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NEW BOUNDS FOR THE PRIME COUNTING FUNCTION

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Abstract

In this paper, we establish a number of new estimates concerning the prime counting function $\pi(x)$, which improve the known results. As an application, we deduce a new result concerning the existence of prime numbers in small intervals.

1. Introduction

After Euclid [8] proved that there are infinitely many primes, the question arose of how fast the prime counting function

$$\pi(x) = \sum_{p \leq x} 1$$

increases as $x \to \infty$. In 1793, Gauss [9] conjectured that

$$\pi(x) \sim \mathrm{li}(x) = \int_0^x \frac{dt}{\log t} \qquad (x \to \infty),$$

which is equivalent to

$$\pi(x) \sim \frac{x}{\log x} \qquad (x \to \infty). \tag{1}$$

In 1896, Hadamard [10] and de la Vallée-Poussin [24] proved, independently, the relation (1), which is actually known as the *Prime Number Theorem*. A more accurate well-known asymptotic formula for $\pi(x)$ is given by

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \ldots + \frac{(n-1)!x}{\log^n x} + O\left(\frac{x}{\log^{n+1} x}\right). \tag{2}$$

Panaitopol [15] provided another asymptotic formula for $\pi(x)$, by proving that

$$\pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \ldots - \frac{k_n(1+\alpha_n(x))}{\log^n x}} \tag{3}$$

for every $n \in \mathbb{N}$, where $\lim_{x\to\infty} \alpha_n(x) = 0$ and positive integers $k_1, k_2, \ldots, k_n$ are given by the recurrence formula

$$k_n + 1!k_{n-1} + 2!k_{n-2} + \ldots + (n-1)!k_1 = n \cdot n!.$$

For instance, we have $k_1 = 1$, $k_2 = 3$, $k_3 = 13$, $k_4 = 71$, $k_5 = 461$ and $k_6 = 3441$.

Since, up to now, no efficient algorithm has been found for computing $\pi(x)$ for large x, we are interested in upper and lower bounds for $\pi(x)$. The first remarkable estimates for the prime counting function are due to Rosser [18]. He used an explicit zero-free region for the Riemann zeta-function $\zeta(s)$ and the verification of the Riemann hypothesis to some given height to estimate Chebyshev's functions

$$\theta(x) = \sum_{p \leq x} \log p, \qquad \psi(x) = \sum_{n=1}^{\infty} \theta(x^{1/n}).$$

Using these estimates for $\theta(x)$ and the well-known fact that $\pi(x)$ and $\theta(x)$ are related by the equation

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t}\, dt \tag{4}$$

which holds for every $x \geq 2$, Rosser [18, Theorem 29] proved that the inequalities

$$\frac{x}{\log x + 2} < \pi(x) < \frac{x}{\log x - 4}$$

hold for every $x \geq 55$. Up to now the sharpest estimates for $\pi(x)$ are due to Berkane and Dusart [2]. In 2015, they proved that the inequality

$$\pi(x) \leq \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{7.57x}{\log^4 x} \tag{5}$$

holds for every $x \geq 110118914$ and that

$$\pi(x) \geq \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{5.2x}{\log^4 x} \tag{6}$$

for every $x \geq 3596143$. According to (2), we prove the following upper and lower bound for $\pi(x)$, which improve the estimates (5) and (6) for large x.

Theorem 1.1. *If $x > 1$, then*

$$\pi(x) < \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6.35x}{\log^4 x} + \frac{24.35x}{\log^5 x} + \frac{121.75x}{\log^6 x} + \frac{730.5x}{\log^7 x} + \frac{6801.4x}{\log^8 x}. \tag{7}$$

Theorem 1.2. *If $x \geq 1332450001$, then*

$$\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{5.65x}{\log^4 x} + \frac{23.65x}{\log^5 x} + \frac{118.25x}{\log^6 x} + \frac{709.5x}{\log^7 x} + \frac{4966.5x}{\log^8 x}.$$

Further, in view of (3), we find the following estimates for $\pi(x)$.

Theorem 1.3. *If* $x \geq e^{3.804}$, *then*

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.35}{\log^2 x} - \frac{12.65}{\log^3 x} - \frac{71.7}{\log^4 x} - \frac{466.1275}{\log^5 x} - \frac{3489.8225}{\log^6 x}}. \quad (8)$$

Theorem 1.4. *If* $x \geq 1332479531$, *then*

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{2.65}{\log^2 x} - \frac{13.35}{\log^3 x} - \frac{70.3}{\log^4 x} - \frac{455.6275}{\log^5 x} - \frac{3404.4225}{\log^6 x}}. \quad (9)$$

As an application of these estimates for $\pi(x)$, we obtain the following result concerning the existence of a prime number in a small interval.

