PRODUCT OF A NUMBER AND ITS MULTIPLICATIVE INVERSE, MOMENTS OF L-FUNCTIONS AND EXPONENTIAL SUMS

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Abstract

In this paper, we study the average size of the product of a number and its multiplicative inverse modulo a prime p. This turns out to be related to moments of L-functions and leads to a curious asymptotic formula for a certain triple exponential sum.

1. Introduction and Main Results

Let p be a prime number. For any a such that $(a, p) = 1$, let $\overline{a}$ be the positive integer less than p such that $a\overline{a} \equiv 1 \pmod{p}$. Of course $a\overline{a}$ can be as small as 1 and as large as $(p-1)^2$. So one may ask how large $a\overline{a}$ is on average. This leads us to study

$$S := \sum_{a=1}^{p-1} a\overline{a} = \sum_{\substack{a=1 \\ ab \equiv 1 \pmod{p}}}^{p-1} \sum_{b=1}^{p-1} ab. \tag{1}$$

More generally, we define

$$S(d) := \sum_{\substack{a=1 \\ ab \equiv d \pmod{p}}}^{p-1} \sum_{b=1}^{p-1} ab. \tag{2}$$

We have

Theorem 1. *For $(d, p) = 1$,*

$$S(d) = \frac{p^3}{4} + O(p^{5/2} \log^2 p).$$

For a Dirichlet character $\chi \pmod p$, let $L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ be the corresponding Dirichlet L-function which has meromorphic continuation over the entire complex plane. As a by-product of our proof of Theorem 1, we have

Corollary 1. *For* $(d,p)=1$,

$$\sum_{\chi \neq \chi_0} \overline{\chi}(d) L(0,\chi)^2 \ll p^{3/2} \log^2 p.$$

One may ask if the error term in Theorem 1 is best possible. For this, we have the following result.

Theorem 2. *For any prime* p,

$$\sum_{d=1}^{p-1} \Big| S(d) - \frac{p^2(p-1)}{4} \Big|^2 = \frac{5}{144} \frac{p^2(p^2-1)^3}{(p^2+1)} + O(p^5 e^{3\log p/\log\log p}).$$

This tells us that, for some $1 \leq d \leq p-1$, we have

$$\Big| S(d) - \frac{p^2(p-1)}{4} \Big| \gg p^{5/2}.$$

So the error term in Theorem 1 is sharp apart from the logarithmic factor.

One can also consider a higher dimensional analogue of (2) by defining

$$S_k(d) := \sum_{\substack{a_1=1 \\ a_1 a_2 \ldots a_k \equiv d}}^{p-1} \sum_{a_2=1}^{p-1} \ldots \sum_{\substack{a_k=1 \\ (\mathrm{mod}\ p)}}^{p-1} a_1 a_2 \ldots a_k$$

and prove

Theorem 3. *For* $k \geq 3$ *and* $(d,p)=1$,

$$S_k(d) = \frac{p^k (p-1)^{k-1}}{2^k} + O_k(p^{3k/2} (\log p)^k).$$

Here O_k *means that the implicit constant may depend on* k.

When $k=3$, one can do slightly better by an exponential sum method and obtain

Theorem 4. *For* $(d,p)=1$,

$$S_3(d) = \frac{p^5}{8} + O(p^{9/2} (\log p)^2).$$

The improvement on the error term may not be significant. However, as a by-product of its proof, we derive an interesting weighted triple exponential sum result, namely

Theorem 5. *For $(l,p)=1$,*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} abc\, e\Big(\frac{labc}{p}\Big) = -\frac{p^5}{8} + O(p^{9/2}\log^3 p)$$

where $e(x) := e^{2\pi i x}$.

We will leave the interested readers to come up with similar exponential sum results in more variables.

2. Some Lemmas

Lemma 1. *For $z \neq 1$ and $z^p = 1$, we have $\sum_{b=1}^{p-1} bz^b = \frac{-p}{1-z}$.*

Proof. As $1 + z + z^2 + ... + z^{p-1} = 0$, one can check directly that

$$(\sum_{b=1}^{p-1} bz^b)(1-z) = z + z^2 + ... + z^{p-1} - (p-1)z^p = -1 - (p-1) = -p$$

which gives the lemma after dividing by $1 - z$. □

Lemma 2. *For $z \neq 1$ and $z^p = 1$, we have $\sum_{b=1}^{p-1} \frac{1}{1-z^b} = \frac{p-1}{2}$.*

