SYMMETRY IN MAXIMAL $(s-1, s+1)$ CORES

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Abstract

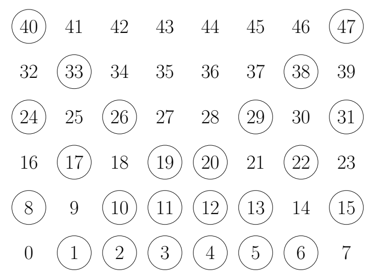
Let s be even and greater than 2. We explain a "curious symmetry" for maximal $(s-1, s+1)$-core partitions first observed by T. Amdeberhan and E. Leven. Specifically, using the s-abacus, we show such partitions have empty s-core and that their s-quotient is comprised of 2-cores. These conditions impose strong conditions on the partition structure, and imply both the Amdeberhan-Leven result and additional symmetry. We conclude by finding the most general family of partitions that exhibit these symmetries, and obtain some new results on maximal $(s-1, s, s+1)$-core partitions.

1. Introduction

The study of simultaneous core partitions, which began fifteen years ago, has seen recent interest due mainly to a conjecture of Armstrong on the average size of an (s,t)-core when $\gcd(s,t) = 1$. R. Stanley and F. Zanello [19] verified the Armstrong conjecture when $t = s+1$; they employ a certain partially ordered set $P_{s,s+1}$ associated to the set of simultaneous $(s, s+1)$-cores which (in this case) exhibits well-understood symmetry. However this poset approach appears difficult to generalize, and P. Johnson [14] recently settled the general case of the Armstrong conjecture using methods from Erhart theory. Amdeberhan and Leven [5] deviate slightly to examine $(s-1, s+1)$-cores in the case where s is even and greater than 2. They do not prove the Armstrong conjecture in this case; they do however explore a"curious symmetry" for the poset $P_{s-1,s+1}$. Our Theorem 6 states their result.

Hidden by the Amdeberhan-Leven proof (which involves integral and fractional parts of a real number) is a connection with the s-core and s-quotient structure viewed on an s-abacus. From this vantage point, the Amdeberhan-Leven theorem is a consequence of symmetry in each runner of the s-abaci of the maximal $(s-1, s+1)$-core; it also reveals additional symmetry in each row not picked up

Figure 1: The 8-abacus of $\kappa_{7,9}$



by the Amdeberhan-Leven formulation. The s-abaci of maximal $(s-1, s+1)$-cores also provide a convenient link to the study of maximal $(s-1, s, s+1)$-cores, objects of recent interest.

We introduce some notation to state the main theorem. Given a partition λ, let λ^0 be its s-core and $(\lambda_{(0)}, \lambda_{(1)}, \cdots, \lambda_{(s-1)})$ be its s-quotient. Let $\kappa_{s\pm 1}$ be the unique maximal simultaneous $(s-1, s+1)$-core partition and $\tau_\ell = (\ell, \ell-1, \ell-2, \cdots, 1)$ be the ℓ-th 2-core partition.

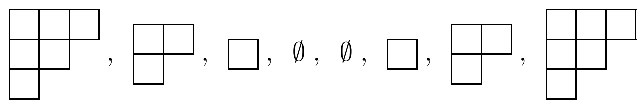
Theorem 1. *Let $s = 2k > 2$. Then $\kappa_{s\pm 1}$ has the following s-core and s-quotient structure:*

1. $(\kappa_{s\pm 1})^0 = \emptyset$.
2. $\kappa_{s\pm 1_{(i)}} = \kappa_{s\pm 1_{(s-1-i)}} = \tau_{k-1-i}$ *where* $0 < i < k-1$.

Example 2. The 8-abacus of $\kappa_{7,9}$ and the associated 8-quotient are shown in **Figure 1** and **Figure 2** respectively. Note: the 8-quotient consists of a sequence of 2-core partitions encoded in the runners of the 8-abacus.

The basic definitions are covered in Section 2. In Section 3.1 we describe the s-abacus of $\kappa_{s\pm 1}$, which we use to prove Theorem 1. We provide an alternate proof of the Amdeberhan-Leven result in Section 3.2. In Section 4.1 we demonstrate an additional symmetry in the rows of the s-abacus of $\kappa_{s-1,s+1}$ and describe the most general family of partitions which satisfy the symmetries exhibited by $\kappa_{s\pm 1}$ in Section 4.2. In Section 5.1 we offer a characterization of the s-abacus of a maximal $(s-1, s, s+1)$-core and examine the relationship between maximal $(s-1, s+1)$ and $(s-1, s, s+1)$-cores, when s is even and greater than 2. We conclude with some questions in Section 5.2.

Figure 2: 8-quotient of $\kappa_{7,9}$



2. Preliminaries

2.1. Basic Definitions

Let $\mathbb{N} = \{0, 1, \cdots\}$ and $n \in \mathbb{N}$. A *partition* λ of n is defined as a finite, non-increasing sequence of positive integers $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ that sums to n. Each λ_α is known as a *component* of λ. Then $\sum_\alpha \lambda_\alpha = n$, and λ is said to have *size* n, denoted $|\lambda| = n$.

The *Young diagram* $[\lambda]$ is a graphic representation of λ in which rows of boxes corresponding to the integer values in the partition sequence are left-aligned. Then λ^* is the *conjugate partition* of λ obtained by exchanging rows and columns of the Young diagram of λ; λ is *self-conjugate* if $\lambda = \lambda^*$. Using matrix notation, a *hook* $h_{\iota\gamma}$ of $[\lambda]$ with *corner* (ι, γ) is the set of boxes to the right of (ι, γ) in the same row, below (ι, γ) in the same column, and (ι, γ) itself. Given $h_{\iota\gamma}$, its *length* $|h_{\iota\gamma}|$ is the number of boxes in the hook. The set $\{h_{\iota 1}\}$ are the *first-column hooks* of λ.

One can *remove* a hook h of λ by deleting boxes in $[\lambda]$ which comprise h and migrating any remaining detached boxes up and to the left. In this way a new partition λ' of size $n - |h_{\iota\gamma}|$ is obtained. An *s-hook* is a hook of length s. An *s-core partition* λ is one in which no hook of length s appears in the Young diagram.

Example 3. Let $\lambda = (4, 3, 2)$. Then the Young diagram of λ is shown in Figure 3. Note that $h_{2,1}$ is of length 4.

