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The Structure of Automorphism Groups of Cayley Graphs and Maps

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Abstract. The automorphism groups $Aut(C(G, X))$ and $Aut(CM(G, X, p))$ of a Cayley graph $C(G, X)$ and a Cayley map $CM(G, X, p)$ both contain an isomorphic copy of the underlying group G acting via left translations. In our paper, we show that both automorphism groups are rotary extensions of the group G by the stabilizer subgroup of the vertex 1_G. We use this description to derive necessary and sufficient conditions to be satisfied by a finite group in order to be the (full) automorphism group of a Cayley graph or map and classify all the finite groups that can be represented as the (full) automorphism group of some Cayley graph or map.

Keywords: Cayley graph, Cayley map, automorphism group

1. Introduction and preliminaries

The only graphs considered in this paper are finite *Cayley graphs* $\Gamma = C(G, X)$ which are finite simple graphs defined for any finite group G and a set of generators $X \subset G$ with the property $1_G \notin X$ and $x^{-1} \in X$ for each $x \in X$. The set $V(\Gamma)$ of vertices of the Cayley graph $\Gamma = C(G, X)$ is the set of elements of G and any two vertices a and b of Γ are adjacent if and only if $b^{-1} \cdot a \in X$. It is easy to see that Cayley graphs defined in this way are simple loop-less non-oriented regular graphs of valency $|X|$.

The (*full*) *automorphism group* $Aut(\Gamma)$ of a graph Γ with the vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ is the group of all permutations of the set $V(\Gamma)$ preserving the edge structure $E(\Gamma)$, i.e., the subgroup of the full symmetric group of all permutations $\varphi \in \mathcal{S}_{V(\Gamma)}$ satisfying the property that $\varphi(u)$ is adjacent to $\varphi(v)$ if and only if u is adjacent to v, for all pairs of vertices $u, v \in V(\Gamma)$. In the case when $\Gamma = C(G, X)$, the automorphism group $Aut(\Gamma)$ can be alternately described as the subgroup of $\mathcal{S}_G$ of all permutations φ with the property $\varphi(a)^{-1}\varphi(a \cdot x) \in X$ for all $a \in G$ and $x \in X$. It easily follows that the set of *left translations* A_a defined for each element $a \in G$ by $A_a(b) = a \cdot b$ constitutes a subgroup of $Aut(\Gamma)$ isomorphic to the underlying group G. As this subgroup acts transitively on the set of vertices $V(\Gamma)$, every Cayley graph $C(G, X)$ is a vertex-transitive graph. Due to their inherent abundance of automorphisms as well as their "compact" description, Cayley graphs have been intensely studied over the last hundred years, and have played an important role in many interesting problems ranging from combinatorial group theory through algebraic combinatorics, extremal graph theory, and, especially lately, applied and theoretical computer science.

Our aim in Section 3 is to describe the structure of the automorphism group $Aut(\Gamma)$ of any Cayley graph $\Gamma = C(G, X)$ in terms of a *rotary extension* [7] of the group G. This will allow us to characterize all finite groups representable as the full automorphism group of some Cayley graph Γ. Related problems have been studied especially in the relation to the classification of the *graphical regular representations*—representations of abstract groups as regular (full) automorphism groups of graphs (which all turn out to be Cayley, due to the regularity requirement). Among the many articles devoted to this problem, let us mention at least the following few: [3, 5, 12, 13].

All the relevant theory concerning rotary extensions of groups will be developed in Section 2.

Section 4 of our paper is devoted to automorphism groups of combinatorial structures closely related to Cayley graphs—the Cayley maps. Automorphism groups of Cayley maps are isomorphic to subgroups of the automorphism groups of their underlying Cayley graphs, and so the problem of characterizing the automorphism groups of Cayley maps is closely tied to the above mentioned problems concerning Cayley graphs.

Let Γ be an arbitrary graph. A 2-cell embedding M of Γ in an orientable surface is called a *map*, and can be simply thought of as a drawing of Γ on an orientable surface with all faces homeomorphic to the open disc. Each of the original edges of the graph Γ can be endowed in M with two opposite directions and gives thereby rise to two oppositely oriented *arcs* of M. We denote the set of all arcs of M by $D(M)$; note that $|D(M)| = 2|E(\Gamma)|$. The *arc-reversing involution* acting on the set $D(M)$ by sending an arc to its oppositely oriented mate is denoted by T. Further, given an arbitrary vertex v of M, the cyclic permutation of the set of arcs emanating from v induced by the chosen orientation of the underlying surface will be denoted by p_v and the product of all cyclic permutations p_v which is a permutation of $D(M)$ called the *rotation* of M will be denoted by R. It is well-known [4] that each map M is completely determined by its underlying graph Γ together with the permutations R and T, and we shall use this fact freely throughout our paper. The (*full*) *automorphism group* $Aut(M)$ of a map M is the group of all permutations of the set $D(M)$ preserving the faces of M, namely, the group of all permutations $\varphi \in \mathcal{S}_{D(M)}$ that commute with both R and T.

In our paper we focus on maps whose underlying graph is a Cayley graph. Let $\Gamma = C(G, X)$, the arc set $D(M)$ of any embedding of a Cayley graph can then be represented as the set of all ordered pairs (g, x), $g \in G$ and $x \in X$, with (g, x) representing the arc emanating from the vertex g and terminating at the vertex $g \cdot x$. Thus, $|D(M)| = |G| \cdot |X|$, the arc-reversing involution T can then be defined by means of $T(g, x) = (g \cdot x, x^{-1})$, and each of the local cyclic permutations ordering the arcs emanating from a vertex g induces a cyclic permutation p_g of the set X defined by the formula $R(g, x) = (g, p_g(x))$.