Theorem 1.5. *For every* $x \geq 58837$ *there is a prime number* p *such that*

$$x < p \leq x\left(1 + \frac{1.1817}{\log^3 x}\right).$$

2. Skewes' Number

One of the first estimates for $\pi(x)$ is due to Gauss. In 1793, he computed that $\pi(x) < \mathrm{li}(x)$ for every $2 \leq x \leq 3000000$ and conjectured that $\pi(x) < \mathrm{li}(x)$ for every $x \geq 2$. However, in 1914, Littlewood [14] proved that $\pi(x) - \mathrm{li}(x)$ changes the sign infinitely many times by showing that there is a positive constant K such that the sets

$$\left\{x \geq 2 \mid \pi(x) - \mathrm{li}(x) > \frac{K\sqrt{x}\log\log\log x}{\log x}\right\}$$

and

$$\left\{x \geq 2 \mid \pi(x) - \mathrm{li}(x) < -\frac{K\sqrt{x}\log\log\log x}{\log x}\right\}$$

are nonempty and unbounded. However, Littlewood's proof is nonconstructive and there is still no example of with $\pi(x) > \mathrm{li}(x)$. Let

$$\Xi = \min\{x \in \mathbb{R}_{\geq 2} \mid \pi(x) > \mathrm{li}(x)\}.$$

The first upper bound for Ξ which was found without the assumption that the of Riemann hypothesis is true is due to Skewes [22] in 1955, namely

$$\Xi < 10^{10^{10^{963}}}.$$

The number on the right-hand side is known as the *Skewes number*. In 1966, Lehman [13] improved this upper bound considerably by showing that $\Xi < 1.65 \cdot 10^{1165}$. After some further improvements the current best upper bound,

$$\Xi < e^{727.951336105} \leq 1.398 \cdot 10^{316},$$

was found by Saouter, Trudgian and Demichel [20]. The first lower bound was given by the calculation of Gauss, namely $\Xi > 3000000$. This lower bound was improved in a series of papers. For details see for example [3], [4], [16], and [19]. For our further inverstigation we use the following improvement:

Proposition 2.1 (Kotnik, [12]). *We have* $\Xi > 10^{14}$.

3. New Estimates for the Prime Counting Function

Before we give our first new estimate for $\pi(x)$, we mention a result [6] about the distance between x and $\theta(x)$, which plays an important role below.

Proposition 3.1 (Dusart, [6]). *Let* $k \in \{1, 2, 3, 4\}$. *Then for every* $x \geq x_0(k)$,

$$|\theta(x) - x| < \frac{\eta_k x}{\log^k x}, \tag{10}$$

where

k	1	2	3	4
η_k	0.001	0.01	0.78	1300
$x_0(k)$	908994923	7713133853	158822621	2

.

By using Tables 6.4 and 6.5 from [6], we obtain the following result.

Proposition 3.2. *If* $x \geq e^{30}$, *then*

$$|\theta(x) - x| < \frac{0.35x}{\log^3 x}.$$

Proof. We set $a = 3600$ and $\varepsilon_\psi = 6.93 \cdot 10^{-12}$. Then we have

$$\frac{1.00007(a+i)^3}{\sqrt{e^{a+i}}} + \frac{1.78(a+i)^3}{(e^{a+i})^{2/3}} + \varepsilon_\psi (a+1+i)^3 < 0.35 \tag{11}$$

for every integer i ranging from 0 to 75. By [7], we can choose

$$\varepsilon_\psi = 6.49 \cdot 10^{-12}$$

for every $e^{3675} \leq x \leq e^{3700}$, and therefore the inequality (11) holds with $\varepsilon_\psi = 6.49 \cdot 10^{-12}$ for every integer i ranging from 75 to 100 as well. From Tables 6.4 and 6.5 in [6], it follows that we can choose $\eta_3 = 0.35$ and $x_0(3) = e^{30}$ in (10). □

Now let $k \in \{1, 2, 3, 4\}$, and choose η_k and $x_1(k)$ so that the inequality

$$|\theta(x) - x| < \frac{\eta_k x}{\log^k x} \tag{12}$$

holds for every $x \geq x_1(k)$. To prove their estimates for $\pi(x)$, Rosser and Schoenfeld [19] introduced the following function, which also plays an important role below.