Proof. Notice that $\frac{1}{1-z} + \frac{1}{1-\overline{z}} = \frac{1-\overline{z}+1-z}{(1-z)(1-\overline{z})} = \frac{1-\overline{z}-z+z\overline{z}}{(1-z)(1-\overline{z})} = 1$ as $|z| = 1$. Therefore

$$\sum_{b=1}^{p-1} \frac{1}{1-z^b} = \frac{1}{2}\sum_{b=1}^{p-1}\Big(\frac{1}{1-z^b} + \frac{1}{1-z^{p-b}}\Big) = \frac{1}{2}\sum_{b=1}^{p-1} 1 = \frac{p-1}{2}.$$

□

Lemma 3. *For $z \neq 1$, $z^p = 1$ and $1 \le d < p$, we have $\sum_{b=1}^{p-1} \frac{z^{-db}}{1-z^b} = \frac{p-1}{2} - d$.*

Proof. Consider

$$\begin{aligned}\sum_{b=1}^{p-1} \frac{1-z^{-db}}{1-z^b} &= \sum_{b=1}^{p-1} \frac{-z^{-db}(1-z^{db})}{1-z^b} = -\sum_{b=1}^{p-1} z^{-db} \sum_{j=0}^{d-1} z^{jb} \\ &= -\sum_{j=0}^{d-1}\sum_{b=1}^{p-1} z^{(j-d)b} = -\sum_{j=0}^{d-1}(-1) = d.\end{aligned}$$

Therefore by Lemma 2,

$$d = \sum_{b=1}^{p-1} \frac{1}{1-z^b} - \sum_{b=1}^{p-1} \frac{z^{-db}}{1-z^b} = \frac{p-1}{2} - \sum_{b=1}^{p-1} \frac{z^{-db}}{1-z^b},$$

which gives the lemma after rearranging terms. □

Lemma 4. *For any prime p and $(k,p)=1$,*

$$\sum_{a=1}^{p-1} a e\Big(\frac{k\overline{a}}{p}\Big) \ll p^{3/2}\log p.$$

Proof. By Weil's bound on incomplete Kloosterman sums (see [1] for example), we have

$$F(u) := \sum_{a=1}^{u} e\Big(\frac{k\overline{a}}{p}\Big) \ll p^{1/2}\log p$$

for $1 \le u < p$. Using this and partial summation,

$$\sum_{a=1}^{p-1} a e\Big(\frac{k\overline{a}}{p}\Big) = \int_{1^-}^{p-1} u dF(u) = (p-1)F(p-1) - \int_{1^-}^{p-1} F(u) du \ll p^{3/2}\log p.$$

□

Lemma 5. *For any integer $p > 1$,*

$$\sum_{k=1}^{p-1} \frac{1}{|1 - e(-k/p)|} \ll p\log p.$$

Proof. Observe that $|1 - e(-k/p)| \ge |\mathrm{Im}(1 - e(-k/p))| = |\sin 2k\pi/p|$. For $0 \le k < p/4$, we have $|\sin 2k\pi/p| \ge k/p$ by observing that the sine function is above the line $y = 2x/\pi$ for $0 \le x \le \pi/2$. So

$$\sum_{k<p/4} \frac{1}{|1 - e(-k/p)|} \le \sum_{k<p/4} \frac{1}{k/p} \ll p\log p.$$

Using $\sin(\pi - x) = \sin x$, we have

$$\sum_{p/4<k\le p/2} \frac{1}{|1 - e(-k/p)|} \ll p\log p.$$

Hence

$$\sum_{k=1}^{p/2} \frac{1}{|1 - e(-k/p)|} \ll p\log p + 1 \ll p\log p, \tag{3}$$

where the 1 may come from the term when $k = p/4$. By complex conjugation,

$$\frac{1}{|1 - e(-k/p)|} = \frac{1}{|1 - e(-(p-k)/p)|}.$$

So from (3),

$$\sum_{k=p/2}^{p-1} \frac{1}{|1 - e(-k/p)|} \ll p\log p \tag{4}$$

and the lemma follows from (3) and (4). □

3. Proofs of Theorems 1 and 3 and Corollary 1

Proof of Theorem 1. We use exponential sums to study (2). By the orthogonality of additive characters (see Equation (4.1) in [2], for example),

$$S(d) = \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} ab \sum_{k=1}^{p} e\Big(\frac{k(d\overline{a}-b)}{p}\Big) = \frac{p(p-1)^2}{4} + \frac{1}{p}\sum_{k=1}^{p-1}\sum_{a=1}^{p-1} ae\Big(\frac{kd\overline{a}}{p}\Big)\sum_{b=1}^{p-1} be\Big(\frac{-kb}{p}\Big)$$

where $e(u) := e^{2\pi i u}$. Hence, by Lemma 1,

$$S(d) = \frac{p(p-1)^2}{4} - \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{a=1}^{p-1} ae\Big(\frac{kd\overline{a}}{p}\Big). \tag{5}$$

