Journal of Algebraic Combinatorics **14** (2001), 199–219

Unimodular Triangulations and Coverings of Configurations Arising from Root Systems

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Received March 15, 2000

Abstract. Existence of a regular unimodular triangulation of the configuration $\Phi^+ \cup \{(0, 0, \ldots, 0)\}$ in $\mathbb{R}^n$, where Φ^+ is the collection of the positive roots of a root system $\Phi \subset \mathbb{R}^n$ and where $(0, 0, \ldots, 0)$ is the origin of $\mathbb{R}^n$, will be shown for $\Phi = \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ and $\mathbf{BC}_n$. Moreover, existence of a unimodular covering of a certain subconfiguration of the configuration $\mathbf{A}_{n+1}^+$ will be studied.

Keywords: initial ideals, unimodular triangulations, unimodular coverings, root systems, positive roots

Introduction

A *configuration* in $\mathbb{R}^n$ is a finite set $\mathcal{A} \subset \mathbb{Z}^n$. Let $\mathrm{conv}(\mathcal{A})$ denote the convex hull of $\mathcal{A}$ in $\mathbb{R}^n$ and write $\sharp(\mathcal{A})$ for the cardinality of $\mathcal{A}$ (as a finite set). A subset $F \subset \mathcal{A}$ is said to be a *simplex belonging to* $\mathcal{A}$ if $\mathrm{conv}(F)$ is a simplex in $\mathbb{R}^n$ of dimension $\sharp(F) - 1$. A *triangulation* of $\mathcal{A}$ is a collection Δ of simplices belonging to $\mathcal{A}$ such that (i) if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$; (ii) $\mathrm{conv}(F) \cap \mathrm{conv}(F') = \mathrm{conv}(F \cap F')$ for all $F, F' \in \Delta$, and (iii) $\mathrm{conv}(\mathcal{A}) = \bigcup_{F \in \Delta} \mathrm{conv}(F)$. Such a triangulation Δ of $\mathcal{A}$ is called *unimodular* if the normalized volume [16, p. 36] of $\mathrm{conv}(F)$ is equal to 1 for each $F \in \Delta$ with $\dim \mathrm{conv}(F) = \dim \mathrm{conv}(\mathcal{A})$. A *unimodular covering* of $\mathcal{A}$ is a collection Δ of simplices belonging to $\mathcal{A}$ such that (i) for each $F \in \Delta$, $\dim \mathrm{conv}(F) = \dim \mathrm{conv}(\mathcal{A})$ and the normalized volume of $\mathrm{conv}(F)$ is equal to 1, and (ii) $\mathrm{conv}(\mathcal{A}) = \bigcup_{F \in \Delta} \mathrm{conv}(F)$.

Let $K[\mathbf{t}, \mathbf{t}^{-1}, s] = K[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}, s]$ denote the Laurent polynomial ring over a field K. We associate each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with the monomial $\mathbf{t}^\alpha s = t_1^{\alpha_1} \cdots t_n^{\alpha_n} s \in K[\mathbf{t}, \mathbf{t}^{-1}, s]$ and write $K[\mathcal{A}]$ for the subalgebra of $K[\mathbf{t}, \mathbf{t}^{-1}, s]$ generated by all monomials $\mathbf{t}^\alpha s$ with $\alpha \in \mathcal{A}$. Let $K[\mathbf{x}] = K[\{x_\alpha;\ \alpha \in \mathcal{A}\}]$ denote the polynomial ring in $\sharp(\mathcal{A})$ variables over K and $I_\mathcal{A} \subset K[\mathbf{x}]$ the kernel of the surjective homomorphism $\pi : K[\mathbf{x}] \to K[\mathcal{A}]$ defined by setting $\pi(x_\alpha) = \mathbf{t}^\alpha s$ for all $\alpha \in \mathcal{A}$. The ideal $I_\mathcal{A}$ is called the *toric ideal* of the configuration $\mathcal{A}$.

Let $<$ be a monomial order [2, p. 53, 17, p. 9] on $K[\mathbf{x}]$ and $in_<(I_\mathcal{A}) \subset K[\mathbf{x}]$ the initial ideal [2, p. 73, 17, p. 10] of $I_\mathcal{A}$ with respect to $<$. Let $\sqrt{in_<(I_\mathcal{A})}$ denote the radical ideal of $in_<(I_\mathcal{A})$ and $\Delta_<(\mathcal{A}) = \{F \subset \mathcal{A};\ \prod_{\alpha \in F} x_\alpha \notin \sqrt{in_<(I_\mathcal{A})}\}$. It then follows that $\Delta_<(\mathcal{A})$ is a triangulation of $\mathcal{A}$, called the *regular triangulation* of $\mathcal{A}$ with respect to the monomial order $<$. It is known [16, Corollary 8.9] that $\Delta_<(\mathcal{A})$ is unimodular if and only if $in_<(I_\mathcal{A})$ is squarefree, i.e., $in_<(I_\mathcal{A}) = \sqrt{in_<(I_\mathcal{A})}$.

Recently, the following six properties on a configulation $\mathcal{A}$ have been investigated by many papers on commutative algebra and combinatorics:

(i) $\mathcal{A}$ is unimodular, i.e., all triangulations of $\mathcal{A}$ are unimodular;
(ii) $\mathcal{A}$ is compressed, i.e., the regular triangulation with respect to any reverse lexicographic monomial order is unimodular;
(iii) $\mathcal{A}$ possesses a regular unimodular triangulation;
(iv) $\mathcal{A}$ possesses a unimodular triangulation;
(v) $\mathcal{A}$ possesses a unimodular covering;
(vi) $\mathcal{A}$ is normal, i.e., the semigroup ring $K[\mathcal{A}]$ is normal.

The hierarchy (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) is easy to prove; while the converse of each of the five implications is false.

Fix $n \geq 2$. Let $\mathbf{e}_i$ denote the i-th unit coordinate vector of $\mathbb{R}^n$. We write $\mathbf{A}_{n-1}^+$, $\mathbf{B}_n^+$, $\mathbf{C}_n^+$, $\mathbf{D}_n^+$ and $\mathbf{BC}_n^+$ for the set of positive roots of root systems $\mathbf{A}_{n-1}, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ and $\mathbf{BC}_n$, respectively [5, pp. 64–65]:

$$
\begin{aligned}
&\mathbf{A}_{n-1}^+ = \{\mathbf{e}_i - \mathbf{e}_j;\, 1 \leq i < j \leq n\};\\
&\mathbf{B}_n^+ = \{\mathbf{e}_i;\, 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j;\, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j;\, 1 \leq i < j \leq n\};\\
&\mathbf{C}_n^+ = \{2\mathbf{e}_i;\, 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j;\, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j;\, 1 \leq i < j \leq n\};\\
&\mathbf{D}_n^+ = \{\mathbf{e}_i + \mathbf{e}_j;\, 1 \leq i < j \leq n\} \cup \{\mathbf{e}_i - \mathbf{e}_j;\, 1 \leq i < j \leq n\};\\
&\mathbf{BC}_n^+ = \mathbf{B}_n^+ \cup \mathbf{C}_n^+.
\end{aligned}
$$

Let, in addition,

$$\tilde{\Phi}^+ = \Phi^+ \cup \{(0, 0, \ldots, 0)\},$$

where $\Phi = \mathbf{A}_{n-1}, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ or $\mathbf{BC}_n$ and where $(0, 0, \ldots, 0)$ is the origin of $\mathbb{R}^n$. An explicit regular unimodular triangulation of the configuration $\tilde{\mathbf{A}}_{n-1}^+$ is constructed in [4, Theorem 6.3]. Moreover, for *any* subconfiguration $\mathcal{A}$ of $\mathbf{A}_{n-1}^+$, the configuration $\tilde{\mathcal{A}} = \mathcal{A} \cup (0, 0, \ldots, 0)$ possesses a regular unimodular triangulation [13, Example 2.4(a)]. Stanley [14, Exercise 6.31(b), p. 234] computed the Ehrhart polynomial of the convex polytope $\mathrm{conv}(\tilde{\mathbf{A}}_{n-1}^+)$. Recently, Fong [3] constructs certain triangulations of the configurations $\tilde{\mathbf{B}}_n^+$ $(= \mathrm{conv}(\tilde{\mathbf{D}}_n^+) \cap \mathbb{Z}^n)$ and $\mathrm{conv}(\tilde{\mathbf{C}}_n^+) \cap \mathbb{Z}^n$ $(= \widetilde{\mathbf{BC}}_n^+)$, and computes the Ehrhart polynomials of $\mathrm{conv}(\tilde{\mathbf{B}}_n^+)$ and $\mathrm{conv}(\tilde{\mathbf{C}}_n^+)$. The triangulations studied in [3] are, however, non-unimodular and it seems to be reasonable to ask if the configurations $\tilde{\mathbf{B}}_n^+$, $\tilde{\mathbf{C}}_n^+$, $\tilde{\mathbf{D}}_n^+$ and $\widetilde{\mathbf{BC}}_n^+$ possess unimodular triangulations.

Our goal is to study the problem (a) which subconfiguration $\tilde{\mathcal{A}} = \mathcal{A} \cup \{(0, 0, \ldots, 0)\}$ of $\widetilde{\mathbf{BC}}_n^+$ possesses a unimodular triangulation; (b) which subconfiguration $\mathcal{A}$ of $\mathbf{BC}_n^+$ possesses a unimodular covering. All subconfigurations of $\{2\mathbf{e}_i;\, 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j;\, 1 \leq i < j \leq n\}$ having unimodular coverings are completely classified [7]. See also [15]. Now, the purpose of the present paper is, as a fundamental step toward this goal, to show the existence of regular unimodular triangulations of the configurations $\tilde{\mathbf{B}}_n^+$, $\tilde{\mathbf{C}}_n^+$, $\tilde{\mathbf{D}}_n^+$ and $\widetilde{\mathbf{BC}}_n^+$ and to study the existence of unimodular coverings of certain subconfigurations of $\mathbf{A}_{n-1}^+$.

In the forthcoming paper [11] we study the existence of unimodular triangulations and coverings of subconfigurations of Φ^+, where $\Phi = \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ or $\mathbf{BC}_n$.

Now, our main result in the present paper is the following

Theorem 0.1

(a) *Fix $n \geq 2$. If a configuration $\mathcal{A} \subset \mathbb{Z}^n$ satisfies the condition*

(0.1.1) $\{\mathbf{e}_i + \mathbf{e}_j;\ 1 \leq i < j \leq n\} \subset \mathcal{A} \subset \mathbf{BC}_n^+$;

(0.1.2) *If $1 \leq i < j < k \leq n$ and if $\mathbf{e}_i - \mathbf{e}_j, \mathbf{e}_j - \mathbf{e}_k \in \mathcal{A}$, then $\mathbf{e}_i - \mathbf{e}_k \in \mathcal{A}$;*

(0.1.3) *Either all $\mathbf{e}_i$ belong to $\mathcal{A}$ or no $\mathbf{e}_i$ belongs to $\mathcal{A}$,*

then the configuration $\tilde{\mathcal{A}} = \mathcal{A} \cup \{(0, 0, \ldots, 0)\} \subset \mathbb{Z}^n$, where $(0, 0, \ldots, 0)$ is the origin of $\mathbb{R}^n$, possesses a regular unimodular triangulation; in other words, the toric ideal $I_{\tilde{\mathcal{A}}}$ possesses a squarefree initial ideal. Thus, in particular, each of the configurations $\tilde{\mathbf{B}}_n^+$, $\tilde{\mathbf{C}}_n^+$, $\tilde{\mathbf{D}}_n^+$ and $\widetilde{\mathbf{BC}}_n^+$ possesses a regular unimodular triangulation.