Definition. For every $x > 1$, we define

$$J_{k,\eta_k,x_1(k)}(x) = \pi(x_1(k)) - \frac{\theta(x_1(k))}{\log x_1(k)} + \frac{x}{\log x} + \frac{\eta_k x}{\log^{k+1} x} + \int_{x_1(k)}^{x} \left(\frac{1}{\log^2 t} + \frac{\eta_k}{\log^{k+2} t} \, dt \right). \tag{13}$$

Proposition 3.3. *If $x \geq x_1(k)$, then*

$$J_{k,-\eta_k,x_1(k)}(x) < \pi(x) < J_{k,\eta_k,x_1(k)}(x). \tag{14}$$

Proof. The claim follows from (4), (12) and (13). □

3.1. Some New Upper Bounds for the Prime Counting Function

In this section we give the proofs of Theorem 1.1 and Theorem 1.3.

Proof of Theorem 1.1. We denote the term on the right-hand side of (7) by $\alpha(x)$ and set

$$\beta(x, y) = \frac{x}{\log^2 y} + \frac{2x}{\log^3 y} + \frac{6x}{\log^4 y} + \frac{24.35x}{\log^5 y} + \frac{121.75x}{\log^6 y} + \frac{730.5x}{\log^7 y} + \frac{6801.4x}{\log^8 y}.$$

Let $x_1 = 10^{14}$. We have

$$\alpha'(x) - J'_{3,0.35,x_1}(x) = \frac{1687.9 \log x - 54411.2}{\log^9 x} \geq 0 \tag{15}$$

for every $x \geq x_1$. Since $\theta(x_1) \geq 99999990573246$ by [6], $\log x_1 \leq 32.2362$, and $\pi(x_1) = 3204941750802$, we obtain

$$\pi(x_1) - \frac{\theta(x_1)}{\log x_1} \leq 102839438084. \tag{16}$$

It follows that

$$\alpha(x_1) - J_{3,0.35,x_1}(x_1) \geq \beta(x_1, e^{32.2362}) - 102839438084 > 0.$$

Using (14) und (15), we get $\alpha(x) > \pi(x)$ for every $x \geq x_1$.

We have

$$\alpha'(x) - \mathrm{li}'(x) = \frac{0.35 \log^5 x - 1.05 \log^4 x + 1687.9 \log x - 54411.2}{\log^9 x} \geq 0$$

for every $x \geq 5 \cdot 10^5$. If we also use $\alpha(5 \cdot 10^5) - \mathrm{li}(5 \cdot 10^5) \geq 2.4 > 0$ and Proposition 2.1, we get $\alpha(x) > \pi(x)$ for every $5 \cdot 10^5 \leq x \leq 10^{14}$.

For every $x \geq 47$, we have $\alpha'(x) \geq 0$. To obtain the required inequality (7) for every $47 \leq x \leq 5 \cdot 10^5$, it suffices to check with a computer that $\alpha(p_i) > \pi(p_i)$ holds for every integer i ranging from $\pi(47)$ to $\pi(5 \cdot 10^5) + 1$, which is really the case.

Since $\pi(46) < \alpha(46)$ and $\alpha'(x) < 0$ is fulfilled for every $1 < x \leq 46$, we obtain $\alpha(x) > \pi(x)$ for every $1 < x \leq 46$.

It remains to consider the case where $46 < x \leq 47$. Here $\alpha(x) > 15 > \pi(x)$, and the theorem is proved. □

Remark. The inequality in Theorem 1.1 improves Berkane's and Dusart's estimate (5) for every $x \geq e^{25.21}$.

By using Proposition 2.1, we prove our third result.

Proof of Theorem 1.3. We denote the right-hand side of the inequality (8) by $\xi(x)$. Let $x_1 = 10^{14}$ and let

$$g(t) = t^7 - t^6 - t^5 - 3.35t^4 - 12.65t^3 - 71.7t^2 - 466.1275t - 3489.8225.$$

Then $g(t) > 0$ for every $t \geq 3.804$. We set

$$\begin{aligned} h(t) = {} & 29470t^{10} + 11770t^9 + 39068t^8 + 164238t^7 + 712906t^6 + 3255002t^5 \\ & + 12190826t^4 + 88308t^3 + 385090t^2 + 846526t - 12787805. \end{aligned}$$

Since $h(t) \geq 0$ for every $t \geq 1$, we obtain

$$\xi'(x) - J'_{3,0.35,x_1}(x) \geq \frac{h(\log x)}{g^2(\log x)\log^4 x} \geq 0 \tag{17}$$

for every $x \geq e^{3.804}$.

