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On a Conjecture of R.P. Stanley; Part II—Quotients Modulo Monomial Ideals

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Abstract. In 1982 Richard P. Stanley conjectured that any finitely generated $\mathbb{Z}^n$-graded module M over a finitely generated $\mathbb{N}^n$-graded $\mathbb{K}$-algebra R can be decomposed as a direct sum $M = \bigoplus_{i=1}^{t} \nu_i S_i$ of finitely many free modules $\nu_i S_i$ which have to satisfy some additional conditions. Besides homogeneity conditions the most important restriction is that the S_i have to be subalgebras of R of dimension at least depth M.

We will study this conjecture for modules $M = R/I$, where R is a polynomial ring and I a monomial ideal. In particular, we will prove that Stanley's Conjecture holds for the quotient modulo any generic monomial ideal, the quotient modulo any monomial ideal in at most three variables, and for any cogeneric Cohen-Macaulay ring. Finally, we will give an outlook to Stanley decompositions of arbitrary graded polynomial modules. In particular, we obtain a more general result in the 3-variate case.

Keywords: Cohen-Macaulay module, combinatorial decomposition, monomial ideal, simplicial complex

1. Introduction

This is the second of two articles studying some aspects of a conjecture formulated by Richard P. Stanley ([14], 5.1), see also ([1], Conjecture 1). In the context of the special case studied in this paper Stanley's Conjecture reads as follows. For the definition of the Stanley depth Sdepth M of a module M we refer to ([1], Definition 1).

Conjecture 1 ([14]) Let $R = \mathbb{K}[X]$ be a polynomial ring in the variables $X = \{x_1, \ldots, x_n\}$ over a field $\mathbb{K}$. Then for any monomial ideal $I \subset R$,

$$\text{Sdepth}\, R/I \geq \text{depth}\, R/I. \tag{1}$$

That is, there exists a finite decomposition (called *Stanley decomposition* throughout this paper) of R/I of the following type

$$R/I = \bigoplus_{i=1}^{k} u_i \mathbb{K}[Z_i], \tag{2}$$

where the u_i are residue classes of (monic) monomials modulo I, $Z_i \subseteq X$ and $|Z_i| \geq$ depth $(XR, R/I) =$ depth R/I for all $i = 1, \ldots, k$.

In this paper we will ask for classes of modules $M = R/I$ for which there exist decompositions (2) satisfying the condition

$$|Z_i| \geq \min_{\mathfrak{p} \in \operatorname{Ass} M} \dim R/\mathfrak{p}. \tag{3}$$

In the most interesting case of Cohen-Macaulay rings M this condition is equivalent to condition (1). However, in general, condition (1) is weaker than (3), and it is not surprising that we will meet ideals, such as in Example 1, where condition (3) fails.

After declaring some notions and notations in Section 2 and deriving a trivial upper bound on the Stanley depth in Section 3 we prove the inequality Sdepth $R/I \leq$ Sdepth $R/\sqrt{I}$ for arbitrary monomial ideals I (Theorem 1) in Section 4. Theorem 2 in Section 5 shows that Stanley's Conjecture holds for R/I whenever I is in a certain sense 'algebraic shellable'. The strong connection between the described algebraic property and shellable simplicial complexes becomes apparent in the squarefree case studied in Section 7 (Corollary 4). The application of Theorem 2 to Borel-fixed and one-dimensional quotients, two cases where the validity of Stanley's Conjecture is well-known, is demonstrated in Section 6.

Subsequently, we apply Theorem 2 in order to show that Stanley's Conjecture holds for the quotient R/I in the following cases: I is a generic monomial ideal (Section 8), I is a monomial ideal in at most three variables (Section 9), or I is a cogeneric monomial Cohen-Macaulay ideal (Section 10).

Finally, we will give an outlook to the general case of arbitrary graded polynomial modules in Section 11 and generalize our results in the 3-variate case.

2. Notation

First we will introduce some notions and notations used throughout this paper. $|A|$ denotes the number of elements of a finite set A.

By R we denote the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ and by T the set of all (monic) monomials $x_1^{i_1} \ldots x_n^{i_n}$ in $\langle x_1, \ldots, x_n \rangle$. $\operatorname{supp} u = \{x_j : 1 \leq j \leq n, i_j > 0\}$ denotes the *support* of the monomial $u = x_1^{i_1} \ldots x_n^{i_n} \in T$.

By a *minimal monomial* t of a set $A \subset R$ we will always mean $t \in A \cap T$ and $\frac{t}{x} \notin A$ for all $x \in \operatorname{supp} t$.

Consider a nonzero monomial ideal $I \subset R$ with irredundant primary decomposition $I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_k$, that is, $\mathfrak{q}_1, \ldots, \mathfrak{q}_k$ are pairwise distinct primary ideals and $I \not\subseteq \mathfrak{q}_i$ for all $i \in \{1, \ldots, k\}$. Note, frequently the additional assumption that the associated prime ideals $\mathfrak{p}_i$ of the primary components $\mathfrak{q}_i$ have to be pairwise distinct is subsumed by the notion irredundant primary decomposition. We neglect this additional assumption since for monomial ideals it is often much more convenient to work with the (up to the order of components uniquely determined) irredundant decomposition in irreducible ideals, i.e. primary ideals generated only by powers of variables. As usual we denote the set of associated primes of $M = R/I$ by $\operatorname{Ass} M = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$, where $\mathfrak{p}_i$ belongs to the primary ideal $\mathfrak{q}_i$, $i \in \{1, \ldots, k\}$.

A set of variables $Y \subseteq X$ is called an *algebraically independent set* of I if and only if $I \cap \mathbb{K}[Y] = \{0\}$. If, in addition, no set of variables properly containing Y is an independent set of I then Y is called a *maximal algebraically independent set* of I. A monomial prime ideal[1] $\mathfrak{p}$ possesses exactly one maximal algebraically independent set, namely the set $Y = \{x \in X \mid x \notin \mathfrak{p}\}$. Any $\mathfrak{p}$-primary ideal $\mathfrak{q}$ has the same uniquely determined maximal algebraically independent set Y as $\mathfrak{p}$, $|Y|$ is called the (Krull-)dimension of $R/\mathfrak{p}$ and denoted by $\dim R/\mathfrak{p}$. Throughout this paper the notation Y_i will always refer to the maximal algebraically independent set of the associated prime $\mathfrak{p}_i$ corresponding to the primary component $\mathfrak{q}_i$ of I, $1 \leq i \leq k$. A set Y is algebraically independent for I if and only if it is algebraically independent for some associated prime ideal of M and it is maximal algebraically independent for I if and only if it is maximal algebraically independent for some minimal associated prime ideal of M. Accordingly, the dimension of M is defined by $\dim M = \max_{\mathfrak{p}_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$. In this paper we will need to refer frequently to $\min_{\mathfrak{p}_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$ which is an upper bound for the depth of M, c.f. [6, 15]. An ideal I satisfying $\dim M = \min_{\mathfrak{p}_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$ is called *pure dimensional* or simply *pure*. Cohen-Macaulay rings M are defined by the condition $\dim M = \operatorname{depth} M$ in which case we call I a Cohen-Macaulay ideal. From the previous observations it follows immediately that any Cohen-Macaulay ideal is pure.

Finally, we will need the notions of generic and cogeneric monomial ideals introduced by Bayer et al. [3] and refined by Miller et al. [9]. A monomial ideal I is called *generic* if for any two distinct minimal generators m and m' which have the same degree in some variable $x \in X$, there is a third minimal generator m'' which strictly divides $\operatorname{lcm}(m, m')$, i.e. $\operatorname{supp} \operatorname{lcm}(m, m') = \operatorname{supp} \frac{\operatorname{lcm}(m,m')}{m''}$.

Generic monomial ideals I allow a simple characterization for R/I being a Cohen-Macaulay ring, which is the case if and only if I is pure dimensional if and only if I has no embedded primary components [9, Theorem 2.5]. If I is generic and $\mathfrak{p}$ is an arbitrary monomial prime ideal then the ideal $I_{(\mathfrak{p})}$ obtained by localization of I at $\mathfrak{p}$ is generic, too [9, Remark 2.1].

A monomial ideal I is called *cogeneric* if whenever two distinct irreducible components $\mathfrak{q}_i$ and $\mathfrak{q}_j$ of I have a minimal generator in common then there exists an irreducible component $\mathfrak{q}_m$ of I which is contained in the ideal sum $\mathfrak{q}_i + \mathfrak{q}_j$ and has no minimal generator in common with this sum. Various characterizations of cogeneric Cohen-Macaulay monomial ideals were given in [9, Theorem 4.9]. In this paper we will make use of one of them, which after a mild reformulation says:

Lemma 1 ([9], Theorem 4.9c) *A cogeneric monomial ideal I with irredundant decomposition $I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_k$ into irreducible components is Cohen-Macaulay iff I is pure dimensional and for any irreducible components $\mathfrak{q}_i$ and $\mathfrak{q}_j$ such that* $\operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_j) > \operatorname{codim} I + 1$ *there exists an irreducible component $\mathfrak{q}_l$ of I satisfying $l \notin \{i, j\}$ and $\mathfrak{q}_l \subset \mathfrak{q}_i + \mathfrak{q}_j$.*

It is easy to observe that we have the following analogue to the generic case. If I is cogeneric and $\mathfrak{p}$ is an arbitrary monomial prime ideal then the ideal $I_{(\mathfrak{p})}$ obtained by localization of I at $\mathfrak{p}$ is cogeneric, too.

