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Flocks and Partial Flocks of Hyperbolic Quadrics via Root Systems

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Abstract. We construct three infinite families of partial flocks of sizes 12, 24 and 60 of the hyperbolic quadric of PG(3, q), for q congruent to -1 modulo 12, 24, 60 respectively, from the root systems of type D_4, F_4, H_4, respectively. The smallest member of each of these families is an exceptional flock. We then characterise these partial flocks in terms of the rectangle condition of Benz and by not being subflocks of linear flocks or of Thas flocks. We also give an alternative characterisation in terms of admitting a regular group fixing all the lines of one of the reguli of the hyperbolic quadric.

Keywords: flock, maximal exterior set, root system, rectangle condition, partial flock, exterior set, exceptional flock

1. Introduction

The study of flocks of hyperbolic quadrics was begun by Thas in 1975 [15]. In this initial paper, Thas already showed that, in characteristic 2, all flocks of the hyperbolic quadric are linear, and that this is false in characteristic not 2, by constructing the flocks now called Thas flocks. By 1987, at a conference in Lincoln, Nebraska, he had shown that for fields of order congruent to 1 modulo 4 (and also for the fields of orders 3 and 7), the only flocks of the hyperbolic quadric are the linear flocks and the Thas flocks [16]. Just before this conference, it had become clear that other flocks existed. (Presumably, this motivated the choice of the congruence condition by Thas.) In 1987, Baker and Ebert in [4] published flocks of the hyperbolic quadric in PG(3, 11) and PG(3, 23); these same flocks were published, together with one in PG(3, 59) by Bader in 1988 [1], Bonisoli in 1988 [6], and Johnson in 1989 [12], and because of their association with exceptional nearfields, they came to be known as the exceptional flocks of the hyperbolic quadric. Then in 1989, Bader and Lunardon in [3] completed the classification of flocks of the hyperbolic quadric, building on fundamental results of Thas [16]. They showed that every flock of the hyperbolic quadric is linear, a Thas flock or one of the three exceptional flocks. A different proof, still based on the results in [16], was published in 1992 by Bonisoli and Korchmàros [8]. In the same year, Durante wrote a thesis [10] studying the exceptional flocks in detail.

The fundamental underlying result of Thas [16] is that, for any flock of the hyperbolic quadric, and any plane of that flock, the reflection about the plane stabilising the hyperbolic quadric also stabilises the flock. Thus results on groups generated by reflections apply. This is the approach of Bonisoli and Korchmàros in [8]. Since the root systems arise from reflection groups, it *should* be no surprise that the exceptional hyperbolic flocks can be connected to root systems in $\mathbf{R}^4$. Studying them in this way, it becomes clear that, while sporadic as flocks, each of them belongs in an infinite family of partial flocks of constant size.

The replacement of each plane of a (partial) flock by its polar point with respect to the hyperbolic quadric turns a flock into a maximal exterior set, and a partial flock into an exterior set. By using a matrix model for PG(3, q), we give an explicit correspondence between the regular subgroups of $PGL(2, q)$ and the flocks of $Q^+(3, q)$. Here we rely on the fundamental result of Bonisoli and Korchmàros [8], derived from the reflection lemma of Thas, which says in our model that each maximal exterior set of $Q^+(3, q)$ is a subgroup of $PGL(2, q)$. Each of the partial flocks of the hyperbolic quadric we constructed corresponds to a semiregular subgroup of $PGL(2, q)$, isomorphic to A_4, S_4 or A_5, in the three cases. Thus an alternative construction from these subgroups of the corresponding exterior sets could be given.

It is easy to give examples of partial flocks of the hyperbolic quadric not satisfying the reflection property, nor forming a group when viewed in our matrix model (for example, delete one plane from a flock). The classification of partial flocks of the hyperbolic quadric is hopeless, and also not worthwhile. However, by keeping the property that the corresponding exterior set forms a group in our matrix model, we achieve a classification of partial flocks with this property. This property can be geometrically characterised in terms of the rectangle condition introduced in 1970 by Benz in his work [5] on Minkowski planes.

In the different sections, we use different models for $Q^+(3, q)$ according to our needs. This is not a whim on the part of the authors. The work with root systems requires the sum of squares quadratic form because of the connection with Euclidean geometry. The work with the rectangle condition requires the determinant quadratic form. The fact that these two forms are not similar over the real numbers, but are similar over finite fields accounts for many of the subtleties and difficulties encountered.

