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The Subconstituent Algebra of an Association Scheme, (Part I)*

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Abstract. We introduce a method for studying commutative association schemes with "many" vanishing intersection numbers and/or Krein parameters, and apply the method to the P- and Q-polynomial schemes. Let Y denote any commutative association scheme, and fix any vertex x of Y. We introduce a non-commutative, associative, semi-simple $\mathbb{C}$-algebra $T = T(x)$ whose structure reflects the combinatorial structure of Y. We call T *the subconstituent algebra of* Y *with respect to* x. Roughly speaking, T is a combinatorial analog of the centralizer algebra of the stabilizer of x in the automorphism group of Y.

In general, the structure of T is not determined by the intersection numbers of Y, but these parameters do give some information. Indeed, we find a relation among the generators of T for each vanishing intersection number or Krein parameter.

We identify a class of irreducible T-modules whose structure is especially simple, and say the members of this class are *thin*. Expanding on this, we say Y is *thin* if every irreducible $T(y)$-module is thin for every vertex y of Y. We compute the possible thin, irreducible T-modules when Y is P- and Q-polynomial. The ones with sufficiently large dimension are indexed by four bounded integer parameters. If Y is assumed to be thin, then "sufficiently large dimension" means "dimension at least four".

We give a combinatorial characterization of the thin P- and Q-polynomial schemes, and supply a number of examples of these objects. For each example, we show which irreducible T-modules actually occur.

We close with some conjectures and open problems.

Keywords: association scheme, P-polynomial, Q-polynomial, distance-regular graph.

1. Introduction

Commutative association schemes provide an elegant framework for the study of codes [6], [15], [19], [54], [56], [59], [60], [62], designs [7], [8], [13], [14], [65], multiplicity-free permutation groups [10], [28], [39], [43], [58], finite geometry [32], [33], [34], [36], [44], and some questions in topology [49], so they are receiving considerable attention. The papers listed above are very recent. To access work done more than a few years ago, we refer the reader to the 1985 book of Bannai and Ito [3], the 1989 book of Brouwer, Cohen, and Neumaier

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[11], and the survey papers of Bannai [5], [6], Bannai and Ito [4], Faradzhev, Ivanov, and Klin [29], and Ma [48].

In this paper we introduce a method for studying commutative association schemes with "many" vanishing intersection numbers and/or Krein parameters. We have in mind the P- and Q-polynomial schemes and their relatives [3, p260, p316], the antipodal P-polynomial schemes [11, p438], the bipartite P-polynomial schemes [11, p211], schemes with more than one P-polynomial or Q-polynomial structure [3, p238], and the "directed" P- and Q-polynomial schemes [38], [45], [46], [51], [78]. We believe the regular near polygons [11, p198], [53] and the distance-transitive P-polynomial schemes with "almost simple" automorphism group [10], [11, p229], [58] will also yield to our approach, due to their extra structure. See Section 7 for some conjectures and open problems on the above topics. To keep things simple, the focus of this paper will be on the P- and Q-polynomial schemes.

Our idea is summarized as follows (formal definitions will begin in Section 3). Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote any commutative association scheme, with $D \ge 3$, intersection numbers p^k_{ij}, and Krein parameters q^k_{ij} $(0 \le i, j, k \le D)$. Let $A_0, A_1, \cdots, A_D$ denote the associate matrices of Y. Then

$$A_i A_j = \sum_{k=0}^{D} p^k_{ij} A_k \qquad (0 \le i, j \le D),$$

and indeed these matrices form a basis for a semi-simple commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$, known as the *Bose-Mesner algebra*. Let $E_0, E_1, \cdots, E_D$ denote the primitive idempotents of M. For the rest of this section, let x denote a fixed vertex in X. For each integer i $(0 \le i \le D)$, let $E^*_i = E^*_i(x)$ (resp. $A^*_i = A^*_i(x)$) denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$, with y, y entry $(A_i)_{xy}$ (resp. $|X|(E_i)_{xy}$) $(y \in X)$. Then E^*_i represents the orthogonal projection onto the ith subconstituent of Y with respect to x (i.e. the span of all $y \in X$ such that $(x, y) \in R_i$), and

