac{j}{k^2}\left(2ki+j\right)\right\rceil &\ge i^2 + \frac{j}{k^2}\left(2ki+j\right) \\
k^2\left(i^2 + \left\lceil \frac{j}{k^2}\left(2ki+j\right)\right\rceil\right) &\ge k^2 i^2 + j\left(2ki+j\right) \\
k^2\left(i^2 + \left\lceil \frac{j}{k^2}\left(2ki+j\right)\right\rceil\right) &\ge \left(ki+j\right)^2 \\
k\sqrt{i^2 + \left\lceil \frac{j}{k^2}\left(2ki+j\right)\right\rceil} &\ge ki+j. \qquad (2)
\end{aligned}$$

From the inequalities (1) and (2), it follows that the sequence $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $i^2 + \beta_{k,j}(i)$.

Next we must show that we have found all possible increases of this sequence. To accomplish this, we note that given any i, there are at most k increases in the finite subsequence $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=i^2-1}^{(i+1)^2-1}$. This follows since the sequence is weakly increasing and there are at most $k+1$ distinct values of $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=i^2-1}^{(i+1)^2-1}$.

First we treat the special case that k is odd and $i = \dfrac{k-1}{2}$. In this case

$$\beta_{k,j}(i) = \left\lceil \frac{2ji}{k} + \frac{j^2}{k^2} \right\rceil - 1 = \left\lceil \frac{j\,(k-1)}{k} + \frac{j^2}{k^2} \right\rceil - 1 = \left\lceil \frac{j^2}{k^2} - \frac{j}{k} \right\rceil + j - 1.$$

Since $0 \le j \le k-1$, we have $0 \le \dfrac{j}{k} < 1$, so $0 \ge \dfrac{j^2}{k^2} - \dfrac{j}{k} > -1$, and thus $\beta_{k,j}(i) = j-1$. We also have $\beta_{k,0}(i) = -1$ and

$$i^2 + \beta_{k,k-1}(i) = i^2 + k - 2 = i^2 + 2i - 1 < (i+1)^2 - 1.$$

Therefore, for the case that k is odd and $i = \dfrac{k-1}{2}$, the integers $i^2 + \beta_{k,0}(i) < i^2 + \beta_{k,1}(i) < \ldots < i^2 + \beta_{k,k-1}(i)$ are k distinct increases, thus all the possible increases, of the subsequence $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=i^2-1}^{(i+1)^2-1}$.

Next assume that $i \ge k/2$. In this case

$$\begin{aligned}
\beta_{k,j+1}(i) &= \left\lceil \frac{2\,(j+1)\,i}{k} + \frac{(j+1)^2}{k^2} \right\rceil - 1 \\
&= \left\lceil \frac{2ji}{k} + \frac{2i}{k} + \frac{j^2}{k^2} + \frac{2j+1}{k^2} \right\rceil - 1 \\
&\ge \left\lceil \frac{2ji}{k} + \frac{2i}{k} + \frac{j^2}{k^2} \right\rceil - 1 \\
&\ge \left\lceil \frac{2ji}{k} + 1 + \frac{j^2}{k^2} \right\rceil - 1 \\
&> \left\lceil \frac{2ji}{k} + \frac{j^2}{k^2} \right\rceil - 1 \\
&= \beta_{k,j}(i).
\end{aligned}$$

We also have $\beta_{k,0}(i) = -1$ and

$$\begin{aligned}
\beta_{k,k-1}(i) &= \left\lceil \frac{2\,(k-1)\,i}{k} + \frac{(k-1)^2}{k^2} \right\rceil - 1 \\
&= \left\lceil 2i - \frac{2i}{k} + \frac{(k-1)^2}{k^2} \right\rceil - 1 \\
&\le \left\lceil 2i - 1 + \frac{(k-1)^2}{k^2} \right\rceil - 1 \\
&= 2i - 1,
\end{aligned}$$

thus $i^2 + \beta_{k,k-1}(i) \le i^2 + 2i - 1 < (i+1)^2 - 1$. Therefore, for $i \ge \frac{k}{2}$, the integers $i^2 + \beta_{k,0}(i) < i^2 + \beta_{k,1}(i) < \ldots < i^2 + \beta_{k,k-1}(i)$ are k distinct increases, thus all the possible increases, of the subsequence $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=i^2-1}^{(i+1)^2-1}$.

