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Coverings of Graphs and Maps, Orthogonality, and Eigenvectors

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Abstract. Lifts of graph and map automorphisms can be described in terms of voltage assignments that are, in a sense, compatible with the automorphisms. We show that compatibility of ordinary voltage assignments in Abelian groups is related to orthogonality in certain $\mathcal{Z}$-modules. For cyclic groups, compatibility turns out to be equivalent with the existence of eigenvectors of certain matrices that are naturally associated with graph automorphisms. This allows for a great simplification in characterizing compatible voltage assignments and has applications in constructions of highly symmetric graphs and maps.

Keywords: graph, map, covering, voltage assignment, orthogonality, eigenvectors, automorphism

1. Introduction

Graph and map coverings are a powerful tool for constructing new graphs and maps from small quotients. The coverings are usually described in terms of various types of voltage assignments on the quotient graphs and maps. A voltage assignment then determines a way of "lifting" the quotient to a "large" graph or map that covers the quotient.

Along with lifting graphs and maps, one is often interested in lifting automorphisms of the quotients in order to obtain lifts that are as symmetric as possible. This is particularly important in constructions of highly transitive graphs and maps. Classical examples are the covering constructions of infinite families of cubic 5-arc-transitive graphs given in [6] and [4]. As regards maps on orientable surfaces, in a certain sense the highest degree of transitivity is achieved when a map is regular, that is, when the group of orientation preserving map automorphisms acts regularly on the incident vertex-edge pairs. Lifting techniques as introduced in [17] or [3, 8] are a natural candidate for constructing new regular maps from old ones. For completeness we mention that finite regular maps of valence m and face length n correspond to torsion-free finite-index normal subgroups of the $(2, m, n)$-triangle groups. As the latter are discrete groups of isometries of the hyperbolic plane or the complex upper half-plane, regular maps are closely related to Riemann surfaces and Galois groups; for a survey of these fascinating connections we recommend the papers [9] and [11].

Voltage assignments that allow the lift of an automorphism of the quotient graph or map have a characteristic algebraic property which we will call "compatibility" with the automorphism. A number of necessary and sufficient conditions for compatibility of voltage assignments have been known in the literature, see e.g. [2, 4, 6, 8, 17]. In a way, all these conditions (recently surveyed in [13]) go back to the classical theory of lifting continuous mappings in algebraic topology (cf. [14]).

The primary object of investigation in this paper are compatible voltage assignments on graphs and maps in *Abelian* groups. This is motivated by the fact that a number of important covering constructions in topological graph theory can be presented in the language of (ordinary) voltage assignments in Abelian groups, as can be seen in Chapters 4 and 5 of the monograph [7] and also in [4, 17].

The main point is that, for voltage assignments in an Abelian group, compatibility with a graph or a map automorphism turns out to be closely related to group automorphisms and to orthogonality in certain $\mathcal{Z}$-modules. In particular, we show that compatible voltage assignments in *cyclic* groups directly correspond to eigenvectors and eigenvalues of certain matrices that are naturally associated with graph automorphisms (and are a generalization of cycle basis matrices as introduced e.g. in [5]). This correspondence can be convenient in various applications, for instance, in constructions of regular maps.

The paper is organized as follows. In Sections 2 and 3 we give a brief description of lifts of graphs, maps, and compatibility, with emphasis on lifts of medial maps and regular maps. A useful representation of automorphisms of connected graphs and maps in terms of certain matrices whose rows and columns are indexed by cotree edges (modulo a chosen spanning tree of the graph) is introduced in Section 4, together with a discussion of basic algebraic properties of the matrices and their relation to flows in graphs. Section 5 contains the main results about the relationship between compatibility of voltage assignments in Abelian groups, orthogonality in $\mathcal{Z}$-modules, and group automorphisms. In the special case when voltages in cyclic groups are considered, compatibility with graph and map automorphisms can be conveniently characterized in the language of eigenvectors of the matrices associated with the automorphisms. Applications of the results to a complete characterization of self-dual regular maps that cyclically branch-cover the tetrahedron in the sphere are given in the final Section 6.

2. Graphs, maps, and their lifts

Graphs in this article will be finite and undirected but otherwise very general in that we allow multiple edges as well as multiple loops. (In topological graph theory graphs are sometimes allowed to have semiedges as well, and our theory can be easily adapted to include them.) It is often convenient to assign directions to edges in order to introduce a certain "coordinate system" in the graph. A directed edge will be called a *dart*. If x is a dart, then x^{-1} denotes the reverse dart to x; the pair x, x^{-1} constitutes an undirected edge of the graph. Let D denote the set of all darts of a graph Γ. The bijection λ of D such that $x\lambda = x^{-1}$ for each $x \in D$ is the *dart-reversing involution*.

Let Γ be a graph with dart set D and let G be a finite group. A mapping $\alpha : D \to G$ is a *voltage assignment on* Γ if $\alpha(x^{-1}) = (\alpha x)^{-1}$ for each dart $x \in D$. The graph Γ endowed with a voltage assignment α in a group G gives rise to a new graph Γ^α, called a *lift* of Γ with respect to α. The dart set of the lift Γ^α is defined to be the set $D^\alpha = D \times G$; for elements of D^α we will use the subscript notation x_g where $x \in D$ and $g \in G$. The darts

x_g and y_h of Γ^α emanate from the same vertex in the lift if and only if $g = h$ and x, y emanate from the same vertex of Γ. The dart-reversing involution λ^α on the lift is defined by $x_g\lambda^\alpha = (x\lambda)_{g\alpha(x)}$. Note that the mapping $p : \Gamma^\alpha \to \Gamma$ given by $p(x_g) = x$ is a graph homomorphism, known in the language of algebraic topology as an *unbranched regular covering* of Γ by the lift Γ^α.

A *walk* W in a graph Γ is a finite sequence of darts $x_1, x_2, \ldots, x_m$ such that the terminal vertex of x_i is the initial vertex of x_{i+1}, $1 \le i < m$. The walk is *closed* and *u-based* if u is both the initial vertex of x_1 and the terminal vertex of x_m. The *reverse* of $W = x_1x_2\ldots x_m$ is the walk $W^{-1} = x_m^{-1}\ldots x_2^{-1}x_1^{-1}$. If α is a voltage assignment on Γ in a group G, the voltage of the walk W is the group element $\alpha(W) = \alpha(x_1)\alpha(x_2)\ldots\alpha(x_m)$; note that $\alpha(W^{-1}) = (\alpha(W))^{-1}$. We will encounter walks and their voltages in subsequent sections, and here we just point out their relation with the connectivity of the lift: If Γ is connected and if u is any vertex of Γ, then the lift Γ^α is connected if and only if the subgroup $G_u = \{\alpha(W);\ W \text{ a closed } u\text{-based walk in } \Gamma\}$ is the entire group G; in such a case the assignment α will be called *proper*.

