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ON FINITE SEMI-REGULAR CONTINUED FRACTIONS

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Abstract

Suppose a finite semi-regular continued fraction (abbreviated as SRCF) is given, and we have to find a regular continued fraction, an even continued fraction or an odd continued fraction whose value is same as that of the given SRCF. In this article, we discuss an algorithm to convert the given SRCF to each of these three types of continued fractions without finding the value of the given SRCF. We also compare the complexity of our algorithms in each case with the direct method which requires computing the actual value of the given SRCF.

1. Introduction

Suppose $\epsilon_i \in \{\pm 1\}$, $a_i \in \mathbb{N}$, for $i \in \mathbb{N}$, and $a_0 \in \mathbb{Z}$. A pair of finite sequences $\{\epsilon_i\}_{i\geq 1}^n$ and $\{a_i\}_{i\geq 0}^n$ is called a *finite semi-regular continued fraction* when $\epsilon_{i+1} + a_i \geq 1$. A pair of infinite sequences $\{\epsilon_i\}_{i\geq 1}^\infty$ and $\{a_i\}_{i\geq 0}^\infty$ is called an *infinite semi-regular continued fraction* when $\epsilon_{i+1} + a_i \geq 1$ and $a_i \geq 2$ for infinitely many values of i (see [2]). A finite semi-regular continued fraction is expressed as

$$a_0 + \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+}\ \frac{\epsilon_3}{a_3+}\ \cdots \frac{\epsilon_n}{a_n} \tag{1}$$

and n is called its *length*. An infinite semi-regular continued fraction is expressed as

$$a_0 + \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+}\ \frac{\epsilon_3}{a_3+}\ \cdots \frac{\epsilon_n}{a_n+} \cdots ; \tag{2}$$

it is called *eventually constant* if there exists $m \in \mathbb{N}$ such that $\epsilon_i = \epsilon_m$ and $a_i = a_m$ for every $i > m$. For an eventually constant semi-regular continued fraction, the smallest m satisfying the above property is treated as its length. We use the abbreviation SRCF for 'semi-regular continued fraction'.

Every a_i $(i \geq 1)$ is called a *partial denominator*, the integer a_0 is called the *integral part*, and every ϵ_i $(i \geq 1)$ is called a *partial numerator* of the continued fraction. The rational number

$$\frac{p_k}{q_k} = a_0 + \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+}\ \frac{\epsilon_3}{a_3+}\ \cdots \frac{\epsilon_k}{a_k}$$

is called the *k-th convergent* of the continued fraction and the sequence $\{p_k/q_k\}_{k\geq 0}$ is called the *sequence of convergents* of this continued fraction. In fact, the sequence of convergents of a finite continued fraction is a finite sequence. The continued fraction

$$a_i + \ \frac{\epsilon_{i+1}}{a_{i+1}+} \cdots$$

is called the *tail at the i-th stage*. The expression

$$y_i = \frac{\epsilon_i}{a_i+}\ \frac{\epsilon_{i+1}}{a_{i+1}+} \cdots \tag{3}$$

is referred to as the *fin at the i-th stage* ([6]). Observe that $\epsilon_i = \operatorname{sign}(y_i)$ and the length of the continued fraction is n if $y_{n+1} = 0$.

The semi-regular continued fraction expressed in (1) or (2) is known as a *regular continued fraction* (abbreviated as RCF) if $\epsilon_i = 1$ for every $i \geq 1$. If $\epsilon_i = \pm 1$, a_i is an even positive integer for $i \geq 1$, and $a_0 \in 2\mathbb{Z}$, then the SRCF is known as an *even continued fraction* (abbreviated as ECF). If $\epsilon_i = \pm 1$, a_i is an odd positive integer for $i \geq 1$, and a_0 is an odd integer, then the SRCF is known as an *odd continued fraction* (abbreviated as OCF) (see [8]). One may refer to [1] for more on continued fractions.