One special case of a Cayley graph embedding into an orientable surface that has received particular attention is the case of an embedding for which all the local permutations p_g are equal in its action on X to a fixed cyclic permutation p of X. Such Cayley graph embeddings are called *Cayley maps* and are denoted by $CM(G, X, p)$. The main reason for the attention they receive, beside the obvious fact that they are easy to describe, is the richness of their automorphism groups. Each element g of G induces a map automorphism A_g defined on the set $D(M)$ via left translation by means of the formula $A_g(a, x) = (g \cdot a, x)$, for all

$g \in G$ and $x \in X$. That A_g is indeed a map automorphism follows easily from the following identities:

$$\begin{aligned} RA_g(a, x) &= R(g \cdot a, x) = (g \cdot a, p_{g \cdot a}(x)) = (g \cdot a, p(x)) = A_g(a, p(x)) \\ &= A_g(a, p_a(x)) = A_g R(a, x), \\ TA_g(a, x) &= T(g \cdot a, x) = (g \cdot a \cdot x, x^{-1}) = A_g(a \cdot x, x^{-1}) = A_g T(a, x), \end{aligned}$$

where the first sequence of identities also clearly indicates why left translations do not induce map automorphisms for arbitrary embeddings of Cayley graphs. Thus, the (full) automorphism group $Aut(M)$ of a Cayley map $M = CM(G, X, p)$ acts transitively on the set of vertices of M via a copy of G, and $|G| \leq |Aut(M)|$. Moreover, it is well-known [2] that the group of orientation preserving automorphisms of any map in an orientable surface (not just of a Cayley map) acts semiregularly on the set of arcs of the map, i.e., the stabilizer of each of the arcs is a trivial group. This implies the upper bound $|Aut(M)| \leq |D(M)| = |G| \cdot |X|$. In the case when the upper bound $|Aut(M)| = |G| \cdot |X|$ is achieved and $Aut(M)$ acts regularly on $D(M)$, we say that the map M is *regular*. Hence, regular Cayley maps are Cayley maps with the richest automorphism group possible and have an eminent position among the class of Cayley maps. For further results on regular Cayley maps see, for instance, [2, 6, 8, 9, 15, 16]. The paper [6] also contains a description of the automorphism groups of Cayley maps in terms of rotary extensions which will allow us in Section 4 to characterize automorphism groups of Cayley maps, and classify the abstract finite groups that can be represented as (full) automorphism groups of Cayley maps.

2. Rotary extensions

The concept of a rotary extension first occurred in relation to automorphism groups of Cayley maps in [6], where it was proved that the automorphism group $Aut(M)$ of any Cayley map $CM(G, X, p)$ is a rotary extension of the underlying group G by a group $\langle \rho \rangle$ generated by a special graph-automorphism ρ stabilizing the identity 1_G and called a "rotary mapping". The main idea behind rotary extensions is a generalization of the semidirect extension of a group H by a subgroup $K \leq Aut(H)$ where the group of automorphisms $Aut(H)$ is replaced by the group of all permutations on H stabilizing the identity 1_H, denoted by $Stab_{\mathcal{S}_H}(1_H)$. Rotary extensions of groups form a special case of a much more general group extension discussed in [14].

Most of the preliminary definitions and ideas for rotary extensions can be found in the article [7], and we include them in this section for the sake of completeness. Also, the paper [7] does not go beyond stating the basic definitions and properties. Although rotary extensions can be defined for both finite and infinite groups, we will mostly restrict ourselves to the finite case.

Let H be a finite group, and let $Stab_{\mathcal{S}_H}(1_H)$ be the subgroup of the full symmetric group $\mathcal{S}_H$ of all permutations φ of the set H with the property $\varphi(1_H) = 1_H$. For each $h \in H$, define a binary operation $\odot_h$ on $Stab_{\mathcal{S}_H}(1_H)$ as follows:

$$(\phi \odot_h \psi)(a) = \phi(h)^{-1} \cdot \phi(h \cdot \psi(a)), \tag{1}$$

for all $a \in H$, where all the multiplications are to be carried out in H. Alternately, $\phi \odot_h \psi = A_{\phi(h)^{-1}} \phi A_h \psi$, where $A_{\phi(h)^{-1}}$ and A_h are left translations by the indicated elements, and the compositions are to be taken from the right. It is easy to verify that each of the operations $\odot_h$ is a non-associative binary operation on $Stab_{\mathcal{S}_H}(1_H)$ with a left identity id_H and a right inverse for each element $\phi \in Stab_{\mathcal{S}_H}(1_H)$, namely the element $h^{-1} \cdot \phi^{-1}(\phi(h) \cdot id_H)$. Moreover, in the case when both ϕ and ψ are group automorphisms of H, the operation $\odot_h$ is the operation of composition of group automorphisms.

Now, instead of extending H by a subgroup of $Aut(H)$, we shall extend it by special subgroups of $Stab_{\mathcal{S}_H}(1_H)$ closed under all binary operations $\odot_h$. A subgroup $K \le Stab_{\mathcal{S}_H}(1_H)$ is said to be *rotary closed* if $\phi \odot_h \psi \in K$, for all $\phi, \psi \in K$ and all $h \in H$. The simplest possible examples of rotary closed groups are the trivial group, $Stab_{\mathcal{S}_H}(1_H)$ and any subgroup of $Aut(H)$, but we shall see soon that there are many more examples of rotary closed subgroups related to automorphism groups of Cayley graphs.