2. Notation and preliminary results

Let q be odd, V be the vector space $GF(q)^4$ and $Q : V \mapsto GF(q)$ be a nondegenerate quadratic form of plus type, so that $\mathcal{Q}^+ = Q^+(3, q) = \{\langle v\rangle : Q(v) = 0\}$ is a hyperbolic quadric of PG$(3, q)$. If P is a point not on $\mathcal{Q}^+$, then for any v such that $\langle v \rangle = P$, $Q(v)$ is either always a square or always a nonsquare. In the first case we say that P is a point of type I, in the second case P is a point of type II. Note that there exist similarities which interchange the points of type I and the points of type II.

A flock of the hyperbolic quadric $\mathcal{Q}^+$ is a partition of the pointset of the quadric into $q+1$ disjoint conics. A maximal exterior set (MES) of $\mathcal{Q}^+$ is a set of $q+1$ points of PG$(3, q)$ such that the line joining any two of them is exterior to $\mathcal{Q}^+$; the polar planes, with respect to $\mathcal{Q}^+$, of the points of a MES determine a flock, and conversely. An exterior set (ES) of $\mathcal{Q}^+$ is a set of points of PG$(3, q)$ such that the line joining any two of them is exterior to $\mathcal{Q}^+$, while a partial flock of $\mathcal{Q}^+$ is a set of pairwise disjoint conics on $\mathcal{Q}^+$. As above, ES and partial flocks are equivalent objects.

The following criterion will be useful later. If the quadratic form Q polarises to the symmetric bilinear form $f(u, v) = Q(u+v) - Q(u) - Q(v)$, then for distinct points $P = \langle u \rangle$ and $R = \langle v \rangle$, $l = \langle u, v \rangle$ is an external line if and only if $f(u, u)f(v, v) - f(u, v)^2 = -\not\square$.

Flocks of the hyperbolic quadric have been classified [3, 16]. Here we list the flocks of $\mathcal{Q}^+$ and the associated MES, rephrasing and combining the results of [1, 3, 10, 13, 15, 16] and pointing out the properties we need in the present paper.

- *Linear.* For l an external line to $\mathcal{Q}^+$, all planes through l determine a flock. The associated maximal exterior set M consists of all points on $l^\perp$, hence $\frac{q+1}{2}$ of the points of M are of type I and the rest of type II.
- *Thas.* Let l be an external line to $\mathcal{Q}^+$, q odd and let m be the polar line to l with respect to $\mathcal{Q}^+$. Then $\mathcal{F} = \{\langle v \rangle^\perp | \langle v \rangle \in l \text{ and } Q(v) = \square \pmod{\square} \text{ or } \langle v \rangle \in m \text{ and } Q(v) = -\not\square \pmod{\square}\}$ is a flock of $\mathcal{Q}^+$. The associated MES consists, for $q \equiv 1 \pmod 4$, of $\frac{q+1}{2}$ points of type I and $\frac{q+1}{2}$ points of type II, while for $q \equiv 3 \pmod 4$ all points are of type I.
- *Exceptional.* There are three exceptional flocks, for $q = 11, 23, 59$. They were found via the associated translation planes, which have coordinates in some exceptional nearfields. For our purposes, we need the geometric properties of the relevant MES.
  - $q = 11$: the points of the MES are all of type I, and they can be split up into three disjoint self-polar tetrahedra which are also desmic tetrahedra.[1]
  - $q = 23$: the points of the MES are all of type I, and can be split up into six disjoint self-polar tetrahedra which can furthermore be uniquely divided into two sets of three desmic tetrahedra.
  - $q = 59$: the points of the MES are all of type I, and can be split up into fifteen disjoint self-polar tetrahedra which can furthermore be uniquely divided into five sets of three desmic tetrahedra.

As noted in the introduction, there are many known examples of partial flocks, hence of ES, of $\mathcal{Q}^+$.

3. Root systems and exterior sets

We start this section with a brief description of some root systems that we will relate to the exceptional flocks of $Q^+(3, q)$.