3. An upper bound for the Stanley depth

Let $m_1, \dots, m_l$ denote the minimal generators of I. We introduce the notion $S = \{u \in T \mid \forall i \in \{1, \dots, l\} : m_i \nmid u\}$ for the set of all standard monomials modulo I. The residue classes of S modulo I form a $\mathbb{K}$-vector space basis of M. In the following we will freely identify M and $\operatorname{span}_{\mathbb{K}} S$. Decomposition (2) has to satisfy $u_i v \in S$ for all monic monomials $v \in \mathbb{K}[Z_i]$, $i = 1, \dots, k$. In particular, for arbitrary $i \in \{1, \dots, k\}$ and $w \in u_i\mathbb{K}[Z_i]$ the set Z_i has to be algebraically independent for the ideal quotient $I : (w)$ and, consequently,

$$\operatorname{Sdepth} M \leq \min_{w \in S} \dim R/(I : (w)) = \min_{\mathfrak{p}_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i \tag{4}$$

which shows that depth M and Sdepth M share the upper bound on the right hand side. Let us start with a simple example illustrating that the above inequality can be strict.

Example 1 Consider the ideal $I = (xz, yz, xu, yu) = (x, y) \cap (z, u) \subset R = \mathbb{K}[x, y, z, u]$. There are two possibilities for two-dimensional subalgebras of R which can appear in a direct summand containing the monomial 1, namely either $\mathbb{K}[z, u]$ or $\mathbb{K}[x, y]$. Choosing $\mathbb{K}[z, u]$ there will be no way to find two-dimensional subalgebras for the direct summands containing the standard monomials x and y. Avoiding this problem by choosing $\mathbb{K}[x, y]$ will only shift the problem to the standard monomials z and u. Hence, $\operatorname{Sdepth} R/I \leq 1 < \min_{\mathfrak{p}_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i = 2$. In fact, we have equality at the very left because $R/I = \mathbb{K}[x, y] \oplus z\mathbb{K}[z, u] \oplus u\mathbb{K}[u]$ is an example for a decomposition of type (2). Moreover, this is already a Stanley decomposition since depth $R/I = 1$.

From

$$I : (u) = \bigcap_{i=1}^{k} (\mathfrak{q}_i : (u)) \quad \text{and} \quad \mathfrak{q}_i : (u) = \begin{cases} \mathfrak{q}_i\colon & \text{if } u \notin \mathfrak{p}_i \\ \mathfrak{q}_i'\colon & \text{if } u \in \mathfrak{p}_i \setminus \mathfrak{q}_i, \\ R\colon & \text{if } u \in \mathfrak{q}_i \end{cases}$$

where $\mathfrak{q}_i'$ is a $\mathfrak{p}_i$-primary ideal properly containing $\mathfrak{q}_i$, we deduce that the maximal algebraically independent sets of $I : (u)$ are exactly the maximal algebraically independent sets Y_i of the associated primes $\mathfrak{p}_i \in \operatorname{Ass} R/I$ which satisfy $u \in (\bigcap_{\mathfrak{p}_j \subset \mathfrak{p}_i} \mathfrak{q}_j) \setminus \mathfrak{q}_i$.

Now, we can describe the problem we met in Example 1 as follows. Let I be a pure dimensional monomial ideal with irredundant decomposition $I = \mathfrak{q}_1 \cap \mathfrak{q}_2$ in maximal primary ideals. Assume, there are two elements $v, w \in \mathfrak{q}_1 \setminus \mathfrak{q}_2$ and two elements $v', w' \in \mathfrak{q}_2 \setminus \mathfrak{q}_1$ such that $\gcd(v, w) = \gcd(v', w') =: m$. Then it follows $m \notin \mathfrak{q}_1 \cup \mathfrak{q}_2$. We will show that under certain conditions we have $\operatorname{Sdepth} R/I < \dim R/I$. Recall, our notation Y_i for the maximal algebraically independent set of $\mathfrak{q}_i$, $i = 1, 2$. In the case $\frac{\operatorname{lcm}(v,w)}{\gcd(v,w)} \in \langle Y_2 \rangle$ it will follow $v\mathbb{K}[Y_2] \cap w\mathbb{K}[Y_2] \neq \{0\}$. Hence, in case $\operatorname{Sdepth} R/I = \dim R/I$ the Stanley decomposition must contain a direct summand $u\mathbb{K}[Y_2]$ containing both v and w and, therefore, also m. Now suppose, we have also $\frac{\operatorname{lcm}(v',w')}{\gcd(v',w')} \in \langle Y_1 \rangle$. Then application of the same arguments shows that the Stanley decomposition must contain a second direct summand $u'\mathbb{K}[Y_1]$ containing v', w' and m. Obviously, this is impossible simultaneously.

In some sense this is the typical scenario for monomial ideals I with the property $\operatorname{Sdepth} R/I < \min_{\mathfrak{p}_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$. This behavior is always caused by contradicting requirements posed by a set of standard monomials modulo I to the direct summand which should contain their greatest common divisor.

4. Stanley depth of the radical

We will study the relationship between the Stanley depth of the residue class ring modulo an arbitrary monomial ideal and the residue class ring modulo its radical. Although our subsequent studies do not rely on the following theorem we decided to include it in the paper since it seems to be an interesting result on its own.

Theorem 1 *Let $I \subset R$ be a monomial ideal and $M = R/I$. Furthermore, let $\sqrt{I} = \bigcap_{\mathfrak{p}\in \operatorname{Ass} M} \mathfrak{p}$ be the radical of I and $\tilde{M} = R/\sqrt{I}$. Then* $\operatorname{Sdepth} M \le \operatorname{Sdepth} \tilde{M}$.

Proof: Let S denote the set of standard monomials modulo I and $\tilde{S}$ the set of standard monomials modulo $\sqrt{I}$. Recall the identifications $M = \operatorname{span}_{\mathbb{K}} S$ and $\tilde{M} = \operatorname{span}_{\mathbb{K}} \tilde{S}$. Consider an arbitrary Stanley decomposition $M = \bigoplus_{i=1}^{m} t_i\mathbb{K}[W_i]$ of M. For each monomial $u \in \tilde{M}$ we define $Z_u := W_i$, where $i \in \{1,\ldots,m\}$ is the uniquely determined index such that $u^\nu \in t_i\mathbb{K}[W_i]$ for all sufficiently large integers ν. Note, $\operatorname{supp} u \subseteq Z_u$ for the so-defined sets Z_u and, therefore, $\tilde{M} \supseteq \sum_{u\in\tilde{M}} u\mathbb{K}[Z_u]$. The other inclusion holds by construction.

Finally, we have to show that any two summands $u\mathbb{K}[Z_u]$ and $v\mathbb{K}[Z_v]$ either intersect trivially or are contained in the summand $\gcd(u,v)\mathbb{K}[Z_{\gcd(u,v)}]$. By construction there exist uniquely determined $i, j \in \{1,\ldots,m\}$ such that $u^r \in t_i\mathbb{K}[W_i]$ and $v^s \in t_j\mathbb{K}[W_j]$ for large enough exponents r and s. Moreover, $Z_u = W_i$ and $Z_v = W_j$. Since the summands originate from a Stanley decomposition either we have $i = j$ or $t_j\mathbb{K}[W_j] \cap t_i\mathbb{K}[W_i] = \{0\}$. In the latter case it follows $v^s\mathbb{K}[Z_v] \cap u^r\mathbb{K}[Z_u] = \{0\}$ and, hence, $v\mathbb{K}[Z_v] \cap u\mathbb{K}[Z_u] = \{0\}$. In the remaining case $i = j$ we have $u^r, v^s \in t_i\mathbb{K}[W_i]$ for all sufficiently large integers r and s. Hence, also $\gcd(u,v)^r = \gcd(u^r, v^r) \in t_i\mathbb{K}[W_i]$ for all sufficiently large exponents r. By construction $Z_{\gcd(u,v)} = W_i$ and, consequently, $u\mathbb{K}[Z_u] + v\mathbb{K}[Z_v] \subseteq \gcd(u,v)\mathbb{K}[Z_{\gcd(u,v)}]$. In conclusion, removing all redundant summands leads to a decomposition $\tilde{M} = \bigoplus_{i=1}^{l} v_i\mathbb{K}[Z_{v_i}]$ which proves $\operatorname{Sdepth} \tilde{M} \ge \min_{i=1}^{l} |Z_{v_i}| \ge \operatorname{Sdepth} M$. □

Of course, there are ideals I for which the inequality is proper. For instance, equality can never hold in the case that I has embedded components and $\operatorname{Sdepth} \tilde{M} = \dim \tilde{M}$. The following example illustrates that we need not to have equality even in the pure case.

