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# SUP-MEASURABLE AND WEAKLY SUP-MEASURABLE MAPPINGS IN THE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We discuss some set–theoretical questions concerning the notion of sup-measurability of functions of two variables and the existence and uniqueness of solutions of ordinary differential equations.

It is well known that the notion of measurability of sets and functions plays an important role in various fields of the classical and modern analysis. For functions of several variables, a related notion of sup-measurability was introduced and investigated (see e.g. [2] and the references given therein). It turned out that this notion can successfully be applied to some topics from analysis and, in particular, to the theory of ordinary differential equations.

Let $\mathbf{N} = \{0, 1, \dots, n, \dots\}$ denote the set of all natural numbers (which is identified with the first infinite ordinal $\omega$) and let $\mathbf{R}$ denote the real line. Let $C_b(\mathbf{R} \times \mathbf{R})$ be the Banach space of all bounded continuous real-valued functions defined on $\mathbf{R} \times \mathbf{R}$. Then, for each function $\Phi$ from this space, we can consider the ordinary differential equation

$$y' = \Phi(x, y)$$

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and, for a point $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$, we can speak on the corresponding Cauchy problem of finding a solution $y = y(x)$ of this equation, satisfying the initial condition $y(x_0) = y_0$. It is well known (see, e.g. [11]) that such a solution does always exist and, since $\Phi$ is bounded, the solution is global, i.e. it is defined on the whole real line $\mathbf{R}$. On the other hand, we cannot assert, in general, the uniqueness of a solution. There are simple examples of continuous bounded functions $\Phi$ on $\mathbf{R} \times \mathbf{R}$ for which the corresponding Cauchy problem admits at least two distinct solutions (in this connection, let us mention the famous work [8] of Lavrentieff where a much stronger result was obtained).

Actually, we need some additional properties of the original function $\Phi \in C_b(\mathbf{R} \times \mathbf{R})$ in order to have the uniqueness of a solution of the differential equation

$$y' = \Phi(x, y) \ (y(x_0) = y_0).$$

For instance, if $\Phi$ satisfies the so-called local Lipschitz condition with respect to the second variable $y$, then we have a unique solution for every Cauchy problem corresponding to $\Phi$.

Let us denote by $Lip_l(\mathbf{R} \times \mathbf{R})$ the family of all those functions from $C_b(\mathbf{R} \times \mathbf{R})$ which satisfy the local Lipschitz condition with respect to $y$. Then, obviously, $Lip_l(\mathbf{R} \times \mathbf{R})$ is a dense vector subspace of $C_b(\mathbf{R} \times \mathbf{R})$. Thus one can conclude that, for all functions $\Phi$ belonging to some dense subset of $C_b(\mathbf{R} \times \mathbf{R})$, the Cauchy problem

$$y' = \Phi(x, y) \ (y(x_0) = y_0, \ x_0 \in \mathbf{R}, \ y_0 \in \mathbf{R})$$

has a unique solution. Orlicz [9] essentially improved this result and showed that it holds true for almost all (in the category sense) functions from the Banach space $C_b(\mathbf{R} \times \mathbf{R})$. More precisely, one can formulate the following statement.

**Theorem 1.** *The set $U$ of all those functions from $C_b(\mathbf{R} \times \mathbf{R})$ for which the corresponding Cauchy problem has a unique solution (for any point $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$) is a dense $G_\delta$-subset of $C_b(\mathbf{R} \times \mathbf{R})$.*

Notice that the proof of Theorem 1 can be obtained by using the well–known Kuratowski lemma from general topology. This lemma states that if $X$ is an arbitrary topological space and $Y$ is a $\sigma$-quasicompact topological space, then the canonical projection

$$\mathrm{pr}_1 \ : \ X \times Y \to X$$

has the property that the image of any $F_\sigma$-subset of the product space $X \times Y$ is an $F_\sigma$-subset of $X$.