Every rational number has a unique RCF expansion having the last partial denominator greater than 1, a unique ECF expansion except when it is a quotient of two odd integers, and a unique OCF expansion except when the last partial quotient is $1/1$ or $-1/1$. In fact, for a rational number which is a quotient of two odd integers, there are two eventually constant ECF expansions. The non-uniqueness of an ECF or an OCF expansion of a rational number is clear from the following identities:

$$\begin{aligned}
\frac{1}{2r} &= \frac{1}{(2r-1)+}\ \frac{1}{1} = \frac{1}{(2r+1)+}\ \frac{-1}{1};\\
\frac{1}{2r+1} &= \frac{1}{2r+}\ \frac{1}{1} = \frac{1}{2(r+1)+}\ \frac{-1}{1};\\
\frac{\pm 1}{1} &= \frac{\pm 1}{2+}\ \frac{-1}{2+}\ \frac{-1}{2+} \cdots .
\end{aligned}$$

Suppose a finite SRCF is given and we have to find an RCF (or ECF or OCF) which has the same value as the given SRCF. One way to do it is by finding the value

x of the given SRCF and then expanding x using the expansion algorithm to find the desired type of continued fraction. We propose in this article that we can bypass finding the value of the given SRCF and convert the SRCF more efficiently into a desired type of continued fraction. Here, the desired types of continued fraction expansion are restricted to RCF, ECF and OCF.

While finding the ECF having the same value as that of the given SRCF, we may find an infinite continued fraction which is eventually constant. The following proposition (c.f. [6, Proposition 6.1]) guarantees that an eventually constant continued fraction whose value is a rational number, can neither be an RCF nor an OCF.

Proposition 1. *Suppose $x \in \mathbb{R}$ has an eventually constant semi-regular continued fraction expansion. Then $x \in \mathbb{Q}$ if and only if all but finitely many partial numerators are -1 and all but finitely many partial denominators are 2.*

The conversion of a semi-regular continued fraction into a regular continued fraction is well known and it goes back to Lagrange [3]. An algorithm for conversion is described in Section 40 of Oskar Perron's classical book [5]. In the proof of Theorem 1, step 1(b) occurs in the foregoing reference to get rid of a partial numerator which is equal to -1. In the method of [5], a couple of identities (cf. [4]) are used to convert an SRCF into an RCF, but one of the identities may produce a zero partial denominator and the other removes a zero partial denominator. However, we did not use any transformation which produces zero as a partial denominator. C. Kraaikamp [2] has defined a process called "singularization" by which a continued fraction can be transformed into another continued fraction by removing $1/1$. In fact, the singularization process has been very useful in the conversion algorithms (c.f. [7]).

In the next section, we discuss algorithms to convert an SRCF into a desired type (ECF, OCF or RCF) of continued fraction without finding the value of the given continued fraction. In fact, we do not use expansion algorithms to find the desired type of continued fraction. In each case, we find the maximum number of basic operations required to perform the complete conversion process (up to a constant independent of the length of the given SRCF). In the last section, we compare the complexity of our algorithms with that of the direct method.

2. Conversion Algorithms

The process of transforming an SRCF into a continued fraction of certain type is by repairing its partial numerators and partial denominators one by one. Repairing is done by applying a set of identities in the forthcoming lemma in certain order. Proving the identities in this lemma is not difficult.

Lemma 1. *Let $a, m \in \mathbb{Z}$, $m \geq 0$, $b \in \mathbb{N}$, $y \in \mathbb{R}$ and let $\epsilon, \epsilon' \in \{\pm 1\}$. Then we have the following identities.*

(1) $a + \frac{\epsilon}{b+y} = a + \epsilon + \frac{-\epsilon}{1+}\ \frac{1}{b-1+y}$.

(2) $a + \frac{\epsilon}{1+}\ \frac{1}{b+y} = a + \epsilon + \frac{-\epsilon}{b+1+y}$.

(3) $a + \frac{\epsilon}{2+}\ \underbrace{\frac{-1}{2+} \cdots \frac{-1}{2+}}_{m\text{-times}}\ \frac{\epsilon'}{b+y} = a + \epsilon + \frac{-\epsilon}{m+2+}\ \frac{-\epsilon'}{b+\epsilon'+y}$.

(4) $a + \frac{\epsilon}{2+}\ \underbrace{\frac{-1}{2+} \cdots \frac{-1}{2+}}_{m\text{-times}}\ \frac{-1}{1+}\ \frac{1}{b+y} = a + \epsilon + \frac{-\epsilon}{(m+2+b)+y}$.