Let $K_1 = 102839438084$, $a = 32.23619$, and $b = 32.236192$. We set

$$\begin{aligned} f(s,t) = {} & K_1 t^7 + (K_1 + s)t^6 + (3.35K_1 + s)t^5 + (12.65K_1 + 3s)t^4 \\ & + (71.7K_1 + 13s)t^3 + (466.1275K_1 + 72.05s)t^2 \\ & + (3489.8225K_1 + 467.3s)t + 3494.25s \end{aligned}$$

and obtain $f(x_1, a) \geq b^8 K_1$. Since $a \leq \log x_1 \leq b$, we have $f(x_1, \log x_1) \geq K_1 \log^8 x_1$ and therefore

$$\begin{aligned} & x_1 \log^6 x_1 + x_1 \log^5 x_1 + 3x_1 \log^4 x_1 + 13x_1 \log^3 x_1 + 72.05x_1 \log^2 x_1 \\ & \qquad + 467.3x_1 \log x_1 + 3494.25x_1 \\ & \quad \geq K_1 \log^8 x_1 - K_1 \log^7 x_1 - K_1 \log^6 x_1 - 3.35K_1 \log^5 x_1 \\ & \qquad - 12.65K_1 \log^4 x_1 - 71.7K_1 \log^3 x_1 - 466.1275K_1 \log^2 x_1 \\ & \qquad - 3489.8225K_1 \log x_1. \end{aligned}$$

It immediately follows that

$$\begin{aligned} x_1 \log^9 x_1 + x_1 \log^8 x_1 + 3x_1 \log^7 x_1 + 13x_1 \log^6 x_1 + 72.05x_1 \log^5 x_1 \\ + 467.3x_1 \log^4 x_1 + 3494.25x_1 \log^3 x_1 + 25.095x_1 \log^2 x_1 \\ + 163.144625x_1 \log x_1 + 1221.437875x_1 \\ > K_1 g(\log x_1) \log^4 x_1. \end{aligned}$$

Since the left-hand side of the last inequality is equal to $x_1(\log^{10} x_1 - (\log^3 x_1 + 0.35)g(\log x_1))$, we have

$$x_1 \log^{10} x_1 > (K_1 \log^4 x_1 + x_1(\log^3 x_1 + 0.35))g(\log x_1).$$

Moreover, $K_1 \geq \pi(x_1) - \theta(x_1)/\log x_1$ by (16), and $g(\log x_1) > 0$. Hence,

$$x_1 \log^{10} x_1 > \left(\left(\pi(x_1) - \frac{\theta(x_1)}{\log x_1} \right) \log^4 x_1 + x_1(\log^3 x_1 + 0.35) \right) g(\log x_1).$$

We divide both sides of this inequality by the positive value $g(\log x_1) \log^4 x_1$, and, by (17) and Proposition 3.2, we get

$$\xi(x) > J_{3,0.35,x_1}(x) \geq \pi(x)$$

for every $x \geq x_1$.

Now let $140000 \leq x \leq x_1$. We compare $\xi(x)$ with $\mathrm{li}(x)$. We set

$$\begin{aligned} r(t) = 0.35t^{11} - 1.75t^{10} + 1.75t^9 - 0.6t^8 - 1.3t^7 - 29492t^6 \\ - 11917t^5 - 40316t^4 - 155136t^3 - 717716t^2 - 3253405t - 12178862. \end{aligned}$$

Then $r(t) \geq 0$ for every $t \geq 10.9$, and we obtain

$$\xi'(x) - \mathrm{li}'(x) \geq \frac{r(\log x)}{g^2(\log x) \log x} \geq 0 \tag{18}$$

for every $x \geq e^{10.9}$. We have $\xi(140000) - \mathrm{li}(140000) > 0.0024$. It remains to use (18) and Proposition 2.1.