2.2. Simultaneous (s, t)-core Partitions

Let s and t be positive integers. A *simultaneous (s,t)-core partition* is one in which no hook of length s or t appears. In 1999, J. Anderson [6] proved when $\gcd(s, t) = 1$, there are exactly $\binom{s+t}{t}/(s + t)$ simultaneous (s, t)-cores.

Subsequent work by B. Kane [1], J. Olsson and D. Stanton [18], and J. Vandehey [21] confirmed the existence of a unique *maximal* (s, t)-core of size $\frac{(s^2-1)(t^2-1)}{24}$, denoted by $\kappa_{s,t}$, which contains all others. Here maximal is meant in terms of size; containment is being able to fit the Young diagram of one partition inside another.

Theorem 4. *(Olsson-Stanton, Theorem 4.1, [18]) Let $gcd(s, t) = 1$. There is a unique maximal simultaneous (s, t)-core $\kappa_{s,t}$ of size $\frac{(s^2-1)(t^2-1)}{24}$. In particular, $\kappa_{s,t}$ is self-conjugate.*

7	4	2	1
4	1		
2			
1			

Figure 3: Young diagram (with hook lengths) of $\kappa_{3,5} = (4, 2, 1, 1)$

Theorem 5. *(Vandehey, [21]) Let gcd*$(s, t) = 1$*. Then* $\kappa_{s,t}$ *contains all other* (s, t)*-cores.*

We note that A. Tripathi [20] and M. Fayers [9] obtained some of the above results using different methods.

A paper of D. Armstrong, C. Hanusa and B. Jones [7] includes a conjecture (the Armstrong conjecture) that the average size of an (s, t)-core is $\frac{(s+t+1)(s-1)(t-1)}{24}$. Stanley and Zanello [19] resolved this conjecture for the case $t = s+1$ by employing a bijection between lower ideals in the poset $P_{s,s+1}$ and simultaneous $(s, s+1)$-cores. We outline this bijection for general s and t. Let $P_{s,t}$ be the partially ordered set whose elements are all positive integers not contained in the numerical semigroup generated by s, t. The partial order requires $z_1 \in P_{s,t}$ to cover $z_2 \in P_{s,t}$ if $z_1 - z_2$ is either s or t. Under this map, a lower ideal I of $P_{s,t}$ corresponds to an (s, t)-core partition whose first-column hook lengths are exactly the values in I. Then $P_{s,t}$ itself corresponds to $\kappa_{s,t}$.

The Armstrong conjecture was verified for self-conjugate partitions by W. Chen, H. Huang, and L. Wang [8] and for $(s, ms+1)$-cores by A. Aggarwal [2] before a full proof was given by P. Johnson [14] using Erhart theory. Since then, V. Wang [22], using an approach of M. Fayers [10], has found a proof of the Armstrong conjecture that avoids Erhart theory; Wang also settles a generalization of the Armstrong conjecture due to M. Fayers [11].

Simultaneous core partitions have also generated interest outside of the Armstrong conjecture. For example, Aggarwal has also proved a partial converse to a theorem of Vandehey on the containment of simultaneous (r, s, t)-cores [3] for distinct r, s, t. In another direction, Amdeberhan [4] proposed several conjectures on maximal $(s-1, s, s+1)$-cores; these have been proved, first by J. Yang, M. Zhong and R. Zhou [24] and later by H. Xiong [23]. We discuss these developments in Section 5.

						47	
33					38		40
	26			29		31	
17		19	20		22		24
	10	11	12	13		15	
1	2	3	4	5	6		8

Figure 4: Amdeberhan-Leven rectangle R for $P_{7,9}$

2.3. A "Curious Symmetry"

For s even, Amdeberhan and Leven examine $P_{s-1,s+1}$ via a rectangle R with $s-2$ rows and s columns, constructed as follows: the bottom-left corner is labelled by 1, the numbers increase from left-to-right and bottom-to-top, and the largest position, in the upper-right corner, is labeled by $(s-2)(s)$. If $x \in P_{s-1,s+1}$ then x is entered into this rectangle; otherwise, the position is left blank. Positions are labeled by pairs (i, j), where i enumerates columns from left-to-right ($1 \le i \le s$), and j rows from bottom-to-top ($1 \le j \le s-2$). They then prove the following result, which they call a "curious symmetry."

Theorem 6. *(Amdeberhan-Leven, Theorem 2.2, [5]) For $s \ge 4$ and even, the (i, j) entry of R is an element of $P_{s-1,s-1}$ if and only if $(i, s-2-j)$ is not. Equivalently, for $1 \le i \le s$ and $1 \le j \le s-2$, $i + s(j-1) \in P_{s-1,s+1}$ if and only if $i + s(s-2-j) \notin P_{s-1,s+1}$.*

There is a precedent for the case Amdeberhan-Leven consider. For $s = 2k > 1$, the maximal simultaneous $(s-1, s+1)$-core is self-conjugate, by Theorem 4. In [12] C. Hanusa and the author showed that it is more natural to think about self-conjugate $(s-1, s+1)$-core partitions than self-conjugate $(s, s+1)$-cores (the latter of which are better behaved in the non-self-conjugate case).

We now review the s-abacus, s-core, and s-quotient constructions.

2.4. Bead-sets

The first column hook lengths uniquely determine a partition λ. We can generalize the set of first column hooks using the notation of a *bead set* X corresponding to λ, where $X = \{0, \cdots, k-1, |h_{11}| + k, |h_{21}| + k, |h_{31}| + k, \cdots\}$ for some non-negative integer k. It can also be seen as a finite set of non-negative integers, represented by *beads* at integral points of the x-axis, i.e. a bead at position x for each x in X and *spacers* at positions not in X. Then $|X|$ is the number of

Figure 5: The minimal bead set for $\lambda = (4, 3, 2)$

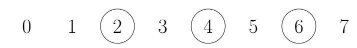


Figure 6: The bead set $X' = \{0, 3, 5, 7\}$ for $\lambda = (4, 3, 2)$

beads that occur after the zero position, wherever that may fall. We say $X = \{0, \cdots, k-1, |h_{11}| + k, |h_{21}| + k, |h_{31}| + k, \cdots\}$ is *normalized with respect to* s if k is the minimal integer such that $|X| \equiv 0 \pmod{s}$. The *minimal* bead-set X of λ is one where 0 labels the first spacer, and is equal to the set of first column hook lengths.