(5) $\pm 1 = \frac{\pm 1}{2+}\ \frac{-1}{2+}\ \frac{-1}{2+} \cdots$.

Writing s/t, where $s, t \in \mathbb{R}$ and $t \neq 0$, as a column $\begin{pmatrix} s \\ t \end{pmatrix}$, the first two identities can be reformulated into the following matrix identities respectively:

$$\begin{aligned} \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ \epsilon & 0 \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix} &= \begin{pmatrix} a+\epsilon & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\epsilon & 0 \end{pmatrix} \begin{pmatrix} b-1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix}, \\ \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \epsilon & 0 \end{pmatrix} \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix} &= \begin{pmatrix} a+\epsilon & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b+1 & 1 \\ -\epsilon & 0 \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix}. \end{aligned}$$

In the process of conversion, finding a sequence of identities to be applied involves certain number of operations. Applying an identity may involve one or more number of basic operations. In fact, while applying the first, the second or the fourth identity, it involves four operations; the third identity requires six operations and the fifth requires only one operation.

Theorem 1. *Every finite SRCF can be converted into an RCF without computing the value of the given SRCF. Further, if n is the length of the given SRCF, then the conversion requires at most $5n$ operations.*

Proof. Suppose the given SRCF is

$$a_0 + \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+}\ \frac{\epsilon_3}{a_3+}\ \cdots \frac{\epsilon_n}{a_n}.$$

Denote its value (which we may not know) by x. Suppose $k \geq 1$ is the smallest positive integer such that $\epsilon_k = -1$.

1. Let $y_{k+1} = 0$ so that $k = n$.

(a) If $a_n = 1$, by a rearrangement, we have

$$x = a_0 + \frac{1}{a_1+}\ \frac{1}{a_2+} \cdots \ \frac{1}{(a_{n-1}-1)}.$$

(b) If $a_n > 1$, using identity (1) of Lemma 1, we have

$$x = a_0 + \frac{1}{a_1+}\ \frac{1}{a_2+} \cdots \ \frac{1}{(a_{n-1}-1)+}\ \frac{1}{1+}\ \frac{1}{a_n - 1}.$$

2. Let $y_{k+1} > 0$ so that $a_k \geq 1$.

(a) If $a_k = 1$, by using identity (2) of Lemma 1, we have

$$x = a_0 + \frac{1}{a_1+}\ \frac{1}{a_2+} \cdots \ \frac{1}{(a_{k-1}-1)+}\ \frac{1}{(a_{k+1}+1)+} \cdots \frac{\epsilon_n}{a_n}.$$

(b) If $a_k > 1$, then by identity (1) of Lemma 1, we have

$$x = a_0 + \frac{1}{a_1+}\ \frac{1}{a_2+} \cdots \ \frac{1}{(a_{k-1}-1)+}\ \frac{1}{1+}\ \frac{1}{(a_k-1)+} \cdots \frac{\epsilon_n}{a_n}.$$

3. Let $y_{k+1} < 0$ so that $a_k \geq 2$.

(a) If $a_k = 2$, suppose s is the smallest positive integer with $s \geq k$ such that $\frac{\epsilon_{s+1}}{a_{s+1}} \neq -1/2$. Then

$$x = a_0 + \frac{1}{a_1+} \cdots \ \frac{1}{a_{k-1}+}\ \underbrace{\frac{-1}{2+} \cdots \frac{-1}{2+}}_{s-k+1\text{-times}}\ \frac{\epsilon_{s+1}}{a_{s+1}+} \cdots \frac{\epsilon_n}{a_n}.$$

If $y_{s+1} \neq 0$ and $\frac{\epsilon_{s+1}}{a_{s+1}} \neq \frac{-1}{1}$, by using identity (3) of Lemma 1, we have

$$x = a_0 + \frac{1}{a_1+} \cdots \ \frac{1}{(a_{k-1}-1)+}\ \frac{1}{(s-k+2)+}\ \frac{-\epsilon_{s+1}}{(a_{s+1}+\epsilon_{s+1})+} \cdots \frac{\epsilon_n}{a_n}.$$