$$A^*_i A^*_j = \sum_{k=0}^{D} q^k_{ij} A^*_k \qquad (0 \le i, j \le D).$$

In fact $A^*_0, A^*_1, \cdots, A^*_D$ form a basis for a semi-simple commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$, and $E^*_0, E^*_1, \cdots, E^*_D$ are the primitive idempotents of M^*. We call M^* the *dual Bose-Mesner algebra with respect to x*. We define the *subconstituent algebra* $T = T(x)$ to be the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M, M^*. T is semi-simple, since it is closed under the conjugate-transpose map, and

$$E^*_i A_j E^*_k = 0 \quad \text{iff} \quad p^k_{ij} = 0 \qquad (0 \le i, j, k \le D), \tag{1}$$

$$E_i A^*_j E_k = 0 \quad \text{iff} \quad q^k_{ij} = 0 \qquad (0 \le i, j, k \le D), \tag{2}$$

are relations in T.

To get an intuitive feel for T, suppose for the moment that the associate classes $R_0, R_1, \cdots, R_D$ are the orbits of the automorphism group Aut(Y) acting on the Cartesian product $X \times X$. Then the Bose-Mesner algebra is the centralizer algebra of Aut(Y) [3, p47]. Whether or not Aut(Y) acts in the above fashion, we may still view the Bose-Mesner algebra as a "combinatorial analog" of this centralizer algebra. Similarily, we may view T as a "combinatorial analog" of the centralizer algebra of the stabilizer of x in Aut(Y). In our main results we assume nothing about Aut(Y).

We may also view T as a homomorphic image of an "abstract subconstituent algebra" $\mathcal{T}$ defined by generators and relations. Recall the *character algebra* C of Y is the $\mathbb{C}$-algebra with basis $x_0, x_1, \cdots, x_D$ such that

$$x_i x_j = \sum_{k=0}^{D} p_{ij}^k x_k \qquad (0 \le i, j \le D)$$

[3, p87]. Then the map $x_i \rightarrow A_i$ $(0 \le i \le D)$ induces a $\mathbb{C}$-algebra isomorphism $C \rightarrow M$. Let e_i denote the primitive idempotent of C mapped to E_i $(0 \le i \le D)$ by this isomorphism. The *dual character algebra* of Y is the $\mathbb{C}$-algebra with basis $x_0^*, x_i^*, \cdots, x_D^*$ such that

$$x_i^* x_j^* = \sum_{k=0}^{D} q_{ij}^k x_k^* \qquad (0 \le i, j \le D)$$

[3, p99]. Then the map $x_i^* \rightarrow A_i^*$ $(0 \le i \le D)$ induces a $\mathbb{C}$-algebra isomorphism $C^* \rightarrow M^*$. Let e_i^* denote the primitive idempotent of C^* mapped to E_i^* $(0 \le i \le D)$ by this second isomorphism. Now let $\mathcal{T}$ denote the $\mathbb{C}$ algebra generated by C, C^*, subject to the relations

$$x_0 = x_0^*,$$
$$e_i^* x_j e_k^* = 0 \quad \text{if} \quad p_{ij}^k = 0 \qquad (0 \le i, j, k \le D),$$
$$e_i x_j^* e_k = 0 \quad \text{if} \quad q_{ij}^k = 0 \qquad (0 \le i, j, k \le D).$$

Then by (1), (2), and since $A_0 = A_0^*$ is the identity in $\text{Mat}_X(\mathbb{C})$, the above mentioned isomorphisms $C \rightarrow M$, $C^* \rightarrow M^*$ extend to a $\mathbb{C}$-algebra homomorphism from $\mathcal{T}$ onto T. The kernel of this homomorphism is non–zero in general, but it seems difficult to describe. It could conceivably vary with the vertex x (if Aut(Y) is not transitive on X), although we have not found any examples where this occurs. We give some open problems concerning $\mathcal{T}$ in Section 7.