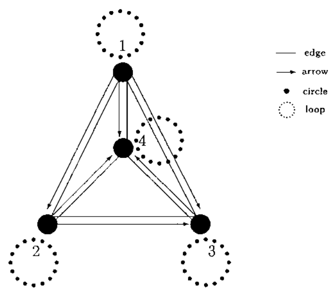
(b) *A subconfiguration $\mathcal{A} \subset \mathbf{A}_{n-1}^+$ with $\dim \mathrm{conv}(\mathcal{A}) = n - 1 \geq 2$ satisfying the condition (0.1.2) possesses a unimodular covering.*

The present paper is organized as follows. First, in Section 1, in order to study triangulations and coverings arising from root systems, certain finite graphs will be introduced. The main purpose of Section 2 is to give a Proof of Theorem 0.1(a). In Section 3, after discussing some questions and conjectures on initial ideals of the configurations $\mathbf{A}_{n-1}^+$ and $\tilde{\mathbf{A}}_{n-1}^+$, we will give a proof of Theorem 0.1(b).

1. Finite graphs and toric ideals

1.1. Finite graphs

Fix $n \geq 2$. Let $[\,n\,] = \{1, 2, \ldots, n\}$ be the vertex set and write Λ_n for the finite graph on $[\,n\,]$ consisting of the edges $\{i, j\}$, $1 \leq i \neq j \leq n$, the arrows (i, j), $1 \leq i < j \leq n$, the circles γ_i, $1 \leq i \leq n$, and the loops δ_i, $1 \leq i \leq n$. Let $E(\Lambda_n)$ denote the set of edges, arrows, circles and loops of Λ_n.



(Finite graph Λ_4)

Let $\rho : E(\Lambda_n) \to \mathbb{Z}^n$ denote the map defined by setting $\rho(\{i, j\}) = \mathbf{e}_i + \mathbf{e}_j$, $\rho((i, j)) = \mathbf{e}_i - \mathbf{e}_j$, $\rho(\gamma_i) = \mathbf{e}_i$ and $\rho(\delta_i) = 2\mathbf{e}_i$.

Let $\mathcal{M}(\Lambda_n)$ denote the $n \times n(n+1)$ $\mathbb{Z}$-matrix

$$\mathcal{M}(\Lambda_n) = (a_{i,\xi})_{i \in [n];\, \xi \in E(\Lambda_n)}$$

with the column vectors

$$(a_{1,\xi}, \ldots, a_{n,\xi})^t = \rho(\xi)^t, \quad \xi \in E(\Lambda_n),$$

where $\rho(\xi)^t$ is the transpose of $\rho(\xi)$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

(Matrix $\mathcal{M}(\Lambda_4)$)

Let, in addition, $\mathcal{M}^*(\Lambda_n)$ denote the $(n+1) \times n(n+1)$ $\mathbb{Z}$-matrix which is obtained by adding the $(n+1)$-th row $(1, 1, \ldots, 1) \in \mathbb{Z}^{n(n+1)}$ to $\mathcal{M}(\Lambda_n)$.

$$\begin{bmatrix} & \mathcal{M}(\Lambda_n) & \\ 1 & & 1 \cdots 1 \end{bmatrix}$$

1.2. Subgraphs

A *subgraph* of Λ_n is a pair $\Sigma = (V(\Sigma), E(\Sigma))$ of $(\emptyset \neq)\, V(\Sigma) \subset [n]$ and $E(\Sigma) \subset E(\Lambda_n)$ such that

(i) if either $\{i, j\} \in E(\Sigma)$ or $(i, j) \in E(\Sigma)$, then $i, j \in V(\Sigma)$;
(ii) if either $\gamma_i \in E(\Sigma)$ or $\delta_i \in E(\Sigma)$, then $i \in V(\Sigma)$;
(iii) each $i \in V(\Sigma)$ is a vertex of some $\xi \in E(\Sigma)$.

Given a subgraph Σ of Λ_n, let $\mathcal{M}(\Sigma)$ denote the submatrix

$$\mathcal{M}(\Sigma) = (a_{i,\xi})_{i \in V(\Sigma);\, \xi \in E(\Sigma)}$$

of $\mathcal{M}(\Lambda_n)$, and let $\mathcal{M}^*(\Sigma)$ denote the submatrix of $\mathcal{M}^*(\Lambda_n)$ which is obtained by adding the $(\sharp(V(\Sigma)) + 1)$-th row $(1, 1, \ldots, 1) \in \mathbb{Z}^{\sharp(E(\Sigma))}$ to $\mathcal{M}(\Sigma)$.

$$\begin{bmatrix} & \mathcal{M}(\Sigma) & \\ 1 & & 1 \cdots 1 \end{bmatrix}$$

We summarize fundamental terminologies on subgraphs of Λ_n.

(a) A *spanning subgraph* of Λ_n is a subgraph Σ of Λ_n with $V(\Sigma) = [\,n\,]$.
(b) A *path* of Λ_n along with vertices $v_0, v_1, \ldots, v_{\ell-1}, v_\ell$, where $\ell \geq 2$ and where $v_p \neq v_q$ if $p < q$ with $(p, q) \neq (0, \ell)$, is a subgraph Σ of Λ_n with $E(\Sigma) = \{\xi_1, \xi_2, \ldots, \xi_\ell\}$, where $\xi_p \neq \xi_q$ if $p < q$ and where ξ_p is either the edge $\{v_{p-1}, v_p\}$ or the arrow $(\min\{v_{p-1}, v_p\}, \max\{v_{p-1}, v_p\})$ for each $1 \leq p \leq \ell$. The *length* of Σ is the integer ℓ and the vertices v_0 and v_ℓ are said to be the *end vertices* of Σ.
(c) A *cycle* of length ℓ of Λ_n is a path of length ℓ of Λ_n two of whose end vertices coincide. Thus, in particular, a cycle of length 2 of Λ_n is a subgraph Σ of Λ_n with $E(\Sigma) = \{(i, j), \{i, j\}\}$, where $1 \leq i < j \leq n$. Each of the edges and the arrows of Λ_n will be regarded as a path of length 1 and, in addition, each of the circles and the loops of Λ_n will be regarded as a cycle of length 1.
(d) For the convenience of the notation, for a cycle Γ of length $\ell \geq 3$ of Λ_n, in the notation $E(\Gamma) = \{\xi_1, \xi_2, \ldots, \xi_\ell\}$, it will be always assumed that, for each $1 \leq k \leq \ell$, ξ_k and ξ_{k+1} possess a common vertex, where $\xi_{\ell+1} = \xi_1$.
(e) A subgraph Σ of Λ_n is called *connected* if, for any vertices v and w of Σ with $v \neq w$, there exists a path of Σ whose end vertices are v and w. A *tree* of Λ_n is a connected subgraph of Λ_n with no cycle. Thus, in particular, a tree possesses neither a circle nor a loop.
(f) If a spanning subgraph Σ of Λ_n with $\sharp(E(\Sigma)) = n$ is connected, then Σ possesses exactly one cycle. If a spanning subgraph Σ of Λ_n with $\sharp(E(\Sigma)) = n + 1$ is connected and possesses a circle γ, then Σ possesses exactly one cycle $\neq \gamma$.

1.3. Determinants

For the purpose of the computation of the normalized volume of the convex hull of a simplex belonging to a configuration, it is required to compute the determinant $\det(\mathcal{M}(\Sigma))$ of a spanning subgraph Σ of Λ_n with $\sharp(E(\Sigma)) = n$ and $\det(\mathcal{M}^*(\Sigma))$ of a spanning subgraph Σ of Λ_n with $\sharp(E(\Sigma)) = n + 1$.

Let Σ be a spanning subgraph of Λ_n with $\sharp(E(\Sigma)) = n$ (resp. $\sharp(E(\Sigma)) = n + 1$). If $\xi \in E(\Sigma)$ is either an edge or an arrow and if one of the vertices of ξ belongs to no $\xi' \in E(\Sigma)$ with $\xi \neq \xi'$, then $|\det(\mathcal{M}(\Sigma))| = |\det(\mathcal{M}(\Sigma \setminus \{\xi\}))|$ (resp. $|\det(\mathcal{M}^*(\Sigma))| = |\det(\mathcal{M}^*(\Sigma \setminus \{\xi\}))|$. This simple observation together with elementary computations of determinants learned in linear algebra yields the following

Proposition 1.1

(a) *If a spanning subgraph* Σ *of* Λ_n *with* $\sharp(E(\Sigma)) = n$ *is connected and if a unique cycle of* Σ *is a circle, then* $|\det(\mathcal{M}(\Sigma))| = 1$.
(b) *If a spanning subgraph* Σ *of* Λ_n *with* $\sharp(E(\Sigma)) = n$ *is connected and if a unique cycle of* Σ *is a loop, or a cycle of length* 2, *or a cycle of odd length* ≥ 3 *with no arrow, then* $|\det(\mathcal{M}(\Sigma))| = 2$.
(c) *Let a spanning subgraph* Σ *of* Λ_n *with* $\sharp(E(\Sigma)) = n + 1$ *and with no arrow be connected and suppose that* Σ *possesses exactly one circle together with either a loop or a cycle of odd length* ≥ 3. *Then* $|\det(\mathcal{M}^*(\Sigma))| = 1$.

(d) *Let $n \geq 2$ be even and Σ the subgraph of Λ_n consisting of two circles γ_1, γ_n and of $n-1$ edges* $\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}$. *Then* $|\det(\mathcal{M}^*(\Sigma))| = 1$.

1.4. Toric ideals

Fix a subgraph G of Λ_n. The map $\rho : E(\Lambda_n) \to \mathbb{Z}^n$ introduced in Section 1.1 enables us to associate G with the configurations

$$\mathcal{A}(G) = \{\rho(\xi); \xi \in E(G)\};$$
$$\tilde{\mathcal{A}}(G) = \mathcal{A}(G) \cup \{(0,0,\ldots,0)\}$$

in $\mathbb{R}^n$. Here $(0,0,\ldots,0)$ is the origin of $\mathbb{R}^n$.