*Remark* 1. Evidently, the Banach space $C_b(\mathbf{R} \times \mathbf{R})$ is not separable. Let $E$ denote the subset of this space, consisting of all those functions which are constant at infinity. Then $E$ is a closed separable vector subspace of $C_b(\mathbf{R}\times\mathbf{R})$ and hence $E$ is a Banach space, as well. Clearly, a direct analogue of Theorem 1 holds true for $E$. Actually, in [9] Orlicz deals with the space $E$. A number of analogues of Theorem 1, for other spaces similar to $C_b(\mathbf{R}\times\mathbf{R})$ or $E$, are discussed in [1].

*Remark* 2. Unfortunately, the set $U$ considered above has a bad algebraic structure. In particular, $U$ is not a subgroup of the additive group of $C_b(\mathbf{R} \times \mathbf{R})$ and, consequently, $U$ is not a vector subspace of $C_b(\mathbf{R} \times \mathbf{R})$. Indeed, suppose for a while that $U$ is a subgroup of $C_b(\mathbf{R} \times \mathbf{R})$. Then $U$ must be a proper subgroup of $C_b(\mathbf{R} \times \mathbf{R})$. Let $\Psi$ be an arbitrary function from $C_b(\mathbf{R} \times \mathbf{R}) \setminus U$. Obviously, we have the equality

$$U \cap (\Psi + U) = \emptyset.$$

But each of the sets $U$ and $\Psi + U$ is the complement of a first category subset of $C_b(\mathbf{R} \times \mathbf{R})$. Therefore the intersection $U \cap (\Psi + U)$ must be the complement of a first category subset of $C_b(\mathbf{R} \times \mathbf{R})$, too, and hence

$$U \cap (\Psi + U) \neq \emptyset.$$

Thus we obtained a contradiction which yields that $U$ cannot be a subgroup of $C_b(\mathbf{R} \times \mathbf{R})$.

For some other properties of $U$ interesting from the set-theoretical and algebraic points of view, see e.g. [1].

Theorem 1 mentioned above shows that, for many functions from the space $C_b(\mathbf{R} \times \mathbf{R})$, we have the existence and uniqueness of a solution of the Cauchy problem. Naturally, we can consider a more general class of functions $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, not necessarily continuous or Lebesgue measurable, and investigate for such functions the corresponding Cauchy problem from the point of view of the existence and uniqueness of a solution.

For this purpose, let us introduce and examine the following three classes of functions acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$: the class of sup-continuous mappings, the class of sup-measurable mappings and the class of weakly sup-measurable mappings.

We shall say that a mapping $\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is sup-continuous (sup-measurable) with respect to the second variable $y$ if, for every continuous (Lebesgue measurable) function $\phi : \mathbf{R} \to \mathbf{R}$, the superposition

$$\Phi_\phi : \mathbf{R} \to \mathbf{R}$$

given by the formula

$$\Phi_\phi(x) = \Phi(x, \phi(x)) \ (x \in \mathbf{R})$$

is also continuous (Lebesgue measurable).

Let us mention that, actually, the first notion yields nothing new: it turns out that the class of all sup-continuous mappings coincides with the class of all continuous mappings acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$. For the sake of completeness, we present here the proof of this simple (and probably well–known) fact.

**Theorem 2.** *Let $\Phi$ be a mapping acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$. Then the following two assertions are equivalent:*
*1) $\Phi$ is continuous;*
*2) $\Phi$ is sup-continuous.*

*Proof.* The implication 1) $\Rightarrow$ 2) is trivial. So it remains to establish only the converse implication 2) $\Rightarrow$ 1). Let $\Phi$ be sup-continuous, and suppose that $\Phi$ is not continuous. Then there exist a point $(x_0, y_0)$ of $\mathbf{R} \times \mathbf{R}$, a real number $\varepsilon > 0$ and a sequence of points

$$\{(x_n, y_n) \ : \ n \in \mathbf{N}, \ n > 0\} \subset \mathbf{R} \times \mathbf{R}$$

such that
a) $\lim_{n \to +\infty} (x_n, y_n) = (x_0, y_0)$;
b) $|\Phi(x_n, y_n) - \Phi(x_0, y_0)| > \varepsilon$ for all $n \in \mathbf{N} \setminus \{0\}$.