Finally, assume that $i < \frac{k-1}{2}$. In this case the integers $i^2 + \beta_{k,0}(i) \le i^2 + \beta_{k,1}(i) \le \ldots \le i^2 + \beta_{k,k-1}(i)$ are not necessarily distinct, and not necessarily less than $(i+1)^2 - 1$. Since $j < k$, we do have that $\beta_{k,j}(i) \le 2i$. If $\beta_{k,k-1}(i) < 2i$, then set $M = k$. Otherwise, let M be the smallest integer such that $0 \le M \le k-1$ and $\beta_{k,M}(i) = 2i$. The inequality (1) implies that $k\sqrt{i^2+2i} < ki + M$, but $k\sqrt{i^2+2i+1} = ki + k$. This means the $k - M$ integers $ki + M, ki + M + 1, \ldots, ki + k - 1$ do not appear in the sequence $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$. For all possible values of M (including $M = k$), we have that there are at most M increases of $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=i^2-1}^{(i+1)^2-1}$. Next we want to show that although the integers $i^2 + \beta_{k,0}(i) \le i^2 + \beta_{k,1}(i) \le \ldots \le i^2 + \beta_{k,M-1}(i)$ may not be distinct, we claim that they do comprise all the increases of $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=i^2-1}^{(i+1)^2-1}$. Suppose that for some $0 \le j \le M-2$ we have $\beta_{k,j}(i) = \beta_{k,j+1}(i)$. From (1) this implies that $k\sqrt{i^2 + \beta_{k,j}(i)} < ki + j$, and from (2) we have $k\sqrt{i^2 + \beta_{k,j+1}(i) + 1} \ge ki + j + 1$, which means that $ki + j$ does not appear in the sequence. This reduces by one the number of possible increases of $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=i^2-1}^{(i+1)^2-1}$. The claim follows from this, since the number of distinct integers among $i^2 + \beta_{k,0}(i) \le i^2 + \beta_{k,1}(i) \le \ldots \le i^2 + \beta_{k,M-1}(i)$ is equal to the maximum number of possible increases of $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=i^2-1}^{(i+1)^2-1}$. □

The following theorem gives simple upper and lower bounds for the distance between increases of the sequence $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$.

Theorem 2. *If $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at n, the next increase will occur no sooner than $n + 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor$, and no later than $n + 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor + 4$.*

Proof. Using Theorem 1, suppose the sequence $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $n = i^2 + \beta_{k,j}(i)$ for some positive integer i and for some $j = 0, 1, \ldots, k-1$. Furthermore, suppose that $i \equiv m \pmod{k}$, i.e., $i = m + tk$ for some nonnegative integer t and some integer m such that $0 \le m \le k-1$. We consider three separate cases, depending on the value of j.

Suppose $j = 0$, thus $n = i^2 - 1$, and $i = \sqrt{n+1} = \lfloor\sqrt{n+1}\rfloor$. Also, $t = \left\lfloor \frac{i}{k} \right\rfloor = \left\lfloor \frac{\lfloor\sqrt{n+1}\rfloor}{k} \right\rfloor$. Since $\beta_{k,1}(i) \ge 0$, the next increase occurs at $i^2 + \beta_{k,1}(i)$. Letting d_1

denote the distance between these increases, we have

$$\begin{aligned} d_1 &= \big(i^2+\beta_{k,1}(i)\big)-\big(i^2-1\big) \\ &= \left\lceil \frac{2i}{k}+\frac{1}{k^2}\right\rceil \\ &= \left\lceil \frac{2(m+tk)}{k}+\frac{1}{k^2}\right\rceil \\ &= 2t+\left\lceil \frac{2m}{k}+\frac{1}{k^2}\right\rceil \\ &= 2\left\lfloor \frac{\lfloor\sqrt{n+1}\rfloor}{k}\right\rfloor+\left\lceil \frac{2m}{k}+\frac{1}{k^2}\right\rceil . \end{aligned}$$