Now consider the T-modules. T acts by left multiplication on the Hermitean space V, $\langle , \rangle$, where $V = \mathbb{C}^{|X|}$ (column vectors) and where $\langle , \rangle$ is the standard Hermitean dot product on V. Then V decomposes into an orthogonal direct sum of irreducible T-modules. We would like to identify all the irreducible T-modules in this sum, but this seems too difficult in general. Therefore, we restrict our attention to the following schemes and modules. Y is said to be *P-polynomial* if for all integers i, j, k $(0 \leq i, j, k \leq D)$, $p_{ij}^k = 0$ (resp. $p_{ij}^k \neq 0$) whenever one of i, j, k is greater than (resp. equal to) the sum of the other two. Similarily, Y is said to be *Q-polynomial* if for all integers i, j, k, $(0 \leq i, j, k \leq D), q_{ij}^k = 0$ (resp. $q_{ij}^k \neq 0$) whenever one of i, j, k is greater than (resp. equal to) the sum of the other two. Also, we say an irreducible T-module W is *thin* whenever dim $E_i^* W \leq 1$ for all integers i $(0 \leq i \leq D)$. Then we find the possible thin, irreducible T-modules when Y is P- and Q-polynomial. To describe them, we introduce the notion of a *Leonard system*, and interpret a theorem of Leonard [3, p263], [47], [70] as a classification of these objects. Although we do not take this view here, a Leonard system of diameter d is essentially an ordered pair of dual q-Racah polynomial sequences with highest degree d, where the limiting cases $q \to \pm 1$ are included as in Leonard's theorem. See [1], [2], [30] for information on the q-Racah polynomials. Now assume Y is P- and Q-polynomial, and let W denote a thin, irreducible T-module. Then we show W is naturally associated with a Leonard system, denoted $LS(W)$. The isomorphism class of W is determined by $LS(W)$, which in turn is determined by a 4-tuple (μ, ν, d, f) of parameters, called the *data sequence* of W. The parameter μ (resp. ν) is the least integer i for which $E_i W \neq 0$ (resp. $E_i^* W \neq 0$), and $d =$ dim $W - 1$. The parameter f is harder to describe. It is either an unordered pair of algebraically related complex numbers, an ordered pair of algebraically related complex numbers, or a complex number, depending on Y. However, in many cases, f takes one of a certain set of values indexed by a bounded integer parameter e. When this occurs we say W is *strong*. It turns out W is strong whenever μ, ν, d satisfy certain bounds, as we now indicate. Let y denote a second vertex in X, and let W' denote a thin, irreducible $T(y)$-module such that W, W' are not orthogonal. Then we show the data sequences of W, W' are algebraically related, determining each other up to a few possibilities. Dually, suppose y, W are as above, but that W' is an irreducible $T(x)$-module such that $A_\rho^*(y)W, W'$ are not orthogonal for some integer ρ $(0 \leq \rho \leq D)$. Then again the data sequences of W, W' are algebraically related, determining each other up to a few possibilities. Combining these facts in an inductive argument, we show W is strong whenever

$$\mu < D/2 \quad \text{or} \quad \nu < D/2.$$

Furthermore, if every irreducible $T(z)$-module is thin for every $z \in X$, then W is strong whenever

$$\mu < D/2 \quad \text{or} \quad \nu < D/2 \quad \text{or} \quad d \geq 3.$$

The above results suggest looking at the following class of schemes. Assume Y is an arbitrary commutative scheme, and let us say Y is *thin with respect to x* if each irreducible $T(x)$-module is thin. Then we give some ways to determine if Y is thin with respect to x. Indeed, suppose that for all $y, z \in X$ where (x, y), (x, z) are in the same associate class, there exists some $g \in$ Aut (Y) such that

$gx = x$, $gy = z$, $gz = y$. Then Y is thin with respect to x. Next suppose that Y is P-polynomial. Then Y is thin with respect to x if and only if for all integers i, j, k $(0 \le i, j, k \le D)$, and all $y, z \in X$ with (x, y), $(x, z) \in R_i$, the number of $w \in X$ with $(x, w) \in R_j$, $(y, w) \in R_1, (z, w) \in R_k$ equals the number of $w' \in X$ with $(x, w') \in R_j, (y, w') \in R_k, (z, w') \in R_1$. A similar result holds if Y is Q-polynomial. Now suppose Y is P- and Q-polynomial. Then Y is thin with respect to x if and only if for all integers i $(2 \le i \le D-1)$, and all $y, z \in X$ with $(x, y), (x, z) \in R_i$, the number of $w \in X$ with $(x, w) \in R_i, (y, w) \in R_1, (z, w) \in R_2$ equals the number of $w' \in X$ with $(x, w') \in R_i, (y, w') \in R_2$, and $(z, w') \in R_1$.