In a way similar to lifts of graphs we may describe lifts of maps; our presentation loosely follows the one given in Section 4 of [2]. A *map* is a 2-cell embedding of a graph Γ in a closed (clockwise) oriented surface; Γ is then called the *underlying graph* of the map. As usual in topological graph theory (cf. [7]), we describe a map by means of two permutations of the dart set D of Γ. The first one is the *rotation* ρ, which at each vertex v cyclically permutes the darts emanating from v in accordance with the orientation of the surface (so that for each dart x emanating from v, the dart $x\rho$ is the clockwise next dart on the surface that emanates from v); the second one is simply the dart-reversing involution λ of Γ. A map M with rotation ρ and dart-reversing involution λ will be denoted by $M(\rho, \lambda)$.

A *corner* of a map $M = M(\rho, \lambda)$ with dart set D is any ordered pair $(x, x\rho)$ or $(x, x\rho^{-1})$ where $x \in D$. For any $x \in D$ the corners $(x, x\rho)$ and $(x\rho, x)$ are mutually *reverse*; in symbols, $(x, x\rho)^{-1} = (x\rho, x)$ and vice versa. A mapping β that assigns elements of a finite group G to corners of M will be called a *corner voltage assignment* on M if $\beta(c^{-1}) = (\beta c)^{-1}$ for each corner c of M. We define two permutations λ^β and ρ^β of the set $D^\beta = D \times G$ by $(x_g)\lambda^\beta = (x\lambda)_g$ and $(x_g)\rho^\beta = (x\rho)_{g\beta(c)}$ where c is the corner $(x, x\rho)$. The corner voltage assignment β is said to be *proper* if the permutation group generated by ρ^β and λ^β is transitive on D^β. In such a case the set D^β is the dart set of a map $M^\beta = M(\rho^\beta, \lambda^\beta)$, called a *lift* of M. The mapping $\pi : D^\beta \to D$ defined by $\pi(x_g) = x$ is a map homomorphism $M^\beta \to M$ which extends to a (possibly branched) covering of the surfaces of M^β and M where branch points may occur in vertices and face centers (with at most one branch point per vertex and per face).

Lifts of a map can equivalently be described in terms of voltage assignments on *darts* (not corners) of the underlying graph of the so-called medial map. Let $M = M(\rho, \lambda)$ be a map with dart set D. The vertex set of the *medial map* $M_\mathrm{m} = M(\rho_\mathrm{m}, \lambda_\mathrm{m})$ is the set of edges of M, which may be identified with the set of unordered pairs $\{x, x\lambda\}$ where $x \in D$. The dart set D_m of M_m consists of all corners of M. From each vertex $\{x, x\lambda\}$ of M_m there emanate exactly four darts of D_m, namely, the ones whose first coordinate is x or $x\lambda$. The rotation ρ_m of the medial map M_m is the permutation of D_m which cyclically permutes the darts emanating from the vertex $\{x, x\lambda\}$ by $((x, x\rho), (x, x\rho^{-1}), (x\lambda, x\lambda\rho), (x\lambda, x\lambda\rho^{-1}))$.

The dart-reversing involution λ_m just permutes corners of the *original* map M with their reverses. Geometrically, the construction of the medial map M_m corresponds to inserting a new vertex in the centre of each edge of M and joining a pair of new vertices by a new edge whenever they lie in a common corner of M.

Now, let β be a proper corner voltage assignment on M in a group G. We define a voltage assignment α on the *underlying graph* Γ_m of the medial map M_m in the same group G by $\alpha(c) = \beta(c)$ for each corner c of M. Note that c as the argument of α denotes a dart of M_m while the same c as the argument of β means a corner of M. Then, α is a proper voltage assignment on Γ_m. We now embed the (connected) lift $(\Gamma_m)^\alpha$ on a surface in such a way that the rotation at each vertex has the form $((x, x\rho)_g, (x, x\rho^{-1})_g, (x\lambda, x\lambda\rho)_g, (x\lambda, x\lambda\rho^{-1})_g)$, $g \in G$, and denote the resulting map by $(M_m)^\alpha$. Then the lift M^β associated with the corner voltage assignment β is related with the map constructed from the assignment α by $(M^\beta)_m \simeq (M_m)^\alpha$. In other words, the medial of the "corner voltage assignment lift" of M is the same map as the "dart voltage assignment lift" of the medial of M.

In this sense, corner voltage assignments on a map M are equivalent with voltage assignments on darts of the underlying graph of the medial map M_m, representing just a different language for describing the lifting phenomena related to maps and their automorphisms. This is part of the folklore in topological graph theory; for details the reader is invited to consult [1, 2].

3. Lifts of graph and map automorphisms and compatibility of voltage assignments

Let α be a voltage assignment on a graph Γ in a group G, let Γ^α be the corresponding lift and let $p : \Gamma^\alpha \to \Gamma$ be the covering projection $x_g \mapsto x$. An automorphism A of the graph Γ (regarded throughout as a permutation of the dart set of Γ) is said to *lift* to an automorphism $\tilde{A}$ of the graph Γ^α if $p\tilde{A} = Ap$, that is, if $p\tilde{A}(x_g) = A(x)$ for each dart x of Γ and any element g of the group G. To present conditions for an automorphism to lift we follow the elementary approach of [8]. We will say that the voltage assignment α is *A-compatible* if

$$\alpha(W) = 1_G \quad \text{if and only if} \quad \alpha(AW) = 1_G \tag{1}$$

for each closed walk W in Γ based at a fixed vertex u (here, AW is the image of the walk W under the automorphism A). If H is a subgroup of the automorphism group $Aut\,(\Gamma)$ of the graph Γ, the assignment α will be called *H-compatible* if (1) holds for each $A \in H$. It is an easy exercise to show that if Γ is a *connected* graph then the compatibility of α does not depend on the choice of the fixed vertex. The results of [8] which we will need later may now be formulated as follows.

Theorem 1 *Let α be a proper voltage assignment on a connected graph Γ in a group G and let A be an automorphism of Γ. Then, A lifts to an automorphism of Γ^α if and only if the assignment α is A-compatible. In such a case there is an automorphism ξ of the group G given by $\xi(\alpha(W)) = \alpha(AW)$, where W is any closed walk in Γ based at a fixed vertex.*

A theory for lifting *map automorphisms* can be found in [2]; we again just sum up the basics needed for understanding the important Theorems 3 and 4 at the end of this Section.