Example 2 Consider the monomial ideal $I = (x_1^2, x_2^2, x_3) \cap (x_2, x_3, x_4) \cap (x_3, x_4^2, x_5) = (x_3, x_2^2x_5, x_2^2x_4^2, x_1^2x_4x_5, x_1^2x_4^2, x_1^2x_2x_5)$ and its radical $\sqrt{I} = (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_3, x_4, x_5)$. Application of Corollary 4 (see page 65) gives $\tilde{M} = \mathbb{K}[x_4, x_5] \oplus x_1\mathbb{K}[x_1, x_5] \oplus x_2\mathbb{K}[x_1, x_2]$. However, $\operatorname{Sdepth} M < 2$ since there is an unresolvable conflict between the elements $x_1^2x_4$ and $x_2^2x_4$ on the one hand side and the elements x_4^2 and x_4x_5 on the other hand side. While the first two elements require a direct summand containing $x_4\mathbb{K}[x_1, x_2]$ the other require $x_4\mathbb{K}[x_4, x_5]$, a contradiction. A Stanley decomposition of M is $M = \mathbb{K}[x_4, x_5]\oplus$

$x_1\mathbb{K}[x_1, x_5] \oplus x_2\mathbb{K}[x_1, x_2] \oplus x_1x_4\mathbb{K}[x_4, x_5] \oplus x_2x_4\mathbb{K}[x_4, x_5] \oplus x_2x_5\mathbb{K}[x_5] \oplus x_1x_2x_4\mathbb{K}[x_4, x_5] \oplus x_1x_2x_5\mathbb{K}[x_5] \oplus x_1^2x_4\mathbb{K}[x_1, x_2] \oplus x_2^2x_4\mathbb{K}[x_2] \oplus x_1x_2^2x_4\mathbb{K}[x_2]$.

The next example demonstrates a situation where the Stanley dimensions of M and $\tilde{M}$ coincide.

Example 3 Let us consider the generic monomial Cohen-Macaulay ideal $I = (x, y^3z^2u) \subset \mathbb{K}[x, y, z, u]$. I has the irredundant decomposition in irreducible components $I = (x, y^3) \cap (x, z^2) \cap (x, u)$ and $M = \mathbb{K}[z, u] \oplus y\mathbb{K}[z, u] \oplus y^2\mathbb{K}[z, u] \oplus y^3\mathbb{K}[y, u] \oplus y^3z\mathbb{K}[y, u] \oplus y^3z^2\mathbb{K}[y, z]$ is a Stanley decomposition of $M = R/I$ according to Theorem 2. Application of the proof of Theorem 1 yields the decomposition $\tilde{M} = \mathbb{K}[z, u] \oplus y\mathbb{K}[y, u] \oplus yz\mathbb{K}[y, z]$ of the quotient modulo the radical $\sqrt{I} = (x, yzu) = (x, y) \cap (x, z) \cap (x, u)$. Since the Stanley dimension of $\tilde{M}$ cannot exceed the dimension of $\tilde{M}$ this is already a Stanley decomposition.

In this example we can go also in the opposite direction, i.e. we can complete the above Stanley decomposition of $\tilde{M}$ to a Stanley decomposition $M = \mathbb{K}[z, u] \oplus y\mathbb{K}[y, u] \oplus yz\mathbb{K}[y, z] \oplus yzu\mathbb{K}[z, u] \oplus y^2zu\mathbb{K}[z, u]$ of M.

Let us discuss the usefulness and the limits of the methods applied in the proof of Theorem 1. Given a Stanley decomposition of M we can decompose $\tilde{M}$ and we have some guaranteed quality of the resulting decomposition, i.e. it is not worse than that of M. However if $\operatorname{Sdepth} M < \operatorname{Sdepth} \tilde{M}$ then we will not obtain a Stanley decomposition of $\tilde{M}$, in general. For instance, replacing the first direct summand in the Stanley decomposition of M in Example 2 according to $\mathbb{K}[x_4, x_5] = \mathbb{K}[x_4] \oplus x_5\mathbb{K}[x_5] \oplus x_4x_5\mathbb{K}[x_4, x_5]$ will maintain the Stanley decomposition property for M but lifting the new decomposition to a decomposition of $\tilde{M}$ will not provide a Stanley decomposition anymore. This is of course not surprising but what one really would like to have is the opposite construction which could be applied successfully in both Examples 2 and 3, namely to complete a given Stanley decomposition of $\tilde{M}$ to a Stanley decomposition of M by adding some direct summands. There seems to be a good chance that this is always possible. However, this question remains open.

5. Main theorem

In this section we will prove a statement which will turn out to be fundamental for the verification of Stanley's Conjecture for large classes of monomial quotient rings in the subsequent sections.

Theorem 2 *Let I be a monomial ideal with irredundant primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$. For $i = 1, \ldots, k$ let $\mathfrak{p}_i$ denote the associated prime ideal and Y_i the maximal independent variable set of $\mathfrak{q}_i$. Further, let $J_1 := R$ and $J_{i+1} := \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_i$ for $i \in \{1, \ldots, k-1\}$. Finally, for each $i \in \{1, \ldots, k\}$ define T_i to be the set of all monomials $t \in J_i \backslash \mathfrak{q}_i$ such that $\frac{t}{x} \notin J_i$ for all variables $x \in (\operatorname{supp} t) \cap Y_i$.*

If any two distinct monomials belonging to the same set T_i differ in the degree of at least one variable $x \in \mathfrak{p}_i$, $i \in \{1, \ldots, k\}$, then it holds $\operatorname{Sdepth} M = \min_{\mathfrak{p}_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$ and $M = \bigoplus_{i=1}^{k} \bigoplus_{t \in T_i} t\mathbb{K}[Y_i]$ is a Stanley decomposition of M.

Proof: The inclusion $M \supseteq \sum_{i=1}^{k} \sum_{t \in T_i} t\mathbb{K}[Y_i]$ follows immediately from $t\mathbb{K}[Y_i] \cap \mathfrak{q}_i = \{0\}$ for all $t \in T_i$. Now, let s be a standard monomial modulo I and $i \in \{1, \dots, k-1\}$ be the uniquely determined integer such that $s \in J_i \setminus J_{i+1}$. Then there exists $t \in T_i$ such that $s \in t\mathbb{K}[Y_i]$. This proves also the other inclusion $M \subseteq \sum_{i=1}^{k} \sum_{t \in T_i} t\mathbb{K}[Y_i]$.

It remains to show that all sums are direct. Each inner sum is direct since any two distinct monomials $t, t' \in T_i$, $i \in \{1, \dots, k\}$, differ in the degree of at least one variable $x \in \mathfrak{p}_i$ and, consequently, we have even either $t\mathbb{K}[Y_i] \cap t'\mathbb{K}[X] = \{0\}$ or $t\mathbb{K}[X] \cap t'\mathbb{K}[Y_i] = \{0\}$.

Next, consider two monomials $t \in T_i$, $t' \in T_j$ for distinct integers $i, j \in \{1, \dots, k\}$. W.l.o.g. assume $i < j$. Then by construction $t' \in \mathfrak{q}_i$ and $t\mathbb{K}[Y_i] \cap \mathfrak{q}_i = \{0\}$. This implies $t'\mathbb{K}[Y_j] \subseteq \mathfrak{q}_i$ and, hence, $t\mathbb{K}[Y_i] \cap t'\mathbb{K}[Y_j] = \{0\}$ proving that also the outer sum is direct.

In conclusion, Sdepth $M \geq \min_{p_i \in \mathrm{Ass}\, M} \dim R/\mathfrak{p}_i$ which has to be equality in view of inequality (4). □

Corollary 1 *Let $I \subset R$ be a monomial ideal which possesses an irredundant primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ such that, for all $i = 2, \dots, k$, $\mathfrak{q}_i$ contains all but one of the minimal generators of the monomial ideal $J_i := \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{i-1}$.*

Then Sdepth $M = \min_{\mathfrak{p}_i \in \mathrm{Ass}\, M} \dim R/\mathfrak{p}_i$, *where $M = R/I$. Moreover, a Stanley decomposition of M can be obtained by removing all redundant summands from the inner sums of*

$$M = \bigoplus_{i=1}^{k} \sum_{t \in J_i \setminus \mathfrak{q}_i} t\mathbb{K}[Y_i], \tag{5}$$

where $J_1 := R$.