Let us say Y is *thin* if Y is thin with respect to every vertex. We show that if Y is P- and Q-polynomial with

$$p^i_{1i} = 0 \quad \text{for all integers } i \qquad (2 \le i \le D-1),$$

or

$$q^i_{1i} = 0 \quad \text{for all integers } i \qquad (2 \le i \le D-1),$$

then Y is thin. Also, many of the known P- and Q-polynomial schemes are thin. Indeed, suppose Y is a known P- and Q-polynomial scheme with $D \ge 6$, but not a Doob scheme [11, p27], the "Bilinear forms" scheme $H_q(N, D)$ [3, p306], the "Alternating forms" scheme $Alt_q(N)$ $(D = \lfloor \frac{N}{2} \rfloor)$ [3, p307], the "Hermitean forms" scheme $Her_q(D)$ [3, p308], or the "Quadratic forms" scheme $Quad_q(N)$ $(D = \lfloor \frac{N+1}{2} \rfloor)$ [3, p308]. Then Y is thin. In each case, we show which of the possible irreducible T-modules actually occur.

The paper is organized as follows. In Section 2, we assemble some results on Leonard systems. Section 3 contains some basic properties of association schemes and subconstituent algebras. In Section 4 we give the structure of a thin irreducible T-module when Y is P- and Q-polynomial. Section 5 considers when Y is thin with respect to a given vertex, and Section 6 is devoted to examples of thin P- and Q-polynomial schemes. Section 7 contains some open problems and suggestions for further research. The paper is self contained, except for the material in Section 2, some preliminary material in Section 3, and Lemma 4.7. Proofs of the material in Section 2 can be found in [70]. The preliminary material in Section 3 can be found in [3, pp52–70], and the proof of Lemma 4.7 can be found in [68].

We will use the following notation. $\mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ will denote the integers, the real numbers, and the complex numbers, respectively. The symbol $\preceq$ will denote an arbitrary but fixed linear order on $\mathbb{C}$. For example, we may take

$$a + bi \preceq c + di \qquad (a, b, c, d \in \mathbb{R}, i^2 = -1)$$

whenever

$$a < c$$

or

$a = c$ and $b \leq d$.

For any nonnegative integer n and any positive integer m,

$$(a)_n = 1 \text{ if } n = 0, \text{ and } a(a-1)\cdots(a-n+1) \text{ if } n > 0.$$
$$(a_1, a_2, \cdots, a_m)_n = (a_1)_n(a_2)_n \cdots (a_m)_n.$$
$$(a;q)_n = 1 \text{ if } n = 0, \text{ and } (1-a)(1-aq)\cdots(1-aq^{n-1}) \text{ if } n > 0.$$
$$(a_1, a_2, \cdots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n.$$

The Kronecker delta δ_{xy} is 1 if $x = y$, and 0 if $x \neq y$. For any real number h, $\lfloor h \rfloor$ will denote the greatest integer less than or equal to h, and $\lceil h \rceil$ will denote the least integer greater than or equal to h. By an *algebra,* we mean an associative algebra with identity. Fix a positive integer n. Then $\mathrm{Mat}_n(\mathbb{C})$ will denote the $\mathbb{C}$-algebra of all $n \times n$ matrices with entries in $\mathbb{C}$. Let $V = \mathbb{C}^n$ (column vectors), and let $\langle , \rangle$ denote the Hermitean form

$$\langle u, v \rangle = u^t \overline{v} \qquad (u, v \in V),$$

where t denotes transpose and $^-$ denotes complex conjugate. Observe $\mathrm{Mat}_n(\mathbb{C})$ acts on V by left multiplication. We call V, $\langle , \rangle$ the *standard module* of $\mathrm{Mat}_n(\mathbb{C})$. If X is a set of order n then we may write $\mathrm{Mat}_X(\mathbb{C})$ instead of $\mathrm{Mat}_n(\mathbb{C})$. In this case we view the coordinates in V, and the corresponding rows and columns in $\mathrm{Mat}_X(\mathbb{C})$, as being indexed by X. For each $x \in X, \hat{x}$ will denote the vector $(0, 0, \cdots, 0, 1, 0, \cdots, 0, 0)^t$, where the 1 is in coordinate x. δ will denote the all 1's vector in V. Now let S denote any subalgebra of $\mathrm{Mat}_n(\mathbb{C})$. Then by an *$S$-module* we mean a subspace W of V such that $SW \subseteq W$. An S-module W is irreducible if it is non–zero, and contains no S-module besides 0, W. Two S-modules W, W' are *isomorphic* if there exists a vector space isomorphism $\sigma : W \rightarrow W'$ such that