We may assume, without loss of generality, that the sequence of points

$$\{x_n \ : \ n \in \mathbf{N}, \ n > 0\} \subset \mathbf{R}$$

is injective and $x_n \neq x_0$ for each $n \in \mathbf{N} \setminus \{0\}$. Indeed, if

$$f_n \ : \ \mathbf{R} \to \mathbf{R} \ (n = 1, 2, ...)$$

denotes the function identically equal to $y_n$, then the function

$$\Phi_{f_n} \ : \ \mathbf{R} \to \mathbf{R}$$

is continuous and $\Phi_{f_n}(x_n) = \Phi(x_n, y_n)$. Therefore, for some positive real number $\delta = \delta(x_n)$ and for all points $x$ belonging to the open interval $]x_n - \delta, \ x_n + \delta[$, we have the inequality

$$|\Phi_{f_n}(x) - \Phi(x_0, y_0)| > \varepsilon$$

or, equivalently,

$$|\Phi(x, y_n) - \Phi(x_0, y_0)| > \varepsilon.$$

From this fact it immediately follows that the above-mentioned sequence $\{x_n \ : \ n \in \mathbf{N}, \ n > 0\}$ can be chosen injective and satisfying the relation

$$(\forall n \in \mathbf{N} \setminus \{0\})(x_n \neq x_0).$$

Now, it is not difficult to define a continuous function $f \ : \ \mathbf{R} \to \mathbf{R}$ such that

$$(\forall n \in \mathbf{N})(f(x_n) = y_n).$$

For this function $f$, we get the continuous superposition

$$\Phi_f \ : \ \mathbf{R} \to \mathbf{R}.$$

Since $\lim_{n\to+\infty} \ x_n = x_0$, we must have the equality

$$\lim_{n\to+\infty} \ \Phi_f(x_n) = \Phi_f(x_0)$$

and, consequently,

$$\lim_{n\to+\infty} \ \Phi(x_n, y_n) = \Phi(x_0, y_0)$$

which is impossible. This contradiction finishes the proof of Theorem 2. □

A completely different situation is for sup-measurable mappings.

On the one hand, simple examples show that if $\Phi \ : \ \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a Lebesgue measurable mapping, then it need not be sup-measurable. Actually, the existence of such examples follows directly from the well-known fact that the composition of Lebesgue measurable functions (acting from $\mathbf{R}$ into $\mathbf{R}$) need not be Lebesgue measurable.

On the other hand, it turns out that there exist (under some additional set–theoretical axioms) various sup-measurable mappings which are not measurable in the Lebesgue sense. In order to present this result, let us first formulate and prove one simple auxiliary statement.

**Lemma 1.** *Suppose that $\Psi$ is a mapping acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$. Then the following two assertions are equivalent:*

*1) $\Psi$ is sup-measurable;*

*2) for every continuous function $\psi \ : \ \mathbf{R} \to \mathbf{R}$, the function $\Psi_\psi$ is Lebesgue measurable.*

*Proof.* The implication 1) ⇒ 2) is trivial. Let us show that the converse implication 2) ⇒ 1) is true, too. Let $\Psi$ satisfy 2) and let $\psi$ be an arbitrary Lebesgue measurable function acting from $\mathbf{R}$ into $\mathbf{R}$. Applying the well-known theorem of Luzin, we can find a countable partition $\{X_k \ : \ k < \omega\}$ of $\mathbf{R}$ and a countable family $\{\psi_k \ : \ k < \omega\}$ of functions from $\mathbf{R}$ into $\mathbf{R}$, such that

a) all sets $X_k$ $(1 \le k < \omega)$ are closed in $\mathbf{R}$ and $X_0$ is of Lebesgue measure zero;

b) all functions $\psi_k$ $(1 \le k < \omega)$ are continuous;

c) for each index $k < \omega$, the restriction of $\psi$ to $X_k$ coincides with the restriction of $\psi_k$ to $X_k$.