Root systems are particular sets of vectors in the Euclidean space $\mathbf{R}^d$. Here we focus on those we need. Let $d = 4$ and $\{e_1, e_2, e_3, e_4\}$ be the standard basis of $\mathbf{R}^4$. The root systems $\bar{D}_4$, $\bar{F}_4$, $\bar{H}_4$ are, respectively, the following sets of vectors in $\mathbf{R}^4$

$$\bar{D}_4 = \{\pm e_i \pm e_j : 1 \leq i < j \leq 4\}, \quad |\bar{D}_4| = 24,$$

$$\bar{F}_4 = \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\} \cup \{\pm e_1, \pm e_2, \pm e_3, \pm e_4\} \cup \bar{D}_4, \quad |\bar{F}_4| = 48,$$

$$\bar{H}_4 = (\bar{F}_4 - \bar{D}_4) \cup \left\{\frac{1}{2}(0, \pm 1, \pm\tau, \pm\tau^{-1})^\sigma, \sigma \in A_4\right\}, \quad |\bar{H}_4| = 120,$$

where $\tau = \frac{1+\sqrt{5}}{2}$. (Here the alternating group A_4 of degree 4 is permuting the coordinates).

For the construction and the classification of root systems see [11]. See also [14].

The root systems $\bar{D}_4$, $\bar{F}_4$, $\bar{H}_4$ can be viewed also in the finite vector space $GF(q)^4$, provided q is odd and provided 5 is a square in the case of $\bar{H}_4$, for which we choose $c \in GF(q)$ with $c^2 = 5$ and let $\tau = \frac{1+c}{2}$. So let $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ be the standard basis of $GF(q)^4$, and let

$$\tilde{D}_4 = \{\pm\tilde{e}_i \pm \tilde{e}_j : 1 \leq i < j \leq 4\},$$

$$\tilde{F}_4 = \left\{\frac{1}{2}(\pm\tilde{e}_1 \pm \tilde{e}_2 \pm \tilde{e}_3 \pm \tilde{e}_4)\right\} \cup \{\pm\tilde{e}_1, \pm\tilde{e}_2, \pm\tilde{e}_3, \pm\tilde{e}_4\} \cup \tilde{D}_4,$$

$$\tilde{H}_4 = (\tilde{F}_4 - \tilde{D}_4) \cup \left\{\frac{1}{2}(0, \pm 1, \pm\tau, \pm\tau^{-1})^\sigma, \sigma \in A_4\right\}.$$

In PG(3, q), let

$$\mathcal{E}(\bar{D}_4) = \{\langle x\rangle : x \in \tilde{D}_4\},$$

$$\mathcal{E}(\bar{F}_4) = \{\langle x\rangle : x \in \tilde{F}_4\},$$

$$\mathcal{E}(\bar{H}_4) = \{\langle x\rangle : x \in \tilde{H}_4\}.$$

Theorem 3.1 *Fix the hyperbolic quadric $Q^+(3, q)$ with equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$. Then*
$\mathcal{E}(\bar{D}_4)$ is an exterior set of size 12 *of $Q^+(3, q)$ iff $q \equiv -1$ (mod* 12);
$\mathcal{E}(\bar{F}_4)$ is an exterior set of size 24 *of $Q^+(3, q)$ iff $q \equiv -1$ (mod* 24);
$\mathcal{E}(\bar{H}_4)$ is an exterior set of size 60 *of $Q^+(3, q)$ iff $q \equiv -1$ (mod* 60).

Proof: Denote by f the bilinear form associated with the quadratic form $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2$. By a direct computation, one can check that $f(x, x)f(y, y) - f(x, y)^2$ is minus a nonsquare for every distinct $\langle x\rangle, \langle y\rangle \in \mathcal{E}(\bar{D}_4)$, $\langle x\rangle, \langle y\rangle \in \mathcal{E}(\bar{F}_4)$ or $\langle x\rangle, \langle y\rangle \in \mathcal{E}(\bar{H}_4)$, under the congruence conditions on q. Indeed, from the congruences on q, -1 is always a nonsquare, so equivalently we can prove that $f(x, x)f(y, y) - f(x, y)^2$ is a square. Here we give only some details. If $q \equiv -1$ (mod 12), then $f(x, x)f(y, y) - f(x, y)^2 \in \{12, 16\}$ for $\langle x\rangle, \langle y\rangle \in \mathcal{E}(\bar{D}_4)$ and both 12 and 16 are squares. If $q \equiv -1$ (mod 24), then