$$(\sigma a - a\sigma)W = 0 \text{ for all } a \in S. \tag{3}$$

S is *symmetric* if $a^t = a$ for all $a \in S$. Now suppose S is closed under the conjugate-transpose map. Then S is *semi-simple* [20, p157]. We will not need the full force of this theory, only the following facts. Let W denote any S-module. Then for each S-module $U \subseteq W$, the orthogonal complement

$$\{w \mid w \in W, \langle u, w \rangle = 0 \text{ for all } u \in U\}$$

of U in W is also an S-module. By induction on the dimension, observe W can be expressed as an orthogonal direct sum of irreducible S-modules. Now suppose each S-module in this sum has dimension 1. Then $ab - ba$ vanishes on W for all $a, b \in S$. If W is a faithful S-module (i.e., $aW = 0 \rightarrow a = 0$ for all $a \in S$), then S is commutative. Conversely, suppose S is commutative. Then every irreducible S-module W has dimension 1. Indeed, pick any nonzero $a \in S$. Since $\mathbb{C}$ is algebraically closed, a has an eigenvector $w \in W$. Let θ denote the associated eigenvalue. Since $Sw = W$ by the irreducibility of W, we have

$$\begin{aligned}(a - \theta I)W &= (a - \theta I)Sw \\ &= S(a - \theta I)w \\ &= 0,\end{aligned}$$

so

$$au \in Span\ (u) \qquad (a \in S,\ u \in W).$$

Now every 1 dimensional subspace of W is an S-module, so W has dimension 1 by irreducibility.

2. Leonard systems

In this section we quote some results on linear algebra that we will use in Sections 4, 5, 6. Proofs can be found in [70]. We first give a version of Leonard's theorem [47], [3, p260], [70] and some related results concerning existence and uniqueness. We then introduce the notion of a *Leonard system,* and view Leonard's theorem as a classification of these objects.

THEOREM 2.1. (Leonard [47], Bannai and Ito [3, p260]). Let d denote a nonnegative integer, and pick any matrices $B, H, B^*, H^* \in \mathrm{Mat}_{d+1}(\mathbb{C})$ of the form

$$B = \begin{pmatrix} a_0 & b_0 & & & & 0 \\ c_1 & a_1 & b_1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & & & & c_d & a_d \end{pmatrix}, \quad H = \mathrm{diag}\ (\theta_0, \theta_1, \cdots, \theta_d), \tag{4}$$

$$B^* = \begin{pmatrix} a_0^* & b_0^* & & & & 0 \\ c_1^* & a_1^* & b_1^* & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & c_{d-1}^* & a_{d-1}^* & b_{d-1}^* \\ 0 & & & & c_d^* & a_d^* \end{pmatrix}, \quad H^* = \mathrm{diag}\ (\theta_0^*, \theta_1^*, \cdots, \theta_d^*), \tag{5}$$

where

$$c_i b_{i-1} \neq 0, \qquad c_i^* b_{i-1}^* \neq 0 \qquad (1 \leq i \leq d), \tag{6}$$

and

$$\theta_i \neq \theta_j, \qquad \theta_i^* \neq \theta_j^* \quad \text{if} \quad i \neq j \quad (0 \leq i, j \leq d). \tag{7}$$

Then the following statements (i), (ii) are equivalent:
(i) There exists an invertible $Q \in \mathrm{Mat}_{d+1}(\mathbb{C})$ such that

$$BQ = QH, \tag{8}$$

$$B^*Q^{-1} = Q^{-1}H^*, \tag{9}$$

the entries in the leftmost column of Q are all equal, (10)

the entries in the leftmost column of Q^{-1} are all equal. (11)