If $\frac{\epsilon_{s+1}}{a_{s+1}} = \frac{-1}{1}$ and $y_{s+2} \neq 0$, by using identity (4) of Lemma 1, we have

$$x = a_0 + \frac{1}{a_1+} \cdots \ \frac{1}{(a_{k-1}-1)+}\ \frac{1}{(s-k+2)+a_{s+2}+}\ \frac{\epsilon_{s+3}}{a_{s+3}+} \cdots \frac{\epsilon_n}{a_n}.$$

If $\frac{\epsilon_{s+1}}{a_{s+1}} = \frac{-1}{1}$ and $y_{s+2} = 0$, by using identity (4) of Lemma 1, we have

$$x = a_0 + \frac{1}{a_1+} \cdots \ \frac{1}{(a_{k-1}-1)}.$$

If $y_{s+1} = 0$, we have

$$x = a_0 + \frac{1}{a_1+}\ \frac{1}{a_2+} \cdots \ \frac{1}{(a_{k-1}-1)+}\ \frac{1}{s-k+2}.$$

(b) If $a_k > 2$, then by identity (1) of Lemma 1, we have

$$x = a_0 + \frac{1}{a_1+}\ \frac{1}{a_2+} \cdots \frac{1}{(a_{k-1}-1)+}\ \frac{1}{1+}\ \frac{1}{(a_k-1)+} \cdots \frac{\epsilon_n}{a_n}.$$

Observe that if the k-th partial numerator ϵ_k is -1, then the next partial numerator, namely, ϵ_{k+1} either vanishes or gets repaired in each step except in sub-case 3(b). To perform a step different from 3(b), it requires at most 8 operations which yields corrections in two consecutive partial numerators at a time. In fact, performing 3(b) successively requires maximum number of operations and each time it involves five operations which yields only one correction at a time. Thus the maximum number of operations to convert an SRCF into an RCF requires $5n$ operations, where n is the length of the given SRCF. □

Theorem 2. *Every finite SRCF can be converted into an ECF without computing the value of the given SRCF. Further, if n is the length of the given SRCF, then the conversion requires at most $6n$ operations.*

Proof. Suppose the given SRCF is

$$a_0 + \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_n}{a_n}.$$

Denote its value by x. Suppose k is the smallest non-negative integer such that a_k is an odd integer.

1. Let $y_{k+2} > 0$ so that $a_{k+1} \geq 1$.

   (a) If $a_{k+1} = 1$, use identity (2) of Lemma 1 to get

   $$x = a_0 + \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_k}{(a_k+\epsilon_{k+1})+}\ \frac{-\epsilon_{k+1}}{(a_{k+2}+1)+}\ \frac{\epsilon_{k+3}}{a_{k+3}+} \cdots \frac{\epsilon_n}{a_n}.$$

   (b) If $a_{k+1} > 1$, then by identity (1) of Lemma 1, we have

   $$x = a_0 + \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_k}{(a_k+\epsilon_{k+1})+}\ \frac{-\epsilon_{k+1}}{1+}\ \frac{1}{(a_{k+1}-1)+}\ \frac{\epsilon_{k+2}}{a_{k+2}+} \cdots \frac{\epsilon_n}{a_n}.$$

   Further, applying identity (1) of Lemma 1 successively $(a_{k+1}-1)$-times and then applying identity (2) of Lemma 1, we get

   $$\begin{aligned} x \;&=\; a_0 + \frac{\epsilon_1}{a_1+} \cdots \frac{\epsilon_k}{(a_k+\epsilon_{k+1})+}\ \frac{-\epsilon_{k+1}}{2+}\ \underbrace{\frac{-1}{2+}\ \frac{-1}{2+} \cdots \frac{-1}{2+}}_{a_{k+1}-2\text{-times}} \\ &\quad \frac{-\epsilon_{k+2}}{(a_{k+2}+\epsilon_{k+2})+}\ \frac{\epsilon_{k+3}}{a_{k+3}+} \cdots \frac{\epsilon_n}{a_n}. \end{aligned}$$