$f(x,x)f(y,y) - f(x,y)^2 \in \{3,4,8\}$ for $\langle x\rangle, \langle y\rangle \in \mathcal{E}(\bar{F}_4)$. Since $q \equiv -1 \pmod{24}$, both 3 and 8 are squares. For $q \equiv -1 \pmod{60}$, the fact can be used that in this case $1 - \frac{\tau^2}{4}$ is a square. □

So we have a construction of an exterior set of size $12m$ of the hyperbolic quadric in PG(3, q) for every $q \equiv -1 \pmod{12m}$, where $m \in \{1,2,5\}$. We denote by $\mathcal{F}(\bar{D}_4)$, $\mathcal{F}(\bar{F}_4)$, $\mathcal{F}(\bar{H}_4)$, respectively, the corresponding partial flocks of size $12m$.

Remark $\mathcal{F}(\bar{D}_4)$, $\mathcal{F}(\bar{F}_4)$, $\mathcal{F}(\bar{H}_4)$ admit the Weyl groups $W(D_4)$, $W(F_4)$, $W(H_4)$ respectively (acting with a kernel of order 2).

Corollary 3.1 *For $q = 11$, $\mathcal{E}(\bar{D}_4)$ is the exceptional MES of $Q^+(3,11)$.*
For $q = 23$, $\mathcal{E}(\bar{F}_4)$ is the exceptional MES of $Q^+(3,23)$.
For $q = 59$, $\mathcal{E}(\bar{H}_4)$ is the exceptional MES of $Q^+(3,59)$.

Proof: It is easy to see that the points of the exterior sets neither are on a line (linear MES) nor on two polar lines (Thas MES). Hence, from the list of flocks of the hyperbolic quadric in Section 2, it follows that the above described MES are the exceptional ones. □

Remark By the same proofs, the sets $\mathcal{E}(\bar{D}_4)$, $\mathcal{E}(\bar{F}_4)$, $\mathcal{E}(\bar{H}_4)$ are *secant* sets with respect to $Q^+(3,q)$, for $q \equiv 1 \pmod{12m}$ where $m \in \{1,2,5\}$ respectively (i.e., for every pair of points in the set, the line joining them is a secant line to the quadric).

4. The matrix model of PG(3, q)

Let (x_1, x_2, x_3, x_4) be homogeneous projective coordinates in PG(3, q), and let $\mathcal{Q}^+ = Q^+(3,q)$ be the hyperbolic quadric in PG(3, q) with equation $Q(X) = x_1x_4 - x_2x_3 = 0$, whose associated polar form is f, with $f(X,Y) = Q(X+Y) - Q(X) - Q(Y) = x_1y_4 + x_4y_1 - x_2y_3 - x_3y_2$. Denote by $\perp$ the polarity defined by $\mathcal{Q}^+$.

Let PM(2, q) be the 3–dimensional projective space associated with the vector space of 2×2 matrices with entries in $GF(q)$. Define the collineation $\psi : \mathrm{PG}(3,q) \to \mathrm{PM}(2,q)$, $P(a,b,c,d) \mapsto \psi(P) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, mapping points of $\mathcal{Q}^+$ to singular matrices, and conversely.

For every point $A \notin \mathcal{Q}^+$, there is a unique collineation, say τ_A, of $PGL(2,q)$ representing the point A.

Note that the point $I(1,0,0,1)$ is not on $\mathcal{Q}^+$, $\psi(I)$ is the identity matrix and, if $Y \notin \mathcal{Q}^+$, $f(X,Y) = \mathrm{tr}(\psi(X)\psi(Y)^{-1})\det(\psi(Y))$.

Let $\mathcal{R}^+ = \{l_{a',b'} \mid (a',b') \in GF(q)^2 - \{(0,0)\}\}$, where $l_{a',b'} = \{(a',b',\lambda a',\lambda b') \mid \lambda \in GF(q)\} \cup \{(0,0,a',b')\}$. Then $\mathcal{R}^+$ is one of the two reguli of $Q^+(3,q)$. Call the other regulus $\mathcal{R}^-$.

The matrix model endows the points not on a hyperbolic quadric with the structure of a group, namely $PGL(2,q)$. (More properly, with the structure of a set on which $PGL(2,q)$ acts regularly.) Thus the subgroup structure and the normal subgroup structure should have geometric consequences. Here we begin to explore these consequences.