Let us denote by $\chi_k$ the characteristic function of $X_k$. Then it is not difficult to check the equality

$$\Psi_\psi = \sum_{k<\omega} \ \chi_k \Psi_{\psi_k}.$$

According to our assumption, all superpositions $\Psi_{\psi_k}$ $(1 \leq k < \omega)$ are Lebesgue measurable. In addition, the function $\Psi_{\psi_0}$ is equivalent to zero. Thus, we easily conclude that the superposition $\Psi_{\psi}$ is Lebesgue measurable, too. □

Now, we can formulate and prove the following statement (cf. [2], [3], [5] and [6]).

**Theorem 3.** *Let* $\mathbf{c}$ *denote the cardinality of the continuum, let* $\lambda$ *denote the standard Lebesgue measure on* $\mathbf{R}$ *and let* $[\mathbf{R}]^{<\mathbf{c}}$ *be the family of all subsets of* $\mathbf{R}$*, whose cardinalities are strictly less than* $\mathbf{c}$*. There exists a subset* $Z$ *of* $\mathbf{R} \times \mathbf{R}$ *such that*

*1) no three distinct points of* $Z$ *belong to a straight line (in other words,* $Z$ *is a set of points in general position);*

*2)* $Z$ *is a Lebesgue nonmeasurable subset of* $\mathbf{R} \times \mathbf{R}$*;*

*3) if* $[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda)$*, then the characteristic function of* $Z$ *is sup-measurable.*

*Proof.* Obviously, we can identify $\mathbf{c}$ with the first ordinal number $\alpha$ such that $\mathrm{card}(\alpha) = \mathbf{c}$. Let $\lambda_2$ denote the standard two-dimensional Lebesgue measure on the plane $\mathbf{R} \times \mathbf{R}$ and let $\{Z_\xi \ : \ \xi < \alpha\}$ be the family of all Borel subsets of $\mathbf{R} \times \mathbf{R}$ having strictly positive $\lambda_2$-measure. In addition, let $\{\phi_\xi \ : \ \xi < \alpha\}$ be the family of all continuous functions acting from $\mathbf{R}$ into $\mathbf{R}$. As usual, we identify any function from $\mathbf{R}$ into $\mathbf{R}$ with its graph lying in the plane $\mathbf{R} \times \mathbf{R}$. Now, using the method of transfinite recursion, we are going to define an $\alpha$-sequence of points

$$\{(x_\xi, y_\xi) \ : \ \xi < \alpha\} \subset \mathbf{R} \times \mathbf{R}$$

satisfying the following conditions:

a) if $\xi < \alpha$, $\zeta < \alpha$ and $\xi \neq \zeta$, then $x_\xi \neq x_\zeta$;

b) for each $\xi < \alpha$, the point $(x_\xi, y_\xi)$ belongs to the set $Z_\xi$;

c) for each $\xi < \alpha$, the point $(x_\xi, y_\xi)$ does not belong to the union of the family $\{\phi_\zeta \ : \ \zeta \leq \xi\}$;

d) for each $\xi < \alpha$, no three distinct points of the set $\{(x_\zeta, y_\zeta) \ : \ \zeta < \xi\}$ belong to a straight line.

Suppose that, for an ordinal $\xi < \alpha$, the partial $\xi$-sequence of points $\{(x_\zeta, y_\zeta) \ : \ \zeta < \xi\}$ has already been defined. Let us consider the set $Z_\xi$. We have $\lambda_2(Z_\xi) > 0$. According to the classical Fubini theorem, we can write

$$\lambda(\{x \in \mathbf{R} \ : \ Z_\xi(x) \in \mathrm{dom}(\lambda) \ \& \ \lambda(Z_\xi(x)) > 0\}) > 0$$

where $Z_\xi(x)$ denotes the section of $Z_\xi$ corresponding to a point $x \in \mathbf{R}$. Taking account of the latter formula, we see that there exists an element

$$x_\xi \in \mathbf{R} \setminus \{x_\zeta \ : \ \zeta < \xi\}$$

for which $\lambda(Z_\xi(x_\xi)) > 0$. In particular, we get the equality

$$\mathrm{card}(Z_\xi(x_\xi)) = \mathbf{c}.$$

Consequently, there exists an element

$$y_\xi \in Z_\xi(x_\xi) \setminus \cup\{\phi_\zeta(x_\xi) \ : \ \zeta \leq \xi\}.$$