An automorphism A of a map $M = M(\rho, \lambda)$ is a permutation of the dart set D of M such that $A(x\rho) = (Ax)\rho$ and $A(x\lambda) = (Ax)\lambda$ for any $x \in D$; note that A induces an orientation-preserving self-homeomorphism of the supporting surface for M, respecting the cell structure of M. Let β be a proper corner voltage assignment on M in a group G and let $\pi : M^\beta \to M$ be the (possibly branched) covering given by $\pi(x_g) = x$. An automorphism A of M *lifts* to a map automorphism $\tilde{A}$ of M^β if $\pi\tilde{A} = A\pi$. For a lifting condition we now need the concept of a *corner walk* W in the map M, which is a sequence $c_1, c_2, \ldots, c_m$ of corners of M such that for each $i < m$ the corners c_i and c_{i+1} are adjacent (meaning that c_i contains a dart x such that either x or x^{-1} is in c_{i+1}); if the corners c_1 and c_m are adjacent as well then W is *closed*. The voltage of the corner walk W is the product $\beta(W) = \beta(c_1)\beta(c_2)\cdots\beta(c_m)$. For any map automorphism A of M the image of a corner $c = (x, y)$ is the corner $Ac = (Ax, Ay)$, and the corner walk AW is the sequence $Ac_1, Ac_2, \ldots, Ac_m$. Again, we say that the corner voltage assignment β is *A-compatible* provided that $\beta(W) = 1_G$ if and only if $\beta(AW) = 1_G$ for each closed corner walk W in M. For a subgroup H of the map automorphism group *Aut* (M), we say that the corner voltage assignment β is *H-compatible* if the above equivalence holds for each $A \in H$. With this terminology, Theorem 9 of [2] may be restated as:

Theorem 2 *Let β be a proper voltage assignment on corners of a map M in a group G and let A be a map automorphism of M. Then, A lifts to a map automorphism of the lifted map M^β if and only if the corner voltage assignment β is A-compatible.*

Theorems 1 and 2 are important in constructions of highly symmetric graphs and maps. As regards maps, it is easy to check that for any two darts $x, y \in D$ of a map M there exists at most one (orientation-preserving) map automorphism of M taking x onto y, and so the automorphism group *Aut*(M) of M contains at most $|D|$ elements. The maps M for which $|Aut(M)| = |D|$ are thus the "most symmetric" maps with respect to (orientation-preserving) automorphisms. As in such a case the group *Aut*(M) acts regularly (i.e., transitively and freely) on the dart set D, the maps with $|Aut(M)| = |D|$ are called *regular maps*. For a good introduction into algebraic theory of maps and regular maps we recommend [10].

We are now ready to explain in detail the importance of the concept of compatibility in covering constructions of regular maps which provide a major motivation for this research. Let M_o be a given regular map endowed with a proper corner voltage assignment β in a group G. It is an immediate consequence of Theorem 9 of [2] that if β is an *Aut*(M_o)-compatible assignment then the lift M_o^β is a regular map that branch covers M_o. Conversely, let $M = M(\rho, \lambda)$ be a regular map with dart set D such that M branch covers our regular map M_o. By a folklore result (see [15] for a proof), there exists a *normal* subgroup K of the group *Aut* (M) such that M_o is isomorphic to the *quotient map* $M/K = M(\rho_K, \lambda_K)$ whose darts are orbits $K(x) = Kx$, $x \in D$, and whose rotation and dart-reversing involution are given by $(Kx)\rho_K = K(x\rho)$ and $(Kx)\lambda_K = K(x\lambda)$.

In a situation as above, we will say that $M \to M_o$ is a *cyclic* covering if K is a cyclic group. We will discuss cyclic coverings in great detail later in Sections 4 and 5.

Continuing our exposition, let $\pi : M \to M/K \simeq M_o$ be the covering projection given by $\pi(x) = Kx$ for any $x \in D$. Now, the group $\{AK;\ A \in Aut\,(M)\} = Aut\,(M)/K \simeq$

Aut $(M/K) \simeq Aut(M_o)$ acts on the darts Kx of the quotient map M/K by $(AK)(Kx) = KA(x)$. Equivalently, denoting AK by A_o and using the projection π, the previous equality translates into $A_o\pi(x) = \pi A(x)$, which just means that *the automorphism* $A_o = AK$ *of the map* $M_o \simeq M/K$ *lifts to the automorphism* A *of* M. By another folklore result in topological graph theory (a proof can easily be obtained from the one of Theorem 2.2.2 of [7]), the map M can be reconstructed from the quotient map M/K by the voltage construction; that is, there exists a corner voltage assignment β on the map $M_o \simeq M/K$ in the group K such that the lift M_o^β is a map isomorphic with M. To sum up, each automorphism of the regular map $M_o \simeq M/K$ lifts to an automorphism of the regular map $M \simeq (M_o)^\beta$; by Theorem 2 this is if and only if our voltage assignment β is $Aut(M_o)$-compatible. This analysis (which is a condensed form of the one that will appear in [15]) provides a short proof of the following result.

Theorem 3 *Let* M *be a regular map. If* β *is a proper* $Aut\,(M)$*-compatible corner voltage assignment on* M, *then* M^β *is a regular map that covers* M. *Conversely, if* M' *is a regular map that is a (possibly branched) covering of the regular map* M, *then* M' *is isomorphic with the lift* M^β *for some* $Aut(M)$*-compatible (proper) corner voltage assignment* β *on* M.

Instead of corner voltage assignments it is often more convenient to work with voltage assignments on darts of medial maps. We therefore include the corresponding restatement of Theorem 3. Note that each automorphism of a map M naturally induces an automorphism A_{m} of the medial map M_{m}; let $Aut_{\mathrm{m}}(M_{\mathrm{m}})$ denote the image of the group $Aut\,(M)$ in the group $Aut\,(M_{\mathrm{m}})$ under the injection $A \mapsto A_{\mathrm{m}}$.

Theorem 4 *Let* M *be a regular map. If* α *is a proper* $Aut_{\mathrm{m}}(M_{\mathrm{m}})$*-compatible voltage assignment on the underlying graph of the medial map* M_{m}, *then there exists a regular map* M' *such that* M' *is a (possibly branched) covering space of* M *and* $(M')_{\mathrm{m}} = (M_{\mathrm{m}})^\alpha$. *Conversely, if* M' *is a regular map that is a (possibly branched) covering of the regular map* M, *then its medial* $(M')_{\mathrm{m}}$ *is isomorphic with the lift* $(M_{\mathrm{m}})^\alpha$ *for some (proper)* $Aut_{\mathrm{m}}(M_{\mathrm{m}})$*-compatible voltage assignment* α *on the underlying graph of* M_{m}.

A still higher degree of map symmetry which we will discuss in Section 6 occurs when a regular map M is *self-dual* in the orientation preserving sense, that is, when there exists an isomorphism from M onto its dual map M^* that preserves the orientation of the common supporting surface for M and M^*. It is easy to see (cf. [3]) that M is a self-dual regular map if and only if its medial map M_{m} is regular. We thus have the following consequence of Theorems 3 and 4 where the group $Aut_{\mathrm{m}}(M_{\mathrm{m}})$ is replaced with $Aut(M_{\mathrm{m}})$.

Corollary 1 *Let* M *be a regular self-dual map branch-covered by a map* M'. *Then* M' *is a self-dual regular map if and only if its medial map is isomorphic to a lift* $(M_{\mathrm{m}})^\alpha$ *for some proper* $Aut\,(M_{\mathrm{m}})$*-compatible voltage assignment* α *on the underlying graph of* M_{m}.

As it is apparent from the preceding two theorems and the corollary, in either the language of corner voltage assignments or the language of voltage assignments on darts of the medial map it is important to be able to recognize and construct compatible voltage assignments. This problem will be studied in detail in Section 4, preceded by a digression into automorphisms and matrices in Section 3.