2. Let $y_{k+2} < 0$ so that $a_{k+1} \geq 2$.

   (a) If $a_{k+1} = 2$, suppose s is the smallest positive integer with $s \geq k+1$ such that $\frac{\epsilon_{s+1}}{a_{s+1}} \neq -1/2$. Then

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots \frac{\epsilon_k}{a_k+} \frac{\epsilon_{k+1}}{2+} \underbrace{\frac{-1}{2+} \cdots \frac{-1}{2+}}_{s-k-1\text{-times}} \frac{\epsilon_{s+1}}{a_{s+1}+} \cdots \frac{\epsilon_n}{a_n}.$$

      i. Let $y_{s+1} \neq 0$ and $\frac{\epsilon_{s+1}}{a_{s+1}} \neq \frac{-1}{1}$, use identity (3) of Lemma 1 to get

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots \frac{\epsilon_k}{(a_k + \epsilon_{k+1})+} \frac{-\epsilon_{k+1}}{(s-k+1)+} \frac{-\epsilon_{s+1}}{a_{s+1} + \epsilon_{s+1}+} \cdots \frac{\epsilon_n}{a_n}.$$

      If $\frac{\epsilon_{s+1}}{a_{s+1}} = \frac{-1}{1}$ and $y_{s+2} \neq 0$, using identity (4) of Lemma 1, we have

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots \frac{\epsilon_k}{(a_k + \epsilon_{k+1})+} \frac{-\epsilon_{k+1}}{(s-k+1) + a_{s+2}+} \frac{\epsilon_{s+3}}{a_{s+3}+} \cdots \frac{\epsilon_n}{a_n}.$$

      If $\frac{\epsilon_{s+1}}{a_{s+1}} = \frac{-1}{1}$ and $y_{s+2} = 0$, then

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots \frac{\epsilon_k}{(a_k + \epsilon_{k+1})}.$$

      ii. If $y_{s+1} = 0$ then $s = n$, so that

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots \frac{\epsilon_k}{(a_k + \epsilon_{k+1})+} \frac{-\epsilon_{k+1}}{(n-k+1)}.$$

   (b) If $a_{k+1} = 3$, by using identity (1) of Lemma 1, we get

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots \frac{\epsilon_k}{(a_k + \epsilon_{k+1})+} \frac{-\epsilon_{k+1}}{1+} \frac{1}{2+} \frac{-1}{a_{k+2}+} \cdots \frac{\epsilon_n}{a_n}.$$

   For this SRCF, the $(k+1)$-th partial denominator is 1, the $(k+2)$-th partial denominator is 2 and the fin at the $((k+1)+2)$-th stage is negative; so we follow step 2(a) to get an SRCF having at most $(n-k-2)$ odd partial denominators.

   (c) If $a_{k+1} > 3$, by using identity (1) of Lemma 1 successively $(a_{k+1} - 4)$-times, we get

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots \frac{\epsilon_k}{(a_k + \epsilon_{k+1})+} \frac{-\epsilon_{k+1}}{2+} \underbrace{\frac{-1}{2+} \cdots \frac{-1}{2+}}_{(a_{k+1}-4)\text{-times}} \frac{-1}{1+} \frac{1}{2+} \frac{-1}{a_{k+2}+} \cdots \frac{\epsilon_n}{a_n}.$$

For this SRCF, the $(k+a_{k+1}-2)$-th partial denominator is 1, the $(k+a_{k+1}-1)$-th partial denominator is 2 and the fin at the $((k+a_{k+1}-2)+2)$-th stage is negative; so we follow step 2(a) to get an SRCF having at most $(n-k-2)$ odd partial denominators.

3. Let $y_{k+2}=0$ so that $k=n-1$.

   (a) If $a_n=1$, then by a rearrangement, we have

   $$x = a_0 + \frac{\epsilon_1}{a_1+}\,\frac{\epsilon_2}{a_2+}\cdots\,\frac{\epsilon_{n-1}}{(a_{n-1}+\epsilon_n)}.$$