Moreover, $y_\xi$ can be chosen in such a way that the corresponding point $(x_\xi, y_\xi)$ does not belong to the union of all straight lines having at least two common points with the set $\{(x_\zeta, y_\zeta) \ : \ \zeta < \xi\}$.

We have thus defined the point $(x_\xi, y_\xi) \in \mathbf{R}\times\mathbf{R}$. Proceeding in this manner, we are able to construct the $\alpha$-sequence $\{(x_\xi, y_\xi) \ : \ \xi < \alpha\}$ satisfying conditions a), b), c) and d). Finally, let us put

$$Z = \{(x_\xi, y_\xi) \ : \ \xi < \alpha\}$$

and let $\Phi$ denote the characteristic function of $Z$ (obviously, $Z$ is considered as a subset of the plane $\mathbf{R} \times \mathbf{R}$). Notice that $Z$ can also be regarded as the graph of a partial function acting from $\mathbf{R}$ into $\mathbf{R}$. Hence the inner $\lambda_2$-measure of $Z$ is equal to zero. On the other hand, the construction of $Z$ immediately yields that $Z$ is a $\lambda_2$-thick subset of the plane. Consequently, $Z$ is nonmeasurable in the Lebesgue sense and the same is true for its characteristic function $\Phi$. It remains to check that $\Phi$ is a sup-measurable mapping under the assumption $[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda)$. Let us take an arbitrary continuous function $\phi \ : \ \mathbf{R} \to \mathbf{R}$. Then $\phi = \phi_\xi$ for some ordinal $\xi < \alpha$. Now, we can write

$$\{x \in \mathbf{R} \ : \ \Phi(x, \phi_\xi(x)) \neq 0\} = \{x \in \mathbf{R} \ : \ (x, \phi_\xi(x)) \in Z\}$$

and it easily follows from condition c) that

$$\mathrm{card}(\{x \in \mathbf{R} \ : \ (x, \phi_\xi(x)) \in Z\}) \leq \mathrm{card}(\xi) + \omega < \mathbf{c}.$$

Since the inclusion $[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda)$ holds, we obtain that the function $\Phi_{\phi_\xi} = \Phi_\phi$ almost vanishes (with respect to $\lambda$) and, in particular, $\Phi_\phi$ is $\lambda$-measurable. Applying Lemma 1, we conclude that $\Phi$ is sup-measurable. $\square$

*Remark* 3. It is reasonable to stress here that the set $Z$ (and, consequently, its characteristic function $\Phi$) is defined within the theory **ZFC**. We used an additional set-theoretical hypothesis only to prove that $\Phi$ is a sup-measurable mapping. Let us also recall that the first construction of a Lebesgue nonmeasurable subset of the Euclidean plane, no three points of which belong to a straight line, is due to Sierpiński (see, for instance, [10]).

Now, starting with the function $\Phi$ defined above, we wish to consider an ordinary differential equation $y' = \Psi(x, y)$ with a Lebesgue nonmeasurable $\Psi$ and we are going to show that, in some situations, it is possible to obtain the existence and uniqueness of a solution of this equation (for any initial conditions).

First of all, we need to determine the class of functions, to which a solution must belong. It is natural to take the class $AC_l(\mathbf{R})$ consisting of all locally absolutely continuous functions on $\mathbf{R}$. In other words, $\psi \in AC_l(\mathbf{R})$ if and only if, for each point $x \in \mathbf{R}$, there exists a neighbourhood $V(x)$ such that the restriction $\psi|V(x)$ is absolutely continuous. Another characterization of locally absolutely continuous functions on $\mathbf{R}$ is the following one: a function $\psi$ belongs to $AC_l(\mathbf{R})$ if and only if there exists a Lebesgue measurable function $f : \mathbf{R} \to \mathbf{R}$ such that $f$ is locally integrable and

$$\psi(x) = \int_0^x f(t)dt + \psi(0)$$

for any $x \in \mathbf{R}$.