Let H be a finite group, and let K be a rotary closed subgroup of $Stab_{\mathcal{S}_H}(1_H)$. The *rotary extension* of H by K, $H \times_{rot} K$, is the set of all ordered pairs $(h, k) \in H \times K$ together with the binary operation

$$(a, \phi) \star (b, \psi) = (a \cdot \phi(b), \phi \odot_b \psi). \tag{2}$$

Note that the product operation in the first coordinate is the "usual" semidirect product multiplication, while the second coordinate multiplication is defined by formula (1). This defines a group structure on $H \times K$:

Theorem 1 *Let H be a group, and let K be a rotary closed subgroup of $Stab_{\mathcal{S}_H}(1_H)$. Then the rotary extension $H \times_{rot} K$ is a group.*

Proof: Although the proof of this theorem is not particularly hard, it is relatively technical, and we shall just state here that the identity element of $H \times_{rot} K$ is the pair $(1_H, id_H)$ and the inverse of the element (a, ϕ) is the pair $(\phi^{-1}(a^{-1}), \phi^{-1}(a^{-1}) \cdot \phi^{-1}(a^{-1} \cdot id_H))$ (where the element of H upon which the second coordinate mapping acts has been omitted). □

We have already mentioned that in the case $K \le Aut(H)$, the rotary extension $H \times_{rot} K$ is a semidirect product of H by K, and in this sense, the rotary extension defined here is a generalization of the concept of a semidirect product. It is well-known that any group product $G = H \cdot K$ with the property $H \cap K = \{1_G\}$ is a semidirect product of H by K if and only if H is a normal subgroup of G. To characterize rotary extensions in a similar vein, consider a group G that can be expressed as a product of two of its subgroups H, K, $G = H \cdot K$ and $H \cap K = \{1_G\}$. Then G is also equal to the product $K \cdot H$, and, moreover, for every pair of elements $h \in H$ and $k \in K$ there exists a unique pair $h_k \in H$ and $k_h \in K$ such that $kh = h_k k_h$. Let Ψ be the mapping from K to $\mathcal{S}_H$ sending elements $k \in K$ to permutations Ψ_k defined by the equation $\Psi_k(h) = h_k$, for all $h \in H$. We can easily see that Ψ is a homomorphism from K to $Stab_{\mathcal{S}_H}(1_H)$. The following is a characterization of rotary extensions in terms of the homomorphism Ψ.

Theorem 2 *Let G be a group and H, K be two subgroups of G such that $G = H \cdot K$ and $H \cap K = \{1_G\}$. If the homomorphism $\Psi : K \longrightarrow Stab_{\mathcal{S}_H}(1_H)$ is injective and the image*

$\Psi(K)$ is rotary closed in $Stab_{\mathcal{S}_H}(1_H)$, then G is isomorphic to a rotary extension of H by K.

Conversely, let $G = H \times_{rot} K$. Then G contains two subgroups H' and K', isomorphic to H and K respectively, such that $H' \cap K' = \{1_G\}$, $G = H' \cdot K'$, the homomorphism $\Psi : K' \longrightarrow Stab_{\mathcal{S}_{H'}}(1_{H'})$ is injective and $\Psi(K')$ is rotary closed in $Stab_{\mathcal{S}_{H'}}(1_{H'})$.

Proof: The proof of this theorem follows along the same lines as the usual proof of the characterization of semidirect products. □

We close this section with a simple observation that immediately follows from the injectivity of Ψ:

Let $G = H \times_{rot} K$ be a rotary extension of H by $K \leq Stab_{\mathcal{S}_H}(1_H)$. Then $K \cap C_G(H) = \{1_G\}$ and $H \not\leq \mathcal{Z}(G)$, where $C_G(H)$ is the centralizer of H in G and $\mathcal{Z}(G)$ is the center of G.

3. Automorphism groups of Cayley graphs

The first theorem of this section relates rotary extensions of groups to the structure of automorphism groups of Cayley graphs.

Theorem 3 *Let $\Gamma = C(G, X)$ be a Cayley graph, and let $K = Stab_{Aut(\Gamma)}(1_G)$ be the stabilizer of the identity vertex in $Aut(\Gamma)$. Then K is a rotary closed subgroup of $Stab_{\mathcal{S}_G}(1_G)$ and $Aut(\Gamma) \cong G \times_{rot} K$.*

Proof: Recall that K is the subgroup of $\mathcal{S}_G$ of all permutations ρ satisfying the properties (i) $\rho(1_G) = 1_G$ and (ii) $\rho(a)^{-1} \cdot \rho(ax) \in X$, for all $a \in G$ and $x \in X$. Thus, K is clearly a subgroup of $Stab_{\mathcal{S}_G}(1_G)$, and to prove the first statement of our theorem it remains to prove that K is rotary closed. Let $\phi, \psi \in K$ and a be an arbitrary element of G. The mapping $\phi \odot_a \psi$ stabilizes the vertex 1_G, as $Stab_{\mathcal{S}_G}(1_G)$ itself is rotary closed. Now, let b be any element of G and x be any element of X. The following series of identities verifies that $\phi \odot_a \psi$ also satisfies the condition (ii).

$$\begin{aligned}((\phi \odot_a \psi)(b))^{-1} \cdot (\phi \odot_a \psi)(bx) &= (\phi(a)^{-1}\phi(a\psi(b)))^{-1} \cdot \phi(a)^{-1}\phi(a\psi(bx)) \\ &= \phi(a\psi(b))^{-1}\phi(a)\phi(a)^{-1}\phi(a\psi(bx)) \\ &= \phi(a\psi(b))^{-1}\phi(a\psi(bx)) \\ &= \phi(a\psi(b))^{-1}\phi(a\psi(b)y) \in X,\end{aligned}$$

where $\psi(b)^{-1}\psi(bx) = y \in X$ follows from the fact that ψ satisfies (ii).