Note that $\{A \in \text{PM}(2, q) : \det(A) = \square\}$ is $PSL(2, q)$ and $\{A \in \text{PM}(2, q) : \det(A) = \not\square\}$ is the coset of $PSL(2, q)$ in $PGL(2, q)$. This corresponds to the points of type I and type II discussed at the beginning of Section 2.

Being a subgroup implies closure under inversion and closure under multiplication. We will explore these independently.

For any points P, R not on $\mathcal{Q}^+$, there is a point S not on $\mathcal{Q}^+$ such that $\psi(S) = \psi(P)\psi(R)$, by matrix multiplication. Also, for any point P not on $\mathcal{Q}^+$ there is a point T not on $\mathcal{Q}^+$ such that $\psi(T) = \psi(P)^{-1}$. In the following, we give a geometric characterisation of S and T in terms of $\mathcal{Q}^+, I, P, R$.

For $A, C \in GL(2, q)$, define $\phi(A, C) : \text{PM}(2, q) \to \text{PM}(2, q)$, $X \mapsto AXC^T$. Note that $\phi(A, C)$ fixes $\mathcal{Q}^+$ for all $A, C \in GL(2, q)$.

Remark This model can be used to establish the isomorphism between $GO^+(4, q)$ and the semidirect product of the central product of $GL(2, q)$ with itself and a cyclic group of order 2. (The cyclic group of order 2 arises from transposition $X \mapsto X^T$. We need only compare orders.)

The plane $I^\perp$ has equation $x_1 + x_4 = 0$. The collineation $\rho_I : \text{PG}(3, q) \to \text{PG}(3, q)$, $(a, b, c, d) \mapsto (-d, b, c, -a)$ fixes $\mathcal{Q}^+$ setwise, fixes the point I and acts as the identity on $I^\perp$, i.e., it is the reflection with center I and axis $I^\perp$. View ρ_I as a collineation of $\text{PM}(2, q)$, denoted by the same symbol. Thus, ρ_I maps any nonsingular matrix A to the matrix -det$(A)A^{-1}$. That is, for any point P not on $\mathcal{Q}^+$ we have $\psi(\rho_I(P)) = \psi(P)^{-1}$. As $A^2 = \text{tr}(A)A - \det(A)I$ for any nonsingular matrix A, it follows that the points I, A, A^2 are collinear. If $A \in I^\perp$, then $\text{tr}(A) = 0$, and so $A^2 = -\det(A)I$. Hence, as projective points, $\psi(A^2) = \psi(I)$ for all nonsingular $A \in I^\perp$.

Hence closure under inversion means closure under the reflection with centre I.

Now define the map

$$\rho_A = \phi(A, I)\rho_I\phi(A, I)^{-1}.$$

Note that if $X \notin \mathcal{Q}^+$, then $\rho_A(X) = AX^{-1}A$ and for $A = I$, the map $\rho_A = \rho_I$ is the same map we have already defined by the same symbol. Then, ρ_A fixes A, maps I to A^2 and maps A^2 to I; moreover, it fixes every point in $A^\perp$. Indeed, for $A(a, b, c, d)$, a point $B(x, y, z, t)$ in $A^\perp$ satisfies $at - bz - cy + dx = 0$. A straightforward computation shows that $\rho_A(B) = B$ as projective points. Also, ρ_A has order 2. So ρ_A is a reflection with centre A, taking X to $AX^{-1}A$ for all points X not on $\mathcal{Q}^+$.

Suppose a subset G of points (containing I) is given in the projective space $\text{PM}(2, q)$, such that G is a group with respect to matrix multiplication. Recall the reflection ρ_I with centre I and axis $I^\perp$ can be viewed projectively as the map $A \mapsto A^{-1}$; so any group G of points of $\text{PM}(2, q)$ is closed under the reflection ρ_I.

Let $A \in G$. Observe G is closed under the reflection ρ_A for every $A \in G$. Indeed, if $A, X \in G$ then $\rho_A(X) = AX^{-1}A \in G$.

Later results in this paper make it clear that the above is a rephrasing of [16, Theorem 2].