Proof: Consider $t, t' \in J_i \setminus \mathfrak{q}_i$, $i \in \{1, \dots, k\}$, such that $\frac{t}{x} \notin J_i$ for all $x \in (\operatorname{supp} t) \cap Y_i$ and $\frac{t'}{x} \notin J_i$ for all $x \in (\operatorname{supp} t') \cap Y_i$. Let t_i denote the uniquely determined minimal generator of J_i which is not contained in $\mathfrak{q}_i$. Then we have $t_i \mid \gcd(t, t')$ and $\deg_x t_i = \deg_x t = \deg_x t'$ for all variables $x \in Y_i$. Hence, either $t = t'$ or $\deg_y t \neq \deg_y t'$ for at least one variable $y \in \mathfrak{p}_i$.

Now, apply Theorem 2 and the assertion will follow. □

Example 4 Consider the generic monomial Cohen-Macaulay ideal $I = (x^2, y^2z^2, yzw, w^2) = (x^2, y, w^2) \cap (x^2, z, w^2) \cap (x^2, y^2, w) \cap (x^2, z^2, w)$ of $R = \mathbb{K}[x, y, z, w]$.

With the notation of Corollary 1 the monomials $t_1 = 1, t_2 = y, t_3 = yz, t_4 = y^2z$ are the unique minimal generators of J_i which do not belong to $\mathfrak{q}_i$, $i = 1, 2, 3, 4$.

Hence, $M = \mathbb{K}[z] \oplus y\mathbb{K}[y] \oplus yz\mathbb{K}[z] \oplus y^2z\mathbb{K}[y] \oplus x\mathbb{K}[z] \oplus w\mathbb{K}[z] \oplus xw\mathbb{K}[z] \oplus yw\mathbb{K}[y] \oplus xy\mathbb{K}[y] \oplus xyw\mathbb{K}[y] \oplus xyz\mathbb{K}[z] \oplus xy^2z\mathbb{K}[y]$ is a Stanley decomposition of $M = R/I$.

However, the corollary will not be applicable to the irredundant decomposition $I = (x^2, y^2, w^2, yw) \cap (x^2, z^2, w^2, zw)$ in maximal primary ideals.

The above example demonstrates that it is sometimes better to work with irreducible rather than with maximal primary ideals and this will become even more apparent in the next sections. However, it should be mentioned that there are also situations where the use of maximal primary ideals is preferable. Consider, for instance, the simple case that I is a primary ideal. While application of Corollary 1 to the primary decomposition $I = \mathfrak{q}$ immediately yield Sdepth $R/I = \dim R/I$ and a Stanley decomposition of R/I it may happen that no permutation of the irreducible components of I allows the application of Theorem 2, a simple example showing this behavior is $I = (x^2, y^2, z, u) \cap (x, y, z^2, u^2)$.

6. Borel-fixed and one-dimensional quotients

Next, we will show how Theorem 2 can be applied in order to confirm Stanley's conjecture in two particular cases where the validity is well-known.

Corollary 2 *Stanley's Conjecture holds for all modules $M = R/I$, where I is a Borel-fixed ideal.*

Proof: If the irreducible components $\mathfrak{q}_1, \ldots, \mathfrak{q}_k$ of I are enumerated according to decreasing dimension of $R/\mathfrak{q}_i$ then for any $l \in \{2, \ldots, k\}$ we have the inclusion $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_{l-1} \subseteq \mathfrak{p}_l$ for the associated prime ideals according to [6, Corollary 15.25]. Hence, we can apply Theorem 2. □

This reflects the well-known fact that Stanley's Conjecture holds for $\mathbb{Z}$-graded R-modules M over $\mathbb{N}$-graded $\mathbb{K}$-algebras R and, hence, also for $\mathbb{Z}^\nu$-graded modules $M = R/I$ over $\mathbb{N}^\nu$-graded $\mathbb{K}$-algebras R when I is in generic coordinates. Under the additional assumption of an infinite field $\mathbb{K}$ such a decomposition was given by [10]. For arbitrary $\mathbb{K}$ a suitable decomposition is due to [2]. For an algorithmic construction of such decompositions we refer to [16].

Corollary 3 *Let $I \subset R$ be a monomial ideal and $M = R/I$. If* $\dim M \leq 1$ *then* Sdepth $M = \min_{\mathfrak{p} \in \operatorname{Ass} M} \dim R/\mathfrak{p}$.

Proof: Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ be an irredundant decomposition in irreducible components and consider an arbitrary fixed $l \in \{2, \ldots, k\}$. Since $\dim R/\mathfrak{q}_l \leq \dim R/I \leq 1$ any two monomials of $(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}) \setminus \mathfrak{q}_l$ which coincide in the degree of all variables belonging to $\mathfrak{p}_l$ can differ only in the degree of at most one variable. Therefore, one is a multiple of the other. Now, the assertion follows from Theorem 2. □

7. Quotients modulo squarefree monomial ideals

Let us consider the particular case of squarefree monomial ideals I in a little more detail. Then all primary components $\mathfrak{q}_i$, $i = 1, \ldots, k$, of I are even prime, i.e. $\mathfrak{q}_i = \mathfrak{p}_i$, and I has no embedded components. Equation (5) of Corollary 1 can be simplified in the squarefree case and one easily observes:

Corollary 4 *Let $I \subset R$ be a squarefree monomial ideal. If the associated primes of the module $M = R/I$ can be ordered in such a way that for each $i = 2, \dots, k$ the set $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{i-1} \setminus \mathfrak{p}_i$ contains exactly one minimal monomial t_i then* $\operatorname{Sdepth} M = \min_{p_i \in \operatorname{Ass} M} \dim R/\mathfrak{p}_i$. *Moreover, setting $t_1 = 1$ we obtain the Stanley decomposition*

$$M = \bigoplus_{i=1}^{k} t_i \mathbb{K}[Y_i]. \tag{6}$$

The next example illustrates that, in general, a suitable order of primary components need not exist.

Example 5 ([11], Remark 3) Consider Reisner's example $I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6) \subset \mathbb{K}[x_1, x_2, x_3, x_4, x_5, x_6]$ of a monomial ideal which is Cohen-Macaulay if and only if the characteristic of $\mathbb{K}$ is not 2. Localization at the prime ideal $\mathfrak{p} = (x_1, x_2, x_3, x_4, x_5)$ yields $I_{(\mathfrak{p})} = (x_1x_4, x_1x_5, x_2x_3, x_2x_5, x_3x_4)$. The associated primes of the module $M_{(\mathfrak{p})} = R_{(\mathfrak{p})}/I_{(\mathfrak{p})}$ are $\mathfrak{p}_1 = (x_1, x_2, x_3)$, $\mathfrak{p}_2 = (x_1, x_2, x_4)$, $\mathfrak{p}_3 = (x_1, x_3, x_5)$, $\mathfrak{p}_4 = (x_2, x_4, x_5)$, $\mathfrak{p}_5 = (x_3, x_4, x_5)$. Application of Corollary 4 yields $M_{(\mathfrak{p})} = \mathbb{K}[x_4, x_5] \oplus x_3\mathbb{K}[x_3, x_5] \oplus x_2\mathbb{K}[x_2, x_4] \oplus x_1\mathbb{K}[x_1, x_3] \oplus x_1x_2\mathbb{K}[x_1, x_2]$.

Now, we ask for a Stanley decomposition of the quotient $M = R/I$. Since no confusion is possible we will denote the associated prime ideal $\mathfrak{p}_i \cap R$, $i = 1, \dots, 5$, by the same symbol $\mathfrak{p}_i$. The missing associated primes of I containing x_6 are $\mathfrak{p}_6 = (x_1, x_4, x_6)$, $\mathfrak{p}_7 = (x_1, x_5, x_6)$, $\mathfrak{p}_8 = (x_2, x_3, x_6)$, $\mathfrak{p}_9 = (x_2, x_5, x_6)$, and $\mathfrak{p}_{10} = (x_3, x_4, x_6)$. One easily observes that the sequence $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4, \mathfrak{p}_5$ cannot be extended by appending one of the primes $\mathfrak{p}_6, \dots, \mathfrak{p}_{10}$ towards a sequence allowing the application of Corollary 4. Moreover, we checked using a computer program that the corollary is not applicable to the intersection of any more than 5 associated prime ideals of M.

Nevertheless, it holds $\operatorname{Sdepth} M = \dim M$ since $M = \mathbb{K}[x_4, x_5, x_6] \oplus x_3\mathbb{K}[x_3, x_5, x_6] \oplus x_2\mathbb{K}[x_2, x_3, x_5] \oplus x_1\mathbb{K}[x_1, x_3, x_6] \oplus x_1x_2\mathbb{K}[x_1, x_2, x_5] \oplus x_1x_4\mathbb{K}[x_1, x_3, x_4] \oplus x_1x_5\mathbb{K}[x_1, x_4, x_5] \oplus x_2x_4\mathbb{K}[x_2, x_4, x_6] \oplus x_2x_6\mathbb{K}[x_1, x_2, x_6] \oplus x_3x_4\mathbb{K}[x_2, x_3, x_4]$ is a Stanley decomposition of M.