Since $\phi \odot_a \psi$ satisfies both (i) and (ii), $\phi \odot_a \psi$ belongs to K which is therefore rotary closed.

Now, let us prove that $Aut(\Gamma)$ is isomorphic to the rotary extension $G \times_{rot} K$. Let ρ be any graph automorphism of Γ. Then $\rho(1_G) \in G$, and so the composition of ρ with the left translation $A_{\rho(1_G)^{-1}}$ is a graph automorphism of Γ that stabilizes the identity: $A_{\rho(1_G)^{-1}} \cdot \rho(1_G) = \rho(1_G)^{-1} \cdot \rho(1_G) = 1_G$. Thus, $A_{\rho(1_G)^{-1}} \cdot \rho$ belongs to K and the mapping Φ sending any graph automorphism ρ to the pair $(\rho(1_G), A_{\rho(1_G)^{-1}} \cdot \rho)$ is a bijective mapping from $Aut(\Gamma)$ onto $G \times_{rot} K$. It remains to prove that Φ is a homomorphism of groups.

Let ρ and ψ be two graph automorphisms of Γ. Then $\Phi(\rho \circ \psi) = ((\rho \circ \psi)(1_G), A_{((\rho\circ\psi)(1_G))^{-1}} \cdot (\rho \circ \psi)) = ((\rho \circ \psi)(1_G), ((\rho \circ \psi)(1_G))^{-1} \cdot (\rho \circ \psi))$. On the other hand, $\Phi(\rho) \star \Phi(\psi) = (\rho(1_G), A_{\rho(1_G)^{-1}} \cdot \rho) \star (\psi(1_G), A_{\psi(1_G)^{-1}} \cdot \psi) = (\rho(1_G), \rho(1_G)^{-1} \cdot \rho) \star (\psi(1_G), \psi(1_G)^{-1} \cdot \psi) = (\rho(1_G) \cdot (\rho(1_G))^{-1} \cdot \rho(\psi(1_G)), (\rho(1_G)^{-1} \cdot \rho(\psi(1_G)))^{-1} \cdot \rho(1_G)^{-1} \cdot \rho(\psi(1_G) \cdot \psi(1_G)^{-1} \cdot \psi) = (\rho(\psi(1_G)), \rho(\psi(1_G))^{-1} \cdot \rho(1_G) \cdot \rho(1_G)^{-1} \cdot (\rho \circ \psi)) = ((\rho \circ \psi)(1_G), ((\rho \circ \psi)(1_G))^{-1} \cdot (\rho \circ \psi))$, which completes the proof of our theorem. □

The above theorem asserts that the full automorphism group of any Cayley graph has the structure of a rotary extension of the underlying group. This result allows for a nice extension of the well-known Cayley theorem.

Corollary 1 *Let G be a finite group of order n. Then G is a rotary factor of the full symmetric group $\mathcal{S}_n$, i.e., $\mathcal{S}_n$ is a rotary extension of G:*

$$\mathcal{S}_n = G \times_{rot} Stab_{\mathcal{S}_n}(1_G).$$

Proof: This is a direct corollary of the previous theorem based on the fact that $\mathcal{S}_n = Aut(C(G, X))$, where X is the set of all non-identity elements of G, and thus, $C(G, X)$ is a complete graph. □

It is not hard to see that Theorem 3 is true for any vertex-transitive automorphism group of a Cayley graph—not just the *full* automorphism group. The connection between automorphism groups of Cayley graphs and rotary extensions goes even deeper.

Theorem 4 *A finite group G can be represented as a vertex-transitive subgroup of the full automorphism group of a Cayley graph if and only if $G \cong H \times_{rot} K$ and there exists a family of orbits $\{\mathcal{O}_i \mid i \in \mathcal{I}\}$ of the action of K on H satisfying the properties $1_H \notin \bigcup \mathcal{O}_i$, $(\bigcup \mathcal{O}_i)^{-1} = \bigcup \mathcal{O}_i$ and $\langle \bigcup \mathcal{O}_i \rangle = H$.*

Proof: One of the implications of the theorem follows from the discussion preceding the theorem. The other implication follows from the simply verifiable fact that G is a vertex-transitive subgroup of the full automorphism group of the graph $C(H, \bigcup \mathcal{O}_i)$. □

Knowing the structure of the (full) automorphism groups of Cayley graphs, we can finally address the problem of classifying all finite groups that are the full automorphism groups of some Cayley graphs, i.e., we will classify all the finite groups G for which there exists a Cayley graph $\Gamma = C(H, X)$ such that $G \cong Aut(\Gamma)$ (note that dropping the requirement that G has to be the *full* automorphism group would make our task trivial: any finite group G is a *subgroup* of the automorphism group of any Cayley graph based on G).