Example 7. Suppose $\lambda = (4, 3, 2)$. Then $\{h_{\iota 1}\} = \{2, 4, 6\}$ is the set of first column hook lengths, and a minimal bead set. Then $X' = \{0, 2+1, 4+1, 6+1\} = \{0, 3, 5, 7\}$ and $X'' = \{0, 1, 2, 3, 2+4, 4+4, 6+4\} = \{0, 1, 2, 3, 6, 8, 10\}$ are two bead sets that also correspond to λ. X' and X'' are normalized with respect to 4 and 7 respectively, since $X' \equiv 0 \pmod 4$ and $X'' \equiv 0 \pmod 7$.

2.5. 2-cores and Staircase Partitions

The results in this section are stated without proof; for more details see Section 2 in [17]. The set of hooks $\{h_{\iota\gamma}\}$ of λ correspond bijectively to pairs (x, y) where $x \in X$, $y \notin X$ and $x > y$; that is, a bead in a bead-set X of λ and a spacer to the left of it. Hooks of length s are those such that $x - y = s$.

Bead x in the minimal bead-set X are in bijection with components of $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$. The following result, which appears as in Lemma 2.4 in [17], allows us to recover the size of the component from its corresponding bead.

Lemma 1. *Let X be a bead set of a partition λ. The size of the component λ_α of λ corresponding to the bead $x' \in X$ is the number of spacers to the left of the bead, that is, $\lambda_\alpha = |y \notin X : y < x'|$.*

Let $\tau_\ell = (\ell, \ell-1, \ell-2, \cdots, 1)$ be the *ℓth staircase partition.* Then $|\tau_\ell| = t_\ell$ where $t_\ell = \binom{\ell+1}{2}$ (the ℓth triangular number). The following two lemmas are well-known.

Lemma 2. *The 2-core partitions are exactly the staircase partitions.*

Lemma 3. *The minimal bead set X for the 2-core τ_ℓ is $\{1, 3, 5, \cdots, 2\ell-3, 2\ell-1\}$. In other words, the 2-core partitions are a sequence of alternating spacers-and-beads of length $2\ell - 1$.*

2.6. The s-abacus

Given a fixed integer s, we can arrange the nonnegative integers in an array of columns and consider the columns as runners:

$$\begin{array}{cccc} ms & ms+1 & & (m+1)s-1 \\ (m-1)s & & & ms-1 \\ \vdots & & \ddots & \\ s & s+1 & & 2s-1 \\ 0 & 1 & \cdots & s-1 \end{array}$$

The column containing i for $0 \le i \le s-1$ will be called *runner* i. The positions $0, 1, 2, \cdots$ on the ith runner corresponding to $i, i+s, i+2s, \cdots$ will be called *row positions on runner* i. Consider a bead set X. Placing a bead at position x for each $x \in X$ gives the *s-abacus diagram* of X. Positions not occupied by beads are *spacers*. A *normalized* abacus will be one whose bead set X is normalized, and the *minimal* abacus one in which X is minimal (or, the first spacer labels the zero position). Note that a bead x in runner i with a spacer y one row below, but also in runner i, corresponds to an s-hook of λ. The following is immediate.

Lemma 4. *An s-abacus in which no spacer appears directly below a bead on the same runner corresponds to an s-core partition.*

2.7. The s-core and s-quotient

By removing a sequence of s-hooks from λ until no s-hooks remain, one obtains its *s-core* λ^0. The s-abacus of λ^0 can be found from the s-abacus of λ by pushing beads in each runner down as low as they can go (see Theorem 2.7.16, [13], with changed orientation). Hence λ^0 is unique since it is independent of the way the s-hooks are removed. For $0 \le i \le s-1$ let $X_i = \{j : i + js \in X\}$ and let $\lambda_{(i)}$ be the partition represented by the bead-set X_i. The *s-quotient* of λ is the sequence $(\lambda_{(0)}, \cdots, \lambda_{(s-1)})$ obtained from X. The next lemma is Proposition 3.5 in [17].

Lemma 5. *Let λ be a partition with s-core λ^0 and s-quotient $(\lambda_{(i)})$, $0 \le i \le s-1$. Then*

1. *Every 1-hook in $\lambda_{(i)}$ corresponds to an s-hook in λ for $0 \le i \le s-1$.*
2. *$n = |\lambda^0| + s \cdot \sum_i |\lambda_{(i)}|$.*

We note that X_i could consist of an interval $[0, m]$ and thus $\lambda_{(i)}$ would be empty (as is the case with $\lambda_{(3)}$ and $\lambda_{(4)}$ in Example 1.2). Lemma 5 implies that there exists a bijection between a partition λ and its s-core and s-quotient, such that each node in some $\lambda_{(i)}$ corresponds to an s-hook in λ. The situation is strengthened when λ is self-conjugate.

Lemma 6. *Suppose* $|X| = 0 \pmod{s}$. *Let* λ^* *be the conjugate of* λ, $(\lambda^*)^0$ *its* s*-core and let* $(\lambda^*_{(i)})$ *be the* s*-quotient of* λ^*, $0 \leq i \leq s-1$. *Then*

1. $(\lambda^*)^0 = (\lambda^0)^*$
2. $(\lambda_{(i)})^* = \lambda_{(s-1-i)}$.

In particular, $\lambda = \lambda^*$ *if and only if* $\lambda^0 = (\lambda^0)^*$ *and* $(\lambda_{(i)})^* = (\lambda^*)_{(i)}$.

2.8. The Axis $\theta(X)$ of a Bead Set of λ

The following results and their proofs can be found in Section 4, [15].

Proposition 1. *Suppose* λ *is a partition of* n *and let* X *be a bead-set for* λ. *Then there exists a half-integer* $\theta(X)$ *(that is, an element of* $\mathbb{Z} + \frac{1}{2}$*) such that the number of beads to the right of* $\theta(X)$ *equals the number of spaces to its left. Conversely, given a bead-spacer sequence and a half-integer such that the number of beads to the right equals the number of spaces to the left, one can recover a partition* λ.

Although the number of beads to the right of $\theta(X)$ and the number of spacers of the left of $\theta(X)$ remain unchanged on any bead-spacer sequence associated to λ, the half-integer value assigned to $\theta(X)$ will depend on X.