Lemma 4.1 *$\{\phi(A, I) : A \in PGL(2, q)\}$ is the linewise stabiliser H of $\mathcal{R}^+$ in $PGO^+(4, q)$. Hence $H \cong PGL(2, q)$.*

Proof: Note that $\phi(A, I)$ fixes every line of $\mathcal{R}^+$. Indeed $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & b' \\ \lambda a' & \lambda b' \end{smallmatrix}\right) = \left(\begin{smallmatrix} aa'+\lambda ba' & ab'+\lambda bb' \\ a'c+\lambda da' & cb'+\lambda dd' \end{smallmatrix}\right)$ $= \left(\begin{smallmatrix} a'(a+\lambda b) & b'(a+\lambda b) \\ a'(c+\lambda d) & b'(c+\lambda d) \end{smallmatrix}\right) = \left(\begin{smallmatrix} a' & b' \\ a'\frac{c+\lambda d}{a+\lambda b} & b'\frac{c+\lambda d}{a+\lambda b} \end{smallmatrix}\right)$ (as projective points). Hence $\phi(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), I)l_{a',b'} = l_{a',b'}$. By comparing orders, equality occurs. □

Remark This lemma gives us another way of relating $Q^+(3, q)$ and $PGL(2, q)$.

5. Exterior sets in the matrix model

Here we explore the matrix model with a view to developing an appropriate extra condition to characterise the exterior sets of Section 3.

A set $\mathcal{S}$ of permutations of a set X is **sharp** if $\forall x, x' \in X$ there is at most one $\sigma \in \mathcal{S}$ with $\sigma(x) = x'$. It is **transitive** if $\forall x, x' \in X$ there is at least one $\sigma \in \mathcal{S}$ with $\sigma(x) = x'$. It is sharply transitive if it is sharp and transitive.

When $\mathcal{S}$ is a group, the standard term for sharp is **semiregular**; while that for sharply transitive is **regular**.

In the following paragraph (and later in the paper), we are working in the matrix model, and we consider $PGL(2, q)$ as a permutation group in its natural action on PG$(1, q)$.

Let $Q^+(3, q)$ be the hyperbolic quadric in the matrix model. Let A and B be two points not on the quadric, i.e., two nonsingular matrices. The line AB is external to $Q^+(3, q)$ iff $\det(A + \lambda B) \neq 0$ for all $\lambda \in GF(q)$ iff $\det(AB^{-1} + \lambda I) \neq 0$ for all λ iff (replacing λ with $-\lambda$) $\det(AB^{-1} - \lambda I) \neq 0$ for all λ iff AB^{-1} has no eigenvalues in $GF(q)$ iff AB^{-1} has no eigenvectors in $GF(q)^2$ iff AB^{-1} has no fixed points on PG$(1, q)$. So an exterior set is sharp and conversely. Moreover, a MES is transitive because it is of order $q + 1$. Also, if both A and B map P to P', then AB^{-1} fixes P : hence a sharply transitive subset is a MES and conversely (see also [6]).

Therefore, by [8, p. 296], a MES containing I is a group. Indeed, it is a regular subgroup of $PGL(2, q)$ (in the natural action on PG$(1, q)$). Conversely, if H is a regular subgroup of $PGL(2, q)$, then H is a MES.

Theorem 5.1 ([8]) *Let X be a subset of $PGL(2, q)$ containing I. Then X is a MES in the matrix model if and only if X is a regular subgroup.*

Remark The presentation of the exceptional flocks in [10], when translated via $(a, b, c, d) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ gives, as subgroups of $PGL(2, q)$ for $q = 11, 23, 59$, A_4, S_4, A_5 respectively. The self-polar tetrahedra correspond to cosets of a Klein four-subgroup; the triads of self-polar tetrahedra correspond to cosets of a subgroup isomorphic to A_4.

The underlying proofs (that a MES is a group) depend upon the fundamental underlying result of Thas [16] that given a MES M, we have $\forall A \in M\ \forall X \in M\ \rho_A(X) \in M$. In studying ES rather than MES the generalisation of these results is not true : there exist ES E that are not groups and for which $\exists A, X \in E$ with $\rho_A(X) \notin E$. However, the examples $\mathcal{E}(\bar{D}_4)$, $\mathcal{E}(\bar{F}_4)$, $\mathcal{E}(\bar{H}_4)$ of Section 3 satisfy both these conditions, as we shall see later. Since the classification of ES is intractable, we choose to study only ES with further conditions, in order to characterise these examples. So which of these two conditions is appropriate here?