There is a strong relationship between Corollary 4 and shellable nonpure simplicial complexes in the sense of [5]. In fact, our assumptions turn out to be an algebraic translation of Björner's and Wachs' notion in terms of the corresponding Stanley-Reisner ring. Our decompositions are directly related to Formula 2.2 of [5] rather than to the decompositions of Stanley-Reisner rings investigated in Section 12 of the second part of their studies. In the pure case the connection can be found also in [14], where it is shown that shellability of the Stanley-Reisner complex implies the Cohen-Macaulay property of the Stanley-Reisner ring independent on the field characteristic. Therefore, the attempt to treat Reisner's Example 5 by Corollary 4 must have failed. Note, that there is also an algebraic counterpart to partitionable complexes [14] which could be applied successfully to the computation of the Stanley decomposition in Example 5. The interested reader might wish to consult Proposition 2 or its nonsquarefree generalization Theorem 2 in (MSRI Preprint 2001-009) for details.

Corollary 4 provides a Stanley decomposition whose summands are in bijection with the associated primes of I. The following proposition shows that in case of pure dimensional squarefree ideals I every Stanley decomposition of R/I has this nice structure.

Proposition 1 *Let $I \subset R = \mathbb{K}[X]$ be a pure squarefree monomial ideal such that* Sdepth $M = \dim M$, *where $M = R/I$. Then the direct summands of an arbitrary Stanley decomposition are in a natural bijection with the associated primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ of M, more precisely, each Stanley decomposition of M has the form*

$$M = \bigoplus_{i=1}^{k} t_i \mathbb{K}[Y_i]. \tag{7}$$

Proof: Consider an arbitrary Stanley decomposition

$$M = \bigoplus_{i=1}^{m} s_i \mathbb{K}[Z_i].$$

From Sdepth $M = \dim M$ we deduce $|Z_i| \geq \dim M$ and, therefore, $Z_i \in \{Y_1, \ldots, Y_k\}$ for all $i = 1, \ldots, m$. Let $1 \leq j \leq k$ and consider a monomial $t \in M$ which belongs to all but the j-th associated prime of M. Furthermore, let i be such that $t \in s_i\mathbb{K}[Z_i]$. Then we must have $Z_i = Y_j$ since, otherwise, $s_i\mathbb{K}[Z_i] \cap I \supseteq t\mathbb{K}[Z_i] \cap I \neq \{0\}$. Hence, for each associated prime $\mathfrak{p}_j$ the Stanley decomposition contains at least one direct summand such that $Z_i = Y_j$.

Now, consider two direct summands $s_i\mathbb{K}[Z_i]$ and $s_l\mathbb{K}[Z_l]$ satisfying $Z_i = Z_l = Y_j$ for some $j \in \{1, \ldots, k\}$. It follows $\operatorname{supp} s_i \subseteq Y_j$ and $\operatorname{supp} s_l \subseteq Y_j$ since, otherwise, $(s_i\mathbb{K}[Z_i] + s_l\mathbb{K}[Z_l]) \cap I \neq \{0\}$. Hence $\operatorname{lcm}(s_i, s_l) \in s_i\mathbb{K}[Z_i] \cap s_l\mathbb{K}[Z_l]$, which implies $i = l$ because the sum is direct. □

In the nonpure squarefree case the number of summands of a Stanley decomposition may exceed the number of prime components even in case Sdepth $R/I = \min_{\mathfrak{p}\in \operatorname{Ass} R/I} \dim R/\mathfrak{p}$ as the following simple example shows.

Example 6 Two Stanley decompositions of $M = R/I$, where $I = (x, y) \cap (y, z, u) \subset \mathbb{K}[x, y, z, u]$, are $M = \mathbb{K}[z, u] \oplus x\mathbb{K}[x]$ and $M = \mathbb{K}[z] \oplus u\mathbb{K}[u] \oplus zu\mathbb{K}[z, u] \oplus x\mathbb{K}[x]$.

8. Quotients modulo generic monomial ideals

The central result of this section consists of the following theorem.

Theorem 3 *Let $I \subset R$ be a generic monomial ideal and $M = R/I$. Then* Sdepth $M = \min_{\mathfrak{p}\in \operatorname{Ass} M} \dim R/\mathfrak{p}$.

Before, we are able to prove the theorem we need to show some preliminary facts.

Remark 1 Let $I \subset \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal and $m_1, \dots, m_l$ its minimal generators ordered in such a way that $\deg_{x_1} m_i \le \deg_{x_1} m_{i+1}$ for all $i \in \{1, \dots, l-1\}$. Furthermore, let $r \in \{1, \dots, l\}$ be such that $\deg_{x_1} m_{r-1} < \deg_{x_1} m_r$ and set $d := \deg_{x_1} m_r$. Then the overideal $I_d \supseteq I$ generated by $\{x_1^d, m_1|_{x_1=1}, \dots, m_{r-1}|_{x_1=1}\}$, where $m_i|_{x_1=1}$ is obtained from m_i by setting $x_1 = 1$ $(1 \le i < r)$, has the following properties:

1. if I is generic, then so is I_d,
2. each irreducible component of I which has x_1^d as a minimal generator is also an irreducible component of I_d,
3. x_1^d is a minimal generator of each irreducible component of I_d.

The given generating set of I_d need not be minimal, nevertheless, the first and the third property are obvious. The second property follows immediately by repeated application of the well-known formula

$$(ts)R + J = (tR + J) \cap (sR + J), \tag{8}$$

which is valid for arbitrary monomial ideals J and arbitrary monomials t and s such that $\gcd(t, s) = 1$. The statement of the remark extends to the ideal $I_\infty := (m_1|_{x_1=1}, \dots, m_l|_{x_1=1})$. In this case the second condition has to be understood as: each irreducible component of I whose associated prime ideal does not contain x_1 is an irreducible component of I_∞. The third condition reads as $x_1 \notin \mathfrak{p}$ for all associated primes of R/I_∞. Obviously, I_∞ can be identified with $I_{(x_2,\dots,x_n)}$, the localization of I at the prime ideal $(x_2, \dots, x_n)$, in a natural way. In this sense Remark 1 generalizes Remark 2.1 from [9].

In the following we will frequently need a certain order between irreducible ideals. We will say that the irreducible ideal $\mathfrak{q}$ is lexicographically smaller than the irreducible ideal $\mathfrak{q}'$ (notation $\mathfrak{q} \prec \mathfrak{q}'$) if there exists $i \in \{1, \dots, n\}$ such that $\{x_j^a \mid 1 \le j < i,\ a \ge 0\} \cap \mathfrak{q} = \{x_j^a \mid 1 \le j < i,\ a \ge 0\} \cap \mathfrak{q}'$ and $x_i^b \in \mathfrak{q} \backslash \mathfrak{q}'$ for a suitable nonnegative integer b.

Lemma 2 *Let the monomial ideal $I \subset \mathbb{K}[x_1, \dots, x_n]$ be generic with irredundant decomposition $I = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_k$ in irreducible components, where $\mathfrak{q}_k \prec \mathfrak{q}_{k-1} \prec \dots \prec \mathfrak{q}_1$. Furthermore, let d be a positive integer such that $x_1^d \notin I$ but x_1^d is a minimal generator for some irreducible component of I, and let $l \in \{2, \dots, k\}$ be minimal satisfying $x_1^d \in \mathfrak{q}_l$. Define $\tilde{I} := \bigcap_{i=1}^{l-1} \mathfrak{q}_i$ as the intersection of all irreducible components which do not contain x_1^d. Then the set $\tilde{I} \backslash \mathfrak{q}$, where $\mathfrak{q}$ is an arbitrary irreducible component of I having x_1^d as a minimal generator, contains exactly one minimal monomial t. Moreover, $x_1^d t$ is a minimal generator of I.*

Proof: Recall by construction $\mathfrak{q} = \mathfrak{q}_l$ is one possible choice, but there may be others.

Consider two minimal monomials t and t' of $\tilde{I} \backslash \mathfrak{q}$. By construction it follows immediately $\tau, \tau' \in I$, where $\tau := \operatorname{lcm}(x_1^d, t) = x_1^d t$ and $\tau' := \operatorname{lcm}(x_1^d, t') = x_1^d t'$. Let m and m' be minimal generators of I such that $m \mid \tau$ and $m' \mid \tau'$. Since $\frac{\tau}{x_1}, \frac{\tau'}{x_1} \notin \mathfrak{q}$ we have $\deg_{x_1} m = \deg_{x_1} m' = d$. By minimality of t and t' it follows $\frac{\tau}{x_i} \notin I$ for all $x_i \in \operatorname{supp} t$ and $\frac{\tau'}{x_i} \notin I$ for all $x_i \in \operatorname{supp} t'$. Hence, $\tau = m$ and $\tau' = m'$.