Now we consider the case where $e^{4.53} \leq x < 140000$. We set

$$s(t) = t^8 - 2t^7 - t^6 - 4.35t^5 - 19.35t^4 - 109.65t^3 - 752.9275t^2 - 5820.46t - 20938.935.$$

Since $s(t) \geq 0$ for every $t \geq 4.53$, we get

$$\frac{g(\log x)^2 \xi'(x)}{\log^5 x} = s(\log x) \geq 0 \tag{19}$$

for every $x \geq e^{4.53}$. Since $g(\log x) > 0$ for every $x \geq e^{3.804}$, using (19) we obtain that $\xi'(x) > 0$ holds for every $x \geq e^{4.53}$. So we check with a computer that $\xi(p_i) > \pi(p_i)$ for every integer i ranging from $\pi(e^{4.53})$ to $\pi(140000) + 1$.

Next, let $45 \leq x < e^{4.52}$. Since we have $s'(t) > 0$ for every $t \geq 3.48$ and $s(4.52) \leq -433$, we get $s(\log x) < 0$. From (19), it follows that $\xi'(x) < 0$ for every $e^{3.804} \leq x \leq e^{4.52}$. Hence $\xi(x) \geq \xi(e^{4.52}) > 26 > \pi(e^{4.52}) \geq \pi(x)$ for every $e^{3.804} \leq x \leq e^{4.52}$.

Finally, $\xi(x) \geq 26 > \pi(x)$ for every $e^{4.52} \leq x \leq e^{4.53}$, and the theorem is proved. □

Remark. Theorem 1.3 leads to an improvement of Theorem 1.1 for every sufficiently large x.

Corollary 3.4. *For every $x \geq 21.95$, we have*

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.35}{\log^2 x} - \frac{12.65}{\log^3 x} - \frac{89.6}{\log^4 x}} .$$

If $x \geq 14.36$, then

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.35}{\log^2 x} - \frac{15.43}{\log^3 x}} ,$$

and for every $x \geq 9.25$ we have

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.83}{\log^2 x}} .$$

If $x \geq 5.43$, then

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1.17}{\log x}} .$$

Proof. The claim follows by comparing each expression on the right-hand side with the right-hand side of (8) and with $\mathrm{li}(x)$. For small x we check the inequalities with a computer. □

3.2. Some New Lower Bounds for the Prime Counting Function

Here we prove the theorems about the lower bounds for $\pi(x)$.

Proof of Theorem 1.4. We denote the denominator on the right-hand side of (9) by $\varphi(x)$. Then $\varphi(x) > 0$ for every $x \geq e^{3.79}$. Let $x_1 = 10^{14}$. We set

$$\phi(x) = \frac{x}{\varphi(x)}$$

and

$$r(t) = 28714t^{10} + 11244t^9 + 36367t^8 + 146093t^7 + 691057t^6 + 3101649t^5 + 11572765t^4 - 77484t^3 - 365233t^2 - 799121t + 12169597.$$

Obviously $r(t) \geq 0$ for every $t \geq 1$. Hence

$$J'_{3,-0.35,x_1}(x) - \phi'(x) \geq \frac{r(\log x)}{(\varphi(x)\log^6 x)^2 \log^5 x} \geq 0 \tag{20}$$

for every $x \geq e^{3.79}$. Since $\theta(10^{14}) \leq 99999990573247$ by Table 6.2 of [6], $\pi(10^{14}) = 3204941750802$, and $32.23619 \leq \log 10^{14} \leq 32.2362$, we get

$$\pi(x_1) - \frac{\theta(x_1)}{\log x_1} \geq 102838475779.$$

Hence, by (13),

$$\begin{aligned} J_{3,-0.35,x_1}(x_1) - \phi(x_1) &\geq 102838475779 + \frac{10^{14}}{32.2362} - \frac{0.35 \cdot 10^{14}}{32.23619^4} - \frac{10^{14}}{\varphi(e^{32.23619})} \\ &\geq 322936. \end{aligned}$$

Using (20) and Proposition 3.2, we obtain $\pi(x) > \phi(x)$ for every $x \geq x_1$.