Let $\Psi$ be a mapping from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$ and let $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$. We say that the corresponding Cauchy problem

$$y' = \Psi(x, y) \ (y(x_0) = y_0)$$

has a unique solution (in the class $AC_l(\mathbf{R})$) if there exists a unique function $\psi \in AC_l(\mathbf{R})$ satisfying the relations:

a) $\psi'(x) = \Psi(x, \psi(x))$ for almost all (with respect to the Lebesgue measure $\lambda$) points $x \in \mathbf{R}$;

b) $\psi(x_0) = y_0$.

For example, if our mapping $\Psi$ is bounded, Lebesgue measurable with respect to $x$ and locally satisfies the Lipschitz condition with respect to $y$, then, for each $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$, the corresponding Cauchy problem has a unique solution. Notice that, in this example, $\Psi$ is necessarily Lebesgue measurable and sup-measurable. Notice also that Theorem 1 can be extended to a certain class of Banach spaces consisting of mappings acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$ which are Lebesgue measurable with respect to $x$ and continuous with respect to $y$.

The next statement shows that the existence and uniqueness of a solution can be fulfilled for some nonmeasurable mappings $\Psi$, too (cf. e.g. [6, p. 82]).

**Theorem 4.** *There is a Lebesgue nonmeasurable mapping*

$$\Psi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

*such that the Cauchy problem*

$$y' = \Psi(x, y) \ (y(x_0) = y_0)$$

*has a unique solution for any point* $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$.

*Proof.* Let $Z$ be the set constructed in the proof of Theorem 3. Denote by $\Phi$ the characteristic function of $Z$. Fix a real number $t$ and put

$$\Psi(x,y) = \Phi(x,y) + t \ (x \in \mathbf{R},\ y \in \mathbf{R}).$$

We assert that $\Psi$ is the required mapping. Indeed, $\Psi$ is Lebesgue nonmeasurable because $\Phi$ is Lebesgue nonmeasurable. Let now $(x_0, y_0)$ be an arbitrary point of the plane $\mathbf{R} \times \mathbf{R}$. Consider a function $\psi : \mathbf{R} \to \mathbf{R}$ defined by the formula

$$\psi(x) = tx + (y_0 - tx_0) \ (x \in \mathbf{R}).$$

The graph of this function is a straight line, so it has at most two common points with the set $Z$. Consequently, the function

$$\Psi_\psi : \mathbf{R} \to \mathbf{R}$$

is equal to $t$ for almost all (with respect to the Lebesgue measure $\lambda$) points from $\mathbf{R}$. We also have $\psi'(x) = t$ for all $x \in \mathbf{R}$. In other words, $\psi$ is a solution of the Cauchy problem

$$y' = \Psi(x,y) \ (y(x_0) = y_0).$$

It remains to show that $\psi$ is a unique solution from the class $AC_l(\mathbf{R})$. For this purpose, let us take an arbitrary solution $\phi$ of the same Cauchy problem, belonging to $AC_l(\mathbf{R})$. Then, for almost all points $x \in \mathbf{R}$, we have the equality

$$\phi'(x) = \Phi(x, \phi(x)) + t.$$

It immediately follows from this equality that the function $\Phi_\phi$ is measurable in the Lebesgue sense. But, as we know,

$$\mathrm{card}(\{x \in \mathbf{R} : \Phi_\phi(x) \neq 0\}) < \mathbf{c}.$$

So we obtain that $\Phi_\phi$ is equivalent to zero and hence $\phi'(x) = t$ for almost all $x \in \mathbf{R}$. Therefore we can conclude that

$$\phi(x) = tx + (y_0 - tx_0) \ (x \in \mathbf{R}).$$

This completes the proof of Theorem 4. □

*Remark* 4. The latter theorem was proved in the theory **ZFC**. However, we do not know whether it is possible to establish within **ZFC** the existence of a sup-measurable mapping which is not measurable in the Lebesgue sense.