Let K be a field. The subalgebra $K[\mathcal{A}(G)]$ of $K[\mathbf{t},\mathbf{t}^{-1},s] = K[t_1,t_1^{-1},\ldots,t_n,t_n^{-1},s]$ is generated by the monomials t_it_js with $\{i,j\} \in E(G)$, $t_it_j^{-1}s$ with $(i,j) \in E(G)$, t_is with $\gamma_i \in E(G)$ and t_i^2s with $\delta_i \in E(G)$. In addition, $K[\tilde{\mathcal{A}}(G)]$ is generated by the above monomials together with s, i.e., $K[\tilde{\mathcal{A}}(G)] = (K[\mathcal{A}(G)])[s]$.

Let $\mathcal{R}_K[\Lambda_n]$ and $\tilde{\mathcal{R}}_K[\Lambda_n]$ denote the polynomial rings

$$\mathcal{R}_K[\Lambda_n] = K[\{y_i\}_{1\leq i\leq n} \cup \{z_i\}_{1\leq i\leq n} \cup \{e_{i,j}\}_{1\leq i<j\leq n} \cup \{f_{i,j}\}_{1\leq i<j\leq n}];$$
$$\tilde{\mathcal{R}}_K[\Lambda_n] = K[\{x\} \cup \{y_i\}_{1\leq i\leq n} \cup \{z_i\}_{1\leq i\leq n} \cup \{e_{i,j}\}_{1\leq i<j\leq n} \cup \{f_{i,j}\}_{1\leq i<j\leq n}]$$

over K, and set

$$\mathcal{R}_K[G] = K\big[\{y_i\}_{\gamma_i\in E(G)} \cup \{z_i\}_{\delta_i\in E(G)} \cup \{e_{i,j}\}_{\{i,j\}\in E(G)} \cup \{f_{i,j}\}_{(i,j)\in E(G)}\big];$$
$$\tilde{\mathcal{R}}_K[G] = K\big[\{x\} \cup \{y_i\}_{\gamma_i\in E(G)} \cup \{z_i\}_{\delta_i\in E(G)} \cup \{e_{i,j}\}_{\{i,j\}\in E(G)} \cup \{f_{i,j}\}_{(i,j)\in E(G)}\big].$$

Write $\pi : \tilde{\mathcal{R}}_K[\Lambda_n] \to K[\mathbf{t},\mathbf{t}^{-1},s]$ for the homomorphism defined by setting $\pi(x) = s$, $\pi(y_i) = t_is$, $\pi(z_i) = t_i^2s$, $\pi(e_{i,j}) = t_it_js$ and $\pi(f_{i,j}) = t_it_j^{-1}s$. If Ker π denote the kernel of π, then the toric ideals $I_{\mathcal{A}(G)}$ of $\mathcal{A}(G)$ and $I_{\tilde{\mathcal{A}}(G)}$ of $\tilde{\mathcal{A}}(G)$ is

$$I_{\mathcal{A}(G)} = \operatorname{Ker}\pi \cap \mathcal{R}_K[G];$$
$$I_{\tilde{\mathcal{A}}(G)} = \operatorname{Ker}\pi \cap \tilde{\mathcal{R}}_K[G].$$

1.5. Reverse lexicographic monomial orders

We fix the reverse lexicographic monomial order $<_{\Lambda_n}$ on the polynomial ring $\tilde{\mathcal{R}}_K[\Lambda_n]$ in n^2+n+1 variables over a field K induced by the ordering of the variables

$$\begin{aligned} &y_1 < y_2 < \cdots < y_n < x < f_{1,2} < f_{1,3} < \cdots < f_{1,n} < f_{2,3} < \cdots < f_{n-1,n} \\ &\quad < e_{1,2} < e_{1,3} < \cdots < e_{1,n} < e_{2,3} < \cdots < e_{n-1,n} < z_1 < z_2 < \cdots < z_n. \end{aligned}$$

If G is a subgraph of Λ_n, then we write $<_G$ for the reverse lexicographic monomial order on $\tilde{\mathcal{R}}_K[G]$ obtained by $<_{\Lambda_n}$ with the elimination of variables; in other words, for monomials u and v of $\tilde{\mathcal{R}}_K[G]$, $u <_G v$ if and only if $u <_{\Lambda_n} v$.

1.6. Normalized volume

Fix a subgraph G of Λ_n with $\dim \operatorname{conv}(\tilde{\mathcal{A}}(G)) = n$. A subgraph Σ of G with $\sharp(E(\Sigma)) = n+1$ is said to be a *facet* of G if $\rho(E(\Sigma))$ is a simplex belonging to $\tilde{\mathcal{A}}(G)$. A subgraph Σ of G with $\sharp(E(\Sigma)) = n$ is said to be a *quasi-facet* of G if $\rho(E(\Sigma)) \cup \{(0, 0, \ldots, 0)\}$ is a simplex belonging to $\tilde{\mathcal{A}}(G)$.

A quasi-facet is a spanning subgraph of G any of whose connected components possesses exactly one cycle. A facet is a spanning subgraph of G any of whose connected components possesses at least one cycle. In addition, except for exactly one connected component, each connected component of a facet of G possesses exactly one cycle.

Let $\sum_{\xi \in E(G)} \mathbb{Z}(\rho(\xi), 1) + \mathbb{Z}(0, 0, \ldots, 0, 1)$ denote the subgroup of the additive group $\mathbb{Z}^{n+1}$ generated by all the vectors $(\rho(\xi), 1) \in \mathbb{Z}^{n+1}$ with $\xi \in E(G)$ together with the vector $(0, 0, \ldots, 0, 1) \in \mathbb{Z}^{n+1}$, and

$$N_G = \left[\mathbb{Z}^{n+1} : \sum_{\xi \in E(G)} \mathbb{Z}(\rho(\xi), 1) + \mathbb{Z}(0, 0, \ldots, 0, 1) \right]$$

the index of $\sum_{\xi \in E(G)} \mathbb{Z}(\rho(\xi), 1) + \mathbb{Z}(0, 0, \ldots, 0, 1)$ in $\mathbb{Z}^{n+1}$. Note that $N_G < \infty$ since dim $\operatorname{conv}(\tilde{\mathcal{A}}(G)) = n$.

Lemma 1.2 *If Σ is a quasi-facet (resp. facet) of G, then the normalized volume of the convex hull of the simplex $\rho(E(\Sigma)) \cup \{(0, 0, \ldots, 0)\}$ (resp. $\rho(E(\Sigma))$) belonging to $\tilde{\mathcal{A}}(G)$ coincides with $|\det(\mathcal{M}(\Sigma))|/N_G$ (resp. $|\det(\mathcal{M}^*(\Sigma))|/N_G$).*

Proof: Let Σ be a quasi-facet (resp. facet) of G and $F = \rho(E(\Sigma)) \cup \{(0, 0, \ldots, 0)\}$ (resp. $\rho(E(\Sigma))$). The normalized volume of $\operatorname{conv}(F)$ coincides with the index of the subgroup $\sum_{\alpha \in F} \mathbb{Z}(\alpha, 1)$ in the additive group $\sum_{\xi \in E(G)} \mathbb{Z}(\rho(\xi), 1) + \mathbb{Z}(0, 0, \ldots, 0, 1) (\subset \mathbb{Z}^{n+1})$. See, e.g., [16, p. 69]. Since the index of the subgroup $\sum_{\alpha \in F} \mathbb{Z}(\alpha, 1)$ in $\mathbb{Z}^{n+1}$ coincides with $|\det(\mathcal{M}(\Sigma))|$ (resp. $|\det(\mathcal{M}^*(\Sigma))|$), the normalized volume of $\operatorname{conv}(F)$ is equal to $|\det(\mathcal{M}(\Sigma))|/N_G$ (resp. $|\det(\mathcal{M}^*(\Sigma))|/N_G$), as desired. □

When we discuss the configuration $\mathcal{A}(G)$ (instead of $\tilde{\mathcal{A}}(G)$), unless there is no confusion, we also say that a subgraph Σ of G with $\sharp(E(\Sigma)) = \dim \operatorname{conv}(\mathcal{A}(G)) + 1$ is a facet of G if $\rho(E(\Sigma))$ is a simplex belonging to $\mathcal{A}(G)$.

1.7. Root systems

The research object of the present paper is the configurations $\tilde{\mathcal{A}}(G)$ ($\subset \tilde{\mathcal{A}}(\Lambda_n)$) associated with root systems $\mathbf{A}_{n-1}$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ and $\mathbf{BC}_n$. To simplify the notation, we write A_{n-1}, B_n, C_n, D_n and BC_n for the subgraphs of Λ_n with

$$E(A_{n-1}) = \{(i, j); 1 \le i < j \le n\};$$
$$E(B_n) = \{\gamma_i; 1 \le i \le n\} \cup \{\{i, j\}; 1 \le i < j \le n\} \cup \{(i, j); 1 \le i < j \le n\};$$

$E(C_n) = \{\delta_i;\ 1 \le i \le n\} \cup \{\{i, j\};\ 1 \le i < j \le n\} \cup \{(i, j);\ 1 \le i < j \le n\};$
$E(D_n) = \{\{i, j\};\ 1 \le i < j \le n\} \cup \{(i, j);\ 1 \le i < j \le n\};$
$E(BC_n) = E(B_n) \cup E(C_n),$

respectively. Note that $BC_n = \Lambda_n$.

2. Existence of regular unimodular triangulations

The purpose of the present section is to give a Proof of Theorem 0.1(a). More precisely, we will show the following

Theorem 2.1 *Fix $n \ge 2$. Let G be a subgraph of Λ_n satisfying the following conditions*:
(2.1.1) *All edges of Λ_n belong to G*;
(2.1.2) *If $1 \le i < j < k \le n$ and if the arrows (i, j) and (j, k) belong to G, then the arrow (i, k) belongs to G*;
(2.1.3) *Either all circles of Λ_n belong to G or no circle of Λ_n belongs to G.*
Then the regular triangulation $\Delta_{<_G}(\tilde{\mathcal{A}}(G))$ of the configuration $\tilde{\mathcal{A}}(G)$ with respect to the reverse lexicographic monomial order $<_G$ is unimodular.

In what follows, we fix a subgraph G of Λ_n satisfying the conditions (2.1.1), (2.1.2) and (2.1.3) of Theorem 2.1. In order to prove Theorem 2.1, what we must do is to study the problem of what can be said about a facet Σ of G with $\rho(E(\Sigma)) \in \Delta_{<_G}(\tilde{\mathcal{A}}(G))$ (such a facet is called a *facet with respect to $<_G$*) as well as a quasi-facet Σ of G with $\rho(E(\Sigma)) \cup \{(0, 0, \ldots, 0)\} \in \Delta_{<_G}(\tilde{\mathcal{A}}(G))$ (such a quasi-facet is called a *quasi-facet with respect to $<_G$*).

One of the preliminary and fundamental steps is to describe some of the quadratic monomials which belong to $in_{<_G}(I_{\tilde{\mathcal{A}}(G)})$.

Lemma 2.2 *Let Σ be a facet or a quasi-facet of G with respect to $<_G$. Then none of the following subgraphs of G appears in Σ*:
(i) $\{(i, j), (j, k)\}$ *with* $i < j < k$;
(ii) $\{(i, j), \{j, k\}\}$ *with* $i < j, i \ne k, j \ne k$;
(iii) $\{(i, j), \gamma_j\}$ *with* $i < j$;
(iv) $\{(i, j), \delta_j\}$ *with* $i < j$.