Let G be an (abstract) finite group. If $G \cong Aut(\Gamma)$ for some Cayley graph $\Gamma = C(H, X)$, then H has to be isomorphic to a subgroup of G. To simplify our notation, let us simply assume that H is a subgroup of G itself. In the case when $H = G$, the action of G on the vertices of Γ is regular, and the Cayley graph $C(G, X)$ is called a *graphical regular representation* of G. Graphical regular representations (GRR's) have been extensively studied in the 70's and 80's and the concentrated effort of several authors resulted in a classification of all finite groups possessing graphical regular representations. A nice overview of these results can be found in [3]. Here, we are particularly interested in the following complete list of finite groups that do not have a graphical regular representation originally introduced by M. Watkins.

Let G be a finite group that does not have a GRR. Then G is an abelian group of exponent greater than 2 or G is a generalized dicyclic group or G is isomorphic to one of the following 13 groups

(1) $\mathcal{Z}_2^2, \mathcal{Z}_2^3, \mathcal{Z}_2^4$
(2) $\mathcal{D}_6, \mathcal{D}_8, \mathcal{D}_{10}$
(3) $\mathcal{A}_4$
(4) $\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab\rangle$
(5) $\langle a, b \mid a^8 = b^2 = 1,\ b^{-1}ab = a^5\rangle$
(6) $\langle a, b, c \mid a^3 = b^3 = c^2 = 1,\ ab = ba,\ (ac)^2 = (bc)^2 = 1\rangle$
(7) $\langle a, b, c \mid a^3 = b^3 = c^3 = 1,\ ac = ca,\ bc = cb,\ b^{-1}ab = ac\rangle$
(8) $\mathcal{Q} \times \mathcal{Z}_3$, $\mathcal{Q} \times \mathcal{Z}_4$, where $\mathcal{Q}$ denotes the quaternion group.

Clearly, any group possessing a GRR can be represented as the full automorphism group of some Cayley graph, namely, the full automorphism group of its GRR. Thus, the only groups for which the question of whether or not they can be represented as the full automorphism group of some Cayley graph needs to be decided are the groups from the above list. Since these groups do not have a GRR, the only way they can possibly be represented as the $Aut(\Gamma)$ of some Cayley graph Γ is via a transitive action on a Cayley graph of some proper subgroup of theirs. These observations lead to the following classification.

Theorem 5 *Let G be a finite group. Then G is isomorphic to the full automorphism group $Aut(\Gamma)$ of a Cayley graph $\Gamma = C(H, X)$ if and only if G is not an abelian group of exponent greater than* 2, *a generalized dicyclic group, or one of the groups* (1), (3), (4), (5), (6), (7), (8) *from the above list.*

Proof: The dihedral groups $\mathcal{D}_6$, $\mathcal{D}_8$, and $\mathcal{D}_{10}$ are well-known to be the full automorphism groups of the Cayley graphs $C(\mathcal{Z}_n, \{1, -1\})$, $n = 3, 4, 5$, of their cyclic subgroups. To prove the theorem we only need to show that none of the groups listed in the theorem can be isomorphic to some $Aut(\Gamma)$, $\Gamma = C(G, X)$.

First, suppose that G is an abelian group that does not have a GRR. Thus, $G \cong Aut(C(H, X))$ would imply $|H| < |G|$ and G would have to act transitively *but not regularly* on the elements of H. This contradicts the well-known theorem that transitive actions of abelian groups have to be regular (see e.g. [18]). Another way of arguing this statement is to observe that if G does not have a GRR and $G \cong Aut(C(H, X))$ then G must be a *non-trivial* rotary extension of H, but no non-trivial rotary extension is abelian as one can deduce from the last note of the previous section.

Next, let G be a generalized dicyclic group. Then G is generated by an abelian group A and an element $b \notin A$ satisfying the relations $b^4 = 1$, $b^2 \in A$ and $b^{-1}ab = a^{-1}$ for all $a \in A$. Suppose (by means of contradiction) that $G \cong Aut(C(H, X))$. G, being a generalized dicyclic group, does not admit a GRR and must be therefore a nontrivial rotary product $G = H \times_{rot} K$ with both H and K nontrivial and $K \cap C_G(H) = \langle 1_G \rangle$. First, $K \subseteq A$ as any element ba, $a \in A$, that would belong to K would also force the element $baba$ belong to A, however, $baba = baa^{-1}b = b^2 \in \mathcal{Z}(G) \subseteq C_G(H)$. Also, any involution $a \in A$, $a^2 = 1_G$, belongs to $\mathcal{Z}(G)$, and thus, K contains no involutions. It follows that there exists an element $k \in K$, $k \neq k^{-1}$. Consider now the mappings Ψ_k and $\Psi_{k^{-1}}$ defined in our characterization of rotary products in the previous section. Clearly, $\Psi_k \mid A = \Psi_{k^{-1}} \mid A$ as A is abelian and $k, k^{-1} \in A$. Furthermore, let ba be any element of H not belonging to A. Then $k \cdot ba = ba_k \cdot k_{ba}$ implies the identity $k^{-1} \cdot ba = k^{-2}ba_k k_{ba} = bk^2 a_k k_{ba} = ba_k \cdot k^2 k_{ba}$. Thus $\Psi_k = \Psi_{k^{-1}}$ on all of H which contradicts the injectivity of Ψ.