If $m = m'$ then it will follow also $t = t'$ since two minimal monomials of $\tilde{I}\backslash\mathfrak{q}$ cannot differ only in the degree of x_1. So let us assume $m \neq m'$. Then by genericity of I there exists a minimal generator m'' of I which satisfies $\deg_{x_i} m'' < \max(\deg_{x_i} m, \deg_{x_i} m')$ for all $x_i \in \operatorname{supp} m''$. Hence, $\deg_{x_1} m'' < d$ and, consequently, $m''|_{x_1=1} \in \mathfrak{q}$. But since m'' divides $\operatorname{lcm}(\tau, \tau')$ we have also that $m''|_{x_1=1}$ divides $\operatorname{lcm}(t, t')$. This implies $\operatorname{lcm}(t, t') \in \mathfrak{q}$ and since $\mathfrak{q}$ is irreducible we must have $t \in \mathfrak{q}$ or $t' \in \mathfrak{q}$, a contradiction.

In summary, we observed that $\tilde{I}\backslash\mathfrak{q}$ contains exactly one minimal monomial t and that $x_1^d t$ is a minimal generator of I. □

Proposition 2 *Let I be a generic monomial ideal and $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ its irredundant decomposition into irreducible components, where $\mathfrak{q}_k \prec \mathfrak{q}_{k-1} \prec \cdots \prec \mathfrak{q}_1$. Then for each $l \in \{2, \ldots, k\}$ the set $(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}) \backslash \mathfrak{q}_l$ contains exactly one minimal monomial.*

Proof: We proceed by induction on the number of variables n. The initial case $n = 1$ is trivial. So let us assume that the statement holds for $(n - 1)$ variables. In particular, the statement holds for the localization I_∞ and for each ideal I_d introduced in Remark 1, in the latter case just consider the quotient $I_d/(x_1)$ and lift the result.

Fix an arbitrary $l \in \{2, \ldots, k\}$ and consider the set $D = (\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1})\backslash\mathfrak{q}_l$. If $x_1 \notin \mathfrak{p}_l$ then the assertion follows immediately by the corresponding property of I_∞. So, consider the case that x_1^d is a minimal generator of $\mathfrak{q}_l$ for some positive integer d. If $x_1^d \notin \mathfrak{q}_{l-1}$ the assertion follows from Lemma 2. So, finally, it remains to consider the case $x_1^d \in \mathfrak{q}_{l-1}$. Let $r \in \{1, \ldots, l - 1\}$ be the minimal index such that $x_1^d \in \mathfrak{q}_r$. If $r = 1$ then the assertion follows immediately by the corresponding property of I_d. Suppose, $1 < r < l$. Then it follows the existence of a uniquely determined minimal monomial $t \in (\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{r-1})\backslash\mathfrak{q}_l$ from Lemma 2. According to Remark 1 I_d has the form $I_d = \tilde{I} \cap \mathfrak{q}_r \cap \cdots \cap \mathfrak{q}_{l'}$, where $l' \in \{l, \ldots, k\}$ is the maximal index such that $x_1^{d-1} \notin \mathfrak{q}_{l'}$ and $\tilde{I}$ is a certain monomial overideal of $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{r-1}$ with the property that x_1^d is a minimal generator for all its irreducible components. Applying the induction assumption to I_d it follows the existence of a uniquely determined minimal monomial $s \in (\tilde{I} \cap \mathfrak{q}_r \cap \cdots \cap \mathfrak{q}_{l-1})\backslash\mathfrak{q}_l$. Since $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{r-1} \cap (\tilde{I} \cap \mathfrak{q}_r \cap \cdots \cap \mathfrak{q}_{l-1}) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}$ any element of D must be a multiple of $\operatorname{lcm}(t, s)$. The simple observation $\operatorname{lcm}(t, s) \in D$ finishes the proof. □

Proof of Theorem 3: The assertion is an immediate consequence of Proposition 2 and Corollary 1. □

9. The 3-variate case

Another class of modules $M = R/I$ for which the validity of Stanley's Conjecture can be proved using Theorem 2 are the quotients modulo monomials ideals in at most three variables.

Proposition 3 *Let $I \subset R = \mathbb{K}[x_1, x_2, x_3]$ be a monomial ideal with irredundant decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ into irreducible components, where* $\dim R/\mathfrak{q}_i \geq \dim R/\mathfrak{q}_j$ *for all integers i and j such that $1 \leq i < j \leq k$. For all $l \in \{2, \ldots, k\}$ set $D_l :=$*

$(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}) \setminus \mathfrak{q}_l$ *and define* $T_l := \{t \in D_l : t$ *is a monomial and* $\forall x \in \operatorname{supp} t \setminus \mathfrak{p}_1 : \frac{t}{x} \notin D_l\}$.

Then for any $l \in \{2, \ldots, k\}$ *and any* $t, t' \in T_l$ *either it holds* $t = t'$ *or there exists* $x \in \mathfrak{p}_l$ *such that* $\deg_x t \neq \deg_x t'$.

Proof: Assume, there exists an ideal $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k \in \mathbb{K}[x_1, x_2, x_3]$ which violates the assertion. Let $l \in \{2, \ldots, k\}$ be minimal such that T_l contains two distinct monomials t and t' which coincide in the degree in all variables belonging to $\mathfrak{p}_l$. Such monomials must differ in the degrees in at least two variables belonging to Y_l, hence, it follows $\dim R/\mathfrak{q}_l \geq 2$. Taking into account the order of irreducible components and the fact that we are in the 3-variate case we can deduce $\dim R/\mathfrak{q}_1 = \cdots = \dim R/\mathfrak{q}_l = 2$. Hence, $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}$ is a principal ideal, obviously this refutes the existence of monomials t and t' with the assumed properties. □

Theorem 4 *Let* $I \subset R = \mathbb{K}[x_1, x_2, x_3]$ *be a monomial ideal and* $M = R/I$. *Then it holds* $\operatorname{Sdepth} M = \min_{\mathfrak{p} \in \operatorname{Ass} M} \dim R/\mathfrak{p}$.

Proof: The assertion is an immediate consequence of Theorem 2 and Proposition 3. □

At the end of this section we present an example which demonstrates that already in four variables we may have $\operatorname{Sdepth} R/I < \dim R/I$ even for pure dimensional monomial ideals which are connected in codimension 1.

Example 7 Consider the cogeneric monomial ideal $I = (x, y^3) \cap (u^2, y^2) \cap (y, z) \cap (u, z^2) \subset \mathbb{K}[x, y, z, u]$. The two monomials y^2u, y^2z^2 belong to all but the first irreducible component, hence, they require a direct summand containing $y^2\mathbb{K}[z, u]$. A similar argument shows that y^3, xy^2 require a direct summand containing $y^2\mathbb{K}[x, y]$. Obviously, both direct summands must be equal and since any module $t\mathbb{K}[Y]$ satisfying $y^2\mathbb{K}[z, u] + y^2\mathbb{K}[x, y] \subseteq t\mathbb{K}[Y]$ will have a nontrivial intersection with I, we have an unresolvable conflict proving $\operatorname{Sdepth} R/I < \dim R/I$.

10. Quotients modulo cogeneric Cohen-Macaulay monomial ideals

Example 7 shows that there is no hope to prove a result for cogeneric monomial ideals which is similarly nice as Theorem 3. However, at least in the Cohen-Macaulay case we will be successful.

Theorem 5 *For any cogeneric Cohen-Macaulay monomial ideal* $I \subset R$ *it holds* $\operatorname{Sdepth} R/I = \dim R/I$.

For the proof we will need the following corollary to Lemma 1.

Corollary 5 *Let* I *be a cogeneric Cohen-Macaulay monomial ideal* I *with irredundant decomposition* $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ *in irreducible components. Furthermore, let* $\mathfrak{q}_i$ *and* $\mathfrak{q}_j$

be irreducible components such that $\operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_j) > \operatorname{codim} I + 1$. *Then there exists an irreducible component* $\mathfrak{q}_m$ *of* I *such that*

1. $\mathfrak{q}_m \subset \mathfrak{q}_i + \mathfrak{q}_j$,
2. $\operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_m) \leq \operatorname{codim} I + 1$,
3. $\mathfrak{q}_i$ *and* $\mathfrak{q}_m$ *have no minimal generator in common*,
4. $x_r^d \notin \mathfrak{q}_m$ *for all minimal generators* x_r^d *of* $\mathfrak{q}_i$ *such that* $x_r^{d-1} \notin \mathfrak{q}_j$.