Clearly, $d_1 \geq 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k}\right\rfloor$. To obtain an upper bound for d_1, note that

$$\frac{2m}{k}+\frac{1}{k^2}\leq \frac{2(k-1)}{k}+\frac{1}{k^2}=2-\frac{2}{k}+\frac{1}{k^2}<2.$$

Thus

$$d_1 \leq 2\left\lfloor \frac{\lfloor\sqrt{n+1}\rfloor}{k}\right\rfloor+2 \leq 2\left\lfloor \frac{\lfloor\sqrt{n}+1\rfloor}{k}\right\rfloor+2 \leq 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor+k}{k}\right\rfloor+2 = 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k}\right\rfloor+4.$$

Next suppose that $n=i^2+\beta_{k,j}(i)$ where $1\leq j\leq k-2$. From the proof of Theorem 1 we have that $i^2\leq n\leq (i+1)^2-1$, thus $i=\lfloor\sqrt{n}\rfloor$. Also, $t=\left\lfloor \frac{i}{k}\right\rfloor = \left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k}\right\rfloor$. Letting $d_2=\big(i^2+\beta_{k,j+1}(i)\big)-\big(i^2+\beta_{k,j}(i)\big)$, it suffices to show that $2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k}\right\rfloor \leq d_2 \leq 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k}\right\rfloor+4$. Indeed we have

$$\begin{aligned} d_2 &= \big(i^2+\beta_{k,j+1}(i)\big)-\big(i^2+\beta_{k,j}(i)\big) \\ &= \left\lceil \frac{2(j+1)i}{k}+\frac{(j+1)^2}{k^2}\right\rceil-\left\lceil \frac{2ji}{k}+\frac{j^2}{k^2}\right\rceil \\ &= \left\lceil \frac{2(j+1)(m+tk)}{k}+\frac{(j+1)^2}{k^2}\right\rceil-\left\lceil \frac{2j(m+tk)}{k}+\frac{j^2}{k^2}\right\rceil \\ &= 2(j+1)t+\left\lceil \frac{2(j+1)m}{k}+\frac{(j+1)^2}{k^2}\right\rceil-\left\lceil \frac{2jm}{k}+\frac{j^2}{k^2}\right\rceil-2jt \\ &= 2t+\left\lceil \frac{2jm}{k}+\frac{j^2}{k^2}+\frac{2m}{k}+\frac{2j+1}{k^2}\right\rceil-\left\lceil \frac{2jm}{k}+\frac{j^2}{k^2}\right\rceil \\ &= 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k}\right\rfloor+\left\lceil \frac{2jm}{k}+\frac{j^2}{k^2}+\frac{2m}{k}+\frac{2j+1}{k^2}\right\rceil-\left\lceil \frac{2jm}{k}+\frac{j^2}{k^2}\right\rceil . \end{aligned}$$

The desired lower bound follows since

$$d_2 \geq 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor + \left\lceil \frac{2jm}{k} + \frac{j^2}{k^2} \right\rceil - \left\lceil \frac{2jm}{k} + \frac{j^2}{k^2} \right\rceil = 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor.$$

The desired upper bound follows since

$$\begin{aligned} d_2 &\leq 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor + \left\lceil \frac{2m}{k} + \frac{2j+1}{k^2} \right\rceil + \left\lceil \frac{2jm}{k} + \frac{j^2}{k^2} \right\rceil - \left\lceil \frac{2jm}{k} + \frac{j^2}{k^2} \right\rceil \\ &\leq 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor + \left\lceil \frac{2k}{k} + \frac{2(k-2)+1}{k^2} \right\rceil \\ &= 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor + 2 + \left\lceil \frac{2k-3}{k^2} \right\rceil \\ &\leq 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor + 3, \end{aligned}$$

where the last inequality follows since $0 \leq k^2 - 2k + 3$ for all real numbers k, which implies that $\frac{2k-3}{k^2} \leq 1$.