Since the paragraph about the abelian case applies also to the groups from line (1) of the list, all that is left to prove is that none of the groups from lines (3) through (8) are isomorphic to a full automorphism group of a Cayley graph. Using the packages "GAP" and "nauty", we have constructed all Cayley graphs $C(H, X)$ satisfying the property that H is a proper subgroup of some group from (3) to (8), and all their automorphism groups. None of the groups listed in lines (3) through (8) appeared on our list. We conclude that none of these groups is isomorphic to the full automorphism group of a Cayley graph. This completes the proof of our classification. □

4. Automorphism groups of Cayley maps

As mentioned in the introduction, automorphism groups of Cayley maps are isomorphic copies of special vertex-transitive subgroups of the automorphism groups of their underlying Cayley graphs. Using Theorem 3, it follows that the automorphism groups of Cayley maps $Aut(CM(G, X, p))$ are rotary extensions of the underlying group G. This has been first observed in [6], where one can also find the following results relevant to the theory developed further in this section.

Let $M = CM(G, X, p)$ be a Cayley map, and let ρ be a bijection of the group G onto itself. We say that ρ is a *rotary mapping* of M if ρ satisfies for all $a \in G$ and $x \in X$ the following three properties:

(i) $\rho(1_G) = 1_G$
(ii) $\rho(a)^{-1}\rho(ax) \in X$
(iii) $\rho(a)^{-1}\rho(ap(x)) = p(\rho(a)^{-1}\rho(ax))$

(i.e., ρ is a graph automorphism of $C(G, X)$ stabilizing the identity, and "commuting" with p on X). For each Cayley map $M = CM(G, X, p)$, there exists a positive integer k, $1 \leq k \leq |X|$ and a rotary mapping ρ_k such that the restriction of ρ_k to X is equal to p^k. Let k be the smallest integer with this property, and let ρ_k be the rotary mapping associated with k ($\rho_k \mid X = p^k$). Then k divides $|X|$ and $Aut(M) \cong G \times_{rot} \langle \rho_k \rangle$, i.e., automorphism groups of Cayley maps are rotary extensions of the underlying group by a cyclic subgroup of order

$|X|/k$. The paper [6] also provides us with a useful formula defining the rotary mapping ρ_k : Let a be an arbitrary element of G, and let $a = x_1 x_2, \ldots, x_n$ be any expression of a in terms of the generators from X. Then

$$\rho_k(a) = \rho_k(x_1 x_2, \ldots, x_n) = b_1 b_2, \ldots, b_n, \tag{3}$$

where $b_1 = p^k(x_1)$, $b_{i+1} = p^{l_i}(b_i^{-1})$, for $1 \le i \le n-1$, and the exponents l_i are the natural numbers determined by the equations $x_{i+1} = p^{l_i}(x_i^{-1})$.

In what follows, we shall use the above results from [6] to classify the finite groups isomorphic to some full automorphism group of a Cayley map.

First we state an analogue of Theorem 4 the proof of which follows from the above stated description of the automorphism groups of Cayley maps and from an argument similar to the one in the proof of Theorem 4.

Theorem 6 *A finite group G can be represented as a vertex-transitive subgroup of the full automorphism group of some Cayley map if and only if $G \cong H \times_{rot} \langle\varphi\rangle$ and there exists a collection of orbits $\{\mathcal{O}_i \mid i \in \mathcal{I}\}$ of φ acting on H such that all orbits are of the same size, their union $X = \bigcup \mathcal{O}_i$ is closed under taking inverses, $1_H \notin X$, and X generates all of H.*

Next, consider the following analogue of the concept of a GRR for a group G. A Cayley map $CM(G, X, p)$ is said to be a *mapical regular representation*, MRR, for a group G if $Aut(CM(G, X, p)) \cong G$. Thus, an (abstract) group G is said to possess an MRR if it can be represented as a vertex-regular full automorphism group of some Cayley map of G. Naturally, a question arises which finite groups allow for an MRR.

The following theorem provides a complete answer to this question together with a classification of all finite groups representable as full automorphism groups of Cayley maps.

Theorem 7 *Let G be a finite group. Then G is isomorphic to the* full *automorphism group $Aut(M)$ of some Cayley map $M = CM(H, X, p)$ if and only if G is not one of the two groups $\mathcal{Z}_3$ and $\mathcal{Z}_2^2$.*

Moreover, each finite group not isomorphic to $\mathcal{Z}_3$ or $\mathcal{Z}_2^2$ also possesses an MRR.

Proof: Let G be a finite group. If $\Gamma = C(G, X)$ is a graphical regular representation for G, then $Aut(CM(G, X, p)) \cong G$, for all cyclic permutations p of X. This is due to the fact that $Aut(CM(G, X, p))$ is isomorphic to a vertex-transitive subgroup of $Aut(C(G, X)) = G$. Thus, any finite group G that has a GRR has also an MRR and is isomorphic to the automorphism group of some Cayley map. Once again, we only need to focus on the groups that do not have a GRR. We shall, however, adopt a different approach this time, and we shall prove the theorem for all sufficiently large finite groups at once, regardless of whether they have a GRR or not. The proof will be slightly different for groups of even and odd order.