   (b) If $a_n>1$, then from identities (1) and (2) of Lemma 1, we see that

   $$x = a_0 + \frac{\epsilon_1}{a_1+}\,\frac{\epsilon_2}{a_2+}\cdots\,\frac{\epsilon_{n-1}}{(a_{n-1}+\epsilon_n)+}\,\frac{-\epsilon_n}{2+}\,\underbrace{\frac{-1}{2+}\,\frac{-1}{2+}\cdots\frac{-1}{2+}}_{a_n-2\text{-times}}.$$

4. Finally, if $k=n$, then $a_n=2r+1$ for some $r\in\mathbb{Z}$, so that

   $$x = a_0 + \frac{\epsilon_1}{a_1+}\,\frac{\epsilon_2}{a_2+}\cdots\,\frac{\epsilon_{n-1}}{a_{n-1}+}\,\frac{\epsilon_n}{(2r+1\pm 1)+(\mp 1)}.$$

   So we apply identity (5) of Lemma 1 and get an eventually constant ECF.

If n is the length of the SRCF, we get an ECF of x by repeating the above steps at most $n/2$ times.

Observe that if the k-th partial denominator a_k is odd, then the next partial denominator, namely, a_{k+1}, gets repaired automatically in each of the four cases above. Observe that the maximum number of operations are required in case (2) when occurrence of 2(c) is followed by occurrence of 2(a) of the algorithm. In fact, the number of operations in this is 12. Hence the maximum number of basic operations to convert an SRCF of length n to an ECF is $6n$. □

Theorem 3. *Every finite SRCF can be converted into an OCF without computing the value of the given SRCF. Further, if n is the length of the given SRCF, then the conversion requires at most $7n$ operations.*

Proof. Suppose the given SRCF is

$$a_0 + \frac{\epsilon_1}{a_1+}\,\frac{\epsilon_2}{a_2+}\cdots\,\frac{\epsilon_n}{a_n}.$$

Denote its value by x. Suppose k is the smallest non-negative integer such that a_k is an even integer.

1. Let $y_{k+2}>0$ so that $a_{k+1}\geq 1$.

(a) First suppose $a_{k+1} = 1$. Using identity (2) of Lemma 1, we have

$$x = a_0 + \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+} \cdots\ \frac{\epsilon_k}{(a_k + \epsilon_{k+1})+}\ \frac{-\epsilon_{k+1}}{(a_{k+2}+1)+}\ \frac{\epsilon_{k+3}}{a_{k+3}+} \cdots \frac{\epsilon_n}{a_n}.$$

(b) Now suppose $a_{k+1} > 1$. Then by identity (1) of Lemma 1, we have

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots\ \frac{\epsilon_k}{(a_k + \epsilon_{k+1})+}\ \frac{-\epsilon_{k+1}}{1+}\ \frac{1}{(a_{k+1}-1)+}\ \frac{1}{a_{k+2}+} \cdots \frac{\epsilon_n}{a_n}.$$

2. Let $y_{k+2} < 0$ so that $a_{k+1} \geq 2$.

(a) Let $a_{k+1} = 2$. Suppose s is the smallest positive integer with $s \geq k+1$ so that $\frac{\epsilon_{s+1}}{a_{s+1}} \neq -1/2$. Then

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots\ \frac{\epsilon_k}{a_k+}\ \frac{\epsilon_{k+1}}{2+}\ \underbrace{\frac{-1}{2+} \cdots \frac{-1}{2+}}_{s-k-1\text{-times}}\ \frac{\epsilon_{s+1}}{a_{s+1}+} \cdots \frac{\epsilon_n}{a_n}.$$

i. If $y_{s+1} \neq 0$ and $\frac{\epsilon_{s+1}}{a_{s+1}} \neq \frac{-1}{1}$, using identity (3) of Lemma 1, we obtain

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots\ \frac{\epsilon_k}{(a_k + \epsilon_{k+1})+}\ \frac{-\epsilon_{k+1}}{(s-k+1)+}\ \frac{-\epsilon_{s+1}}{a_{s+1}+\epsilon_{s+1}+} \cdots \frac{\epsilon_n}{a_n}.$$

If $\frac{\epsilon_{s+1}}{a_{s+1}} = \frac{-1}{1}$ and $y_{s+2} \neq 0$, using identity (4) of Lemma 1, we have