Moreover, if all circles of Λ_n belongs to G, then none of the cycles of length 2, i.e., $\{(i, j), \{i, j\}\}$ with $i < j$, appears in Σ.

Proof: Since the binomials $f_{i,j} f_{j,k} - x f_{i,k}$ with $i < j < k$, $f_{i,j} e_{j,k} - x e_{i,k}$ with $i < j$, $i \ne k$, $j \ne k$, $f_{i,j} y_j - x y_i$ with $i < j$, and $f_{i,j} z_j - x e_{i,j}$ with $i < j$ belong to $I_{\tilde{\mathcal{A}}(G)}$, their initial monomials $f_{i,j} f_{j,k}$, $f_{i,j} e_{j,k}$, $f_{i,j} y_j$, and $f_{i,j} z_j$ belong to $in_{<_G}(I_{\tilde{\mathcal{A}}(G)})$. Moreover, if all circles of Λ_n belong to G, then the binomial $e_{i,j} f_{i,j} - y_i^2$ with $i < j$ belongs to $I_{\tilde{\mathcal{A}}(G)}$ and its initial monomial $e_{i,j} f_{i,j}$ belongs to $in_{<_G}(I_{\tilde{\mathcal{A}}(G)})$. □

A simple, however, indispensable result which follows immediately from the above Lemma 2.2 is the following

Corollary 2.3 *Let Σ be a facet (resp. a quasi-facet) of G with respect to $<_G$ and suppose that no cycle of length 2 appears in Σ. Then each row of $\mathcal{M}^*(\Sigma)$ (resp. $\mathcal{M}(\Sigma)$) is either a nonnegative integer vector (i.e., a vector any of whose components is a nonnegative integer) or a nonpositive integer vector. Hence, for the purpose of the computation of $|\det(\mathcal{M}^*(\Sigma))|$ (resp. $|\det(\mathcal{M}(\Sigma))|$), one can assume that each non-zero component of $\mathcal{M}^*(\Sigma)$ (resp. $\mathcal{M}(\Sigma)$) is positive.*

The role of the cycles appearing in facets or quasi-facets of G with respect to $<_G$ will turn out to be important. Recall that the cycles of length 1 are the circles and the loops, and that the cycles of length 2 are the subgraphs Σ with $E(\Sigma) = \{(i, j), \{i, j\}\}$, $1 \leq i < j \leq n$.

Lemma 2.4 *Every cycle of length ≥ 3 appearing in either a facet or a quasi-facet of G with respect to $<_G$ is of odd length and possesses at least one edge.*

Proof: Let Σ be either a facet or a quasi-facet of G with respect to $<_G$ and Γ a cycle of length ℓ (≥ 3) appearing in Σ with $E(\Gamma) = \{\xi_1, \xi_2, \ldots, \xi_\ell\}$. In case ℓ is even, Corollary 2.3 yields the equality $\sum_{k=1}^{\ell/2} \rho(\xi_{2k}) = \sum_{k=1}^{\ell/2} \rho(\xi_{2k-1})$, which contradicts the fact that either $\rho(E(\Sigma))$ or $\rho(E(\Sigma)) \cup \{(0, 0, \ldots, 0)\}$ belongs to $\Delta_{<_G}(\tilde{\mathcal{A}}(G))$. Hence ℓ is odd. If no edge of Σ belongs to Γ, then for some $1 \leq k \leq \ell$ one has either $\xi_k = (u, v)$, $\xi_{k+1} = (v, w)$, or $\xi_k = (v, w)$, $\xi_{k+1} = (u, v)$, where $u, v, w \in [n]$ with $u < v < w$ and where $\xi_{\ell+1} = \xi_1$. However, Lemma 2.2 says that this is impossible. □

Even though Remark 2.5 below will be not necessarily required to complete a proof of Theorem 2.1, we state it here for its usefulness in our forthcoming papers.

Remark 2.5 If all arrows of Λ_n belong to G, then every cycle of length ≥ 3 appearing in either a facet or a quasi-facet of G with respect to $<_G$ is of length 3.

Proof: Let Γ be a cycle with $E(\Gamma) = \{\xi_1, \xi_2, \xi_3, \ldots, \xi_\ell\}$, where $\ell \geq 5$, appearing in either a facet or a quasi-facet of G with respect to $<_G$. Suppose that, say, ξ_2 is *weakest* in $E(\Gamma)$ with respect to $<_G$, i.e.,

$$\pi^{-1}\left(\mathbf{t}^{\rho(\xi_2)}s\right) <_G \pi^{-1}\left(\mathbf{t}^{\rho(\xi)}s\right)$$

for all $\xi_2 \neq \xi \in E(\Gamma)$. First, if ξ_2 is an arrow (i, j) with j being a vertex of ξ_1, then, for some $w, v \in [n]$, $\xi_1 = (w, j)$ and either $\xi_3 = (i, v)$ or $\xi_3 = \{i, v\}$. If $\xi_3 = (i, v)$, then $i < w < j < v$ and $\rho(\xi_1) + \rho(\xi_3) = \rho(\xi_2) + \rho(\xi)$, where $\xi = (w, v)$. If $\xi_3 = \{i, v\}$, then $i < w < j$, $v \notin \{i, w, j\}$ and $\rho(\xi_1) + \rho(\xi_3) = \rho(\xi_2) + \rho(\xi)$, where $\xi = \{w, v\}$. Second, if ξ_2 is an edge $\{i, j\}$ with j being a vertex of ξ_1, then, for some $w, v \in [n]$, $\xi_1 = \{w, j\}$, $\xi_3 = \{i, v\}$ and $\rho(\xi_1) + \rho(\xi_3) = \rho(\xi_2) + \rho(\xi)$, where $\xi = \{w, v\}$. □

Somewhat surprisingly, if a cycle Γ of odd length ≥ 3 appearing in a facet or a quasi-facet Σ of G with respect to $<_G$ possesses at least one arrow, then no edge is contained in $E(\Sigma)\backslash E(\Gamma)$. Namely,

Lemma 2.6 *Let Σ be either a facet or a quasi-facet of G with respect to $<_G$ and Γ a cycle of odd length ≥ 3 appearing in Σ with at least one arrow. Then all edges of Σ belong to Γ.*

Proof: Let $\xi = \{i, j\} \in E(\Sigma)$ with $\xi \notin E(\Gamma) = \{\xi_1, \xi_2, \ldots, \xi_{2\ell-1}\}$. Let, say, $\xi_2 = (v, w)$ be the arrow which is weakest in $E(\Gamma)$ with respect to $<_G$. Let $\xi_1 = (u, v)$ with $u < v$ by Lemma 2.2. Now, the contradiction $\{\xi_1, \xi_3, \ldots, \xi_{2\ell-1}, \xi\} \not\subset E(\Sigma)$ arises, since Corollary 2.3 yields

$$\sum_{k=1}^{\ell-1} \rho(\xi_{2k}) + \rho(\{i, u\}) + \rho(\{j, u\}) = \sum_{k=1}^{\ell} \rho(\xi_{2k-1}) + \rho(\xi).$$

□

We now come to one of the crucial and fundamental facts.

Lemma 2.7 *Let Σ be either a facet or a quasi-facet of G with respect to $<_G$. Let Γ_1 and Γ_2 be cycles appearing in Σ. If each of Γ_1 and Γ_2 is a loop, or a cycle of length 2, or a cycle of odd length ≥ 3, then $V(\Gamma_1) \cap V(\Gamma_2) \neq \emptyset$.*

Proof: First, suppose that both Γ_1 and Γ_2 are cycles of odd length ≥ 3 with $V(\Gamma_1) \cap V(\Gamma_2) = \emptyset$, $E(\Gamma_1) = \{\xi_1, \xi_2, \ldots, \xi_{2\ell-1}\}$ and $E(\Gamma_2) = \{\eta_1, \eta_2, \ldots, \eta_{2m-1}\}$. Then by Lemmas 2.4 and 2.6 it may be assumed that all ξ_k and all η_p are edges of G. Let, say, $\xi_2 = \{i, j\}$ be the weakest edge in $E(\Gamma_1) \cup E(\Gamma_2)$ with respect to $<_G$. Let $\xi_1 = \{w, i\}$ and $\eta_1 = \{u, v\}$. Now, the contradiction $\{\xi_1, \xi_3, \ldots, \xi_{2\ell-1}, \eta_1, \eta_3, \ldots, \eta_{2m-1}\} \not\subset E(\Sigma)$ arises, since

$$\sum_{k=1}^{\ell-1} \rho(\xi_{2k}) + 2\rho(\{w, u\}) + \sum_{p=1}^{m-1} \rho(\eta_{2p}) = \sum_{k=1}^{\ell} \rho(\xi_{2k-1}) + \sum_{p=1}^{m} \rho(\eta_{2p-1}).$$

Second, let Γ_1 be a cycle of odd length ≥ 3 with $E(\Gamma_1) = \{\xi_1, \xi_2, \ldots, \xi_{2\ell-1}\}$, and let Γ_2 be a cycle of length 2 consisting of an arrow (i, j) and an edge $\{i, j\}$ with $i \notin V(\Gamma_1)$ and $j \notin V(\Gamma_1)$. Since the edge $\{i, j\}$ belongs to Γ_2, all ξ_k are edges of G by Lemma 2.6. Let w be a vertex of ξ_1. Then the contradiction $\{(i, j), \{i, j\}, \xi_1, \xi_3, \ldots, \xi_{2m-1}\} \not\subset E(\Sigma)$ arises, since

$$\rho((i, j)) + \rho(\{i, j\}) + \sum_{k=1}^{\ell} \rho(\xi_{2k-1}) = (0, 0, \ldots, 0) + 2\rho(\{i, w\}) + \sum_{k=1}^{\ell-1} \rho(\xi_{2k}).$$

When Γ_1 is a cycle of odd length ≥ 3 with $E(\Gamma_1) = \{\xi_1, \xi_2, \ldots, \xi_{2\ell-1}\}$ and Γ_2 is a loop δ_i with $i \notin V(\Gamma_1)$, assuming that ξ_2 is weakest in $E(\Gamma_1)$ with respect to $<_G$ and choosing a vertex w of ξ_1, the contradiction $\{\xi_1, \xi_3, \ldots, \xi_{2\ell-1}, \delta_i\} \not\subset E(\Sigma)$ arises, since

$$\sum_{k=1}^{\ell-1} \rho(\xi_{2k}) + 2\rho(\{i, w\}) = \sum_{k=1}^{\ell} \rho(\xi_{2k-1}) + \rho(\delta_i).$$

If both Γ_1 and Γ_2 are cycles of length 2 with $V(\Gamma_1) \cap V(\Gamma_2) = \emptyset$, $E(\Gamma_1) = \{\{i, j\}, (i, j)\}$ and $E(\Gamma_2) = \{\{i', j'\}, (i', j')\}$, then a contradiction arises, since

$$\rho(\{i, j\}) + \rho((i, j)) + \rho(\{i', j'\}) + \rho((i', j')) = 2(0, 0, \ldots, 0) + 2\rho(\{i, i'\}).$$

Let Γ_1 be a cycle of length 2 with $E(\Gamma_1) = \{\{i, j\}, (i, j)\}$ and Γ_2 a loop δ_k with $i \neq k$, $j \neq k$. Again, a contradiction arises, since

$$\rho(\{i, j\}) + \rho((i, j)) + \rho(\delta_k) = (0, 0, \ldots, 0) + 2\rho(\{i, k\}).$$

Finally, if Γ_1 is a loop δ_i and Γ_2 is a loop δ_j with $i \neq j$, then a contradiction arises, since

$$\rho(\delta_i) + \rho(\delta_j) = 2\rho(\{i, j\}). \qquad \square$$

Now, what can be said about quasi-facets of G with respect to $<_G$?