Example 8. Consider $\lambda = (4, 3, 2)$. Then $\theta(X) = 2.5$ and $\theta(X') = 3.5$ for $X = \{2, 4, 6\}$ and $X' = \{0, 3, 5, 7\}$ respectively. See Figure 6 and Figure 7.

We call $\theta(X)$ the *axis* of a bead set X of λ. Each X_i has an axis $\theta(X_i)$; when $\lambda^0 = \emptyset$, the value does not change as i runs from 0 to $s-1$.

Lemma 7. *Suppose* X *is normalized with respect to* s *and* $|X| = ms$. *Then the following are equivalent:*

1. $\lambda^0 = \emptyset$
2. *each runner has exactly* m *beads*
3. $\theta(X_i) = m - \frac{1}{2}$ *for all* $0 \leq i \leq s-1$.

Example 9. The maximal $(5, 7)$-core $\kappa_{5,7}$ has empty 8-core. In the normalized (minimal) 8-abacus in Figure 1, $|X| = 3 \cdot 8$, whereas each runner has 3 beads, and X_i has axis $\theta(X_i) = \frac{5}{2}$ for $0 \leq i \leq 7$.

Figure 7: The minimal bead set X for $\kappa_{3,5}$.

If λ is self-conjugate we say X has an *axis of symmetry.*

Corollary 1. *Let X be a bead-set for λ. Then λ is a self-conjugate partition if and only if there exists a half-integer $\theta(X)$ such that beads and spacers in X to the right of $\theta(X)$ are reflected respectively to spacers and beads in X to its left.*

Example 10. The maximal $(3, 5)$-core $\kappa_{3,5}$ is self-conjugate, with minimal bead set $X = \{1, 2, 4, 7\}$, and $\theta(X) = 3.5$. Then beads and spacers to the right of 3.5 are reflected to spacers and beads to the left of it. See Figure 8.

Note that when a bead set is not minimal, the sequence of beads in positions $[0, 1, \cdots, k]$ will be reflected onto spacers in positions greater than the last bead.

3. The Structure of $\kappa_{s\pm1}$

3.1. The s-abacus $\alpha(s)$

To recover the Amdeberhan-Leven result we first construct the s-abacus of $\kappa_{s\pm1}$.

Definition 11. Let $s = 2k > 2$, and consider s runners, indexed from left-to-right by $0 \le i \le s-1$ and $s-2$ rows, indexed from bottom-to-top by $0 \le j \le s-3$. We construct the s-abacus $\alpha(s)$ as follows: for each $i \in [0, k-2]$, runners i and $s-i-1$ are composed of beads in the first i rows; spacers-and-beads alternate in rows $j > i$ until the total number of beads in each runner is $k-1$. Spacers fill the remainder of the rows.

Example 12. Consider the 8-abacus $\alpha(8)$. It has three beads in each runner. Runners 3 and 4 consist of three beads below three spacers; runners 2 and 5 have two beads followed by a spacer-and-bead, then two spacers; runners 1 and 6 have one bead followed by spacer-bead-spacer-bead-spacer; and runners 0 and 7 have an alternating sequence of spacers-and-beads. [See Figure 1.]

Lemma 8. *The s-abacus $\alpha(s)$ is normalized with respect to s.*

Proof. The total number of beads in $\alpha(s)$ is $2k(k-1) = s(\frac{s-2}{2})$. □

Lemma 9. *The following holds for the s-abacus $\alpha(s)$*

1. *There is a bead in row j of runner* 0 *if and only if there is a bead in row* $j-1$ *of runner* 1.

2. *There is a bead in row j of runner* $2k-1$ *if and only if there is a bead in row* $j-1$ *of runner* $2k-2$.

3. *There is a spacer in row j of runner* 0 *if and only if there is a spacer in row* $j+1$ *of runner* 1.

4. *There is a spacer in row j of runner* $2k-1$ *if and only if there is a spacer in row* $j+1$ *of runner* $2k-2$.

Proof. By Definition 11, runner $i = 0$ begins in row $j = 0$ with a spacer, and continues upwards with alternating beads-and-spacers. Runner $i = 1$ begins with a bead in row 1, and continues upwards, alternating spacers-and-beads. Since both columns have $2k-2$ rows, (1) and (3) follow. For (2) and (4), a similar argument holds. □

Lemma 10. *The* $(s+2)$*-abacus* $\alpha(s+2)$ *can be obtained from the* s*-abacus* $\alpha(s)$ *using the following procedure:*

1. *Append a new row of* $2k$ *beads below* $\alpha(s)$.

2. *Append a new row of* $2k$ *spacers above* $\alpha(s)$.

3. *Append a new runner of length* $2k-2$ *consisting of alternating beads-and-spacers to the left, and an identical column to the right, of* $\alpha(s)$*. [Both of these columns start with a bead in the bottom row.]*

4. *Append a single spacer to the bottom, and a single bead at the top of, both new runners in step (3). [The total number of beads in all runners, both the two new runners, as well as the* $s = 2k$ *previous runners, will now be* k*.]*

5. *Renumber the runners with* i' *so* $0 \leq i' \leq 2k+1$ *and the rows with* j' *so that* $0 \leq j' \leq 2k-1$*. Renumber the abacus positions, with 0 in the bottom left-most corner, increasing from left-to-right and bottom-to-top, with final position* $(2k+1)(2k-1)$ *in the upper-right-hand corner.*

Proof. It is enough to see that the result of these five steps satisfies Definition 11 for $\alpha(s+2)$. □

Example 13. To see how Lemma 10 is used to obtain $\alpha(10)$ from $\alpha(8)$, see Appendix A, Figure 12 and Figure 13.

Recall λ^0 and $(\lambda_{(i)})$ for $0 \le i \le s-1$ are the s-core and s-quotient of λ respectively, and that τ_ℓ is the ℓth 2-core partition. For the following two lemmas we abuse notation and let $\alpha(s)$ refer to both the s-abacus and its corresponding partition.

Lemma 11. *Suppose $s = 2k > 2$. Then*

1. $\alpha(s)^0 = \emptyset$

2. $\alpha(s)_{(i)} = \alpha(s)_{(s-i-1)} = \tau_{k-i+1}$.

Proof. We prove each condition separately.

1. Since each runner $\alpha(s)_i$ has $k-1$ beads and $k-1$ spacers, the removal of all s-hooks throughout all the runs will result in an s-abacus with each runners having $k-1$ beads beneath $k-1$ spacers. This arrangement corresponds to the empty partition.