Next, let $x_2 = 8 \cdot 10^9$ and $x_2 \leq x \leq x_1$. We set

$$h(t) = -0.01t^{15} + 0.39t^{14} - 1.78t^{13} + 1.763t^{12} + 0.033t^{11} - 2.997t^{10}.$$

For every $29 \leq t \leq 33$, we get $h(t) \geq 0.443t^{12} - 2.997t^{10} > 0$. For every $23 \leq t \leq 29$, we obtain $h(t) \geq 13.723t^{12} - 2.997t^{10} > 0$. Therefore,

$$J'_{2,-0.01,x_2}(x) - \phi'(x) \geq \frac{h(\log x)}{(\varphi(x)\log^6 x)^2 \log^4 x} \geq 0 \tag{21}$$

for every $e^{23} \leq x_2 \leq x \leq x_1 \leq e^{33}$. Since $\theta(x_2) \leq 7999890793$ (see Table 6.1 of [6]), $\pi(x_2) = 367783654$ and $22.8027 \leq \log x_2$, we obtain

$$\pi(x_2) - \frac{\theta(x_2)}{\log x_2} \geq 367783654 - \frac{7999890793}{22.8027} \geq 16952796.$$

Using $22.8 \leq \log x_2 \leq 22.8028$, we get

$$J_{2,-0.01,x_2}(x_2) - \phi(x_2) \geq 16952796 + \frac{x_2}{22.8028} - \frac{0.01x_2}{22.8^3} - \frac{x_2}{\varphi(e^{22.8})} \geq 2360.$$

Using (21) and Proposition 3.3, we prove the required inequality for every $x_2 \leq x \leq x_1$.

It remains to consider the case where $1332479531 \leq x \leq x_2$. We set

$$s(t) = t^8 - 2t^7 - t^6 - 3.65t^5 - 18.65t^4 - 110.35t^3 - 736.8275t^2 - 5682.56t - 20426.535.$$

Since $s(t) \geq 0$ for every $t \geq 4.6$, we obtain

$$\phi'(x) = \frac{s(\log x)\log^5 x}{(\varphi(x)\log^6 x)^2} \geq 0$$

for every $x \geq e^{4.6}$. And again we use a computer to check that the inequality $\pi(p_i) \geq \phi(p_{i+1})$ for every integer i ranging from $\pi(1332479531)$ to $\pi(x_2) + 1$. □

Using a computer and Theorem 1.4, we obtain the following weaker estimates for $\pi(x)$.

Corollary 3.5. *If $x \geq x_0$, then*

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{a}{\log^2 x} - \frac{b}{\log^3 x} - \frac{c}{\log^4 x} - \frac{d}{\log^5 x}},$$

where

a	2.65	2.65	2.65	2.65	2.65	2.65
b	13.35	13.35	13.35	13.35	13.35	13.1
c	70.3	70.3	45	34	5	0
d	276	69	0	0	0	0
x_0	1245750347	909050897	768338551	547068751	374123969	235194097
a	2.65	2.65	2.65	2.62	2.1	0
b	8.6	7.7	4.6	0	0	0
c	0	0	0	0	0	0
d	0	0	0	0	0	0
x_0	93811339	65951927	38168363	16590551	6690557	468049

Proof. By comparing each right-hand side with the right-hand side of (9), we see that each inequality holds for every $x \geq 1332479531$. For smaller x we check the asserted inequalities using a computer. □

Now we prove Theorem 1.2 by using Theorem 1.4.

Proof of Theorem 1.2. For $y > 0$ we set

$$R(y) = 1 + \frac{1}{y} + \frac{2}{y^2} + \frac{5.65}{y^3} + \frac{23.65}{y^4} + \frac{118.25}{y^5} + \frac{709.5}{y^6} + \frac{4966.5}{y^7}$$

and

$$S(y) = y - 1 - \frac{1}{y} - \frac{2.65}{y^2} - \frac{13.35}{y^3} - \frac{70.3}{y^4} - \frac{455.6275}{y^5} - \frac{3404.4225}{y^6}.$$

Then $S(y) > 0$ for every $y \geq 3.79$, and moreover, $y^{13}R(y)S(y) = y^{14} - T(y)$, where

$$\begin{aligned} T(y) = {} & 11017.9625y^6 + 19471.047875y^5 + 60956.6025y^4 + 250573.169y^3 \\ & + 1074985.621875y^2 + 4678311.7425y + 16908064.34625. \end{aligned}$$

Using Theorem 1.4, we get

$$\pi(x) > \frac{x}{S(\log x)} > \frac{x}{S(\log x)}\left(1 - \frac{T(\log x)}{\log^{14} x}\right) = \frac{xR(\log x)}{\log x}$$

for every $x \geq 1332479531$. So it remains to check the required inequality for every $1332450001 \leq x \leq 1332479531$. Let

$$U(x) = \frac{xR(\log x)}{\log x}$$

and $u(y) = y^8 - 0.35y^5 + 1.05y^4 - 39732$. Since $u(y) \geq 0$ for every $y \geq 3.8$, it follows that $U'(x) = u(\log x)/\log^9 x \geq 0$ for every $x \geq e^{3.8}$. So we use a computer to check that the inequality $\pi(p_i) > U(p_{i+1})$ holds for every integer i ranging from $\pi(1332450001)$ to $\pi(1332479531)$. □

Remark. Obviously, Theorem 1.2 yields an improvement of Dusart's estimate (6).