Assuming Martin's Axiom and using an argument similar to the proof of Theorem 3, one can show that there exists a mapping

$$\Phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

satisfying the conditions:

1) for every Lebesgue measurable function $\phi : \mathbf{R} \to \mathbf{R}$, the superpositions

$$x \to \Phi(\phi(x), x) \ (x \in \mathbf{R}),$$
$$x \to \Phi(x, \phi(x)) \ (x \in \mathbf{R})$$

are also Lebesgue measurable;

2) $\Phi$ is not measurable in the Lebesgue sense.

In fact, for the existence of $\Phi$, we do not need the whole power of Martin's Axiom. It suffices to apply a certain set-theoretical hypothesis weaker than Martin's Axiom (cf. Theorem 5 below).

On the other hand, it is not difficult to prove that if a mapping

$$\Psi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

has the property that, for any two Borel functions $f$ and $g$ acting from $\mathbf{R}$ into $\mathbf{R}$, the superposition

$$x \to \Psi(f(x), g(x)) \ (x \in \mathbf{R})$$

is Lebesgue measurable, then $\Psi$ is Lebesgue measurable, too (see e.g. [5]). Moreover, one can assert that if the above–mentioned superposition is Lebesgue measurable for all continuous functions $f$ and $g$ acting from $\mathbf{R}$ into $\mathbf{R}$, then $\Psi$ is Lebesgue measurable (cf. Lemma 1).

Let $\Phi$ be a mapping acting from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$. We shall say that $\Phi$ is weakly sup-measurable if, for any continuous function $\phi : \mathbf{R} \to \mathbf{R}$ differentiable almost everywhere (with respect to $\lambda$), the superposition $\Phi_\phi$ is Lebesgue measurable.

Evidently, from the point of view of the theory of ordinary differential equations, the notion of a weakly sup-measurable mapping is more important than the notion of a sup-measurable mapping, because any solution of an ordinary differential equation must be continuous and differentiable almost everywhere.

Clearly, Theorem 3 can be formulated in terms of weakly sup-measurable mappings. In this connection, the following question arises naturally: does there exist a weakly sup-measurable mapping which is not sup-measurable? In order to give a partial answer to this question, we need one auxiliary statement due to Jarník (see [4]).

**Lemma 2.** *There exists a continuous function* $f : \mathbf{R} \to \mathbf{R}$ *nowhere approximately differentiable.*

In fact, Jarník proved in [4] that the set of all those functions from the Banach space $C([0, 1])$, which are nowhere approximately differentiable, is residual in $C([0, 1])$. In our further considerations, we need only one such a function.

**Theorem 5.** *Suppose that*

*1)* $[\mathbf{R}]^{<\mathbf{c}} \subset \mathrm{dom}(\lambda)$*;*

*2) for any cardinal number* $\kappa < \mathbf{c}$ *and for any family* $\{X_\xi \ : \ \xi < \kappa\}$ *of* $\lambda$*-measure zero subsets of* $\mathbf{R}$*, we have*

$$\cup\{X_\xi \ : \ \xi < \kappa\} \neq \mathbf{R}.$$

*Then there exists a weakly sup-measurable mapping* $\Phi$ *which is not sup-measurable.*