First, let G be a finite group of an odd order greater than or equal to 13. Let X be the set of all non-identity elements of G, $X = \{a \mid a \in G, a \neq 1_G\}$. We will construct a cyclic permutation $p = (p_1, p_2, \ldots, p_{|X|})$ of X such that $Aut(CM(G, X, p)) \cong G$. Since

$|X| \geq 12$ and $|G|$ is odd, we can find two distinct elements x and y from X such that all five elements x, x^{-1}, y, y^{-1} and $x \cdot y$ are different. Now, let p be any cyclic permutation of X with the first five elements defined as follows: $p_1 = x$, $p_2 = x^{-1}$, $p_3 = y$, $p_4 = y^{-1}$ and $p_5 = x \cdot y$, that satisfies the property that each element of X is listed in p next to its inverse (i.e., p_i^{-1} is equal to either the predecessor or the successor of p_i). Such a cyclic permutation of X clearly exists as X contains no involutions. Recall now that the results from [6] yield that $Aut(CM(G, X, p)) \cong G$ if and only if the smallest divisor of $|X|$ associated to a rotary mapping is $|X|$ itself in which case the rotary mapping $\rho_{|X|}$ is simply equal to id_G and $Aut(CM(G, X, p))$ is a rotary extension of G by a trivial group. We are going to alter the permutation p in such a way that will guarantee that none of the bijections ρ_k defined by formula (3) and associated with a divisor k of $|X|$, $k \neq |X|$, will be equal to p^k on X. Thus, the automorphism group of the resulting Cayley map will be a rotary extension of G by a trivial group and will therefore be isomorphic to G. Let $\mathcal{J} = \{1, k_1, k_2, \ldots, k_j\}$ be the list of divisors of $|X|$ smaller than $|X|$ listed in an increasing order. First, we are going to "disable" the rotary mapping ρ_1. Consider the image of $x \cdot y$ under the mapping ρ_1 defined by formula (3) : $\rho_1(x \cdot y) = p^1(x) \cdot p^{l_1}((p^1(x))^{-1})$, where l_1 is the solution of $y = p^{l_1}(x_1^{-1})$, i.e., $l_1 = 1$ (since y follows immediately after x^{-1} in p). Hence, $\rho_1(x \cdot y) = p(x) \cdot p^1((p(x))^{-1}) = x^{-1} \cdot p((x^{-1})^{-1}) = x^{-1} \cdot p(x) = x^{-1} \cdot x^{-1}$. On the other hand, $p(x \cdot y) = p_5$. In the case when $p_5 \neq x^{-1} \cdot x^{-1}$, we obtain $\rho_1(x \cdot y) \neq p(x \cdot y)$, hence, $\rho_1 \mid X \neq p$ and therefore the smallest divisor of $|X|$ for which the corresponding ρ_k equals p^k on X is not 1 (and $|Aut(CM(G, X, p))| < |G| \cdot |X|/1$). A more interesting situation occurs when $p_5 = x^{-1} \cdot x^{-1}$. In this case there is a chance for $\rho_1 \mid X$ to be equal to p, which would cause the automorphism group to be too big. To avoid that, we will alter the permutation p by swapping the fifth and sixth element of p, i.e., if $p_5 = x^{-1} \cdot x^{-1}$ and $p_6 = b$, we will set $p_5 = b$ and $p_6 = x^{-1} \cdot x^{-1}$. If we consider the rotary mapping ρ_1 defined by the new permutation p and formula (3), we still obtain $\rho(x \cdot y) = x^{-1} \cdot x^{-1}$ (as the first four elements of p have not been changed!), while $p(x \cdot y) = p_5 = b$ is not equal to $x^{-1} \cdot x^{-1}$ anymore, and $\rho_1 \mid X \neq p$. Thus, in both cases (p_5 equal to $x^{-1} \cdot x^{-1}$ or not), we obtain a permutation p such that $|Aut(CM(G, X, p))| < |G| \cdot |X|/1$.

In order to "disable" all the possible rotary mappings other than $\rho_{|X|}$, we just need to repeat the above described swapping process for all $\rho_k, k \in \mathcal{J}$. We will do it using induction. We have already shown a way to disable the rotary mapping ρ_1 without changing the order of the first five elements. Now suppose (the induction hypothesis) that $\rho_{k_j} \mid X \neq p^{k_j}$ for all $j \leq n$. We will alter the permutation p in such a way that will disable $\rho_{k_{n+1}}$ while at the same time the alteration will not affect the fact that $\rho_{k_j} \mid X \neq p^{k_j}$ for $j \leq n$. Consider the image of $x \cdot y$ under $\rho_{k_{n+1}}$ as defined by formula (3): $\rho_{k_{n+1}}(x \cdot y) = p^{k_{n+1}}(x) \cdot p^{l_1}((p^{k_{n+1}}(x))^{-1})$. The exponent l_1 is equal to 1 again (we have not changed the order of the first five elements), and thus, $\rho_{k_{n+1}}(x \cdot y) = p^{k_{n+1}}(x) \cdot p((p^{k_{n+1}}(x))^{-1})$. If $p^{k_{n+1}}(x \cdot y) \neq p^{k_{n+1}}(x) \cdot p((p^{k_{n+1}}(x))^{-1})$, then $\rho_{k_{n+1}} \mid X \neq p^{k_{n+1}}$, and we do not need to do any changes. If $p^{k_{n+1}}(x \cdot y) = p^{k_{n+1}}(x) \cdot p((p^{k_{n+1}}(x))^{-1})$, then we need to swap the element $p^{k_{n+1}}(x \cdot y) = p_{k_{n+1}+5}$ with its right neighbor. It is obvious that this swap will disable $\rho_{k_{n+1}}$. Moreover, none of the computations that disabled the mappings ρ_j, $j \leq n$, will be affected by this change, as all the images $\rho_j(x \cdot y) = p^j(x) \cdot p((p^j(x))^{-1})$ and $p^j(x \cdot y)$, as well as all the elements used in their computation are positioned left of the swap, and are not changed by the swap (notice that the fact that $p((p^j(x))^{-1})$ is to the left of the swap is due to the fact that we have started with a permutation p where elements and their inverses were close one to another).