Proof: Lemma 1 yields the existence of $\mathfrak{q}_{l'}$ such that $\mathfrak{q}_{l'} \subset \mathfrak{q}_i + \mathfrak{q}_j$. Moreover, taking into account the cogenerity of I we can arrange that $\mathfrak{q}_{l'}$ does not contain any of the common minimal generators of $\mathfrak{q}_j$ and $\mathfrak{q}_i + \mathfrak{q}_j$. If $\operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_{l'}) > \operatorname{codim} I + 1$ we repeat this process and after finitely many iterations we obtain an irreducible component $\mathfrak{q}_l$ such that $\mathfrak{q}_l \subset \mathfrak{q}_i + \mathfrak{q}_j$ and $\operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_l) < \operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_j)$. By induction on $\operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_j)$ we can even assume $\operatorname{codim}(\mathfrak{p}_i + \mathfrak{p}_l) \leq \operatorname{codim} I + 1$. Now, suppose $\mathfrak{q}_i$ and $\mathfrak{q}_l$ have a minimal generator in common. Then by definition of cogeneric monomial ideals there exists $\mathfrak{q}_{m'} \subset \mathfrak{q}_i + \mathfrak{q}_l \subseteq \mathfrak{q}_i + \mathfrak{q}_j$ having no minimal generator in common with $\mathfrak{q}_i + \mathfrak{q}_l$. If $\mathfrak{q}_i$ and $\mathfrak{q}_{m'}$ still have common minimal generators we repeat the argument with $l = m'$. Since $\mathfrak{q}_i$ cannot share any of its minimal generators with more than one of the intermediate components appearing in this process, eventually, we obtain $\mathfrak{q}_m$ satisfying the first three conditions of the corollary. Finally, the validity of the fourth condition is an immediate consequence of the validity of the first and the third condition. □

Proposition 4 *Let* I *be a cogeneric Cohen-Macaulay monomial ideal and* $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ *its irredundant decomposition in irreducible components, where the components are enumerated according to lexicographically descending order.*

For each $l \in \{2, \ldots, k\}$ *set* $D_l := (\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}) \setminus \mathfrak{q}_l$ *and define* $T_l := \{t \in D_l : t$ *is a monomial and* $\forall x_i \in \operatorname{supp} t \setminus \mathfrak{p}_l : \frac{t}{x_i} \notin D_l\}$.

Then for all $l \in \{2, \ldots, k\}$ *and all* $t, t' \in T_l$ *either it holds* $t = t'$ *or there exists* $x_i \in \mathfrak{p}_l$ *such that* $\deg_{x_i} t \neq \deg_{x_i} t'$.

Proof: In analogy to the proof of Proposition 2 we proceed by induction on the number of variables n. Again the initial case $n = 1$ is trivial. So let us assume that the statement holds for $(n-1)$ variables.

Suppose there exists $l \in \{2, \ldots, k\}$ such that T_l contains two distinct elements t and t' which have the same degree in all variables belonging to $\mathfrak{p}_l$. Let $\hat{t}$ and $\hat{t'}$ denote the monomials obtained by setting all variables of $\mathfrak{p}_l$ equal to 1 in t and t', respectively. By construction we have $\operatorname{lcm}(s, \hat{t}) = t$ and $\operatorname{lcm}(s, \hat{t}') = t'$, where $s := \gcd(t, t')$ is the greatest common divisor of t and t'. From $t, t' \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}$ we conclude that any irreducible component $\mathfrak{q}_j$, $1 \leq j < l$, which does not contain s has to contain both $\hat{t}$ and $\hat{t'}$ but cannot contain $\gcd(\hat{t}, \hat{t'})$. In conclusion,

$$\forall j \in \{1, \ldots, l-1\} : s \notin \mathfrak{q}_j \Rightarrow \operatorname{codim}(\mathfrak{p}_j + \mathfrak{p}_l) \geq \operatorname{codim} I + 2. \tag{9}$$

To have $s \in \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_{l-1}$ is impossible by construction of T_l. Hence, there exists $j \in \{1, \ldots, l-1\}$ such that $s \notin \mathfrak{q}_j$. Application of Corollary 5 yields the existence of an irreducible component $\mathfrak{q}_m$ with the following three properties: (i) $\mathfrak{q}_m \subset \mathfrak{q}_l + \mathfrak{q}_j$, (ii) codim $(\mathfrak{p}_m + \mathfrak{p}_l) \leq$ codim $I + 1$, and (iii) $x_r^a \notin \mathfrak{q}_m$ for all minimal generators x_r^a of $\mathfrak{q}_l$ such that $x_r^{a-1} \notin \mathfrak{q}_j$. I_∞ as a localization of I at a monomial prime ideal is cogeneric. Hence, application of the induction hypothesis to I_∞ shows $x_1 \in \mathfrak{p}_l$. Therefore, $\mathfrak{q}_l$ possesses a minimal generator of the form x_1^d and by the assumption on the order of the irreducible components of I this implies $x_1^{d-1} \notin \mathfrak{q}_j$ and, therefore, $x_1^d \notin \mathfrak{q}_m$ according to (iii). Consequently, $m < l$ and in view of (9) and (ii) we obtain $s \in \mathfrak{q}_m$. Application of (i) yields $s \in \mathfrak{q}_m \subset \mathfrak{q}_j + \mathfrak{q}_l$, in contradiction to $s \notin \mathfrak{q}_j$ and $s \notin \mathfrak{q}_l$. In summary, the supposition that T_l contains distinct elements t, t' which have the same degree in all variables belonging to $\mathfrak{p}_l$ must have been wrong. □

Proof of Theorem 5: The assertion follows immediately from the above proposition and Theorem 2. □

11. Outlook to graded polynomial modules

Finally, let us generalize our investigations in two directions. As before, let $R = \mathbb{K}[X]$ be a polynomial ring but equipped with a $\mathbb{N}^\nu$-grading for some $\nu \in \{1, \ldots, n\}$. Furthermore, let F be a free R-module with basis $\{e_1, \ldots, e_m\}$ and equipped with a $\mathbb{Z}^\nu$-grading which extends the $\mathbb{N}^\nu$-grading of R in the sense that $\deg_{\mathbb{Z}^\nu} ue_j = \deg_{\mathbb{N}^\nu} u + \deg_{\mathbb{Z}^\nu} e_j$ for arbitrary homogeneous elements $u \in R$ and $1 \leq j \leq m$. Finally, assume $\sum_{i=1}^{\nu} d_i = 1$ for all $x \in X$, where $\deg_{\mathbb{N}^\nu} x = (d_1, \ldots, d_\nu)$. Throughout this section we will always assume that the field $\mathbb{K}$ is infinite.

We consider modules $M = F/N$, where N is a $\mathbb{Z}^\nu$-graded submodule of F. In the particular case that N is monomial, what here should mean, that N is generated by elements of the form ue_j, where $u \in R$ is a monomial and $1 \leq j \leq m$, the previous results extend easily. We subdivide the set B of minimal generators of N in m pairwise disjoint (not necessarily nonempty) sets $B_1, \ldots, B_m$, where B_i consists of all minimal generators of the form ue_i, $1 \leq i \leq m$. Let I_i, $i \in \{1, \ldots, m\}$, denote the monomial ideal of R generated by the set $\{u : ue_i \in B_i\}$. Since, depth $(A \oplus B) = \min(\text{depth } A, \text{depth } B)$ for arbitrary R-modules A and B we can construct a Stanley decomposition for M by plugging in Stanley decompositions for each module R/I_i in the direct sum decomposition $M = \bigoplus_{i=1}^{m} (R/I_i)\, e_i$.

Now, let N be an arbitrary $\mathbb{Z}^\nu$-graded submodule of F and consider an admissible term order $\prec$ of F refining the $\mathbb{Z}^\nu$-grading, i.e. for any monic $\mathbb{Z}^\nu$-homogeneous elements $a, b \in F$ such that $\deg_{\mathbb{Z}^\nu} a < \deg_{\mathbb{Z}^\nu} b$ it holds $\text{in}_\prec a \prec \text{in}_\prec b$. As usual, $\text{in}_\prec a$ denotes the initial term of $a \in F$ with respect to $\prec$. The modules $F/\text{in}_\prec N$ and F/N are isomorphic graded $\mathbb{K}$-spaces via the linear mapping induced by the assignment $a + \text{in}_\prec N \mapsto a + N$ for all module monomials $a = ue_i$. In view of this isomorphism any Stanley decomposition of $\text{in}_\prec M := F/\text{in}_\prec N$ is also a Stanley decomposition of $M = F/N$ whenever it holds depth $M \leq$ depth $\text{in}_\prec M$. Gräbe proved the inequality depth $M \geq$ depth $\text{in}_\prec M$ for arbitrary homogeneous M and there are examples where the inequality is proper ([7], Theorem 4.1). But we have the additional freedom of applying two preparatory transformations before we fix the order $\prec$ and the corresponding initial module $\text{in}_\prec M$. First, we can apply a variable transformation which has to respect the $\mathbb{N}^\nu$-grading of R. Second, we can transform the basis of F in a $\mathbb{Z}^\nu$-grading respecting way. There are examples where both transformations are necessary, but it is open if they are also sufficient.