By Lemmas 4 and 5,

$$S(d) = \frac{p(p-1)^2}{4} + O\Big(p^{3/2}\log p \sum_{k=1}^{p-1} \frac{1}{|1-e(-k/p)|}\Big) = \frac{p^3}{4} + O(p^{5/2}\log^2 p). \tag{6}$$

□

Another way to study (2) is through character sums.

Proof of Corollary 1. By the orthogonality of Dirichlet characters (see [2], Chapter 4), we have

$$\begin{aligned} S(d) =& \frac{1}{\phi(p)} \sum_{\chi \pmod p} \overline{\chi}(d) \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} ab\chi(a)\chi(b) \\ =& \frac{1}{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} ab + \frac{1}{\phi(p)} \sum_{\chi \neq \chi_0} \overline{\chi}(d) \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} ab\chi(a)\chi(b) \\ =& \frac{p^2(p-1)}{4} + \frac{1}{p-1} \sum_{\chi\neq\chi_0} \overline{\chi}(d)\Big(\sum_{a=1}^{p-1} a\chi(a)\Big)^2. \end{aligned}$$

As $\sum_{a \pmod p} a\chi(a) = -pL(0,\chi)$ (see [2], page 310) and combine with the functional equation for Dirichlet L-functions), we have

$$S(d) = \frac{p^2(p-1)}{4} + \frac{p^2}{p-1} \sum_{\chi\neq\chi_0} \overline{\chi}(d) L(0,\chi)^2. \tag{7}$$

Comparing (7) and (6), we deduce

$$\sum_{\chi\neq\chi_0} \overline{\chi}(d) L(0,\chi)^2 \ll p^{3/2}\log^2 p.$$

□

The character sum method can be used to study a higher dimensional analogue of Theorem 1.

Proof of Theorem 3. By the orthogonality of Dirichlet characters (see [2], Chapter 4), we have

$$\begin{aligned} S_k(d) &= \frac{1}{\phi(p)} \sum_{\chi \pmod p} \overline{\chi}(d) \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \dots \sum_{a_k=1}^{p-1} a_1 a_2 \dots a_k \chi(a_1)\chi(a_2)\dots\chi(a_k) \\ &= \frac{1}{p-1} \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \dots \sum_{a_k=1}^{p-1} a_1 a_2 \dots a_k \\ &\quad + \frac{1}{\phi(p)} \sum_{\chi \neq \chi_0} \overline{\chi}(d) \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \dots \sum_{a_k=1}^{p-1} a_1 a_2 \dots a_k \chi(a_1)\chi(a_2)\dots\chi(a_k) \\ &= \frac{p^k(p-1)^{k-1}}{2^k} + \frac{1}{p-1} \sum_{\chi \neq \chi_0} \overline{\chi}(d) \Big(\sum_{a=1}^{p-1} a\chi(a) \Big)^k . \end{aligned}$$

As $\sum_{a \pmod p} a\chi(a) \ll p^{3/2} \log p$ by the Polya-Vinogradov inequality (see Theorem 9.18 in [2] for example) and partial summation, we have

$$S_k(d) = \frac{p^k(p-1)^{k-1}}{2^k} + O_k(p^{3k/2} (\log p)^k)$$

which gives Theorem 3. □

4. Proof of Theorem 2

Proof of Theorem 2. Define

$$M := \sum_{d=1}^{p-1} \Big| S(d) - \frac{p^2(p-1)}{4} \Big|^2 .$$

By (7),

$$\begin{aligned} M &= \sum_{d=1}^{p-1} \Big| \frac{p^2}{p-1} \sum_{\chi \neq \chi_0} \overline{\chi}(d) L(0,\chi)^2 \Big|^2 \\ &= \frac{p^4}{(p-1)^2} \sum_{\chi_1 \neq \chi_0} \sum_{\chi_2 \neq \chi_0} L(0,\chi_1)^2 \overline{L(0,\chi_2)^2} \sum_{d=1}^{p-1} \overline{\chi_1}(d) \chi_2(d) \\ &= \frac{p^4}{(p-1)} \sum_{\chi_1 \neq \chi_0} |L(0,\chi_1)|^4 = \frac{p^4}{(p-1)} \sum_{\substack{\chi_1 \pmod p \\ \chi_1(-1)=-1}} |L(0,\chi_1)|^4 \end{aligned}$$