Finally, suppose that $j = k-1$, thus $n = i^2 + \beta_{k,k-1}(i)$, and $i = \lfloor\sqrt{n}\rfloor$. Again, $t = \left\lfloor \frac{i}{k} \right\rfloor = \left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor$. Letting $d_3 = \big((i+1)^2 - 1\big) - \big(i^2 + \beta_{k,k-1}(i)\big)$, it suffices to show that $2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor \leq d_3 \leq 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor + 4$. Indeed we have

$$\begin{aligned} d_3 &= \big((i+1)^2 - 1\big) - \big(i^2 + \beta_{k,k-1}(i)\big) \\ &= 2i - \left\lceil \frac{2(k-1)i}{k} + \frac{(k-1)^2}{k^2} \right\rceil + 1 \\ &= 2(m+tk) - \left\lceil \frac{2(k-1)(m+tk)}{k} + \frac{(k-1)^2}{k^2} \right\rceil + 1 \\ &= 2m + 2tk - \left\lceil \frac{2km + 2tk^2 - 2m - 2tk}{k} + \frac{(k-1)^2}{k^2} \right\rceil + 1 \\ &= 2t - \left\lceil \frac{(k-1)^2}{k^2} - \frac{2m}{k} \right\rceil + 1 \\ &= 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor - \left\lceil \frac{(k-1)^2}{k^2} - \frac{2m}{k} \right\rceil + 1. \end{aligned}$$

The desired lower bound follows since

$$d_3 \geq 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor - \left\lceil \frac{(k-1)^2}{k^2} \right\rceil + 1 = 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor.$$

The desired upper bound follows since

$$d_3 \le 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor - \left\lceil \frac{(k-1)^2}{k^2} - \frac{2k}{k} \right\rceil + 1 = 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor - \left\lceil \frac{(k-1)^2}{k^2} \right\rceil + 3 = 2\left\lfloor \frac{\lfloor\sqrt{n}\rfloor}{k} \right\rfloor + 2.$$

□

Direct computation from Theorem 1 yields the following formulas.

Corollary 1. *Let $k = 2$. The sequence $\left(\lfloor 2\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at n if and only if $n = i^2 - 1$ or $n = i^2 + i$ for some positive integer i.*

Corollary 2. *For all nonnegative integers t we have $\beta_{k,j}(tk + i) = 2jt + \beta_{k,j}(i)$ for all positive integers k and i, and for each $j = 0, 1, \ldots, k-1$.*

Corollary 3. *If k is even, then for all nonnegative integers t we have $\beta_{k,j}\left(i + \frac{tk}{2}\right) = \beta_{k,j}(i) + jt$ for all positive integers k and i, and for each $j = 0, 1, \ldots, k-1$. In particular, we have $\beta_{k,j}\left(i + \frac{k}{2}\right) = \beta_{k,j}(i) + j$.*

To illustrate the usefulness of Corollary 2 in conjunction with Theorem 1, we (again) consider the example with $k = 3$. We will see that if the value of $i \bmod 3$ is specified, then we can express each $\beta_{3,j}(i)$ without the ceiling function, as was needed in Remark 2.

Example 1. The sequence $\left(\lfloor 3\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases as follows.