(ii) Both

$$a_i = \theta_0 - b_i - c_i \qquad (0 \le i \le d) \quad (c_0 = b_d = 0), \tag{12}$$

$$a_i^* = \theta_0^* - b_i^* - c_i^* \qquad (0 \le i \le d) \quad (c_0^* = b_d^* = 0), \tag{13}$$

where at least one of the following cases I, IA, II, IIA, IIB, IIC, III hold if $d \ge 1$ (The expressions $LS(\ldots)$ below are labels, see Definition 2.3):

(I) $\qquad LS(\mathrm{I}, q, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d)$

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})/q^i \qquad (0 \le i \le d) \tag{14}$$

$$b_0 = \frac{h(1-q^{-d})(1-r_1q)(1-r_2q)}{1-s^*q^2}$$

$$b_i = \frac{h(1-q^{i-d})(1-s^*q^{i+1})(1-r_1q^{i+1})(1-r_2q^{i+1})}{(1-s^*q^{2i+1})(1-s^*q^{2i+2})} \qquad (1 \le i \le d-1)$$

$$c_i = \begin{cases} \dfrac{h(1-q^i)(1-s^*q^{i+d+1})(r_1-s^*q^i)(r_2-s^*q^i)}{s^*q^d(1-s^*q^{2i})(1-s^*q^{2i+1})} & \text{if } s^* \neq 0 \\ h(1-q^i)(sq-r_1q^{i-d}-r_2q^{i-d}) & \text{if } s^* = 0 \end{cases} \qquad (1 \le i \le d-1)$$

$$c_d = \begin{cases} \dfrac{h(1-q^d)(r_1-s^*q^d)(r_2-s^*q^d)}{s^*q^d(1-s^*q^{2d})} & \text{if } s^* \neq 0 \\ h(1-q^d)(sq-r_1-r_2) & \text{if } s^* = 0, \end{cases}$$

where

$$q \neq 0,$$

$$q \succ q^{-1} \quad \text{if} \quad ss^* \neq 0,$$

and

$$r_1r_2 = ss^*q^{d+1}.$$

To get θ_i^*, b_i^*, c_i^*, exchange (θ_0, h, s) and (θ_0^*, h^*, s^*), and preserve (r_1, r_2, q).

(IA) $LS(\mathrm{IA}, q, h^*, r, s, \theta_0, \theta_0^*, d)$

$$\begin{aligned}
\theta_i &= \theta_0 - sq(1-q^i) && (0 \le i \le d) \\
b_i &= -rq^{i+1}(1-q^{i-d}) && (0 \le i \le d-1) \\
c_i &= q(1-q^i)(s - rq^{i-d-1}) && (1 \le i \le d) \\
\theta_i^* &= \theta_0^* + h^*(1-q^i)/q^i && (0 \le i \le d) \\
b_i^* &= h^* r(1-q^{i-d})/sq^{2i+1} && (0 \le i \le d-1) \\
c_i^* &= h^*(1-q^i)(r - sq^i)/sq^{2i} && (1 \le i \le d)
\end{aligned}$$

(II) $LS(\mathrm{II}, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d)$

$$\begin{aligned}
\theta_i &= \theta_0 + hi(i+1+s) && (0 \le i \le d) \\
b_0 &= \frac{-dh(1+r_1)(1+r_2)}{2+s^*} && \\
b_i &= \frac{h(i-d)(i+1+s^*)(i+1+r_1)(i+1+r_2)}{(2i+1+s^*)(2i+2+s^*)} && (1 \le i \le d-1) \\
c_i &= \frac{hi(i+s^*+d+1)(i+s^*-r_1)(i+s^*-r_2)}{(2i+1+s^*)(2i+s^*)} && (1 \le i \le d-1) \\
c_d &= \frac{hd(d+s^*-r_1)(d+s^*-r_2)}{2d+s^*},
\end{aligned}$$

where

$$r_1 + r_2 = s + s^* + d + 1.$$

To get θ_i^*, b_i^*, c_i^*, exchange (θ_0, h, s) and (θ_0^*, h^*, s^*), and preserve r_1, r_2.