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots\ \frac{\epsilon_k}{(a_k + \epsilon_{k+1})+}\ \frac{-\epsilon_{k+1}}{(s-k+1)+a_{s+2}+}\ \frac{\epsilon_{s+3}}{a_{s+3}+} \cdots \frac{\epsilon_n}{a_n}.$$

If $\frac{\epsilon_{s+1}}{a_{s+1}} = \frac{-1}{1}$ and $y_{s+2} = 0$, then

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots\ \frac{\epsilon_k}{(a_k + \epsilon_{k+1})}.$$

ii. If $y_{s+1} = 0$ then $s = n$ and we have

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots\ \frac{\epsilon_k}{(a_k + \epsilon_{k+1})+}\ \frac{-\epsilon_{k+1}}{(n-k+1)}.$$

(b) Now let $a_{k+1} > 2$. Then by identity (1) of Lemma 1, we have

$$x = a_0 + \frac{\epsilon_1}{a_1+} \cdots\ \frac{\epsilon_k}{(a_k + \epsilon_{k+1})+}\ \frac{-\epsilon_{k+1}}{1+}\ \frac{1}{(a_{k+1}-1)+}\ \frac{\epsilon_{k+2}}{a_{k+2}+} \cdots \frac{\epsilon_n}{a_n}.$$

3. Let $y_{k+2} = 0$ so that $k = n-1$.

(a) If $a_n = 1$, by a suitable rearrangement, we have

$$x = a_0 + \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_{n-1}}{(a_{n-1}+\epsilon_n)}.$$

(b) If $a_n > 1$, we use identity (1) of Lemma 1 to obtain

$$x = a_0 + \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_{n-1}}{(a_{n-1}+\epsilon_n)+}\ \frac{-\epsilon_n}{1+}\ \frac{1}{a_n - 1}.$$

4. Let $k = n$. Then,

$$x = a_0 + \frac{\epsilon_1}{a_1+}\ \frac{\epsilon_2}{a_2+} \cdots \frac{\epsilon_{n-1}}{a_{n-1}+}\ \frac{\epsilon_n}{(a_n \pm 1)+}\ \frac{\mp 1}{1}.$$

If n is the length of the SRCF, we get the OCF of x by repeating the above steps at most n times.

Observe that if the k-th partial denominator a_k is even, $y_{k+2} > 0$ and $a_{k+1} > 1$, then repairing requires seven operations whereas all the other cases involve a fewer number of operations. Thus, the number of operations to convert an SRCF into an OCF is at most $7n$, where n is the length of the given SRCF. □

3. Concluding Remarks

Suppose n is the length of the given SRCF. To find the value x of the SRCF, it requires $3n$ operations. If m is the length of the desired type (RCF, ECF, OCF) of SRCF expansion of x, then the expansion requires $5m$ operations. Thus, to convert the given SRCF into the desired type of SRCF by finding (the value of) x and then expanding x, it requires $3n + 5m$ basic operations.

The algorithm in Theorem 1, which converts an SRCF of length n into an RCF (having the same value), requires at most $5n$ operations. Thus, our algorithm is efficient if $5n < 3n + 5m$ or $m > 0.4n$. Further, the algorithm in Theorem 2, which converts an SRCF of length n into an RCF (having the same value), requires at most $6n$ operations. So our algorithm is efficient provided $6n < 3n+5m$ or $m > 0.6n$. To find an OCF corresponding to a given SRCF of length n, the algorithm in Theorem 3 requires at most $7n$ operations. So the algorithm is efficient whenever $7n < 3n+5m$, that is, $m > 0.8n$.

If the length m of the resulting continued fraction is smaller than the length n of the given SRCF, it is clear in the proof of the theorems that the conversion process will follow certain steps each of which repairs several undesired partial numerators or partial denominators in one step. Thus, the number of steps required for the conversion will decrease proportionately.

We have used identities in Lemma 1 to convert an SRCF into an RCF, an ECF and an OCF, expecting a more efficient way than the direct method. Note that a finite SRCF is a rational number and every rational number has an ECF expansion, an OCF expansion and an RCF expansion. By utilizing identities in Lemma 1, we believe one can write an algorithm to convert a finite SRCF to any other type of SRCF provided the value of the given SRCF has an expansion of the desired type.

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