2. We use induction on k. For $k = 2$ it is true. Assume it is for k. We obtain the $\alpha(s+2)$ from $\alpha(s)$ by Lemma 11. By construction, for $1 \le i' \le 2k$ we have $|\alpha(s)_{(i'-1)}| = |\alpha(s+2)_{(i')}|$; hence, by the inductive hypothesis and since $i+1 = i'$, $|\alpha(s+2)_{(i')}| = \tau_{(k+1)-i'-1}$. It only remains to check $i' = 0$ and $2k+1$. The proof is finished using Lemma 3, and (3) and (4) of Lemma 10.

□

Example 14. $\alpha(8)$ has 8-quotient $(\lambda_{(0)}, \cdots, \lambda_{(s-1)}) =$

$$((3,2,1),(2,1),(1),\emptyset,\emptyset,(1),(2,1),(3,2,1)).$$

[See Appendix A, Figure 12 and Appendix B, Figure 16.]

Recall that when τ_ℓ is the ℓth 2-core partition, we let $t_\ell = |\tau_\ell|$.

Lemma 12. *Let $s = 2k > 2$. Then $\alpha(s)$ is the minimal s-abacus for $\kappa_{s-1,s+1}$.*

Proof. By construction, $\alpha(s)$ is minimal, since zero labels the first spacer. We must show:

1. $|\alpha(s)| = \frac{((2k-1)^2-1)((2k+1)^2-1)}{24}$,

2. $\alpha(s)$ contains no $(2k-1)$-hooks or $(2k+1)$-hooks.

Then by the uniqueness implied by Theorem 4, $\alpha(s) = \kappa_{s\pm1}$. We use the structure of $\alpha(s)$ and induction on k.

By Lemma 5, $|\lambda| = |\lambda^0| + s \cdot \sum |\lambda_{(i)}|$. Since $\alpha(s)^0 = \emptyset$, to prove (1), it is enough to calculate $2k \cdot \sum_i |\alpha(s)_{(i)}|$, which equals $2k \cdot 2\sum_{i=1}^{k-1} t_i = (4k)\frac{(k-1)(k)(k+1)}{6}$. In particular $4k\frac{(k)(k^2-1)}{6} = \frac{16k^4-16k^2}{24} = \frac{(4k^2-4k)(4k^2+4k)}{24}$. Finally, after completing-the-square, one obtains

$$\frac{((2k-1)^2-1)((2k+1)^2-1)}{24}.$$

To prove (2), we use induction on $k > 2$. For the basic case, s=4, it holds: $\alpha(4)$ has no 3-hooks or 5-hooks. [See Appendix A, Figure 10.] By the inductive hypothesis we know the $2k$-abacus of $\kappa_{2k\pm1}$ contains no $(2k-1)$-hooks or $(2k+1)$-hooks. More specifically, no bead in $\alpha(s)$ has a spacer either $2k+1$ or $2k-1$ positions below it. Apply Lemma 10 to obtain $\alpha(s+2)$; this adds two additional positions between the beads and spacers arising from $\alpha(s)$. Hence there are no $(2k+1)$-hooks or $(2k+3)$-hooks arising from bead-spacer pairs (x, y) where both x and y are in runners $1 < i' < 2k-2$. It remains to examine the beads and spacers introduced by runners $i' = 0, 2k+1$.

If a bead in row j' of runner $i' = 0$ were to add a new $(2k+3)$-hook, a spacer would have to appear in row $j'-2$ of the runner $i' = 2k+1$. By construction, such positions are occupied by beads, since runners 0 and $2k+1$ are identical. If a bead in row j' of $i' = 0$ were to add a new $(2k+1)$-hook, a spacer would have to appear in row $j'-1$ of runner $i' = 1$; by the Lemma 9(1), this position is always occupied by a bead.

If a bead in row j' on runner $i' = 2k+1$ were to add a new $(2k+3)$-hook, a spacer would appear in row $j'-1$ of runner $i' = 2k$; by Lemma 9(2) this position is always occupied by a bead. If a bead in row j' of runner $i' = 2k+1$ were to add a new $(2k+1)$-hook, a spacer would have to appear in the same row in the runner $i' = 0$. By construction, the two runners are identical, so a bead in one implies a bead in the other.

If a spacer in row j' of runner $i' = 0$ were to add a new $(2k+3)$-hook, a bead would have to appear in row $j'+1$ of runner $i' = 1$; by Lemma 9(3), this position is always occupied by a spacer. If a spacer in row j' of $i' = 0$ were to add $(2k+1)$-hook, a bead would have to appear in the same row of runner $i' = 2k+1$. By construction, the two runners are identical, so a spacer in one implies a spacer in the other.

If a spacer in row j' of runner $i' = 2k+1$ were to add a new $(2k+3)$-hook, a bead would have to appear in row $j'+2$ in runner $i' = 0$; by construction, since both runners are identical alternating sequences of spacer-and-beads, such positions are occupied by spacers. If a spacer in row j' of runner $i' = 2k+1$ were to add a new $(2k+1)$-hook, a bead would have to appear in row $j'+1$ of runner $i' = 2k$; by Lemma 9(4) this position is occupied by a spacer. □

3.2. An Alternative Proof of Amdeberhan-Leven

Using the results of the previous section, and a few lemmas, we can provide an alternative proof to Theorem 6. We begin with a classical result of Sylvester.

Lemma 13. *The largest integer in* $P_{s,t}$ *is* $st - s - t$.

Let R be the rectangle described in Section 2.3.

Corollary 2. *R does not contain 0 or* $s^2 - 2s$.

Proof. R does not contain 0 by construction. By Lemma 13, $s^2 - 2s - 1$ is the largest integer in $P_{s-1,s+1}$, hence also in R. □

Definition 15. We say $(i, j) \in \alpha(s)$ if $i + js \in P_{s-1,s+1}$ where $0 \leq i \leq s - 1$ and $0 \leq j \leq s - 3$.

Lemma 14. *Let* $0 \leq i \leq s - 1$ *and* $0 \leq j \leq s - 3$. *Then* $(i, j) \in \alpha(s)$ *if and only if* $i + js \in R$.