1. If $i \equiv 1 \pmod 3$, i.e., $i = 3t + 1$ for some nonnegative integer t, then using Corollary 2 we have,

$$\beta_{3,0}(i) = \beta_{3,0}(3t+1) = -1,$$
$$\beta_{3,1}(i) = \beta_{3,1}(3t+1) = 2t + \beta_{3,1}(1) = 2t,$$
$$\beta_{3,2}(i) = \beta_{3,2}(3t+1) = 4t + \beta_{3,2}(1) = 4t + 1,$$

   Thus $\left(\lfloor 3\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $(3t+1)^2 - 1, (3t+1)^2 + 2t, (3t+1)^2 + 4t + 1$ for all nonnegative integers t. That is, $\left(\lfloor 3\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $i^2 - 1, i^2 + \frac{2}{3}i - \frac{2}{3}$, and at $i^2 + \frac{4}{3}i - \frac{1}{3}$ for all positive integers i such that $i \equiv 1 \pmod 3$.

2. If $i \equiv 2 \pmod 3$, i.e., $i = 3t + 2$ for some nonnegative integer t, then we can compute $\beta_{3,j}(i)$ as follows:

$$\beta_{3,0}(i) = \beta_{3,0}(3t+2) = -1,$$
$$\beta_{3,1}(i) = \beta_{3,1}(3t+2) = 2t + \beta_{3,1}(2) = 2t + 1,$$
$$\beta_{3,2}(i) = \beta_{3,2}(3t+2) = 4t + \beta_{3,2}(2) = 4t + 3$$

   Thus $\left(\lfloor 3\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $(3t+2)^2 - 1, (3t+2)^2 + 2t + 1, (3t+2)^2 + 4t + 3$ for all nonnegative integers t. That is, $\left(\lfloor 3\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $i^2 - 1, i^2 + \frac{2}{3}i - \frac{1}{3}$, and at $i^2 + \frac{4}{3}i + \frac{1}{3}$ for all positive integers i such that $i \equiv 2 \pmod 3$.

3. If $i \equiv 0 \pmod 3$, i.e., $i = 3t+3$ for some nonnegative integer t, then we can compute $\beta_{3,j}(i)$ as follows:

$$\beta_{3,0}(i) = \beta_{3,0}(3t+3) = -1,$$

$$\beta_{3,1}(i) = \beta_{3,1}(3t+3) = 2t + \beta_{3,1}(3) = 2t+2,$$

$$\beta_{3,2}(i) = \beta_{3,2}(3t+3) = 4t + \beta_{3,2}(3) = 4t+4.$$

Thus $\left(\lfloor 3\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $(3t+3)^2-1, (3t+3)^2+2t+2, (3t+3)^2+4t+4$ for all nonnegative integers t. That is, $\left(\lfloor 3\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $i^2-1, i^2+\frac{2}{3}i$, and at $i^2+\frac{4}{3}i$ for all positive integers i such that $i \equiv 0 \pmod 3$.

To illustrate Corollary 3, we consider an example with $k = 4$.

Example 2. The sequence $\left(\lfloor 4\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases as follows.

1. If $i \equiv 1 \pmod 4$, i.e., $i = 4t+1$ for some nonnegative integer t, then we can use Corollary 2 to compute $\beta_{4,j}(i)$ in a method similar to the previous example. We find that $\left(\lfloor 4\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $(4t+1)^2-1, (4t+1)^2+2t+0, (4t+1)^2+4t+1, (4t+1)^2+6t+2$ for all nonnegative integers t. That is, $\left(\lfloor 4\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at i^2-1, $i^2+\frac{i}{2}-\frac{1}{2}$, i^2+i, and at $i^2+\frac{3}{2}i+\frac{1}{2}$ for all positive integers i such that $i \equiv 1 \pmod 4$.