Observation 5.1 In the presentation of [10], the MES of the exceptional flocks are groups. Unfortunately, not all the sets of points S closed under reflections ρ_A for all $A \in S$ are groups. Indeed, put $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The set $S = \{I, B, C\}$ is not a group, as $BC \notin S$ but all the ρ_X for $X \neq I$, act as the identity on S and ρ_I fixes S. Also, note that if S is a set of nonsingular 2×2 matrices closed under the reflections ρ_A for all $A \in S$, if $B, C \in S$, and $BC \in S$, then $CB \in S$ because $\rho_B \rho_C(BC) = CB$. This explains why we later emphasise the rectangle condition rather than reflections as an appropriate extra condition on partial flocks.

Benz [5, Configuration (G)] has linked the property that a subset is a group with the so-called rectangle condition (see also [7]). Recall a rectangle in $Q^+(3, q)$ is a quadruple (P_1, P_2, P_3, P_4) of points of $Q^+(3, q)$ such that the lines $\langle P_1, P_2\rangle$, $\langle P_3, P_4\rangle$ belong to the same regulus, while the lines $\langle P_1, P_3\rangle$, $\langle P_2, P_4\rangle$ belong to the opposite one.

Rectangle condition For $s = 1, 2, 3, 4$, let $(P_1^s, P_2^s, P_3^s, P_4^s)$ be a rectangle. Assume that for $i = 1, 2, 3$ the points $P_i^1, P_i^2, P_i^3, P_i^4$ are pairwise distinct and lie on a common conic on $Q^+(3, q)$. Then, $P_4^1, P_4^2, P_4^3, P_4^4$ are pairwise distinct points and lie on a common conic on $Q^+(3, q)$.

Define $\|^+$ (and $\|^-$) on $Q^+(3, q)$ by $P \|^+ R \iff \exists l \in \mathcal{R}^+(\mathcal{R}^-)$ with $P, R \in l$. Given 3 distinct conics C_1, C_2, C_3 of $Q^+(3, q)$, for any choice of distinct points P_1^1, P_1^2, P_1^3 on C_1, define P_2^1, P_2^2, P_2^3 by $P_1^i \|^+ P_2^i \in C_2$ and P_3^1, P_3^2, P_3^3 by $P_1^i \|^- P_3^i \in C_3$ and P_4^1, P_4^2, P_4^3 by $P_2^i \|^- P_4^i \|^+ P_3^i$. Then there is a fourth conic C_4 such that $P_4^1, P_4^2, P_4^3 \in C_4$. This conic C_4 is independent of the choice of P_1^1, P_1^2, P_1^3, which follows from the fact that $Q^+(3, q)$, viewed as a Minkowski plane, satisfies the rectangle condition of Benz in [5] (see also [7]).

A set $\mathcal{F}$ of conics satisfies the **rectangle condition** if whenever $C_1, C_2, C_3 \in \mathcal{F}$ are distinct, the conic C_4 above is also an element of $\mathcal{F}$.

An exterior set satisfies the **rectangle condition** if and only if the corresponding partial flock satisfies the rectangle condition.

Remark In PM$(2, q)$, if $C_1 = I^\perp \cap Q^+(3, q)$, $C_2 = P^\perp \cap Q^+(3, q)$, $C_3 = R^\perp \cap Q^+(3, q)$ then $C_4 = (PR)^\perp \cap Q^+(3, q)$. (See [5, Theorem 4].) Hence an exterior set E containing I satisfies the rectangle condition if and only if E is a subgroup of $PGL(2, q)$.

Lemma 5.1 *Let E be an exterior set containing I. The following are equivalent*:
1. *E is a subgroup of $PGL(2, q)$*;
2. *$\{\phi(A, I) : A \in E\}$ is a group*;
3. *E admits a group G fixing every line of $\mathcal{R}$ and acting regularly on E*;
4. *E satisfies the rectangle condition.*

Proof: 1.$\Longleftrightarrow$2. Note that $\phi(A_1, I)\phi(A_2, I) = \phi(A_1A_2, I) \in G$ iff $A_1, A_2 \in E$.

1.$\Longrightarrow$3. $G = \{\phi(A, I) : A \in E\}$ is a group which, by Lemma 4.1, fixes every line of $\mathcal{R}^+$. From $\phi(A, I)I = A$ it follows that G acts regularly on E.