Proof. By Lemma 12 $\alpha(s)$ is the minimal s-abacus for $\kappa_{s\pm1}$. By the discussion in Section 2.7, it contains exactly the same values as $P_{s\pm1}$. Since R has the same values as $P_{s\pm1}$ by construction, we are done. □

Proof of Theorem 6. By Lemma 14 the contents of R and $\alpha(s)$ are identical; in particular by Corollary 2 we do not lose anything by inserting 0 and removing $s^2 - 2s$ from the diagram. This has the effect of shifting the rightmost column of R to the first column of $\alpha(s)$ and up one row. Hence it is enough to prove the following condition: $(i, j) \in \alpha(s)$ if and only if $(i, s-3-j) \in \alpha(s)$ for $0 \leq i \leq s-1$ and $0 \leq j \leq s-3$.

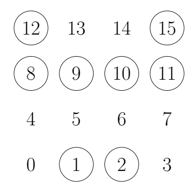
By induction on k. For $k = 2$ it is clear. Suppose $(i, j) \in \alpha(s)$ if and only if $(i, s - 3 - j) \notin \alpha(s)$ holds for $s = 2k$. Consider now $s = 2(k + 1)$. The inductive hypothesis and Lemma 10 imply that $(i', j') \in \alpha(s)$ if and only if $(i', s-1-j') \notin \alpha(s)$ for $1 \leq i' \leq s$ and $1 \leq j' \leq s - 2$. It remains to show the property holds for (i, j) when $j' = 0$ or $s - 1$ and when $i = 0$ or $s + 1$. However by construction when $j' = 0$ and $1 \leq i' \leq s$, $(i', 0) \in \alpha(s)$ and $(i', s - 1) \notin \alpha(s)$. When $i = 0$ or $s + 1$, the runner consists of alternating sequence of spacers-and-beads, hence the property holds. □

4. Generalizations

4.1. Additional Symmetry for Maximal $(s - 1, s + 1)$-cores

Using Theorem 1 we can strengthen the Amdeberhan-Leven result to include additional symmetry.

Figure 8: The minimal 4-abacus of $\lambda = (8, 6, 6, 6, 6, 6, 1, 1)$



Theorem 16. *Let $s = 2k > 2$ and let $\alpha(s)$ be the s-abacus of $\kappa_{s\pm1}$. Let $0 \le i \le s-1$ and $0 \le j \le s-3$. Then the following are equivalent:*

1. $(i, j) \in \alpha(s)$
2. $(i, s-3-j) \notin \alpha(s)$
3. $(s-1-i, j) \in \alpha(s)$.

Proof. By Theorem 6 it is sufficient to prove (1) $\iff$ (3). This follows from Lemmas 9, 11, and 12, and an induction argument similar to the proof Theorem 6. □

4.2. $(UD, -)$ and $(RL, +)$ Symmetry

The symmetries exhibited by the s-abacus of $\kappa_{s\pm1}$ can be formalized and generalized to a larger family of partitions. For the remainder of this section we assume that the bead-set X of λ is normalized with respect to s. Let $0 \le i \le s-1$. Suppose that the s-abacus of λ has maximum value $i + (q-1)s$. In particular, the normalized s-abacus of λ has s columns and q rows, indexed by pairs (i, j) where $0 \le j \le q-1$.

Definition 17. We say the s-abacus of λ exhibits $(UD, -)$ symmetry if a there is a bead in the (i, j) position if and only if there is a spacer in the $(i, q-1-j)$ position. [UD here refers to *up-down.*]

Lemma 15. *The s-abacus of λ exhibits $(UD, -)$ symmetry if and only if q is even, $\lambda_{(i)} = \lambda^*_{(i)}$ for all $0 \le i \le s-1$, and $\lambda^0 = \emptyset$.*

Proof. Suppose the s-abacus π of λ exhibits $(UD, -)$ symmetry. Then if $(i, j) \in \pi$ if and only if $(i, q-1-j) \notin \pi$. This is equivalent to each runner i having axis $\theta(X_i) = \frac{q-1}{2}$ such that beads and spacers less than $\theta(X_i)$ are reflected across to spacers and beads. Hence q must be even, so beads and spacers can be paired. By Corollary 1, this also implies that $\lambda_{(i)} = \lambda^*_{(i)}$ for each $0 \le i \le s-1$. Finally, by Lemma 7, $\lambda^0 = \emptyset$. The proof in the other direction is clear. □

Definition 18. We say the s-abacus of λ exhibits $(RL,+)$ symmetry if there is a bead in the (i,j) position if and only if there is a bead in the $(s-1-i,j)$ position. [RL here refers to *right-left*.]

Lemma 16. *The s-abacus of λ exhibits $(RL,+)$ symmetry if and only if runner i and runner $s-i-1$ have the same number of beads, and $\lambda_{(i)} = \lambda_{(s-1-i)}$ for $0 \le i \le s-1$.*

Proof. Suppose the s-abacus of λ exhibits $(RL,+)$ symmetry. Then each runner i and $s-i-1$ must be identical. This means runners i and $s-i-1$ have the same number of beads and $\lambda_{(i)} = \lambda_{(s-i-1)}$ for each $0 \le i \le s-1$. The proof in the other direction is clear. □

Theorem 19. *λ exhibits both $(UD,-)$ and $(RL,+)$ symmetry with respect to s if and only if q is even and the following three conditions hold for all $0 \le i \le s-1$:*

1. $\lambda^0 = \emptyset$
2. $\lambda_{(i)} = \lambda^*_{(i)}$
3. $\lambda_{(i)} = \lambda_{(s-i-1)}$.

Proof. This follows from Lemma 7, Lemma 15, and Lemma 16. □

Example 20. The minimal 4-abacus of $\lambda = (8,6,6,6,6,6,1,1)$ exhibits $(UD,-)$ and $(RL,+)$ symmetry, but is neither a 3-core nor a 5-core. See Figure 5.

The following corollary is immediate.

Corollary 3. *Let $s = 2k > 1$. The s-abacus of $\kappa_{s\pm1}$ exhibits $(UD,-)$ and $(RL,+)$ symmetry.*

Corollary 4. *If the s-abacus of λ exhibits both $(UD,-)$ and $(RL,+)$ then λ is self-conjugate and has empty s-core.*

Proof. By Theorem 19, since $\lambda_{(i)} = \lambda_{(s-i-1)}$ and $\lambda_{(i)} = \lambda^*_{(i)}$, we have $\lambda_{(i)} = \lambda^*_{(s-i-1)}$. Since $\lambda^0 = \emptyset$, and by assumption $|X| = 0 \pmod{s}$, we have $\lambda = \lambda^*$ by Lemma 6. □

5. Simultaneous $(s-1, s, s+1)$-cores

5.1. An s-abacus Characterization of the Longest $(2k-1, 2k, 2k+1)$-core

A conjecture of Amdeberhan [5] on the size of a maximal $(s-1, s, s+1)$-core has recently been verified.