4. Automorphisms and matrices

Let Γ be a connected graph in which a spanning tree T has been chosen. Let $E = \{e_1, e_2, \ldots, e_r\}$ be the set of all cotree edges, that is, edges of Γ that are not in T, and let $\{e_{r+1}, e_{r+2}, \ldots, e_{r+t}\}$ be the set of all edges of T. For any edge e_i, $1 \le i \le r+t$, we fix one of the two darts corresponding to e_i; let this fixed dart be denoted by x_i. The set $X = \{x_i;\ 1 \le i \le r+t\}$ of the chosen darts will be called an *orientation* of the graph Γ.

With each automorphism $A \in Aut(\Gamma)$ we associate an $r \times (r+t)$ matrix $C_T(A)$ with entries c_{ij} defined as follows. For $1 \le i \le r$ let C_i be the unique directed cycle of the graph $T + e_i$ containing the dart x_i; that is, all the remaining darts of C_i come from edges in T. Consider the image AC_i of our directed cycle. Each dart of AC_i is either of the form x_j or of the form x_j^{-1} for some j, $1 \le j \le r+t$. Let AC_i contain exactly $s = s_i$ darts and let their complete list be $x_{j_1}^{\varepsilon_1}, x_{j_2}^{\varepsilon_2}, \ldots, x_{j_s}^{\varepsilon_s}$, where $\varepsilon_m \in \{+1, -1\}$ are suitable exponents. Then we let $c_{i j_m} = \varepsilon_m$ for $1 \le m \le s$, and $c_{ij} = 0$ for the remaining indices j. The $r \times (r+t)$ matrix $C_T(A) = (c_{ij})$ defined in this way will be called the *cycle basis matrix of* Γ *corresponding to* A *and* T. The $r \times r$ matrix $L_T(A)$ formed by the first r columns of $C_T(A)$ will be called the T*-reduced matrix of* A; it is one of the central concepts of our paper.

Before proceeding further we point out that the matrices $C_T(A)$ and $L_T(A)$ depend on four parameters: The automorphism A, the spanning tree T, the orientation of Γ and the enumeration of darts. Out of these four parameters only the first two appear explicitly in the notation; the remaining two will tacitly be assumed to be as introduced above, except when stated otherwise.

The goal of this section is to investigate basic properties of T-reduced matrices of graph automorphisms. Consider first the special case when $A = id$, the identity automorphism of Γ. Instead of $C_T(id)$ we will just write C_T; the reader will quickly recognize that (up to permutation of columns) the $r \times (r+t)$ matrix C_T is, in the terminology of [5], the *basis matrix of the cycle space of* Γ *corresponding to* T. It is well known that C_T is a unimodular matrix, that is, determinants of all its $r \times r$ submatrices are equal either to 0 or to ± 1. Moreover, an $r \times r$ submatrix of C corresponding to columns $j_1, j_2, \ldots, j_r$ has determinant equal to ± 1 if and only if the edges $e_{j_1}, e_{j_2}, \ldots, e_{j_r}$ are *cotree edges* for *some* spanning tree of Γ; see e.g. [5].

Any given automorphism A of our graph Γ can be represented by means of a $(r+t) \times (r+t)$ matrix $D(A)$ with entries d_{ij} where $d_{ij} = 1$ if $Ax_i = x_j$, $d_{ij} = -1$ if $Ax_i = x_j^{-1}$, and $d_{ij} = 0$ otherwise. It is obvious that $D(A)$ is a "dart analogue" of the usual representation of an automorphism in form of a permutation matrix that reflects the action of the automorphism on the *vertex set*. Again, the important information for us is contained in the $(r+t) \times r$ matrix $D_T(A)$ consisting of the first r columns of $D(A)$. Observe that the matrix $D_T(A)$ contains exactly one non-singular $r \times r$ submatrix; its rows correspond to the edges in the set $A^{-1}E$, that is, to the cotree edges of the spanning tree $A^{-1}T$ of the graph Γ.

The T-reduced matrix $L_T(A)$ is related with the matrix $D_T(A)$ and the cycle space basis matrix C_T by means of the equality

$$L_T(A) = C_T \cdot D_T(A) \tag{2}$$

whose easy verification is left to the reader. We will use this identity to prove the following fact about T-reduced matrices which may not be obvious at a first glance.

Lemma 1 *Let* Γ *be a connected graph, let* T *be a spanning tree of* Γ, *and let* A *be an automorphism of* Γ. *Then, the determinant of the* T*-reduced matrix* $L_T(A)$ *is equal to* ± 1.

Proof: Let $L = L_T(A)$, $C = C_T$ and $D = D_T(A)$ be the matrices introduced above. Using the well known Cauchy-Binet formula for determinants we obtain from (2) that $\det(L) = \det(CD) = \sum_\sigma \det(C_\sigma)\det(D_\sigma)$ where the summation ranges over all r-subsets σ of the set $\{1, 2, \ldots, r+t\}$, and C_σ and D_σ are the $r \times r$ submatrices of C and D induced by the columns and rows whose indices are in σ, respectively. As noted earlier, the matrix D contains only one non-singular submatrix D' determined by the r-subset σ' that corresponds to the edge set $A^{-1}E$. The matrix D' is essentially (up to signs) a permutation matrix, and therefore we have $\det(D') = \pm 1$. Further, $A^{-1}E$ is the set of cotree edges with respect to the spanning tree $A^{-1}T$, and so the $r \times r$ submatrix C' of C determined by the column indices in σ' has determinant equal to ± 1. It follows that $\det(L) = \det(CD) = \sum_\sigma \det(C_\sigma)\det(D_\sigma) = \det(C')\det(D') = \pm 1$. □

Our next aim is to give an alternative description of the matrices $C_T(A)$ and $L_T(A)$ in terms of a special kind of integral flows on graphs, called circulations; this will prove useful especially in the proofs that follow. We start with recalling a few basic facts; for a more substantial discussion we refer the reader to [5].

A *circulation* on our graph Γ with the chosen orientation $X = \{x_i;\ 1 \le i \le r+t\}$ is any integer valued function ϕ on the darts $x_i \in X$ for which the "flow conservation property" at each vertex is satisfied, that is, for each vertex v of Γ the sum of the values of ϕ on all darts of X emanating from v is equal to the sum of the ϕ-values on all darts terminating at v. For each i such that $1 \le i \le r$ there exists a *unique* circulation ϕ_i such that $\phi_i(x_i) = 1$ and ϕ_i is equal to zero for all darts outside $T + x_i$; we call ϕ_i the *elementary circulation* associated with x_i. The circulations ϕ_i form a basis of the space of all circulations in the sense that any circulation ϕ on Γ can be uniquely expressed in the form $\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_r\phi_r$ with integer coefficients c_i. Let ϕ_T denote the $\bar{T}$*-trace* of ϕ, that is, the restriction of the mapping ϕ onto the set of cotree darts x_i, $1 \le i \le r$. Viewing both ϕ as well as ϕ_T as row vectors of length $r+t$ and r, respectively, we have $\phi_T = (c_1, c_2, \ldots, c_r)$, and $\phi = \phi_T \cdot C_T$ where C_T is the basis matrix of the cycle space of Γ corresponding to T. It follows that for $1 \le i \le r$, the i-th row of the matrix C_T coincides with the vector ϕ_i.