*Proof.* We can identify $\mathbf{c}$ with the first ordinal number $\alpha$ such that $\mathrm{card}(\alpha) = \mathbf{c}$. Let $f$ be a function from Lemma 2. Let $\{B_\xi \ : \ \xi < \alpha\}$ be some Borel base of the $\sigma$-ideal of all Lebesgue measure zero subsets of $\mathbf{R}$ and let $\{\phi_\xi \ : \ \xi < \alpha\}$ be the family of all continuous functions acting from $\mathbf{R}$ into $\mathbf{R}$ and differentiable almost everywhere in $\mathbf{R}$. We are going to construct (by transfinite recursion) an injective $\alpha$-sequence

$$\{(x_\xi, y_\xi) \ : \ \xi < \alpha\} \subset \mathbf{R} \times \mathbf{R}$$

of points belonging to the graph of $f$. Suppose that, for an ordinal $\xi < \alpha$, the partial $\xi$-sequence $\{(x_\zeta, y_\zeta) \ : \ \zeta < \xi\}$ has already been defined. Notice that, for each $\zeta \leq \xi$, the closed set

$$P_\zeta = \{x \in \mathbf{R} \ : \ \phi_\zeta(x) = f(x)\}$$

is of Lebesgue measure zero. Indeed, if $\lambda(P_\zeta) > 0$, then we can find a density point $x$ of $P_\zeta$ belonging to $P_\zeta$, such that there exists an approximate derivative $f'_{ap}(x)$, which is impossible. Consequently, $\lambda(P_\zeta) = 0$ for all $\zeta \leq \xi$, and the set

$$\mathbf{R} \setminus (\{B_\zeta \ : \ \zeta \leq \xi\} \cup \{P_\zeta \ : \ \zeta \leq \xi\} \cup \{x_\zeta \ : \ \zeta < \xi\})$$

is not empty. Let $x_\xi$ be an arbitrary point from this set and let $y_\xi = f(x_\xi)$.

Proceeding in such a manner, we are able to define the required family of points $\{(x_\xi, y_\xi) \ : \ \xi < \alpha\}$. Now, we put

$$Z = \{(x_\xi, y_\xi) \ : \ \xi < \alpha\}, \ X = \{x_\xi \ : \ \xi < \alpha\}$$

and denote by $\Phi$ the characteristic function of $Z$. Then it can easily be seen that $\Phi$ is a weakly sup-measurable mapping (cf. the proof of Theorem 3). On the other hand, let us consider the superposition $\Phi_f$. Obviously, we have

$$\Phi_f(x) = 1 \Leftrightarrow (x, f(x)) \in Z \Leftrightarrow x \in X.$$

It follows from our construction that $X$ is a Sierpiński type subset of the real line $\mathbf{R}$ (for the definition and various properties of Sierpiński sets, see e.g. [10]). In particular, $X$ is not measurable in the Lebesgue sense and, therefore, $\Phi_f$ is not Lebesgue measurable, too. We thus conclude that $\Phi$ is not a sup-measurable mapping. □

*Remark* 5. It is well known that assumptions 1) and 2) of Theorem 5 are logically independent (see, for instance, [7]). Slightly changing the argument presented above, one can show (under the assumptions of Theorem 5) that there exists a weakly sup-measurable mapping which is not sup-measurable and, in addition, is not Lebesgue measurable.

We do not know whether Theorem 5 is valid in the theory **ZFC**.

*Remark* 6. Evidently, the notion of sup-measurability can be formulated in terms of the Baire property instead of the measurability in the Lebesgue sense. It is easy to verify that, for the Baire property, a direct analog of Theorem 3 holds true. The corresponding analog of Theorem 5 holds true, too (in this case, we do not need Lemma 2; it suffices to apply the existence of a continuous nowhere differentiable function acting from $\mathbf{R}$ into $\mathbf{R}$).

*Remark* 7. Taking Theorem 4 into account, it is reasonable to pose a problem of finding appropriate analogs of Theorem 1 for sup-measurable (weakly sup-measurable) mappings. More precisely, it would be interesting to describe all those topological vector spaces of sup-measurable (weakly sup-measurable) mappings, for which Theorem 1 is valid. We see that, according to Theorems 3 and 4, some of the above-mentioned spaces can contain Lebesgue nonmeasurable mappings.

Finally, let us point out that several logical and set-theoretical aspects of the classical Cauchy-Peano theorem (on the existence of solutions of ordinary differential equations) are discussed in [12].

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