2. If $i \equiv 2 \pmod 4$, i.e., $i = 4t+2$ for some nonnegative integer t, then we again use Corollary 2 to compute $\beta_{4,j}(i)$. We find that $\left(\lfloor 4\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $(4t+2)^2-1, (4t+2)^2+2t+1, (4t+2)^2+4t+2, (4t+2)^2+6t+3$ for all nonnegative integers t. That is, $\left(\lfloor 4\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at i^2-1, $i^2+\frac{i}{2}$, i^2+i, and at $i^2+\frac{3}{2}i$ for all positive integers i such that $i \equiv 2 \pmod 4$.

3. If $i \equiv 3 \pmod 4$, i.e., $i = 4t+3$ for some nonnegative integer t, we now use Corollary 3 to compute $\beta_{4,j}(i)$. Indeed

$$\beta_{4,j}(4t+3) = \beta_{4,j}\left(4t+1+\frac{4}{2}\right) = \beta_{4,j}(4t+1)+j.$$

Thus $\left(\lfloor 4\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $(4t+3)^2-1, (4t+3)^2+2t+1, (4t+3)^2+4t+3, (4t+3)^2+6t+5$ for all nonnegative integers t. That is, $\left(\lfloor 4\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at i^2-1, $i^2+\frac{i}{2}-\frac{1}{2}$, i^2+i, and at $i^2+\frac{3}{2}i+\frac{1}{2}$ for all positive integers i such that $i \equiv 3 \pmod 4$.

4. If $i \equiv 0 \pmod 4$, i.e., $i = 4t+4$ for some nonnegative integer t, we again use Corollary 3 to compute $\beta_{4,j}(i)$. Indeed

$$\beta_{4,j}(4t+4) = \beta_{4,j}\left(4t+2+\frac{4}{2}\right) = \beta_{4,j}(4t+2)+j.$$

Thus $\left(\lfloor 4\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $(4t+4)^2-1, (4t+4)^2+2t+2, (4t+4)^2+4t+4, (4t+4)^2+6t+6$ for all nonnegative integers t. That is, $\left(\lfloor 4\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at i^2-1, $i^2+\frac{i}{2}$, i^2+i, and at $i^2+\frac{3}{2}i$ for all positive integers i such that $i\equiv 0 \pmod 4$.

For certain values of $i \bmod k$ we can in fact express $\beta_{k,j}(i)$ without the ceiling function for arbitrary values of k, as demonstrated by the following propositions.

Proposition 1. *Let k be any positive integer.*

(i) *Suppose $i\equiv k-1 \pmod k$, i.e., $i=k-1+tk$ for some nonnegative integer t. Then $\beta_{k,j}(i)=2jt+2j-1$ for $j=0,1,\dots,k-1$. Thus $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at $i^2-1, i^2+2t+1, i^2+4t+3, \dots, i^2+2(k-1)t+2k-3$.*

(ii) *Suppose $i\equiv 0 \pmod k$, i.e., $i=tk$ for some positive integer t. Then $\beta_{k,j}(i)=2jt$ for $j=1,2,\dots,k-1$. Thus $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at*

$$i^2-1, i^2+2t, i^2+4t, \dots, i^2+2(k-1)t.$$

Proof. To prove (i), observe that

$$\beta_{k,j}(k-1+tk)=\left\lceil \frac{2j(k-1+tk)}{k}+\frac{j^2}{k^2}\right\rceil-1=\left\lceil\frac{j^2}{k^2}-\frac{2j}{k}\right\rceil+2j+2jt-1.$$

We claim that $\left\lceil\frac{j^2}{k^2}-\frac{2j}{k}\right\rceil=0$. Indeed

$$j\le 2k \Leftrightarrow j^2\le 2kj \Leftrightarrow \frac{j^2-2kj}{k^2}\le 0 \Leftrightarrow \frac{j^2}{k^2}-\frac{2j}{k}\le 0,$$

and

$$(j-k)^2>0 \Leftrightarrow j^2-2kj+k^2>0 \Leftrightarrow j^2-2kj>-k^2 \Leftrightarrow \frac{j^2}{k^2}-\frac{2j}{k}>-1.$$