Let ϕ be a circulation on our graph Γ endowed with the orientation $X = \{x_i;\ 1 \le i \le r+t\}$ and let A be an automorphism of Γ. Intuitively, A maps the circulation ϕ onto a new circulation which we denote here by ϕA. However, it is not possible to say that the value of ϕA on the dart Ax should simply be equal to $\phi(x)$ because our graph is *undirected* and hence the dart Ax may not be in X. Therefore we define the mapping ϕA on the darts $y \in X$

by letting $(\phi A)(y) = \phi(A^{-1}y)$ if $A^{-1}y \in X$, and $(\phi A)(y) = -\phi(A^{-1}y^{-1})$ if $A^{-1}y \notin X$. It is easy to check that ϕA is indeed a circulation on the graph Γ. We will often treat the circulation ϕA as a row vector $((\phi A)(x_1), (\phi A)(x_2), \ldots, (\phi A)(x_{r+t}))$ of length $r + t$. We also let the $\bar{T}$-trace of ϕA be the r-dimensional vector $(\phi A)_T$ formed by the first r entries of ϕA.

Let us now return to the matrices $C_T(A)$ and $L_T(A)$. First, our definitions imply that for $1 \leq i \leq r$ the i-th row of the matrix $C_T(A)$ coincides with the vector $\phi_i A$ where ϕ_i is the elementary circulation associated with the cotree dart x_i; in other terms, $\phi_i A = (\phi_i)_T \cdot C_T(A)$. Consequently, rows of the T-reduced matrix $L_T(A)$ are simply the $\bar{T}$-traces $(\phi_i A)_T$ of A-images of the elementary circulations ϕ_i, $1 \leq i \leq r$. It also follows that if ϕ is an arbitrary circulation on Γ, then

$$\phi A = \phi_T \cdot C_T(A) \quad \text{and} \quad (\phi A)_T = \phi_T \cdot L_T(A). \tag{3}$$

Indeed, as we know, ϕ has a unique decomposition of the form $\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_r\phi_r$ where ϕ_i are elementary circulations, and therefore $\phi A = c_1\phi_1 A + c_2\phi_2 A + \cdots + c_r\phi_r A = (c_1(\phi_1)_T + c_2(\phi_2)_T + \cdots + c_r(\phi_r)_T) \cdot C_T(A) = \phi_T \cdot C_T(A)$; the equality $(\phi A)_T = \phi_T \cdot L_T(A)$ is immediate.

With help of the above facts it is now easy to extract more properties of reduced matrices of graph automorphisms. Recall that the multiplicative group of all $r \times r$ integer matrices with determinant ± 1 is known as the *unitary group* and is usually denoted by $U(r, \mathcal{Z})$. In Lemma 1 we have seen that reduced matrices of graph automorphisms are elements of $U(r, \mathcal{Z})$; in fact, the following stronger result holds.

Proposition 1 *Let Γ be a connected graph, let T be a spanning tree of Γ, and let Γ have r cotree edges. Then the mapping $A \mapsto L_T(A)$ defines a homomorphism from the group Aut (Γ) into the unitary group $U(r, \mathcal{Z})$.*

Proof: We keep the previously introduced notation regarding the graph Γ, that is, its (fixed) orientation is $X = \{x_i;\ 1 \leq i \leq r + t\}$ where the last t darts belong to the spanning tree T. Let A and B be two automorphisms of Γ and let $L_T(A)$ and $L_T(B)$ be their T-reduced matrices. Since $L_T(id)$ is the identity matrix, it remains to be shown that for the T-reduced matrix $L_T(AB)$ of the composition of A and B we have $L_T(AB) = L_T(A)L_T(B)$.

Let ϕ be an arbitrary circulation on Γ and let ϕA be the image of ϕ under the automorphism A, as introduced before. Using the obvious fact that $\phi(AB) = (\phi A)B$ and the second part of (3) we obtain $\phi_T \cdot (L_T(A)L_T(B)) = (\phi_T \cdot L_T(A))L_T(B) = (\phi A)_T \cdot L_T(B) = ((\phi A)B)_T = (\phi(AB))_T = \phi_T \cdot L_T(AB)$. As the equation $\phi_T \cdot (L_T(A)L_T(B)) = \phi_T \cdot L_T(AB)$ holds for *any* integer circulation ϕ, we conclude that $L_T(AB) = L_T(A)L_T(B)$. □

Of course, it is a direct consequence of the homomorphism property $L_T(AB) = L_T(A)$ $L_T(B)$ that the determinants of the T-reduced matrices of graph automorphisms are equal to ± 1, because $\det(L_T(A))\det(L_T(A^{-1})) = \det(L_T(AA^{-1})) = 1$ and, obviously, the determinants of both $L_T(A)$ and $L_T(A^{-1})$ are integers. Nevertheless we preferred to have a more structural proof of this fact, as presented in Lemma 1.

We conclude with pointing out that the determinant of the T-reduced matrix $L_T(A)$ of an automorphism A of a connected graph Γ depends just on the automorphism A and *not* on the choice of the spanning tree T or on the way the cotree edges are enumerated and oriented. Indeed, if the matrix $L_T(A)$ is based on a given enumeration $x_1, x_2, \ldots, x_r$ of cotree darts, then a different enumeration corresponds to a conjugate of $L_T(A)$ by a suitable permutation matrix, preserving thus the determinants. If a cotree dart x_i is re-directed, that is, replaced by its reverse x_i^{-1}, the corresponding matrix is the conjugate of $L_T(A)$ by the $r \times r$ matrix whose all off-diagonal entries are zeros, with i-th diagonal entry equal to -1 and the remaining diagonal entries equal to 1. Independence of the determinant of $L_T(A)$ on the choice of the spanning tree is more challenging and we state it as a separate result.

Proposition 2 *Let T and T' be two spanning trees of a connected graph Γ and let A be an automorphism of Γ. Then, the reduced matrices $L_T(A)$ and $L_{T'}(A)$ have the same determinant.*

Proof: Let $L_T(A)$ and $L_{T'}(A)$ be reduced matrices of A based on enumerations $x_1, x_2, \ldots, x_r$ and $x'_1, x'_2, \ldots, x'_r$ of cotree edges relative to the spanning trees T and T', respectively. We introduce an auxiliary $r \times r$ matrix $K = (k_{ij})$ as follows. As before, for $1 \le i \le r$ let ϕ_i be the elementary circulation associated with the dart x_i with respect to the spanning tree T. The entry k_{ij} of the matrix K is defined as the j-th entry of the $\bar{T}'$-trace of the circulation vector $\phi_i A^{-1}$. Equivalently, the vector $(\phi_i A^{-1})_{T'}$ simply constitutes the i-th row of the matrix K. An easy reflection shows that for $1 \le i \le r$ the vectors $(\phi_i A^{-1})_{T'}$ are linearly independent over the rationals, and hence the matrix K is nonsingular. Note also that for any circulation ϕ on Γ we have $\phi_T \cdot K = (\phi A^{-1})_{T'}$; this follows from a calculation similar to the one before the statement of Proposition 1.