by the orthogonality of Dirichlet characters (see [2], Chapter 4) and the fact that $L(0,\chi) = 0$ when $\chi(-1) = 1$ (see [2], Chapter 10). As $L(0,\chi) = \frac{\tau(\chi)}{\pi} L(1,\chi)$ and $|\tau(\chi)| = p^{1/2}$ where $\tau(\chi) = \sum_{a=1}^{p} \chi(a)e(a/p)$ is the Gauss sum,

$$M = \frac{p^6}{\pi^4(p-1)} \sum_{\substack{\chi_1 \pmod p \\ \chi_1(-1)=-1}} |L(1,\chi_1)|^4 = \frac{5}{144} \frac{p^2(p^2-1)^3}{(p^2+1)} + O(p^5 e^{3\log p/\log\log p})$$

by Lemma 2 of Zhang [3]. This tells us that for some $1 \le d \le p-1$, we have

$$\Big| S(d) - \frac{p^2(p-1)}{4} \Big| \gg p^{5/2}.$$

So the error term in (6) is optimal apart from the logarithmic factor. □

5. Double Exponential Sum

In this section, we want to study

$$D := \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} ab\, e\Big(\frac{lab}{p}\Big). \tag{8}$$

First, observe that

$$D = \sum_{d=1}^{p-1} e\Big(\frac{ld}{p}\Big) \sum_{\substack{a=1 \\ ab \equiv d \pmod p}}^{p-1} \sum_{b=1}^{p-1} ab.$$

By (5),

$$D = -\frac{p(p-1)^2}{4} - \sum_{d=1}^{p-1} e\Big(\frac{ld}{p}\Big) \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{a=1}^{p-1} a e\Big(\frac{kd\overline{a}}{p}\Big).$$

Now the above sum can be rewritten as

$$\begin{aligned} &\sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{a=1}^{p-1} a \sum_{d=1}^{p-1} e\Big(\frac{d(l+k\overline{a})}{p}\Big) \\ &= -\sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{a=1}^{p-1} a + \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{a=1}^{p-1} a \sum_{d=1}^{p} e\Big(\frac{d(l+k\overline{a})}{p}\Big) \\ &\quad - \frac{p(p-1)^2}{4} + p \sum_{a=1}^{p-1} \frac{a}{1-e(al/p)} \end{aligned}$$

by Lemma 2. Therefore

$$D = -p \sum_{a=1}^{p-1} \frac{a}{1-e(al/p)}. \tag{9}$$

6. Triple Exponential Xum: proof of Theorem 5

Proof of Theorem 5. In this section, we consider

$$T := \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} abc\, e\Big(\frac{labc}{p}\Big)$$

where $0 < l < p$. One can rearrange it as

$$T = \sum_{c=1}^{p-1} c \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} ab\, e\Big(\frac{lcab}{p}\Big) = -p\sum_{c=1}^{p-1} c \sum_{a=1}^{p-1} \frac{a}{1-e(acl/p)}$$

by (9). Grouping the sums according to $ac \equiv d \pmod p$, we have

$$T = -p\sum_{d=1}^{p-1} \frac{1}{1-e(dl/p)} \sum_{\substack{a=1 \\ ac\equiv d \pmod p}}^{p-1}\sum_{c=1}^{p-1} ac.$$

By (5) and Lemma 2,

$$\begin{aligned} T &= -p\sum_{d=1}^{p-1} \frac{1}{1-e(dl/p)}\Big[\frac{p(p-1)^2}{4} - \sum_{k=1}^{p-1}\frac{1}{1-e(-k/p)}\sum_{a=1}^{p-1} ae\Big(\frac{kd\bar{a}}{p}\Big)\Big] \\ &= -\frac{p^2(p-1)^3}{8} + p\sum_{d=1}^{p-1}\frac{1}{1-e(dl/p)}\sum_{k=1}^{p-1}\frac{1}{1-e(-k/p)}\sum_{a=1}^{p-1} ae\Big(\frac{kd\bar{a}}{p}\Big). \qquad (10)\end{aligned}$$

Theorem 5 follows immediately by observing that (10) has a main term $-p^5/8$ and an error term $O(p^{9/2}\log^3 p)$ using Lemmas 4 and 5. □