To complete this proof by induction we just need to argue that the last swap (the one disabling ρ_{k_j}) will not accidentally spill over to the beginning of the permutation and change the element $p_1 = x$. This follows from our choice of the size of X, $|X| \geq 12$. The last two elements that might possibly be swapped are p_{k_j+5} and p_{k_j+6}, where k_j is the largest divisor of $|X|$ not equal to X. Hence, the swap will not spill over to p_1 if $k_j + 6 \leq |X|$. Since G is an odd degree group, $|X|$ is even, and the largest divisor k_j of $|X|$ is at most $|X|/2$. It follows that $k_j + 6 \leq |X|$ if $(|X|/2) + 6 \leq |X|$, i.e., $|X| \geq 12$ or $|G| \geq 13$, and this requirement is enough to guarantee that we can perform all the changes.

This completes the proof by induction, and we conclude that any finite group G of odd order $\geq$13 allows for the existence of a cyclic permutation p of the set $X = G - \{1_G\}$ such that $Aut(CM(G, X, p)) \cong G$.

Now, suppose that G is a finite group of an even order greater than or equal to 8. Let X again be the set of all non-identity elements of G. Since G is of even order, it contains at least one involution x, and since $|G| \geq 8$, it also contains an element y different from x. Let $p = (p_1, p_2, \ldots, p_{|X|})$ be again a cyclic permutation of X. There are two possibilities to define the beginning of p this time, depending on whether y can be chosen to be an involution (i.e., whether G contains more involutions than just x) or not. If y can also be chosen to be an involution, set $p_1 = x$, $p_2 = y$ and $p_3 = x \cdot y$. If there are no more involutions beside x, choose the element y in such a way so that the four elements $x, y, y^{-1}, x \cdot y$ are all different (this is possible since $|G| \geq 8$) and choose the beginning of p to be $p_1 = x$, $p_2 = y$, $p_3 = y^{-1}$ and $p_4 = x \cdot y$. In both cases, complete the permutation p so that the elements that are not involutions stand next to their inverses. Next, starting from the above described permutation p disable the non desirable rotary mappings just like we did in the case of odd order groups. This can be done by induction as long the last swapped element does not spill over to p_1. The last two elements that might possibly be swapped are p_{k_j+3} and p_{k_j+4} or p_{k_j+4} and p_{k_j+5} depending on which of the two possibilities for p we are using (where k_j is once again the largest divisor). Thus, the last swap will not effect p_1 if $k_j + 5 \leq |X|$. Since $|X|$ is odd, k_j is at most $|X|/3$, which finally implies $|X| \geq 7.5$ or $|G| \geq 10$. Finally, in the case when $|G| = 8$, the set X is of size 7. The only divisor of 7 smaller than 7 is 1, and so we only need to disable ρ_1. There is obviously enough room to do that, which extends our arguments to all even order groups G of size at least 8. We will leave the details of this part of the proof out as they are quite similar to the odd order part.

The above proofs leave us with only finitely many groups that may not be isomorphic to the automorphism group of any Cayley map, namely, the odd order groups $\mathcal{Z}_{11}$, $\mathcal{Z}_9$, $\mathcal{Z}_3^2$, $\mathcal{Z}_7$, $\mathcal{Z}_5$, $\mathcal{Z}_3$, and $\mathcal{Z}_1$, and the even order groups $\mathcal{Z}_6$, $\mathcal{S}_3$, $\mathcal{Z}_4$, $\mathcal{Z}_2^2$, and $\mathcal{Z}_2$. Following the above ideas about choosing the permutation p, one can easily find permutations p such that $Aut(CM(G, X, p)) \cong G$ for all the groups in this list but $\mathcal{Z}_3$ and $\mathcal{Z}_2^2$. Finally, one can easily construct all the Cayley maps based on the remaining two groups—there is only one Cayley map for $\mathcal{Z}_3$ (even if we drop the requirement that X must generate the group, we only obtain one more map that way), and only two isomorphic classes for $\mathcal{Z}_2^2$ (four, if we drop the requirement for X to be a generating set). None of the maps has either $\mathcal{Z}_3$ or $\mathcal{Z}_2^2$ as its automorphism group. Moreover, none of the two groups can be the automorphism group of a Cayley map of a smaller group as that would lead to a non-trivial rotary extension that is non-abelian. We can conclude that the only groups that are not isomorphic to the full automorphism group of some Cayley map and that do not have an MRR are $\mathcal{Z}_3$ and $\mathcal{Z}_2^2$. □

It follows from the above theorem, that each finite group G different from $\mathcal{Z}_3$ or $\mathcal{Z}_2^2$ allows for the existence of a Cayley map of a complete graph based on G with the automorphism group being as small as possible. The opposite side of the spectrum, namely the finite groups G that give rise to the existence of a Cayley map based on a complete graph of G that has a *regular* automorphism group have been studied by James and Jones in [10] who have shown that the only regular Cayley maps whose underlying graphs are complete are balanced Cayley maps of order p^n, p a prime.

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