Lemma 2.8 *If all circles of Λ_n belong to G and if Σ is a quasi-facet of G with respect to $<_G$, then*

(a) *no edge belongs to* Σ;
(b) *all cycles appearing in* Σ *are circles*;
(c) $|\det(\mathcal{M}(\Sigma))| = 1$.

Proof:

(a) Since the binomial $xe_{i,j} - y_i y_j$ belongs to $I_{\tilde{\mathcal{A}}(G)}$, its initial monomial $xe_{i,j}$ belongs to $in_{<_G}(I_{\tilde{\mathcal{A}}(G)})$. Hence $\{i, j\} \notin E(\Sigma)$ for all quasi-facets Σ of G with respect to $<_G$.
(b) Since no edge belongs to Σ, by Lemma 2.4 each cycle appearing in Σ is either a circle or a loop. If $\delta_i \in E(G)$, then $xz_i \in in_{<_G}(I_{\tilde{\mathcal{A}}(G)})$ since $xz_i - y_i^2 \in I_{\tilde{\mathcal{A}}(G)}$. Hence $\delta_i \notin E(\Sigma)$ for all quasi-facets Σ of G with respect to $<_G$.
(c) It follows from (b) that a unique cycle of each connected component of Σ is a circle of Λ_n. It then follows from Proposition 1.1(a) that $|\det(\mathcal{M}(\Sigma))| = 1$, as required. $\square$

Lemma 2.9 *If no circle of Λ_n belongs to G, then*

(a) *there exists no facet of G with respect to $<_G$*;
(b) $|\det(\mathcal{M}(\Sigma))| = 2$ *for all quasi-facets Σ of G with respect to $<_G$.*

Proof:

(a) Since no y_i belongs to $\tilde{\mathcal{R}}_K[G]$, the variable x is weakest with respect to the reverse lexicographic monomial order $<_G$. It then follows that x never appears in each monomial belonging to the (unique) minimal set of monomial generators of $in_{<_G}(I_{\tilde{\mathcal{A}}(G)})$. In other words, the origin $(0, 0, \ldots, 0) \in \mathbb{R}^n$ belongs to each simplex $F \in \Delta_{<_G}(\tilde{\mathcal{A}}(G))$ with dim conv$(F) = \dim \mathrm{conv}(\tilde{\mathcal{A}}(G))$ $(= n)$.
(b) Let Σ be a quasi-facet of G with respect to $<_G$. Since each connected component of Σ possesses a unique cycle and since G possesses no circle, it follows from Lemmas 2.4 and 2.7 that Σ is connected. Since a unique cycle appearing in Σ is a loop, or a cycle of

length 2 or a cycle of odd length ≥ 3, we know $|\det(\mathcal{M}(\Sigma))| = 2$ by Proposition 1.1(b), as required. □

We turn to the discussion of what can be said about facets of G with respect to $<_G$, where all circles of Λ_n belong to G. Note that, by Lemma 2.2, no cycle of length 2 appears in G.

For a while, suppose that all circles of Λ_n belong to G, and let Σ be a facet of G with respect to $<_G$ whose connected components are $\Sigma_1, \ldots, \Sigma_{h-1}$ and Σ_h, where Σ_h possesses at least two cycles. By rearranging the rows and columns of $\mathcal{M}^\star(\Sigma)$,

$$|\det(\mathcal{M}^\star(\Sigma))| = \left(\prod_{k=1}^{h-1} |\det(\mathcal{M}(\Sigma_k))|\right)|\det(\mathcal{M}^\star(\Sigma_h))| \ \ (\neq 0).$$

Lemma 2.10 *One of the cycles appearing in Σ_h is a circle.*

Proof: Since no cycle of length 2 appears in G, Corollary 2.3 enables us to assume that all non-zero components of $\mathcal{M}^\star(\Sigma_h)$ are positive. If no circle belongs to Σ_h, then the sum of the components of each column of $\mathcal{M}^\star(\Sigma_h)$ is three and the last component of each column of $\mathcal{M}^\star(\Sigma_h)$ is 1. Thus $\det(\mathcal{M}^\star(\Sigma_h)) = 0$. □

Two cases arise: Either Σ_h possesses exactly one circle, or Σ_h possesses at least (hence exactly) two circles.

Lemma 2.11 *If Σ_h possesses exactly one circle, then* $|\det(\mathcal{M}^\star(\Sigma))| = 1$.

Proof: Since Σ_h possesses either a loop or a cycle of odd length ≥ 3, by Lemma 2.7 a unique cycle appearing in each of the connected components $\Sigma_1, \ldots, \Sigma_{h-1}$ must be a circle. By Proposition 1.1(a), $|\det(\mathcal{M}(\Sigma_k))| = 1$ for all $1 \leq k \leq h-1$. Moreover, by Proposition 1.1 (c) and Corollary 2.3, we have $|\det(\mathcal{M}^\star(\Sigma_h))| = 1$. Hence $|\det(\mathcal{M}^\star(\Sigma))| = 1$, as desired. □

Lemma 2.12 *If Σ_h possesses exactly two circles, then* $|\det(\mathcal{M}^\star(\Sigma))| = 1$.

Proof: Let γ_v and γ_w with $v < w$ be circles of Σ_h, and fix a path L of length ℓ of Σ_h joining v with w. Let $v = v_0, v_1, \ldots, v_{\ell-1}, v_\ell = w$ be the vertices of L and $E(L) = \{\xi_1, \xi_2, \ldots, \xi_\ell\}$, where each ξ_k is either the edge $\{v_{k-1}, v_k\}$ or the arrow $(\min\{v_{k-1}, v_k\}, \max\{v_{k-1}, v_k\})$. Then ℓ must be odd. Because, if ℓ is even, then the contradiction $\{\xi_1, \xi_3, \ldots, \xi_{\ell-1}, \gamma_w\} \not\subset E(\Sigma)$ arises, since Corollary 2.3 yields

$$\rho(\gamma_v) + \rho(\xi_2) + \rho(\xi_4) + \cdots + \rho(\xi_\ell) = \rho(\xi_1) + \rho(\xi_3) + \cdots + \rho(\xi_{\ell-1}) + \rho(\gamma_w).$$

If an edge $\eta = \{i, j\}$ appears in one of $\Sigma_1, \Sigma_2, \ldots \Sigma_{h-1}$, then the contradiction $\{\xi_1, \xi_3, \ldots, \xi_\ell, \gamma_w, \eta\} \not\subset E(\Sigma)$ arises, since Corollary 2.3 yields

$$\begin{aligned}&\rho(\gamma_v) + \rho(\xi_2) + \rho(\xi_4) + \cdots + \rho(\xi_{\ell-1}) + \rho(\{w, i\}) + \rho(\{w, j\})\\ &\quad = \rho(\xi_1) + \rho(\xi_3) + \cdots + \rho(\xi_\ell) + \rho(\gamma_w) + \rho(\eta).\end{aligned}$$

Thus a unique cycle appearing in each Σ_k, $1 \le k \le h-1$, is either a circle or a loop. If a loop δ_i belongs to Σ_k, then the contradiction $\{\xi_1, \xi_3, \dots, \xi_\ell, \gamma_w, \delta_i\} \not\subset E(\Sigma)$ arises, since

$$\begin{aligned}&\rho(\gamma_v) + \rho(\xi_2) + \rho(\xi_4) + \cdots + \rho(\xi_{\ell-1}) + 2\rho(\{w, i\})\\ &\quad = \rho(\xi_1) + \rho(\xi_3) + \cdots + \rho(\xi_\ell) + \rho(\gamma_w) + \rho(\delta_i).\end{aligned}$$

Hence a unique cycle appearing in each Σ_k, $1 \le k \le h-1$, is a circle. Thus, in particular, $\prod_{k=1}^{h-1} |\det(\mathcal{M}(\Sigma_k))| = 1$.

Now, to prove $|\det(\mathcal{M}^\star(\Sigma))| = 1$, it remains to show that $|\det(\mathcal{M}^\star(\Sigma_h))| = 1$. Recall that if $\xi \in E(\Sigma_h)$ is either an edge or an arrow and if one of the vertices of ξ belongs to no $\xi' \in E(\Sigma_h)$ with $\xi \ne \xi'$, then $|\det(\mathcal{M}^\star(\Sigma_h))| = |\det(\mathcal{M}^\star(\Sigma_h \backslash \{\xi\}))|$. Thus $|\det(\mathcal{M}^\star(\Sigma_h))| = |\det(\mathcal{M}^\star(L))|$. In addition, by Proposition 1.1(d) and Corollary 2.3, $|\det(\mathcal{M}^\star(L))| = 1$, as required. □

We are now in the position to complete our proof of Theorem 2.1.

Proof of Theorem 2.1: Since $\mathbf{e}_i + \mathbf{e}_j \in \mathcal{A}(G)$ for all $1 \le i < j \le n$, it follows that dim conv$(\tilde{\mathcal{A}}(G)) = n$. Moreover, the subgroup $\sum_{\xi \in E(G)} \mathbb{Z}(\rho(\xi), 1) + \mathbb{Z}(0, 0, \dots, 0, 1)$ of the additive group $\mathbb{Z}^{n+1}$ coincides with $\mathbb{Z}^{n+1}$ if all γ_i belong to G, and coincides with $\{(a_1, \dots, a_n, a_{n+1}) \in \mathbb{Z}^{n+1};\ \sum_{i=1}^n a_i \in 2\mathbb{Z}\}$ if no γ_i belongs to G. Hence the index N_G is equal to 1 (resp. 2) if and only if every (resp. no) circle of Λ_n belongs to G. Hence, by virtue of Lemma 1.2, Lemma 2.8(c), Lemma 2.9(b) together with Lemmas 2.11 and 2.12, the normalized volume of the convex hull conv(F) of each simplex $F \in \Delta_{<_G}(\tilde{\mathcal{A}}(G))$ with dim conv$(F) = n$ is equal to 1. Thus the regular triangulation $\Delta_{<_G}(\tilde{\mathcal{A}}(G))$ is unimodular. □

Remark 2.13 Let $n \ge 3$, and let G be a subgraph of Λ_n which possesses all edges of Λ_n and at least one circle of Λ_n. Let $<_{rev}$ denote the reverse lexicographic monomial order on $\tilde{\mathcal{R}}_K[G]$ induced by an arbitrary ordering of the variables with $x < y_i$, $x < z_i$, $x < e_{i,j}$ and $x < f_{i,j}$ for all y_i, z_i, $e_{i,j}$ and $f_{i,j}$ belonging to $\tilde{\mathcal{R}}_K[G]$. Then the regular triangulation $\Delta_{<_{rex}}(\tilde{\mathcal{A}}(G))$ of the configuration $\tilde{\mathcal{A}}(G)$ is not unimodular.