3.$\Longrightarrow$1. By Lemma 4.1, $\exists X \leq PGL(2, q)$ such that $G = \{\phi(A, I) : A \in X\}$. From $\phi(A, I)I = A$ it follows that $X = E$.

2.$\Longleftrightarrow$4. By [5, Theorem 4], viewing an exterior set E as the set $\{\phi(A, I) : A \in E\}$, E is a group if and only if the rectangle condition is satisfied for any choice of the (pairwise distinct) points P_A^s on the conic $C_A = \{X \in \mathrm{PM}(2, q) : \det(X) = 0, f(X, A) = 0\}$ for $A \in E$. □

Remarks

1. When all of the cases of Lemma 5.1 are satisfied, the group E acts semiregularly on $\mathrm{PG}(1, q)$.
2. In the statements of the above two lemmas it would be more formally correct to write τ_A instead of A. We have chosen to emphasise the direct use of the matrices instead.
3. Note that $\mathcal{F}(\bar{D}_4)$, $\mathcal{F}(\bar{F}_4)$, $\mathcal{F}(\bar{H}_4)$ satisfy the rectangle condition as they arise from groups.
4. Alternative 3 is included as: 1) it is purely group-theoretic; and 2) it allows us to use the main theorem in a sequel paper [2] on BLT-sets.

6. Partial flocks satisfying the rectangle condition

We can now prove:

Theorem 6.1 *Let $\mathcal{F}$ be a partial flock of $Q^+(3, q)$. Then $\mathcal{F}$ admits a group G fixing every line of one of the reguli and acting regularly on $\mathcal{F}$ if and only if*
(1) $\exists r : |\mathcal{F}| = \frac{q+1}{r}$, *$G$ is cyclic and $\mathcal{F}$ is a subflock of a linear flock.*
(2) $\exists r : |\mathcal{F}| = \frac{q+1}{r}$ *is even, G is dihedral and $\mathcal{F}$ is a subflock of a Thas flock (so $q \equiv 3$ (mod 4) if r is even) and $\mathcal{F}$ is not a subflock of a linear flock.*
(3) $|\mathcal{F}| = 12$, $G = A_4$ *and* $\mathcal{F} = \mathcal{F}(\bar{D}_4)$ (*so* $q \equiv -1$ (*mod* 12)).
(4) $|\mathcal{F}| = 24$, $G = S_4$ *and* $\mathcal{F} = \mathcal{F}(\bar{F}_4)$ (*so* $q \equiv -1$ (*mod* 24)).
(5) $|\mathcal{F}| = 60$, $G = A_5$ *and* $\mathcal{F} = \mathcal{F}(\bar{H}_4)$ (*so* $q \equiv -1$ (*mod* 60)).

Proof: By Lemma 5.1 the corresponding ES is a semiregular subgroup E of $PGL(2, q)$. Since E is semiregular, $|E|$ divides $(q + 1)$. By the list of subgroups of $PGL(2, q)$ [9], $E \cong G$ is one of the possibilities in the statement of the theorem.

In case (1), $G \leq C_{q+1}$, so by [8] $\mathcal{F}$ is a subflock of the linear flock.

In case (2), $G \leq D_{q+1}$, so by [8] $\mathcal{F}$ is a subflock of the Thas flock.

In each of the cases (3), (4) and (5) the uniqueness, up to conjugacy, of the subgroups forces the partial flocks to be unique (subject to satisfying the rectangle condition).

Hence $\mathcal{F}$ must be equivalent to one of the partial flocks constructed in Section 3. The arguments of [8] then apply to show that the partial flocks correspond to A_4, S_4, A_5 respectively. □

We may now characterise partial flocks satisfying the rectangle condition.

Corollary 6.1 *Let $\mathcal{F}$ be a partial flock of $Q^+(3, q)$. Then $\mathcal{F}$ satisfies the rectangle condition iff it is one of the examples in* (1), (2), (3), (4), (5) *of Theorem* 6.1.

Observation 6.1 Theorem 6.1 characterises $\mathcal{F}(\bar{D}_4)$, $\mathcal{F}(\bar{F}_4)$, $\mathcal{F}(\bar{H}_4)$ as those partial flocks which are not subflocks of linear flocks or Thas flocks and which satisfy the rectangle condition.

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Note

1. Three tetrahedra are desmic if any two of them are in perspective from each vertex of the third one.

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