To prove (ii), observe that for $j=1,2,\dots,k-1$ we have

$$\beta_{k,j}(tk)=\left\lceil\frac{2jtk}{k}+\frac{j^2}{k^2}\right\rceil-1=\left\lceil\frac{j^2}{k^2}\right\rceil+2jt-1=2jt.$$

□

Proposition 2. *Let k be an even positive integer.*

(i) *Suppose $i\equiv \frac{k}{2}-1 \pmod k$, i.e., $i=\frac{k}{2}-1+tk$ for some nonnegative integer t. Then $\beta_{k,j}(i)=2jt+j-1$ for $j=0,1,\dots,k-1$. Thus $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ increases at*

$$i^2-1, i^2+2t, i^2+4t+1, \dots, i^2+2(k-1)t+k-2.$$

(ii) *Suppose* $i \equiv \frac{k}{2} \pmod{k}$*, i.e.,* $i = \frac{k}{2} + tk$ *for some nonnegative integer* t*. Then* $\beta_{k,j}(i) = 2jt + j$ *for* $j = 1, 2, \ldots, k-1$*. Thus* $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ *increases at*

$$i^2 - 1, i^2 + 2t + 1, i^2 + 4t + 2, \ldots, i^2 + 2(k-1)t + k - 1.$$

Proof. To prove (i), we first use Corollary 3 to obtain

$$\beta_{k,j}\left(\frac{k}{2} - 1 + tk\right) + j = \beta_{k,j}\left(\frac{k}{2} - 1 + tk + \frac{k}{2}\right) = \beta_{k,j}(k - 1 + tk).$$

Substituting the result from Proposition 1 we have

$$\beta_{k,j}\left(\frac{k}{2} - 1 + tk\right) + j = 2j - 1 + 2jt,$$

and (i) follows from this.

To prove (ii), we again start with Corollary 3 to obtain

$$\beta_{k,j}\left(\frac{k}{2} + tk\right) + j = \beta_{k,j}\left(\frac{k}{2} + tk + \frac{k}{2}\right) = \beta_{k,j}(k(t+1)).$$

Substituting the result from Proposition 1 we have

$$\beta_{k,j}\left(\frac{k}{2} + tk\right) + j = 2j(t+1),$$

and (ii) follows from this. □

Proposition 3. *Let* k *be a positive odd integer. Suppose* $i \equiv \frac{k-1}{2} \pmod{k}$*, i.e.,* $i = \frac{k-1}{2} + tk$ *for some nonnegative integer* t*. Then* $\beta_{k,j}(i) = 2jt + j - 1$ *for* $j = 0, 1, \ldots, k-1$*. Thus* $\left(\lfloor k\sqrt{n}\rfloor\right)_{n=1}^{\infty}$ *increases at*

$$i^2 - 1, i^2 + 2t, i^2 + 4t + 2, \ldots, i^2 + 2(k-1)t + k - 2.$$

Proof. By Theorem 1 we have

$$\begin{aligned}
\beta_{k,j}\left(\frac{k-1}{2} + tk\right) &= \left\lceil \frac{2j\left(\frac{k-1}{2} + tk\right)}{k} + \frac{j^2}{k^2}\right\rceil - 1 \\
&= \left\lceil \frac{jk - j + 2jtk}{k} + \frac{j^2}{k^2}\right\rceil - 1 \\
&= \left\lceil \frac{-j}{k} + \frac{j^2}{k^2}\right\rceil + 2jt + j - 1.
\end{aligned}$$

Since $0 \le \frac{j}{k} < 1$, we have $0 \ge \frac{j^2}{k^2} - \frac{j}{k} > -1$. Thus $\left\lceil \frac{-j}{k} + \frac{j^2}{k^2}\right\rceil = 0$, and $\beta_{k,j}\left(\frac{k-1}{2} + tk\right) = 2jt + j - 1$. □

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