Proof: Let $n = 3$. If all loops of Λ_3 belong to G, then $y_i^2 - xz_i \in I_{\tilde{\mathcal{A}}(G)}$ and $y_i^2 \in in_{<_{rev}}(I_{\tilde{\mathcal{A}}(G)})$ for all i with $\gamma_i \in E(G)$. Thus $in_{<_{rev}}(I_{\tilde{\mathcal{A}}(G)})$ is not squarefree. Suppose that at least one loop of Λ_3 does not belong to G and write Σ for the subgraph of G with $E(\Sigma) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Then $\rho(\Sigma) \cup \{(0, 0, 0)\} \in \Delta_{<_{rex}}(\tilde{\mathcal{A}}(G))$. In fact, if $\rho(\Sigma) \cup \{(0, 0, 0)\} \notin \Delta_{<_{rex}}(\tilde{\mathcal{A}}(G))$, then $xe_{1,2}e_{1,3}e_{2,3} \in \sqrt{in_{<_{rev}}(I_{\tilde{\mathcal{A}}(G)})}$ and $(e_{1,2}e_{1,3}e_{2,3})^m \in in_{<_{rev}}(I_{\tilde{\mathcal{A}}(G)})$ for some $m > 0$. However, no binomial of the form

$$(e_{1,2}e_{1,3}e_{2,3})^m - \prod_{\gamma_i \in E(G)} y_i^{b_i} \prod_{\delta_i \in E(G)} z_i^{c_i} \prod_{\{i,j\} \in E(G)} e_{i,j}^{p_{i,j}} \prod_{(i,j) \in E(G)} f_{i,j}^{q_{i,j}}$$

belongs to $I_{\tilde{\mathcal{A}}(G)}$. Since the convex hull of $\rho(\Sigma) \cup \{(0, 0, 0)\}$ is of dimension 3 and its normalized volume is 2, the regular triangulation $\Delta_{<_{rex}}(\tilde{\mathcal{A}}(G))$ is not unimodular.

Let $n \geq 4$ and $\gamma_i \in E(G)$. If G' is an induced subgraph of G with $\gamma_i \in E(G')$ and with exactly three vertices, then $K[\tilde{\mathcal{A}}(G')]$ is a combinatorial pure subring [6] of $K[\tilde{\mathcal{A}}(G)]$. Hence $\Delta_{<_{rex}}(\tilde{\mathcal{A}}(G))$ is not unimodular, as desired. □

3. Unimodular coverings of subconfigurations of $\mathbf{A}_{n-1}^{+}$

Even though the purpose of the present section is to prove Theorem 0.1(b) concerning the existence of unimodular coverings of subconfigurations of $\mathbf{A}_{n-1}^{+}$, we begin with questions and conjectures on initial ideals of $\tilde{\mathbf{A}}_{n-1}^{+}$ and $\mathbf{A}_{n-1}^{+}$.

First of all, we study the unimodular triangulation of $\tilde{\mathbf{A}}_{n-1}^{+}$ constructed in [4]. Let $<_{lex}$ denote the lexicographic monomial order on the polynomial ring

$$\tilde{\mathcal{R}}_K[A_{n-1}] = K[\{x\} \cup \{f_{i,j}\}_{1 \leq i < j \leq n}]$$

over K induced by the ordering of the variables

$$f_{1,2} > f_{1,3} > \cdots > f_{1,n} > f_{2,3} > \cdots > f_{n-1,n} > x,$$

and let $<_{rev}$ denote the reverse lexicographic monomial order on $\tilde{\mathcal{R}}_K[A_{n-1}]$ induced by the ordering of the variables

$$x < f_{1,n} < f_{1,n-1} < \cdots < f_{1,2} < f_{2,n} < \cdots < f_{n-1,n}.$$

Then each of the initial ideals $in_{<_{lex}}(I_{\tilde{\mathbf{A}}_{n-1}^{+}})$ and $in_{<_{rev}}(I_{\tilde{\mathbf{A}}_{n-1}^{+}})$ of the toric ideal $I_{\tilde{\mathbf{A}}_{n-1}^{+}}$ is generated by the squarefree quadratic monomials $f_{i,k}f_{j,\ell}$ with $1 \leq i < j < k < \ell \leq n$ and $f_{i,j}f_{j,k}$ with $1 \leq i < j < k \leq n$.

We say that, in general, a monomial ideal I of the polynomial ring $K[x_1, x_2, \ldots, x_n]$ *comes from a poset* if I is generated by squarefree quadratic monomials and if there is a partial order on the finite set $[\,n\,]$ such that $x_i x_j \in I$ if and only if i and j are incomparable in the partial order. See [6, 10, 12].

By using standard techniques [6], it is not difficult to show that, for all $n \geq 5$, the initial ideal $in_{<_{lex}}(I_{\tilde{\mathbf{A}}_{n-1}^{+}})$ $(=in_{<_{rev}}(I_{\tilde{\mathbf{A}}_{n-1}^{+}}))$ does *not* come from a poset.

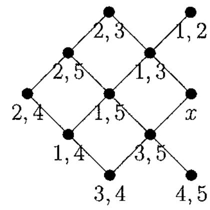
Question 3.1 *Does there exist a monomial order $<$ on $\tilde{\mathcal{R}}_K[A_{n-1}]$ such that the initial ideal $in_<(I_{\tilde{\mathbf{A}}_{n-1}^{+}})$ comes from a poset?*

If the answer to Question 3.1 is "yes," then it follows from [12, Corollary 3.6] that the infinite divisor poset of the semigroup ring $K[\tilde{\mathbf{A}}_{n-1}^{+}]$ is shellable.

Example 3.2 Let $n = 5$ and let $<$ be the lexicographic monomial order on $\tilde{\mathcal{R}}_K[A_4]$ induced by the ordering of the variables

$$f_{1,2} > f_{4,5} > x > f_{1,3} > f_{3,5} > f_{2,3} > f_{1,5} > f_{3,4} > f_{2,5} > f_{1,4} > f_{2,4}.$$

Then $in_<(I_{\tilde{\mathbf{A}}_4^+})$ comes from a poset. The Hasse diagram of the poset is drawn below.



We do not know, for $n = 6, 7, \ldots$, if there exists a monomial order $<$ on $\tilde{\mathcal{R}}_K[A_{n-1}]$ such that the initial ideal $in_<(I_{\tilde{\mathbf{A}}_{n-1}^+})$ comes from a poset. It can be, however, proved without difficulty that, if $<$ is a monomial order on $\tilde{\mathcal{R}}_K[A_{n-1}]$, where $n \geq 5$, such that x appears in no monomial belonging to a unique minimal system of monomial generators of $in_<(I_{\tilde{\mathbf{A}}_{n-1}^+})$, the initial ideal $in_<(I_{\tilde{\mathbf{A}}_{n-1}^+})$ does not come from a poset.

A completely different and powerful technique in order to show that the infinite divisor poset of a semigroup ring is shellable is also known [1, Theorem 3.1]. We refer the reader to [1] for the detailed information about extendable sequentially Koszul semigroup rings and combinatorics on shellable infinite divisor posets.

Conjecture 3.3

(a) The semigroup ring $K[\tilde{\mathbf{A}}_{n-1}^+]$ of the configuration $\tilde{\mathbf{A}}_{n-1}^+$ is extendable sequentially Koszul.
(b) (follows from (a)) The infinite divisor poset of the semigroup ring $K[\tilde{\mathbf{A}}_{n-1}^+]$ is shellable.

Let G be a subgraph of A_{n-1}. Since the matrix $\mathcal{M}(A_{n-1})$ is totally unimodular and since $(0, 0, \ldots, 0) \in \tilde{\mathcal{A}}(G)$, by virtue of [13, Example 2.4(a)] the initial ideal $in_<(I_{\tilde{\mathcal{A}}(G)})$ is squarefree for *any* reverse lexicographic monomial order $<$ on the polynomial ring $\tilde{\mathcal{R}}_K[G]$ with $x < f_{i,j}$ for all $(i, j) \in E(G)$. However, in general, the toric ideal $I_{\tilde{\mathcal{A}}(G)}$ cannot be generated by quadratic binomials. For example, if $n = 6$ and G is the subgraph of A_5 with $E(G) = \{(1, 2), (2, 4), (4, 6), (1, 3), (3, 5), (5, 6)\}$, then $I_{\tilde{\mathcal{A}}(G)} = (f_{1,2}f_{2,4}f_{4,6} - f_{1,3}f_{3,5}f_{5,6})$.

Question 3.4

(a) *For which subgraphs G of A_{n-1}, is the toric ideal $I_{\tilde{\mathcal{A}}(G)}$ generated by quadratic binomials*?
(b) *For which subgraphs G of A_{n-1}, does the toric ideal $I_{\tilde{\mathcal{A}}(G)}$ possess an initial ideal generated by quadratic monomials*?

The situation for $\mathcal{A}(G)$ is, however, completely different and, in general, $\mathcal{A}(G)$ is not normal. For example, if $n = 5$ and G is a subgraph of A_4 with $E(G) = \{(1, 2), (2, 3), (3, 4), (4, 5), (1, 5), (1, 4)\}$, then $I_{\mathcal{A}(G)} = (f_{1,2}f_{1,5}^2f_{2,3}f_{3,4} - f_{1,4}^3f_{4,5}^2)$ and $\mathcal{A}(G)$ is non-normal.

For a subgraph G of Λ_n with $E(G) \subset \{\{i, j\}; 1 \leq i < j \leq n\}$, a combinatorial characterization for the toric ideal of the configuration $\mathcal{A}(G)$ to be generated by quadratic binomials is obtained in [9, Theorem 1.2]. It is known [8] that if G is, in addition, bipartite, then the toric ideal of the configuration $\mathcal{A}(G)$ possesses an initial ideal generated by quadratic monomials if and only if every cycle Γ of even length ≥ 6 appearing in G possesses at least one "chord," i.e., an edge $\xi = \{v, w\} \in E(G)$ with $v \in V(\Gamma)$, $w \in V(\Gamma)$ and $\xi \notin E(\Gamma)$.