Now, let ϕ be an arbitrary integer circulation on the graph Γ and let ϕ_T be the $\bar{T}$-trace of ϕ, viewed as a row vector of length r. Then $\phi_T L_T(A)$ is the $\bar{T}$-trace of the circulation ϕA, and $(\phi_T L_T(A))K$ is the $\bar{T}'$-trace of the circulation $(\phi A)A^{-1}$, that is, the $\bar{T}'$-trace of ϕ. On the other hand, $\phi_T K$ is the $\bar{T}'$-trace of the circulation ϕA^{-1}, and $(\phi_T K)L_{T'}(A)$ is the $\bar{T}'$-trace of the circulation $(\phi A^{-1})A = \phi$. It follows that for any integer circulation ϕ on Γ we have $\phi_T(L_T(A)K) = \phi_T(KL_{T'}(A))$, that is, $L_T(A)K = KL_{T'}(A)$. By nonsingularity of K we have $L_{T'}(A) = K^{-1}L_T(A)K$, and hence $\det(L_{T'}(A)) = \det(L_T(A))$. □

Combining Propositions 1 and 2 we have the following interesting result.

Corollary 2 *Let Γ be a connected graph and let T be an arbitrary spanning tree of Γ. Then the mapping $A \mapsto \det(L_T(A))$ depends only on the automorphism A and is a homomorphism from the group $Aut(\Gamma)$ into the group $\mathcal{Z}_2 = \{\pm 1\}$.*

If the above homomorphism is *onto* then $Aut(\Gamma)$ contains a normal subgroup of index two. The converse is not true, as can be seen by examples of unicyclic graphs (connected graphs containing exactly one cycle) Γ having the additional property that $Aut(\Gamma) \simeq \mathcal{Z}_n$ for even n.

5. Compatibility and eigenvectors

The importance of recognizing and constructing compatible voltage assignments was outlined in Section 2. As we shall see, there are interesting connections between compatibility of voltage assignments with respect to graph automorphisms, orthogonal complements of $\mathcal{Z}$-modules, and eigenvectors of reduced basis matrices of automorphisms. In this section we explain the connections and present related results.

We start with some folklore facts in topological graph theory. Let Γ be a connected graph and let T be a spanning tree of Γ. A voltage assignment α on Γ in a group G will be called *T-reduced* if $\alpha(x) = 1_G$ for each dart x that belongs to T. It is well known that to any voltage assignment β on Γ in the group G there exists a T-reduced voltage assignment α on Γ in the same group G such that the lifts Γ^β and Γ^α are *equivalent* in the following sense: There exists a graph isomorphism $f : \Gamma^\beta \to \Gamma^\alpha$ such that $p_\alpha f = p_\beta$ where $p_\beta : \Gamma^\beta \to \Gamma$ and $p_\alpha : \Gamma^\alpha \to \Gamma$ are the corresponding projections. From now on we therefore concentrate on T-reduced voltage assignments only, as this is without loss of generality.

At this point we return to reduced matrices of graph automorphisms. As in the previous section, let Γ be a connected graph with a spanning tree T and let $E = \{e_i;\ 1 \le i \le r\}$ be the collection of all edges not in T. We again choose for each e_i a dart x_i by picking a fixed orientation of the edge e_i. Let α be a T-reduced voltage assignment on a graph Γ in an *Abelian* group G. For brevity, let $\alpha_i = \alpha(x_i)$, $1 \le i \le r$, and let $\bar{\alpha}$ denote the $r \times 1$ column vector $(\alpha_1, \alpha_2, \ldots, \alpha_r)^\top$. We will refer to $\bar{\alpha}$ as the *T-reduced voltage vector*. Note that α is a proper voltage assignment if and only if the entries of $\bar{\alpha}$ generate the group G. If ξ is an automorphism of the voltage group G, then by $\xi(\bar{\alpha})$ we denote the $r \times 1$ column vector $(\xi(\alpha_1), \xi(\alpha_2), \ldots, \xi(\alpha_r))^\top$. Further, let A be an automorphism of Γ and let $L = L_T(A)$ be the corresponding T-reduced basis matrix of A as introduced in the preceding section. Throughout, both L and $\bar{\alpha}$ will be associated with the *same* enumeration of the edges in E and the *same* choice of their direction (i.e., the cotree *darts*).

We are almost ready to state the main result that relates A-compatibility with orthogonality in the $\mathcal{Z}$-module G^r; it just remains to introduce a few more terms. Let G be an Abelian group and let $G^r = G \times G \times \cdots \times G$ (r times) be the direct product of r copies of G, considered as an r-dimensional $\mathcal{Z}$-module. We will assume that the elements of G^r are *column vectors* $\bar{a} = (a_1, a_2, \ldots, a_r)^\top$ of length r, with entries in G. If K is a submodule of G^r, by the symbol $K^\perp$ we denote the set of all *row vectors* $\bar{y}$ of length r with *integer* entries such that $\bar{y}\bar{a} = y_1a_1 + y_2a_2 + \cdots + y_ra_r = 0$ for each $\bar{a} \in K$ (here and in what follows we use the symbol 0 for the unit element of G). It is obvious that $K^\perp$ is a *submodule* of the $\mathcal{Z}$-module $\mathcal{Z}^r$; in what follows we will call $K^\perp$ the *orthogonal complement* of K. If K is a submodule of G^r generated by a single column vector $\bar{a}$, we simply write $\bar{a}^\perp$ instead of $K^\perp$.

Theorem 5 *Let Γ be a connected graph, let T be a spanning tree of Γ and let α be a proper T-reduced voltage assignment on Γ in an Abelian group G. Let A be an automorphism of the graph Γ, let $L = L_T(A)$ be the associated T-reduced matrix of A, and let $\bar{\alpha}$ be the corresponding T-reduced voltage vector. Then, the following three statements are equivalent*:

(a) *The voltage assignment α is A-compatible.*
(b) *There exists an automorphism ξ of G such that $L\bar{\alpha} = \xi(\bar{\alpha})$.*
(c) *The orthogonal complements $\bar{\alpha}^{\perp}$ and $(L\bar{\alpha})^{\perp}$ are identical.*

Proof: (a) *implies* (b): Let L_i denote the i-th row of the matrix L; here $1 \leq i \leq r$ where r is the number of cotree darts in our graph. Recalling the definition of $L = L_T(A)$ and using the fact that the voltage group is Abelian, it is easy to check that the dot product $L_i\bar{\alpha}$ is equal to $\alpha(AC_i)$ where AC_i is the A-image of the directed basis cycle C_i containing the cotree dart x_i. Let u be a fixed vertex of Γ and let W_i be the unique shortest u-based closed walk in Γ whose only dart not in T is x_i; clearly $\alpha(W_i) = \alpha(C_i) = \alpha_i$. Now, let ξ be the automorphism of the group G from Theorem 1 given by $\xi(\alpha W) = \alpha(AW)$ for each closed u-based walk in Γ; it follows that $\xi(\alpha_i) = \alpha(AW_i)$. But invoking commutativity of G again we have $\alpha(AW_i) = \alpha(AC_i)$. Thus, $L_i\bar{\alpha} = \alpha(AC_i) = \xi(\alpha_i)$, and hence $L\bar{\alpha} = \xi(\bar{\alpha})$.