(IIA) $LS(\mathrm{IIA}, h, r, s, s^*, \theta_0, \theta_0^*, d)$

$$\begin{aligned}
\theta_i &= \theta_0 + hi(i+1+s) && (0 \le i \le d) \\
b_i &= h(i-d)(i+1+r) && (0 \le i \le d-1) \\
c_i &= hi(i+r-s-d-1) && (1 \le i \le d) \\
\theta_i^* &= \theta_0^* + s^* i && (0 \le i \le d) \\
b_0^* &= \frac{-ds^*(1+r)}{2+s} &&
\end{aligned}$$

$$b_i^* = \frac{s^*(i-d)(i+1+s)(i+1+r)}{(2i+1+s)(2i+2+s)} \qquad (1 \le i \le d-1)$$

$$c_i^* = \frac{-s^*i(i+s+d+1)(i+s-r)}{(2i+1+s)(2i+s)} \qquad (1 \le i \le d-1)$$

$$c_d^* = \frac{-s^*d(d+s-r)}{2d+s}$$

(IIB) $LS(\text{IIB}, h^*, r, s, s^*, \theta_0, \theta_0^*, d)$

$$\theta_i = \theta_0 + si \qquad (0 \le i \le d)$$

$$b_0 = \frac{-ds(1+r)}{2+s^*}$$

$$b_i = \frac{s(i-d)(i+1+s^*)(i+1+r)}{(2i+1+s^*)(2i+2+s^*)} \qquad (1 \le i \le d-1)$$

$$c_i = \frac{-si(i+s^*+d+1)(i+s^*-r)}{(2i+1+s^*)(2i+s^*)} \qquad (1 \le i \le d-1)$$

$$c_d = \frac{-sd(d+s^*-r)}{2d+s^*}$$

$$\theta_i^* = \theta_o^* + h^*i(i+1+s^*) \qquad (0 \le i \le d)$$

$$b_i^* = h^*(i-d)(i+1+r) \qquad (0 \le i \le d-1)$$

$$c_i^* = h^*i(i+r-s^*-d-1) \qquad (1 \le i \le d)$$

(IIC) $LS(\text{IIC}, r, s, s^*, \theta_0, \theta_0^*, d)$

$$\theta_i = \theta_0 + si \qquad (0 \le i \le d)$$

$$b_i = (i-d)r/s^* \qquad (0 \le i \le d-1)$$

$$c_i = i(r-ss^*)/s^* \qquad (1 \le i \le d).$$

To get θ_i^*, b_i^*, c_i^*, exchange (θ_0, s) and (θ_0^*, s^*).

(III) $LS(\text{III}, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d)$

$$\theta_i = \theta_0 + h(s-1+(1-s+2i)(-1)^i) \qquad (0 \le i \le d)$$

$$b_i = h(2i+2+r_2-s^*+(-1)^i(r_2+s^*)) \times \frac{(2i+r_1-d+1-(-1)^{i+d}(r_1+d+1))}{2(2i+2-s^*)} \qquad (0 \le i \le d-1)$$

$$c_i = -h(2i-r_2-s^*+(-1)^i(r_2+s^*)) \times \frac{(2i-2s^*-r_1+d+1-(-1)^{i+d}(r_1+d+1))}{2(2i-s^*)} \qquad (1 \le i \le d)$$

where

$$r_1 + r_2 = -s - s^* + d + 1.$$

To obtain θ_i^*, b_i^*, c_i^*, exchange (θ_0, h, s) and (θ_0^*, h^*, s^*), and preserve r_1, r_2.

Note 2.2. The denominators in the above formulae for b_i, b_i^* $(0 \leq i \leq d-1)$, c_i, c_i^* $(1 \leq i \leq d)$ are non-zero whenever (7) holds. Indeed, pick any $\eta \in \{\text{I, IA, II, IIA, IIB, IIC, III}\}$, and assume $\theta_0, \theta_1, \cdots, \theta_d$, $\theta_0^*, \theta_1^*, \cdots, \theta_d^*$ are given in Case η of Theorem 2.1, for some positive integer d, and some $q, h, h^*, s, s^* \in \mathbb{C}$. Then

(i) For all integers i, j $(0 \leq i, j \leq d), \theta_i - \theta_j =$

$$\begin{array}{ll} \text{Case I} & hq^{-i}(1-q^{i-j})(1-sq^{i+j+1}) \\ \text{Case IA} & -sq^{j+1}(1-q^{i-j}) \\ \text{Case II, IIA} & h(i-j)(i+j+1+s) \\ \text{Case IIB, IIC} & s(i-j) \\ \text{Case III} & \begin{cases} 2h(-1)^i(i-j) & \text{if } i-j \text{ is even} \\ 2h(-1)^i(i+j+1-s) & \text{if } i-j \text{ is odd.} \end{cases} \end{array} \tag{15}$$