4. On the Existence of Prime Numbers in Short Intervals

Let $m \in \mathbb{N}_0$ and $r > 0$. This section deals with finding an explicit constant $x_0 = x_0(m, r)$ so that for every $x \geq x_0$ there exists a prime number in the interval

$$\left(x, x\left(1 + \frac{r}{\log^m x}\right)\right].$$

Remark. The prime number theorem guarantees the existence of such an x_0.

Before proving Theorem 1.5, we mention some known results starting from $m = 0$. The first result is due to Schoenfeld [21]. He gave the value $x_0(0, 1/16597) = 2010759.9$. In 2003, this was improved as follows:

Proposition 4.1 (Ramaré and Saouter, [17]). *For every $x \geq 10726905041$ the interval*

$$\left(x, x\left(1 + \frac{1}{28313999}\right)\right].$$

contains a prime number.

In 2014, Kadiri and Lumley [11, Table 2] found a series of improvements of Proposition 4.1. For the proof of Theorem 1.5, we need the following result which easily follows from the last row of Table 2 in [11].

Proposition 4.2 (Kadiri and Lumley, [11]). *For every $x \geq e^{150}$ the interval*

$$\left(x, x\left(1 + \frac{1}{2442159713}\right)\right]$$

contains a prime number.

For $m = 2$, Dusart [5] proved, that for every $x \geq 3275$ there exists a prime number p such that

$$x < p \leq x\left(1 + \frac{1}{2\log^2 x}\right).$$

In 2010, Dusart [6] improved his own result by finding $x_0(2, 1/25) = 396738$. For $m = 2$ and $r = 1/111$, we have the following

Proposition 4.3 (Trudgian, [23]). *For every $x \geq 2898239$ the interval*

$$\left(x, \left(1 + \frac{1}{111\log^2 x}\right)\right]$$

contains a prime number.

Now let $a, b \in \mathbb{R}$. We define $z_1(a), z_2(b) \in \mathbb{N} \cup \{\infty\}$ by

$$z_1(a) = \min\left\{k \in \mathbb{N} \mid \pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{a}{\log^2 x}} \text{ for every } x \geq k\right\}$$

and

$$z_2(b) = \min\left\{k \in \mathbb{N} \mid \pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{b}{\log^2 x}} \text{ for every } x \geq k\right\}.$$

To prove Theorem 1.5, we start with

Lemma 4.4. *Let $z_0 \in \mathbb{R} \cup \{-\infty\}$ and let $c\colon (z_0, \infty) \to [1, \infty)$ be a map. Then,*

$$\begin{aligned}
&\pi(c(x)x) - \pi(x) \\
&\quad > \frac{x((c(x)-1)(\log x - 1 - \frac{1}{\log x}) - \log c(x) - \frac{c(x)\log c(x) + bc(x) - a}{\log^2 x})}{(\log(c(x)x) - 1 - \frac{1}{\log(c(x)x)} - \frac{a}{\log^2(c(x)x)})(\log x - 1 - \frac{1}{\log x} - \frac{b}{\log^2 x})} \\
&\qquad - \frac{x(\frac{2bc(x)\log c(x)}{\log^3 x} + \frac{bc(x)\log^2 c(x)}{\log^4 x})}{(\log(c(x)x) - 1 - \frac{1}{\log(c(x)x)} - \frac{a}{\log^2(c(x)x)})(\log x - 1 - \frac{1}{\log x} - \frac{b}{\log^2 x})}
\end{aligned}$$

for every $x \geq \max\{\lfloor z_0 \rfloor + 1, z_2(b), z_3(a)\}$, where $z_3(a) = \min\{k \in \mathbb{N} \mid k\,c(k) \geq z_1(a)\}$.