(b) *implies* (c): Let $\bar{y}$ be an r-dimensional integer row vector. Then, $\bar{y}\bar{\alpha} = 0$ if and only if $\xi(\bar{y}\bar{\alpha}) = 0$, which holds if and only if $\bar{y}\xi(\bar{\alpha}) = 0$; using (b) this is finally equivalent with $\bar{y}(L\bar{\alpha}) = 0$. This shows that the orthogonal complements $\bar{\alpha}^{\perp}$ and $(L\bar{\alpha})^{\perp}$ coincide.

(c) *implies* (a): Let $W = \cdots x_{i_1}^{\varepsilon_1} \cdots x_{i_2}^{\varepsilon_2} \cdots \ \cdots x_{i_m}^{\varepsilon_m} \cdots$ be a closed walk in the graph Γ based at a fixed vertex u, where x_{i_l} are the (not necessarily distinct) cotree darts, $\varepsilon_l \in \{+1, -1\}$, and the dotted spaces correspond to darts in the spanning tree T. Using our notation $\alpha(x_i) = \alpha_i$ we have $\alpha(W) = \varepsilon_1\alpha_{i_1} + \varepsilon_2\alpha_{i_2} + \cdots + \varepsilon_m\alpha_{i_m}$. As G is Abelian, we may write $\alpha(W) = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_r\alpha_r = \bar{y}\bar{\alpha}$, where the i-th entry y_i of the row vector $\bar{y} = (y_1, y_2, \ldots, y_r)$ is the integer determined by the number of times the walk W traverses the dart x_i minus the number of times W traverses the reverse dart x_i^{-1}.

The first of the two equations above can also be derived in a different way which will prove useful in the next paragraph. As before, for $1 \leq i \leq r$ let W_i be the shortest u-based walk in Γ whose unique dart not in T is x_i. Viewing now Γ as a 1-dimensional cell complex, the walk W is easily seen to be homotopic (with base point u) to the walk $W' = W_{i_1}^{\varepsilon_1} W_{i_2}^{\varepsilon_2} \cdots W_{i_m}^{\varepsilon_m}$ for $\varepsilon_1, \ldots, \varepsilon_m \in \{\pm 1\}$. As homotopic walks in a 1-complex have the same voltages, we have $\alpha(W) = \alpha(W') = \alpha(W_{i_1}^{\varepsilon_1}) + \alpha(W_{i_2}^{\varepsilon_2}) + \cdots + \alpha(W_{i_m}^{\varepsilon_m}) = \varepsilon_1\alpha_{i_1} + \varepsilon_2\alpha_{i_2} + \cdots + \varepsilon_m\alpha_{i_m}$, as seen above.

Applying the automorphism A we see that the walk AW is homotopic to the walk AW' relative to the base point Au, and it follows again that $\alpha(AW) = \alpha(AW') = \varepsilon_1\alpha(AW_{i_1}) + \varepsilon_2\alpha(AW_{i_2}) + \cdots + \varepsilon_m\alpha(AW_{i_m})$. By the definition of the T-reduced basis matrix $L = L_T(A)$, for each individual u_i-based walk W_i we know that $\alpha(AW_i)$ is equal to the i-th entry $(L\bar{\alpha})_i$ of the column vector $L\bar{\alpha}$. Therefore, since the voltage group G is Abelian, we may rewrite the above sum in the form $\alpha(AW) = y_1(L\bar{\alpha})_1 + y_2(L\bar{\alpha})_2 + \cdots + y_r(L\bar{\alpha})_r = \bar{y}(L\bar{\alpha})$, where $\bar{y}$ is the same row vector as before.

This analysis shows that for the walk W we have $\alpha(W) = 0$ if and only if $\bar{y}\bar{\alpha} = 0$, which is (by (c)) equivalent with $\bar{y}(L\bar{\alpha}) = 0$, and this holds if and only if $\alpha(AW) = 0$. Consequently, the assignment α is A-compatible. □

From the form of the previous result one can expect that compatibility is related also to eigenvectors and eigenvalues. Given an Abelian group G and an $r \times r$ integer matrix L, we say that a nonzero column vector $\bar{a} \in G^r$ is a *G-eigenvector of L* if $L\bar{a} = \lambda\bar{a}$ for some integer λ. Observe that if $G = \mathcal{Z}_n$, the equality $L\bar{a} = \lambda\bar{a}$ is equivalent with $L\bar{a} = (\lambda + kn)\bar{a}$ for any $k \in \mathcal{Z}$. Therefore, in the special case $G = \mathcal{Z}_n$, when writing $L\bar{a} = \lambda\bar{a}$ we will always assume that $\lambda \in \mathcal{Z}_n$; we will call such λ the *eigenvalue* corresponding to the G-eigenvector $\bar{a}$.

The equation $L\bar{a} = \lambda\bar{a}$ may be written in the form $(L - \lambda I)\bar{\alpha} = \bar{0}$ where I is the $r \times r$ identity matrix and $\bar{0}$ is the column of r zeros (i.e., unit elements of G). But even in the special case when $G = \mathcal{Z}_n$, it is *not* always true that λ is then a root in G of the equation $\det(L - \lambda I) = 0$. However, the implication $L\bar{a} = \lambda\bar{a} \Rightarrow \det(L - \lambda I) = 0$ does hold in the important special case when the entries of $\bar{a}$ form a generating set for the group $\mathcal{Z}_n$ [16].

With help of these facts we now prove a stronger version of Theorem 5 in the case when the voltage group is cyclic.

Theorem 6 *Let Γ be a connected graph, let T be a spanning tree of Γ and let α be a proper T-reduced voltage assignment on Γ in a cyclic group $G = \mathcal{Z}_n$. Let A be an automorphism of the graph Γ, let $L = L_T(A)$ be the associated T-reduced matrix of A, and let $\bar{\alpha}$ be the corresponding T-reduced voltage vector. Then, the following four statements are equivalent:*

(a) *The voltage assignment α is A-compatible.*
(b) *There exists an automorphism ξ of G such that $L\bar{\alpha} = \xi(\bar{\alpha})$.*
(c) *The orthogonal complements $\bar{\alpha}^{\perp}$ and $(L\bar{\alpha})^{\perp}$ are identical.*
(d) *The vector $\bar{\alpha}$ is a G-eigenvector of the matrix L corresponding to an eigenvalue λ that is a generator of G.*

Moreover, if either of (a)–(d) *holds then the eigenvalue λ satisfies the equation* $\det(L - \lambda I) = 0$ *in G.*

Proof: The equivalence of (a), (b) and (c) is Theorem 5. In order to prove that, say, (b) implies (d), let G, A, L and α be as above, and let $L\bar{\alpha} = \xi(\bar{\alpha})$ for an automorphism ξ of the group G. It is well known that all automorphisms of a cyclic group $G = \mathcal{Z}_n$ have the form $g \mapsto \lambda g$ where $(\lambda, n) = 1$. Therefore, (b) translates to $L\bar{\alpha} = \lambda\bar{\alpha}$, which is (d). The fact that (d) implies (b) is obvious. The last statement follows from [16]. □