Theorem 21. *(Yang-Zhong-Zhou, [24]; Xiong, [23]) The size of the largest* $(s-1, s, s+1)-$*core is*

1. $k\binom{k+1}{3}$ *if* $s = 2k > 2$
2. $(k+1)\binom{k+1}{3} + \binom{k+2}{3}$ *if* $s = 2k+1 > 2$.

The result is proved in two different ways: Yang, Zhong and Zhou extend the ideas of Stanley and Zanello to examine a poset $P_{s-1,s,s+1}$ associated to $(s-1, s, s+1)$-cores; for Xiong it is a consequence of numerical properties of bead sets associated to $(s-1, s, s+1, s+2, \cdots, s+k)$-cores. Here we find a characterization of the longest s-abacus, that is, the one corresponding the the $(s-1, s, s+1)$-core with the most components, and show that it corresponds to a maximal $(s-1, s, s+1)$-core.

We say that an s-abacus $\alpha'(s)$ is a *sub-abacus* of $\alpha(s)$ if they have the same number of runners and $(i, j) \in \alpha'(s)$ implies that $(i, j) \in \alpha(s)$. Let $\bar{\alpha}(s)$ be the sub-abacus of $\alpha(s)$ obtained by deleting any bead in $\alpha(s)$ which has a spacer directly below it on the same runner.

Lemma 17. *The s-abacus $\bar{\alpha}(s)$ corresponds to an s-core partition.*

Proof. This follows from Lemma 4, since, by construction, there is no bead in any runner with a spacer below it. □

Lemma 18. *Let $0 \le i \le k-1$. The s-abacus $\bar{\alpha}(s)$ consists of consecutive beads in the rows $j = 0, 1, \cdots, i$ of the i and $s-1-i$ runners.*

Proof. This follows from the construction of $\alpha(s)$ in Definition 11, where the i and $s-i-1$ runners have beads in the rows $0, 1, \cdots, i$, followed by alternating spacer-bead sequences. □

Example 22. Consider the 10-abacus $\bar{\alpha}(10)$ in Appendix C, Figure 21. Runners 0 and 9 have no beads, runners 1 and 8 have one bead, runners 2 and 7 have two beads and so on.

Lemma 19. *Let $0 \le j \le k-2$. Reading from left-to-right, row j of the s-abacus $\bar{\alpha}(s)$ consists o $j+1$ spacers, followed by $s-2(j+1)$ beads, followed by $j+1$ spacers.*

Proof. This follows from Lemma 18. □

Example 23. Consider the 10-abacus $\bar{\alpha}(10)$ in Appendix C, Figure 21. Row 0 has a spacer followed by eight beads, followed by a spacer. Row 1 has two spacers followed by six beads, followed by two spacers, and so on.

Lemma 20. *Let $j > 0$. If $(i, j) \in \bar{\alpha}(s)$, then $(i-1, j-1) \in \bar{\alpha}(s)$ and $(i+1, j-1) \in \bar{\alpha}(s)$.*

Proof. This is equivalent to saying that each bead in the second row of $\alpha(s)$ or above has a bead one row below and one column to the right, and one row below to the left, which follows from Lemma 19. □

Example 24. Consider the 10-abacus $\bar{\alpha}(10)$ in Appendix C, Figure 21. The bead in position (2,1) (with bead-value 12) is flanked by beads in positions (1,0) and (3,0) (with bead-values 1 and 3 respectively).

Let $\kappa_{s-1,s,s+1}$ be the $(s-1,s,s+1)$-core with the largest number of components.

Lemma 21. *The s-abacus $\bar{\alpha}(s)$ corresponds to the $\kappa_{s-1,s,s+1}$, that is, the one with the most components.*

Proof. By Remark 1 of [6], the s-abacus of any $(s-1,s+1)$-core partition must be a sub-abacus of $\alpha(s)$. Then $\bar{\alpha}(s)$ must be the s-abacus of $\kappa_{s-1,s,s+1}$ since it is obtained by deleting any bead in $\alpha(s)$ with a spacer immediately below it. It also must be the sub-abacus with the most beads, since including another bead would mean an s-hook is introduced.

Since $\alpha(s)$ was a minimal bead set, so too is $\bar{\alpha}(s)$; since each bead corresponds to a component, this means $\bar{\alpha}(s)$ is the $(s-1,s,s+1)$-core with the most components. □

We denote the $(s-1,s,s+1)$-core corresponding to $\bar{\alpha}(s)$ by $\kappa_{s-1,s,s+1}$.

Lemma 22. *Each bead in row j of $\bar{\alpha}(s)$ corresponds to a size $(j+1)^2$ component of $\kappa_{s-1,s,s+1}$.*

Proof. By Lemma 1, a bead x corresponds to a partition component whose size is the number of spacers less than x. We use induction on j. It is clear that each bead in the row 0 corresponds to a component of size $1=(0+1)^2$. Suppose it is true for $j-1$. Then, by the inductive hypothesis, there are j^2 spacers less than any bead in row $j-1$. By Lemma 19, the number of spacers between a bead in row $j-1$ and a bead in row j is $j+(j+1)$. Since the number of spacers less than a bead in row j is $j^2+j+(j+1)=(j+1)^2$ we are done. □

The next corollary follows from Lemma 22. [See Theorem 2.5 in [16] for more details.]

Corollary 5. *Let $s=2k>2$. Then $\kappa_{s-1,s,s+1}=$*

$$(((k-1)^2)^2,((k-2)^2)^4,...,16^{2k-8},9^{2k-6},4^{2k-4},1^{2k-2}).$$

We are now in a position to prove that $\kappa_{s-1,s,s+1}$ the longest $(s-1,s,s+1)$-core is a maximal one.