All normal subconfigurations of $\{2\mathbf{e}_i;\ 1 \leq i \leq n\} \cup \{\mathbf{e}_i + \mathbf{e}_j;\ 1 \leq i < j \leq n\}$ are completely classified [7, 15]. More precisely, [7, Corollary 2.3] says that, for a connected subgraph G of Λ_n with $E(G) \subset \{\{i, j\};\ 1 \leq i < j \leq n\} \cup \{\delta_i;\ 1 \leq i \leq n\}$, the following conditions are equivalent:

(i) $\mathcal{A}(G)$ is normal;
(ii) $\mathcal{A}(G)$ possesses a unimodular covering;
(iii) If each of Γ_1 and Γ_2 is either a loop or an odd cycle of length ≥ 3 appearing in G and if Γ_1 and Γ_2 possess no common vertex, then there is a "bridge" between Γ_1 and Γ_2, i.e., an edge $\{v_1, v_2\} \in E(G)$ with $v_1 \in V(\Gamma_1)$ and $v_2 \in V(\Gamma_2)$.

Question 3.5 *Find a combinatorial characterization of subgraphs G of A_{n-1} such that the configuration $\mathcal{A}(G)$ possesses a unimodular covering.*

We now turn to the discussion of the existence of a unimodular covering of a subconfiguration $\mathcal{A} \subset \mathbf{A}_{n-1}^{+}$, where $n \geq 3$, satisfying the condition (0.1.2). The fundamental technique to prove Theorem 0.1(b) is already developed in [7].

Let Γ be a cycle of length ℓ appearing in A_{n-1} with $V(\Gamma) = \{v_0, v_1, \ldots, v_{\ell-1}\}$ and $E(\Gamma) = \{\xi_1, \xi_2, \ldots, \xi_\ell\}$, where each ξ_k is the arrow $(\min\{v_{k-1}, v_k\}, \max\{v_{k-1}, v_k\})$ and where $v_\ell = v_0$. Let $E_{\rightarrow}(\Gamma) = \{\xi_k \in E(\Gamma);\ \xi_k = (v_{k-1}, v_k)\}$ and $E_{\leftarrow}(\Gamma) = \{\xi_k \in E(\Gamma);\ \xi_k = (v_k, v_{k-1})\}$. Let

$$\delta(\Gamma) = |\sharp(E_{\rightarrow}(\Gamma)) - \sharp(E_{\leftarrow}(\Gamma))|.$$

A cycle Γ appearing in A_{n-1} is called *homogeneous* if $\delta(\Gamma) = 0$. The following fact can be proved easily by similar techniques as in the proof of [7, Proposition 1.3].

Lemma 3.6 *If G is a subgraph of A_{n-1}, then* $\dim \operatorname{conv} \mathcal{A}(G) = n - 1$ *if and only if G is a connected and spanning subgraph of G with at least one nonhomogeneous cycle.*

For a while, we work with a fixed spanning subgraph G of A_{n-1} with at least one nonhomogeneous cycle. Let m_G denote the greatest common divisor of the positive integers $\delta(\Gamma)$, where Γ is any cycle appearing in G which is nonhomogeneous:

$$m_G = \mathrm{GCD}(\{\delta(\Gamma)\ ;\ \Gamma \text{ is a nonhomogeneous cycle appearing in } G\}).$$

As in Section 1.6 a subgraph Σ of G is said to be a facet of G if $\rho(E(\Sigma))$ is a simplex belonging to the configuration $\mathcal{A}(G) \subset \mathbb{Z}^n$ with $\dim \operatorname{conv}(\rho(E(\Sigma))) = n - 1$. Again, as in [7, Lemma 1.4], we easily obtain the following

Lemma 3.7 *A subgraph Σ of G is a facet of G if and only if Σ is a connected and spanning subgraph of G with n arrows such that Σ possesses exactly one cycle and the cycle is nonhomogeneous.*

How can we compute the normalized volume of conv $(\rho(E(\Sigma)))$ for a facet Σ of G?

Lemma 3.8 *If Σ is a facet of G, then the normalized volume of* $\mathrm{conv}(\rho(E(\Sigma)))$ *is equal to $\delta(\Gamma)/m_G$, where Γ is a unique cycle appearing in Σ.*

Proof: Recall that the normalized volume of $\mathrm{conv}(\rho(E(\Sigma)))$ coindides with the index of the subgroup $\sum_{\xi\in E(\Sigma)} \mathbb{Z}(\rho(\xi), 1)$ in the additive group $\sum_{\xi\in E(G)} \mathbb{Z}(\rho(\xi), 1) \subset \mathbb{Z}^{n+1}$. Let $\mathbf{e}_{n+1} = (0, 0, \dots, 0, 1) \in \mathbb{Z}^{n+1}$. To obtain the required result, we now show that

$$\sum_{\xi\in E(G)} \mathbb{Z}(\rho(\xi), 1) \Big/ \sum_{\xi\in E(\Sigma)} \mathbb{Z}(\rho(\xi), 1) = \left\{0, m_G\mathbf{e}_{d+1}, 2m_G\mathbf{e}_{d+1}, \dots, \left(\frac{\delta(\Gamma)}{m_G} - 1\right)m_G\mathbf{e}_{d+1}\right\}.$$

Choose an arbitrary arrow $(i, j) \in E(G)\setminus E(\Sigma)$. Since Σ is a connected and spanning subgraph of G, we can find a cycle Γ' with $E(\Gamma') = \{\xi_1, \xi_2, \dots, \xi_{\ell-1}, \xi_\ell\}$, where each of the arrows $\xi_1, \dots, \xi_{\ell-1}$ belongs to Σ, such that $\xi_\ell = (i, j)$, i is a vertex of ξ_1 and j is a vertex of $\xi_{\ell-1}$. Then

$$(\mathbf{e}_i - \mathbf{e}_j, 1) \pm \delta(\Gamma')\mathbf{e}_{n+1} \in \sum_{\xi\in E(\Sigma)} \mathbb{Z}(\rho(\xi), 1).$$

Let $\delta(\Gamma) = am_G$ and $\delta(\Gamma') = bm_G$, where a and b are nonnegative integers. Let $c = b - da \le a - 1$ with $0 \le d \in \mathbb{Z}$. Then

$$(\mathbf{e}_i - \mathbf{e}_j, 1) \pm d\delta(\Gamma)\mathbf{e}_{n+1} \pm cm_G\mathbf{e}_{n+1} \in \sum_{\xi\in E(\Sigma)} \mathbb{Z}(\rho(\xi), 1).$$

Since $\delta(\Gamma)\mathbf{e}_{n+1} \in \sum_{\xi\in E(\Sigma)} \mathbb{Z}(\rho(\xi), 1)$, it follows that

$$(\mathbf{e}_i - \mathbf{e}_j, 1) \in \pm cm_G\mathbf{e}_{n+1} + \sum_{\xi\in E(\Sigma)} \mathbb{Z}(\rho(\xi), 1).$$

Now, the desired result follows immediately since $cm_G\mathbf{e}_{n+1} \in \sum_{\xi\in E(\Sigma)} \mathbb{Z}(\rho(\xi), 1)$ if and only if cm_G is divided by $\delta(\Gamma)$ $(= am_G)$. $\square$

One of the direct consequences of Lemma 3.8 is

Proposition 3.9 *The configuration $\mathcal{A}(A_{n-1})$ associated with the root system $\mathbf{A}_{n-1}$ possesses a regular unimodular triangulation. More precisely, if $<$ is the reverse lexicographic* *monomial order on $\mathcal{R}_K[A_{n-1}]$ induced by the ordering of the variables*

$$f_{1,2} < f_{1,3} < \cdots < f_{1,n} < f_{2,3} < \cdots < f_{n-1,n},$$

then the initial ideal $in_<(I_{\mathcal{A}(A_{n-1})})$ of the toric ideal $I_{\mathcal{A}(A_{n-1})}$ with respect to $<$ is squarefree.

Proof: Noting that $m_{A_{n-1}} = 1$, in case that the regular triangulation $\Delta_<(\mathcal{A}(A_{n-1}))$ of $\mathcal{A}(A_{n-1})$ with respect to $<$ is not unimodular, by Lemmas 3.7 and 3.8 a cycle Γ with $\delta(\Gamma) \geq 2$ such that

$$\prod_{(i,j)\in E(\Gamma)} f_{i,j} \notin \sqrt{in_<\big(I_{\mathcal{A}(A_{n-1})}\big)}$$

appears in A_{n-1}. Let f_{v_0,v_1} be the weakest variable among all the variables $f_{v,w}$ with $(v, w) \in E(\Gamma)$. Let $V(\Gamma) = \{v_0, v_1, \ldots, v_{\ell-1}\}$, where $\ell \geq 4$ since $\delta(\Gamma) \geq 2$, and $E(\Gamma) = \{\xi_1, \xi_2, \ldots, \xi_\ell\}$, where each ξ_k is the arrow $(\min\{v_{k-1}, v_k\}, \max\{v_{k-1}, v_k\})$ with $v_\ell = v_0$. Since f_{v_0,v_1} is weakest, it follows that $v_0 < v_1 < v_{\ell-1}$. Assuming $\sharp(E_\rightarrow(\Gamma)) > \sharp(E_\leftarrow(\Gamma))$, let g denote the binomial

$$f_{v_0,v_{\ell-1}}^{\delta(\Gamma)} \prod_{\xi_k\in E_\rightarrow(\Gamma)} f_{v_{k-1},v_k} - f_{v_0,v_1}^{\delta(\Gamma)} f_{v_1,v_{\ell-1}}^{\delta(\Gamma)} \prod_{\xi_k\in E_\leftarrow(\Gamma)} f_{v_k,v_{k-1}} \in I_{\mathcal{A}(A_{n-1})}.$$

Since $\delta(\Gamma) \geq 2$, the initial monomial of g is

$$f_{v_0,v_{\ell-1}}^{\delta(\Gamma)} \prod_{\xi_k\in E_\rightarrow(\Gamma)} f_{v_{k-1},v_k} \in in_<\big(I_{\mathcal{A}(A_{n-1})}\big).$$

This contradicts $\prod_{(i,j)\in E(\Gamma)} f_{i,j} \notin \sqrt{in_<(I_{\mathcal{A}(A_{n-1})})}$. Hence the regular triangulation $\Delta_<(\mathcal{A}(A_{n-1}))$ is unimodular. □

At present, we do not know if each of the configurations $\mathcal{A}(\mathbf{B}_n)$, $\mathcal{A}(\mathbf{C}_n)$, $\mathcal{A}(\mathbf{D}_n)$ and $\mathcal{A}(\mathbf{BC}_n)$ possesses a regular unimodular triangulation.

By virtue of Lemma 3.8 it is now easy to characterize all unimodular configurations $\mathcal{A} \subset \mathbf{A}_{n-1}^+$ with $\dim \mathrm{conv}(\mathcal{A}) = n - 1$.

Proposition 3.10 *Let G be a connected and spanning subgraph of A_{n-1} with at least one nonhomogeneous cycle. Then the configuration $\mathcal{A}(G)$ is unimodular if and only if $\delta(\Gamma) = \delta(\Gamma')$ for all nonhomogeneous cycles Γ and Γ' appearing in G.*

A *chord* of a cycle Γ appearing in A_{n-1} is an arrow (v, w), where v and w are vertices of Γ, with $(v, w) \notin E(\Gamma)$.