7. Proof of Theorem 4

Proof of Theorem 4. By definition, we have $S_3(d) = \sum_{\substack{a=1 \\ abc\equiv d \pmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} abc$. By the orthogonality of additive characters,

$$\begin{aligned} S_3(d) &= \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} abc \sum_{l=1}^{p} e\Big(\frac{l(abc-d)}{p}\Big) \\ &= \frac{p^2(p-1)^3}{8} + \frac{1}{p}\sum_{l=1}^{p-1} e\Big(-\frac{dl}{p}\Big)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} abce\Big(\frac{labc}{p}\Big).\end{aligned}$$

By (10) and Lemma 3,

$$\begin{aligned}
S_3(d) = \frac{p^2(p-1)^3}{8} &+ \sum_{l=1}^{p-1} e\Big(-\frac{dl}{p}\Big)\Bigg[-\frac{p(p-1)^3}{8} \\
&\qquad + \sum_{t=1}^{p-1} \frac{1}{1-e(tl/p)} \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{a=1}^{p-1} ae\Big(\frac{kt\overline{a}}{p}\Big)\Bigg] \\
&= \frac{p(p+1)(p-1)^3}{8} + \sum_{t=1}^{p-1}\sum_{l=1}^{p-1} \frac{e(-lt\overline{t\overline{d}}/p)}{1-e(tl/p)} \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{a=1}^{p-1} ae\Big(\frac{kt\overline{a}}{p}\Big) \\
&= \frac{p(p+1)(p-1)^3}{8} + \sum_{t=1}^{p-1}\Big(\frac{p-1}{2} - \overline{t\overline{d}}\Big) \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{a=1}^{p-1} ae\Big(\frac{kt\overline{a}}{p}\Big) =: S_1 + S_2.
\end{aligned}$$

Hence

$$\begin{aligned}
S_2 &= \frac{p-1}{2} \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{a=1}^{p-1} a \sum_{t=1}^{p-1} e\Big(\frac{kt\overline{a}}{p}\Big) + \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{t=1}^{p-1} \overline{t\overline{d}} \sum_{a=1}^{p-1} ae\Big(\frac{kt\overline{a}}{p}\Big) \\
&= -\frac{p-1}{2} \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{a=1}^{p-1} a + \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{t=1}^{p-1} t \sum_{a=1}^{p-1} ae\Big(\frac{kd\overline{t}\overline{a}}{p}\Big) \\
&= -\frac{p(p-1)^3}{8} + \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{t=1}^{p-1}\sum_{a=1}^{p-1} ate\Big(\frac{kd\overline{t}\overline{a}}{p}\Big) \\
&= -\frac{p(p-1)^3}{8} + \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{c=1}^{p-1} e\Big(\frac{kd\overline{c}}{p}\Big) \sum_{\substack{t=1 \\ at\equiv c \pmod p}}^{p-1} \sum_{a=1}^{p-1} at.
\end{aligned}$$

By (5),

$$\begin{aligned}
S_2 = -\frac{p(p-1)^3}{8} &+ \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{c=1}^{p-1} e\Big(\frac{kd\overline{c}}{p}\Big)\Bigg[\frac{p(p-1)^2}{4} \\
&\qquad - \sum_{l=1}^{p-1} \frac{1}{1-e(-l/p)} \sum_{a=1}^{p-1} ae\Big(\frac{lc\overline{a}}{p}\Big)\Bigg] \\
= -\frac{p(p-1)^3}{4} &- \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{c=1}^{p-1} e\Big(\frac{kd\overline{c}}{p}\Big) \sum_{l=1}^{p-1} \frac{1}{1-e(-l/p)} \sum_{a=1}^{p-1} ae\Big(\frac{lc\overline{a}}{p}\Big) \\
= -\frac{p(p-1)^3}{4} &- \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{l=1}^{p-1} \frac{1}{1-e(-l/p)} \sum_{a=1}^{p-1} a \sum_{c=1}^{p-1} e\Big(\frac{lc\overline{a} + kd\overline{c}}{p}\Big) \\
= -\frac{p(p-1)^3}{4} &- \sum_{k=1}^{p-1} \frac{1}{1-e(-k/p)} \sum_{l=1}^{p-1} \frac{1}{1-e(-l/p)} \sum_{a=1}^{p-1} aS(l\overline{a}, kd; p),
\end{aligned}$$

where $S(a, b; p)$ is the Kloosterman sum. Using Weil's bound on Kloosterman sums [1] and Lemma 5, we have

$$S_2 = -\frac{p(p-1)^3}{4} + O(p^{9/2}\log^2 p).$$

Consequently,

$$S_3(d) = \frac{p(p-1)^4}{8} + O(p^{9/2}\log^2 p),$$

which gives Theorem 4. □

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