Theorem 25. *Let $s=2k>2$. Then $\kappa_{s-1,s,s+1}$ is a maximal $(s-1,s,s+1)$-core.*

Proof. By Lemma 22, each bead in row j corresponds to a size $(j+1)^2$ component of $\kappa_{s-1,s,s+1}$. By Lemma 19, there are are $(s-2(j+1))$ beads in row j. Hence

$$|\kappa_{s-1,s,s+1}| = \sum_{j=0}^{k-1}(s-2(j+1))(j+1)^2.$$

Since

$$\begin{aligned}\sum_{j=0}^{k-1}(s-2(j+1))(j+1)^2 &= 2k\sum_{j=0}^{k-1}(j+1)^2 - 2\sum_{j=0}^{k-1}(j+1)^3 \\ &= 2k\sum_{j=1}^{k} j^2 - 2\sum_{j=1}^{k} j^3 \\ &= \frac{2k(k)(k+1)(2k+1)}{6} - \frac{3k^2(k+1)^2}{6} \\ &= \frac{k^2(k^2-1)}{6} \\ &= k\binom{k+1}{3},\end{aligned}$$

we are done. □

5.2. Further Directions

Theorem 21 allows us to compare $|\kappa_{s\pm1}|$ with $|\kappa_{s-1,s,s+1}|$ when $s = 2k > 2$.

Proposition 2. *Let $s = 2k > 2$. Then $|\kappa_{s\pm1}| > |\kappa_{s-1,s,s+1}|$. In particular, $|\kappa_{s\pm1}| = 4|\kappa_{s-1,s,s+1}|$*

Proof. Since s is even, by Theorem 21(1) above $|\kappa_{s-1,s,s+1}| = \frac{k^4-k^2}{6}$. However by Theorem 4, $|\kappa_{s\pm1}| = \frac{((s-1)^2-1)((s+1)^2-1)}{24}$. This simplifies to $\frac{4(k^4-k^2)}{6}$. The result follows. □

Corollary 6. *$\kappa_{(s-1,s+1)}$ is never an s-core.*

Corollary 6 also follows from Theorem 1: since $\kappa_{s\pm1}$ is expressed only in its s-quotient (its s-core is empty), and each 1-hook in the s-quotient corresponds to an s-hook of $\kappa_{s\pm1}$, the maximal $(s-1, s+1)$-core is comprised completely of s-hooks.

Several questions arise from the analysis in this section. Firstly, is there interpretation of the factor of 4 that appears in Proposition 2, either in the geometry of the s-abacus or in the manipulation of Young diagrams? A cursory comparison between $\bar{\alpha}(s)$ and $\alpha(s)$ does not suggest an obvious one (compare Appendix A and Appendix C, for example). Secondly, Aggarwal, Yang-Zhong-Zhou and Xiong all note that when $s = 2k > 2$, there are two maximal $(s-1, s, s+1)$-cores, and in particular,

$\kappa_{s-1,s,s+1}$ is **not** self-conjugate. What then is the size of maximal **self-conjugate** $(s-1, s, s+1)$-core in this case?

Finally, Yang-Zhong-Zhou spend several pages establishing that the longest $(s-1, s, s+1)$-core is of maximal size. Is there a shorter abacus proof that this is the case? If so, then our characterization of $\bar{\alpha}(s)$ could be employed to develop a new proof of Theorem 21. Aggarwal has commented that it is not known for general, distinct, s, t, u when the longest (s, t, u)-core is a maximal one.

These questions are beyond the scope of this paper; we leave their investigation to other venues.

Note: since these results first appeared, a combinatorial explanation for the factor of 4 that appears in Proposition 2 has been found by the author and J. Sellers [16].

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APPENDIX A

The s-abaci $\alpha(s)$ of $\kappa_{s\pm1}$

Figure 9: $s = 4$

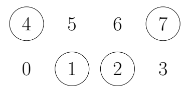


Figure 10: $s = 6$

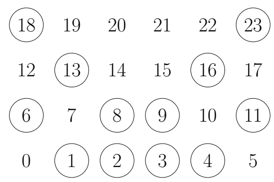


Figure 11: $s = 8$

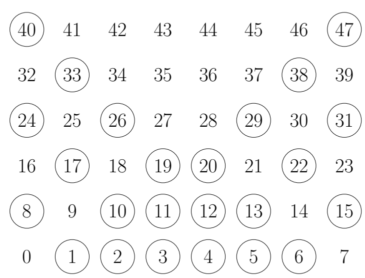
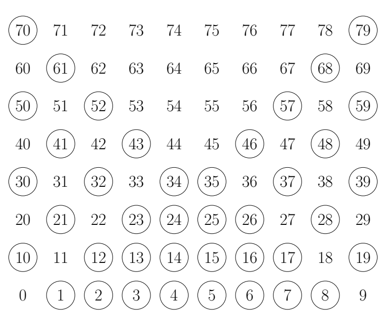


Figure 12: $s = 10$



APPENDIX B

The s-quotients of $\kappa_{s\pm 1}$

Figure 13: 4-quotient of $\kappa_{3,5}$

Figure 14: 6-quotient of $\kappa_{5,7}$

Figure 15: 8-quotient of $\kappa_{7,9}$

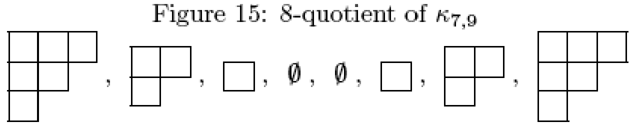
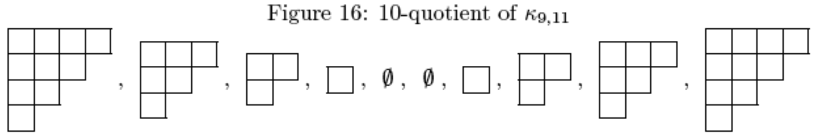


Figure 16: 10-quotient of $\kappa_{9,11}$



APPENDIX C

The s-abaci $\bar{\alpha}(s)$ of $\kappa_{s-1,s,s+1}$

Figure 17: $s = 4$

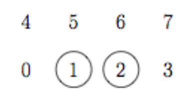


Figure 18: $s = 6$

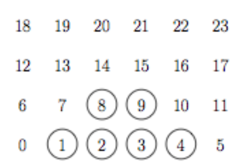


Figure 19: $s = 8$

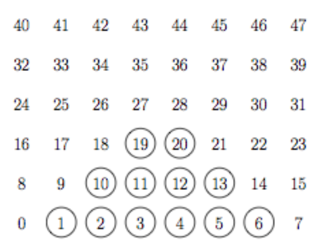


Figure 20: $s = 10$