Lemma 3.11 *Fix $n \geq 3$. Let G be a connected and spanning subgraph of A_{n-1} with at least one nonhomogeneous cycle. Let Ω denote the set of all facets Σ of G such that a* *unique cycle appearing in* Σ *has no chord. Then*

$$\operatorname{conv}(\mathcal{A}(G)) = \bigcup_{\Sigma\in\Omega} \operatorname{conv}(\rho(E(\Sigma))).$$

Proof: Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \operatorname{conv}(\mathcal{A}(G))$ and choose a facet Σ of G with $\alpha \in \operatorname{conv}(\rho(E(\Sigma)))$. Write

$$\alpha = \sum_{\xi\in E(\Sigma)} a_\xi \rho(\xi)$$

with each $0 \le a_\xi \in \mathbb{R}$ and $\sum_{\xi\in E(\Sigma)} a_\xi = 1$. Let us assume that a unique cycle Γ appearing in Σ possesses a chord. Let $V(\Gamma) = \{v_0, v_1, \ldots, v_{\ell-1}\}$, where $\ell \ge 3$ and $E(\Gamma) = \{\xi_1, \xi_2, \ldots, \xi_\ell\}$, where each ξ_k is the arrow $(\min\{v_{k-1}, v_k\}, \max\{v_{k-1}, v_k\})$ with $v_\ell = v_0$. Let $\eta = (v_0, v_q)$ be a chord of Γ, where $2 \le q < \ell - 1$ and $v_0 < v_q$. Let Γ_1 and Γ_2 denote the cycles with $V(\Gamma_1) = \{v_0, v_1, \ldots, v_q\}$, $E(\Gamma_1) = \{\xi_1, \xi_2, \ldots, \xi_q, \eta\}$ and $V(\Gamma_2) = \{v_q, v_{q+1}, \ldots, v_0\}$, $E(\Gamma_2) = \{\xi_{q+1}, \xi_{q+2}, \ldots, \xi_\ell, \eta\}$. Since the cycle Γ is nonhomogeneous, it follows that either Γ_1 or Γ_2 is nonhomogeneous.

First, suppose that Γ_1 is homogeneous and let

$$a = \min\{a_\xi;\, \xi \in E_\rightarrow(\Gamma_1)\} \ge 0.$$

Replacing a_ξ with $a_\xi - a$ if $\xi \in E_\rightarrow(\Gamma_1)$, replacing a_ξ with $a_\xi + a$ if $\eta \ne \xi \in E_\leftarrow(\Gamma_1)$, and setting $a_\eta = a$, the expression

$$\alpha = \sum_{\xi\in E(\Sigma)\cup\{\eta\}} a_\xi \rho(\xi)$$

arises, where at least one arrow $\xi \in E_\rightarrow(\Gamma_1)$ satisfies $a_\xi = 0$. Fix such an edge ξ and write Σ' for the subgraph obtained by deleting ξ from Σ and by adding η to Σ. Then Σ' is a facet of G with a unique cycle Γ_2 and $\alpha \in \operatorname{conv}(\rho(E(\Sigma')))$.

Second, let $\sharp(E_\rightarrow(\Gamma)) < \sharp(E_\leftarrow(\Gamma))$ and suppose that both Γ_1 and Γ_2 are nonhomogeneous. Then either $\sharp(E_\rightarrow(\Gamma_1)) < \sharp(E_\leftarrow(\Gamma_1))$ or $\sharp(E_\rightarrow(\Gamma_2)) < \sharp(E_\leftarrow(\Gamma_2))$. Let, say, $\sharp(E_\rightarrow(\Gamma_1)) < \sharp(E_\leftarrow(\Gamma_1))$. Note that $\eta \in E_\leftarrow(\Gamma_1)$ and $\eta \in E_\rightarrow(\Gamma_2)$. In what follows we use the notation f_ξ instead of $f_{v,w}$ if $\xi = (v, w)$. Let

$$g_1^{(+)} = \prod_{\xi\in E_\rightarrow(\Gamma_1)} f_\xi, \quad g_1^{(-)} = \prod_{\xi\in E_\leftarrow(\Gamma_1)} f_\xi,$$
$$g_2^{(+)} = \prod_{\xi\in E_\rightarrow(\Gamma_2)} f_\xi, \quad g_2^{(-)} = \prod_{\xi\in E_\leftarrow(\Gamma_2)} f_\xi,$$
$$h^{(+)} = \prod_{\xi\in E_\rightarrow(\Gamma)} f_\xi, \quad h^{(-)} = \prod_{\xi\in E_\leftarrow(\Gamma)} f_\xi.$$

Then $f_\eta h^{(+)} = g_1^{(+)} g_2^{(+)}$ and $f_\eta h^{(-)} = g_1^{(-)} g_2^{(-)}$. Now, the binomial

$$\left(g_1^{(+)}\right)^{\delta(\Gamma)}\left(h^{(-)}\right)^{\delta(\Gamma_1)} - \left(g_1^{(-)}\right)^{\delta(\Gamma)}\left(h^{(+)}\right)^{\delta(\Gamma_1)}$$

belongs to $I_{\mathcal{A}(G)}$. Hence we can find a binomial

$$g = g^{(+)} - g^{(-)} \in I_{\mathcal{A}(G)}$$

such that

(i) $\operatorname{supp}(g^{(+)}) \cup \operatorname{supp}(g^{(-)}) = \{f_\xi; \xi \in E(\Gamma) \cup \{\eta\}\}$;
(ii) $\operatorname{supp}(g^{(+)}) \cap \operatorname{supp}(g^{(-)}) = \emptyset$;
(iii) $\eta \in \operatorname{supp}(g^{(-)})$,

where $\operatorname{supp}(g^{(+)})$ is the support of the monomial $g^{(+)}$. Let

$$g^{(+)} = \prod_{f_\xi \in \operatorname{supp}(g^{(+)})} f_\xi^{b_\xi}, \quad g^{(-)} = \prod_{f_\xi \in \operatorname{supp}(g^{(-)})} f_\xi^{c_\xi}.$$

Then

$$\sum_{f_\xi \in \operatorname{supp}(g^{(+)})} b_\xi \rho(\xi) = \sum_{f_\xi \in \operatorname{supp}(g^{(-)})} c_\xi \rho(\xi), \quad \sum_{f_\xi \in \operatorname{supp}(g^{(+)})} b_\xi = \sum_{f_\xi \in \operatorname{supp}(g^{(-)})} c_\xi.$$

Let

$$a = \min\left\{a_\xi / b_\xi;\ f_\xi \in \operatorname{supp}\left(g^{(+)}\right)\right\} \geq 0.$$

Replacing a_ξ with $a_\xi - ab_\xi$ (≥ 0) if $f_\xi \in \operatorname{supp}(g^{(+)})$, replacing a_ξ with $a_\xi + ac_\xi$ if $f_\eta \neq f_\xi \in \operatorname{supp}(g^{(-)})$, and setting $a_\eta = ac_\eta$, the expression

$$\alpha = \sum_{\xi \in E(\Sigma) \cup \{\eta\}} a_\xi \rho(\xi)$$

arises, where at least one arrow $\xi \in E(\Gamma)$ with $f_\xi \in \operatorname{supp}(g^{(+)})$ satisfies $a_\xi = 0$. Fix such an edge ξ and write Σ' for the subgraph obtained by deleting ξ from Σ and by adding η to Σ. Then Σ' is a facet of G with a unique cycle, which coincides with either Γ_1 or Γ_2, and $\alpha \in \operatorname{conv}(\rho(E(\Sigma')))$.

Hence repeated applications of such techniques enable us to find a facet Σ of G with $\Sigma \in \Omega$ and with $\alpha \in \operatorname{conv}(\rho(E(\Sigma)))$. Thus $\operatorname{conv}(\mathcal{A}(G)) = \bigcup_{\Sigma \in \Omega} \operatorname{conv}(\rho(E(\Sigma)))$, as desired. $\square$

We are approaching a proof of Theorem 0.1(b). A much more general result for the existence of unimodular coverings of subconfigurations of $\mathbf{A}_{n-1}^{+}$ is the following

Theorem 3.12 *Fix $n \geq 3$. Let G be a connected and spanning subgraph of A_{n-1} with at least one nonhomogeneous cycle and suppose that every nonhomogeneous cycle Γ appearing in G with $\delta(\Gamma) \neq m_G$ has a chord. Then the configuration $\mathcal{A}(G)$ possesses a unimodular covering; in particular, $\mathcal{A}(G)$ is normal.*

Proof: Work with the same notation as in Lemma 3.11. Let Ω' denote the set of all facets Σ of G such that a unique cycle Γ appearing in Σ satisfies $\delta(\Gamma) = m_G$. If every nonhomogeneous cycle Γ appearing in G with $\delta(\Gamma) \neq m_G$ has a chord, then since $\Omega \subset \Omega'$ it follows from Lemma 3.11 that $\mathrm{conv}(\mathcal{A}(G)) = \bigcup_{\Sigma \in \Omega'} \mathrm{conv}(\rho(E(\Sigma)))$. Lemma 3.8 guarantees that, for a facet Σ of G, the normalized volume of $\mathrm{conv}(\rho(\Sigma))$ is equal to 1 if and only if $\Sigma \in \Omega'$. Hence the collection $\{\rho(E(\Sigma));\ \Sigma \in \Omega'\}$ of simplices belonging to $\mathcal{A}(G)$ turns out to be a unimodular covering of $\mathcal{A}(G)$. □

Proof of Theorem 0.1(b): Let G be a connected and spanning subgraph of A_{n-1} with at least one nonhomogeneous cycle. Since G satisfies the condiditon (0.1.2), if $1 \leq i < j < k \leq n$ and if the arrows (i, j) and (j, k) belong to G, then the arrow (i, k) belongs to G. Hence every cycle of length ≥ 4 appearing in G with no chord is homogeneous of even length. Thus a nonhomogeneous cycle Γ appearing in G with no chord is of length 3. Note that $\delta(\Gamma) = 1$ if Γ is a cycle of length 3. Hence $m_G = 1$ and by Theorem 3.12 the configuration $\mathcal{A}(G)$ possesses a unimodular covering, as desired. □

Let, in general, $\mathcal{A} \subset \mathbb{Z}^n$ be a configuration with $\dim \mathrm{conv}(\mathcal{A}) = d - 1$. We introduce the finite (ordinary) graph on the vertex set consisting of all simplices F belonging to $\mathcal{A}$ with $\dim \mathrm{conv}(F) = d - 1$ $(= \sharp(F) - 1)$ with the edge set consisting of those 2-element subsets $\{F, F'\}$ of the vertex set such that $\sharp(F \cap F') = d - 1$. Then the technique discussed in the proof of Lemma 3.11 guarantees that if $\alpha \in \mathrm{conv}(F)$ then there is an edge $\{F, F'\}$ of the finite graph with $\alpha \in \mathrm{conv}(F')$.

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