Remark 2 Let $\mathbb{K}$ be infinite. Suppose that the following two questions have a positive answer:

1. Does Conjecture 1 hold for arbitrary monomial ideals $I \subset R$?
2. Given an arbitrary $\mathbb{Z}^\nu$-graded module $M = F/N$. Does there exist grading respecting transformations of R and F and an admissible term order $\prec$ of the transformed module monomials such that depth $M =$ depth $\text{in}_\prec M$?

Then Stanley's Conjecture holds for arbitrary $\mathbb{Z}^\nu$-graded polynomial modules M.

The proof is an immediate consequence of our above observations. This article is a step towards the solution of the first question. Similar to generic and cogeneric monomial ideals also the initial ideals of toric ideals have a well-studied decomposition in irreducible components [8]. This makes them a next promising candidate for investigating Stanley's Conjecture using our methods.

In the case $\nu = 1$ and $F = R$ a positive answer to the second question was given by Bayer and Stillman who proved that the equality depth $M =$ depth $\text{in}_\prec M$ holds when $\prec$ is a reverse lexicographical term order and M is in generic coordinates ([4], Theorem 2.4), see also ([6], Section 15.7). In the extremal case $F = R$ and $\nu = n$ the second question has obviously a positive answer since $M = \text{in}_\prec M$ even without application of a preparatory transformation. This gives reason to the hope that question 2 has a general positive answer in the case $F = R$, when each group of variables of some fixed $\mathbb{N}^\nu$-degree is transformed generically and an admissible term order $\prec$ is used which acts as a reverse lexicographical term order on each set of transformed variables of a fixed $\mathbb{N}^\nu$-degree. Note, if the variables are enumerated in such a way that variables of the same $\mathbb{N}^\nu$-degree have consecutive numbers then the variable transformations which respect the $\mathbb{N}^\nu$-grading of R correspond to the subgroup of $GL(n, \mathbb{K})$ formed by all block-diagonal matrices with blocks corresponding to groups of variables of common $\mathbb{N}^\nu$-degree. In the 3-variate case it is easy to generalize Theorem 4 in this way.

Theorem 6 *Let $\mathbb{K}$ be an infinite field and $R = \mathbb{K}[x_1, x_2, x_3]$ a $\mathbb{N}^\nu$-graded polynomial ring over $\mathbb{K}$, where $\nu \in \{1, 2, 3\}$. Assume that for each of the variables x_1, x_2, x_3 the sum of the coordinates of its $\mathbb{N}^\nu$-degree is* 1.

Then Sdepth $R/I = \min_{\mathfrak{p}\in \text{Ass}\, R/I} \dim R/\mathfrak{p}$ *for any $\mathbb{N}^\nu$-homogeneous ideal I of R.*

Proof: The assertion follows from [4, Theorem 2.4] in case $\nu = 1$ and from Theorem 4 in case $\nu = 3$. So it remains to consider only the case $\nu = 2$, where we can assume without loss of generality that $\deg_{\mathbb{N}^\nu} x_1 = (1, 0)$ and $\deg_{\mathbb{N}^\nu} x_2 = \deg_{\mathbb{N}^\nu} x_3 = (0, 1)$. Using exactly the same arguments as in [6, Ch. 15] one can show that there is a linear variable transformation fixing x_1 and mapping x_2 and x_3 to linearly independent linear combinations of them such that the initial ideal of the transformed ideal with respect to an arbitrary degree compatible admissible term order satisfying $x_3 \prec x_2$ possesses a minimal generator $x_1^\alpha x_2^\beta$ which has strictly larger x_2-degree than any other minimal generator and which has minimal x_1-degree among all minimal generators. In fact, concentrating only on the transformation of x_2 and x_3 there is a Zariski open subset of $GL(2, \mathbb{K})$ describing suitable transformations. Further, according to [6, Proposition 15.15] it suffices to check the equality $\operatorname{depth} R/I = \operatorname{depth} R/\mathrm{in}_\prec I$ for the case that each variable occurs in some minimal generator of $\mathrm{in}_\prec I$. In this case one easily observes $\operatorname{depth} R/I \leq 1$. Hence, it remains to prove that whenever $\mathrm{in}_\prec I$ possesses a trivial irreducible component then also I possesses a trivial primary component (i.e. $(x_1, x_2, x_3) \in \operatorname{Ass} R/I$).

Now, assume that (x_1^a, x_2^b, x_3^c) is an irreducible component of $\mathrm{in}_\prec I$ and let G denote the reduced Gröbner basis of I with respect to $\prec$. By homogeneity of I each element of G can be written in the form $x_1^{\alpha'}(x_2^{\beta'} + p(x_2, x_3))x_3^{\gamma'}$, where $p(x_2, x_3)$ is a homogeneous polynomial in x_2 and x_3 of $\mathbb{N}$-degree β' without $x_2^{\beta'}$-term. By J we denote the ideal generated by all elements $g|_{x_1=1}$ which are obtained by substituting 1 for x_1 in the Gröbner basis elements $g \in G$ which satisfy $\deg_{x_1} \mathrm{in}_\prec g < a$ and $\deg_{x_3} \mathrm{in}_\prec g < c$. The preparatory transformation ensures that J contains a homogeneous element whose initial term is a power of x_2. Moreover, for any homogeneous element $h \in J$ we have $\deg_{x_2} \mathrm{in}_\prec h \geq b$ or $\deg_{x_3} \mathrm{in}_\prec h \geq c$ since, otherwise, there must be a nonnegative integer $a' < a$ such that $x^{a'}h \in I$ which is impossible because of $x^{a'}\mathrm{in}_\prec h \notin (x_1^a, x_2^b, x_3^c) \supseteq \mathrm{in}_\prec I$. $(x_1^a, x_3^c) + J$ is a (x_1, x_2, x_3)-primary ideal containing I and by the above observations I is not contained in any higher dimensional primary subideal of $(x_1^a, x_3^c) + J$. Hence, $(x_1, x_2, x_3) \in \operatorname{Ass} R/I$ and we are through. □

Finally, let us demonstrate the additional problems arising in the case $\operatorname{rank} F > 1$ using two examples.

Example 8 Consider $R = \mathbb{K}[x_1, x_2, x_3]$ with an arbitrary $\mathbb{N}^\nu$-grading. Further, let $\operatorname{rank} F = 3$ and $\deg_{\mathbb{Z}^\nu} e_1 = \deg_{\mathbb{Z}^\nu} e_2 = \deg_{\mathbb{Z}^\nu} e_3 = (0, \ldots, 0)$. Assume $\operatorname{char} \mathbb{K} \neq 2$ and let $M = F/N$, where $N = (x_1e_1 + x_1e_2, x_2e_1 + x_2e_3, x_3e_2 + x_3e_3)$. Then the elements $x_1e_1, x_2e_1, x_3e_2, x_1x_2e_2, x_1x_2x_3e_3$ generate the initial module $\mathrm{in}_\prec M = F/\mathrm{in}_\prec N$, where $\prec$ is an arbitrary admissible term order such that $e_3 \prec e_2 \prec e_1$. Hence, $\operatorname{depth} \mathrm{in}_\prec M = 1 < \operatorname{depth} M$ and by symmetry reasons the same is true for any admissible term order $\prec$. However, after an obvious basis transformation we obtain $N = (x_1\hat{e}_1, x_2\hat{e}_2, x_3\hat{e}_3)$ and, hence, $\operatorname{depth} \mathrm{in}_\prec M = \operatorname{depth} M = 2$.

Example 9 Consider $R = \mathbb{K}[x_1, \ldots, x_n]$ with respect to an arbitrary $\mathbb{N}^\nu$-grading. Furthermore, let $\operatorname{rank} F = n + 1$ and $\deg_{\mathbb{Z}^\nu} e_1 = (0, \ldots, 0)$, $\deg_{\mathbb{Z}^\nu} e_i = \deg_{\mathbb{N}^\nu} x_{i-1}$ $(i = 2, \ldots, n + 1)$ the degrees of the free generators of F. Then the module $M = F/N$, where N is generated by the elements $x_ie_1 + e_{i+1}$, $i = 1, \ldots, n$, is free of rank 1 and has depth n.

If two module monomials ue_i and ve_j have the same $\mathbb{Z}^\nu$-degree we have to break ties by the order $\prec$. If ties are broken by $ue_i \prec ve_j \Leftrightarrow i < j$ then we obtain $\mathrm{in}_\prec N = (e_2, \ldots, e_{n+1})$ and $\operatorname{depth} \mathrm{in}_\prec M = n = \operatorname{depth} M$. However, $\operatorname{depth} \mathrm{in}_\prec M = 0 < \operatorname{depth} M$ if we break ties according to $ue_i \prec ve_j \Leftrightarrow i > j$.

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Notes

1. Actually this implies that all minimal generators of $\mathfrak{p}$ are variables.
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