Proof. For every $x \geq \max\{\lfloor z_0 \rfloor + 1, z_2(b), z_3(a)\}$, we have

$$\begin{aligned}
&\pi(c(x)x) - \pi(x) \\
&> \frac{c(x)x}{\log(c(x)x) - 1 - \frac{1}{\log(c(x)x)} - \frac{a}{\log^2(c(x)x)}} - \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{b}{\log^2 x}} \\
&= x\,\frac{(c(x)-1)(\log x - 1) - \log c(x) - \frac{c(x)-1}{\log(c(x)x)} - \frac{c(x)\log c(x)}{\log x \log(c(x)x)} - \frac{bc(x)-a}{\log^2(c(x)x)}}{(\log(c(x)x) - 1 - \frac{1}{\log(c(x)x)} - \frac{a}{\log^2(c(x)x)})(\log x - 1 - \frac{1}{\log x} - \frac{b}{\log^2 x})} \\
&\quad - x\,\frac{\frac{2bc(x)\log c(x)}{\log x \log^2(c(x)x)} + \frac{bc(x)\log^2 c(x)}{\log^2 x \log^2(c(x)x)}}{(\log(c(x)x) - 1 - \frac{1}{\log(c(x)x)} - \frac{a}{\log^2(c(x)x)})(\log x - 1 - \frac{1}{\log x} - \frac{b}{\log^2 x})}.
\end{aligned}$$

Since $c(x) \geq 1$, the lemma is proved. □

Now we prove Theorem 1.5, where for the first time for $m = 3$ we find an explicit value $x_0(m, r)$ and which leads to an improvement of Proposition 4.3 for every $x \geq e^{131.1687}$.

Proof of Theorem 1.5. We set $a = 2.65$ and $b = 3.83$. By Corollary 3.5 and Corollary 3.4, we obtain $z_1(a) \leq 38168363$ and $z_2(b) = 10$. As in the proof of Theorem 1.4, we check with a computer that $z_1(a) = 36917641$. Further, we define

$$c(x) = 1 + \frac{1.1817}{\log^3 x}$$

and $z_0 = 1$. Then $z_3(a) = 36909396$. We consider the function

$$\begin{aligned}
g(x) = 0.0017x^2 - 2.3634x - 1.1817 - \frac{5.707611}{x} - \frac{9.051822}{x^2} - \frac{1.39641489}{x^4} \\
- \frac{10.6965380574}{x^5} - \frac{5.3482690287}{x^6} - \frac{6.32004951121479}{x^9},
\end{aligned}$$

and get $g(x) \geq 0.056$ for every $x \geq 1423.728$. We set

$$\begin{aligned}
f(x) = (c(x)-1)(\log^5 x - \log^4 x - \log^3 x) - \log^4 x \log c(x) \\
- (c(x)\log c(x) + 3.83c(x) - 2.65)\log^2 x \\
- 2 \cdot 3.83c(x)\log c(x)\log x - 3.83c(x)\log^2 c(x)
\end{aligned}$$

and substitute $c(x) = 1 + 1.1817/\log^3 x$ in $f(x)$. Using the inequality $\log(1+t) \leq t$ which holds for every $t > -1$, we get $f(x) \geq g(\log x) \geq 0.056$ for every $x \geq e^{1423.728}$. By Lemma 4.4, we obtain

$$\begin{aligned}
&\pi\left(x\left(1 + \frac{1.1817}{\log^3 x}\right)\right) - \pi(x) \\
&\qquad > \frac{f(x)/\log^4(x)}{(\log(c(x)x) - 1 - \frac{1}{\log(c(x)x)} - \frac{2.65}{\log^2(c(x)x)})(\log x - 1 - \frac{1}{\log x} - \frac{3.83}{\log^2 x})} \\
&\qquad \geq 0
\end{aligned}$$

for every $x \geq e^{1423.728}$. For every $e^{150} \leq x \leq e^{1423.728}$, the theorem follows directly from Proposition 4.2. Then we use Propositions 4.1 and 4.3 to obtain the result for every $2898239 \leq x < e^{150}$. Next we check with a computer that

$$p_n \left(1 + \frac{1.1817}{\log^3 p_n}\right) > p_{n+1}$$

for every integer n ranging from $\pi(58889)$ to $\pi(2898239) + 1$. Finally, we confirm that

$$\pi\left(x + \frac{1.1817x}{\log^3 x}\right) > 5949 = \pi(x)$$

is true for every $58837 \leq x < 58889$. □

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