6. Applications

The theory explained in the preceding section can be easily applied and the associated computations are reasonably simple. As an example, we now use Theorem 6 to characterize, up to covering equivalence, *all* (in the orientation preserving sense) self-dual regular maps that cyclically cover the spherical regular map of a tetrahedron. (Cyclic coverings have been defined in Section 2.) To this end, by Theorem 3 it is sufficient to determine all corner voltage assignments in cyclic groups that are compatible with the map automorphism group of the tetrahedron. Rather than working with corner voltage assignments we will pass to ordinary voltage assignments on darts of the corresponding medial map, which in our case is the spherical map of an octahedron. It is easy to see (cf. Corollary 1 in Section 2) that the medial maps of self-dual regular maps that (cyclically) branch-cover the tetrahedron exactly correspond to the regular maps of valence 4 that (cyclically) branch-cover the octahedron. In what follows we therefore concentrate on determining (up to covering equivalence) all

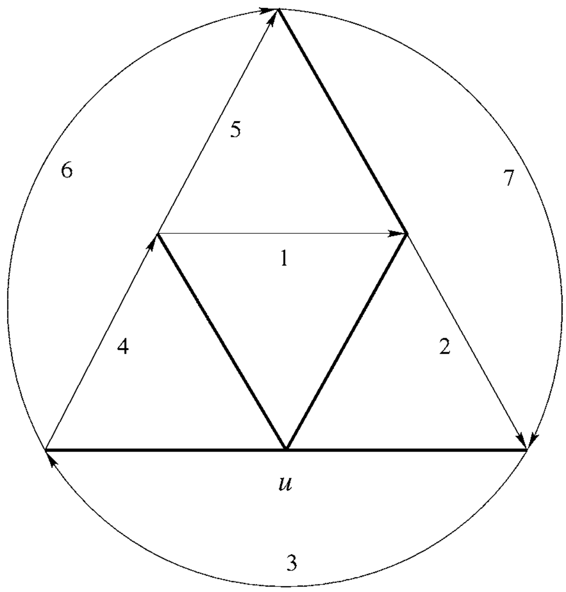

Figure 1. The map of an octahedron.

voltage assignments on darts of an octahedron in cyclic groups that generate a connected lift and are compatible with the (orientation preserving) map automorphism group of an octahedron.

Let M be the spherical map of an octahedron as in figure 1 where thick lines represent a chosen spanning tree T and the cotree darts are numbered i instead of x_i, $1 \leq i \leq 7$.

Let Γ be the underlying graph of M and let A and B be automorphisms of both Γ and M such that A clockwise rotates the map M about the centre of the picture by the angle of $2\pi/3$ whereas B clockwise rotates M about the vertex u (by one face). A short calculation shows that the T-reduced basis matrices $L_A = L_T(A)$ and $L_B = L_T(B)$ of the automorphisms A and B are given by

$$L_A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}; \quad L_B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

Let α be a T-reduced voltage assignment on Γ in a cyclic group $\mathcal{Z}_n$ and let α_i be the voltage on the dart i, $1 \leq i \leq 7$. We know that α generates a connected lift if and only if $\{\alpha_i;\ 1 \leq i \leq 7\}$ is a generating set for $\mathcal{Z}_n$. As the automorphisms A, B generate the group

$Aut(M)$, in order to characterize all proper voltage assignments α in $\mathcal{Z}_n$ that are $Aut(M)$-compatible it is sufficient to determine all T-reduced voltage vectors $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_7)^\top$ that are both A- and B-compatible. Invoking Theorem 6, this all is equivalent with looking for a column vector $\bar{\alpha}$ as above, with entries that generate $\mathcal{Z}_n$, such that $L_A\bar{\alpha} = \lambda\bar{\alpha}$ and $L_B\bar{\alpha} = \mu\bar{\alpha}$ where λ and μ are $\mathcal{Z}_n$-eigenvalues of the matrices L_A and L_B, respectively. This is a system of 14 equations in 9 unknowns λ, μ, and entries of $\bar{\alpha}$; a routine calculation shows that all its solutions are given by $\lambda = 1$ and $\bar{\alpha} = \alpha_1(1, \mu, 1, \mu, 1 + \mu, 2 + 2\mu, 1 + \mu)^\top$, where α_1 generates $\mathcal{Z}_n$ and μ satiefies $\mu^2 = 1$ and $4\alpha_1(1 + \mu) = 0$ in $\mathcal{Z}_n$.

Since $(\alpha_1, n) = 1$, the element α_1 has a multiplicative inverse α_1^{-1} in $\mathcal{Z}_n$. Now, the column vector $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_7)^\top$ is an eigenvector of both L_A and L_B if and only if the vector $\bar{\beta} = \alpha_1^{-1}\bar{\alpha}$ is. From the characterization theorem of equivalence of two coverings [12] one immediately concludes that the coverings induced by the vectors $\bar{\alpha}$ and $\bar{\beta}$ are equivalent. The above analysis and Theorem 6 together with the theory of coverings of regular maps explained in Sections 2 and 3 imply the following result.

Theorem 7 *Let $M' \to M$ be a cyclic branched covering of the spherical regular map M of a tetrahedron by a map M'. Then M' is a self-dual regular map (in the orientation preserving sense) if and only if $M' \simeq M^\alpha$ where α is a corner voltage assignment on M in a cyclic group $\mathcal{Z}_n$ such that the T-reduced voltage vector $\bar{\alpha}$ corresponding to the situation in figure* 1 *has the form $\bar{\alpha} = (1, \mu, 1, \mu, 1 + \mu, 2(1 + \mu), 1 + \mu)^\top$ where $\mu^2 \equiv 1$ and* $4(1 + \mu) \equiv 0 \pmod{n}$.

As a by-product we obtain a characterization of all regular maps of valence 4 that cyclically cover the spherical map of an octahedron.

Corollary 3 *Let M' be a map of valence* 4 *that cyclically covers the octahedron in a sphere. Then M' is regular if and only if $M' \simeq M^\alpha$ where α is a voltage assignment on the darts of the octahedron as in figure* 1 *in a cyclic group $\mathcal{Z}_n$ such that the corresponding T-reduced voltage vector $\bar{\alpha}$ has the form $\bar{\alpha} = (1, \mu, 1, \mu, 1 + \mu, 2(1 + \mu), 1 + \mu)^\top$ where* $\mu^2 \equiv 1$ *and* $4(1 + \mu) \equiv 0 \pmod{n}$.

Theorem 7 and Corollary 3 carry a complete information about the regular (self-dual) maps that cyclically cover the octahedron (tetrahedron). By standard tools in topological graph theory (cf. [7]) one can determine the valence and face length and hence the genus of the maps. Checking the covering equivalence condition of [12] one can easily see that the coverings described in the two results above are pairwise inequivalent. However, it was not our intention to exhibit all such details here (including all solutions of the above congruences). Our point was to demonstrate the power of the theory explained in Sections 3 and 4, in particular, Theorems 5 and 6. We believe that these methods will find further application in constructing highly symmetric graphs and maps.

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