(ii) For all integers i, j $(0 \leq i, j \leq d), \theta_i^* - \theta_j^* =$

$$\begin{array}{ll} \text{Case I} & h^*q^{-i}(1-q^{i-j})(1-s^*q^{i+j+1}) \\ \text{Case IA} & h^*q^{-i}(1-q^{i-j}) \\ \text{Case II, IIB} & h^*(i-j)(i+j+1+s^*) \\ \text{Case IIA, IIC} & s^*(i-j) \\ \text{Case III} & \begin{cases} 2h^*(-1)^i(i-j) & \text{if } i-j \text{ is even} \\ 2h^*(-1)^i(i+j+1-s^*) & \text{if } i-j \text{ is odd.} \end{cases} \end{array}$$

(iii) Suppose (7) holds. Then

Case I	$q^i \neq 1$	$(1 \leq i \leq d)$,	$hh^* \neq 0$,		(16)
	$sq^i \neq 1$,	$s^*q^i \neq 1$	$(2 \leq i \leq 2d)$,		(17)
Case IA	$q^i \neq 1$	$(1 \leq i \leq d)$,	$h^*sq \neq 0$,		
Case II	$s \neq -i$,	$s^* \neq -i$	$(2 \leq i \leq 2d)$,	$hh^* \neq 0$,	
Case IIA	$s \neq -i$,	$(2 \leq i \leq 2d)$,	$hs^* \neq 0$,		
Case IIB	$s^* \neq -i$	$(2 \leq i \leq 2d)$,	$h^*s \neq 0$,		
Case IIC	$s \neq 0$,	$s^* \neq 0$,			
Case III	$s \neq 2i$	$s^* \neq 2i$,	$(1 \leq i \leq d)$,	$hh^* \neq 0$.	

Definition 2.3. Let d denote a non-negative integer, and pick any matrices $B, H, B^*, H^* \in \mathrm{Mat}_{d+1}(\mathbb{C})$ of the form (4)–(7) that satisfy (8)–(11). Then we refer to the 4-tuple (B, H, B^*, H^*) as a *Leonard system over* $\mathbb{C}$. The system is *over* $\mathbb{R}$ if the entries of B, H, B^*, H^* are all in $\mathbb{R}$. The integer d is the *diameter* of the system. $\theta_0, \theta_1, \cdots, \theta_d$ is the *eigenvalue sequence* of the system, and $\theta_0^*, \theta_1^*, \cdots, \theta_d^*$ is the *dual eigenvalue sequence* of the system. In each case I, IA, II, IIA, IIB, IIC, III of part (ii) in Theorem 2.1, the heading $LS(\cdots)$ refers to the Leonard system given beneath it.

One might ask if the label $LS(\cdots)$ is determined by the Leonard system it represents. This is essentially the case if the diameter is at least 3, as the following lemma shows.

LEMMA 2.4. [70]. Let $\psi = (B, H, B^*, H^*)$ denote a Leonard system over $\mathbb{C}$ with diameter $d \geq 1$. Referring to part (ii) of Theorem 2.1, suppose the eigenvalue sequence and dual eigenvalue sequence of ψ are as in

(i) Case I, for some $q, h, h^*, s, s^* \in \mathbb{C}$ ($q \succ q^{-1}$ if $ss^* \neq 0$). (18)
(ii) Case IA, for some $q, h^*, s \in \mathbb{C}$. (19)
(iii) Case II, for some $h, h^*, s, s^* \in \mathbb{C}$. (20)
(iv) Case IIA, for some $h, s, s^* \in \mathbb{C}$. (21)
(v) Case IIB, for some $h^*, s, s^* \in \mathbb{C}$. (22)
(vi) Case IIC, for some $s, s^* \in \mathbb{C}$. (23)
(vii) Case III, for some $h, h^*, s, s^* \in \mathbb{C}$. (24)

Then there exists

(i) $r_1, r_2 \in \mathbb{C}$ ($r_1 r_2 = ss^* q^{d+1}$) such that
$\psi = LS(\mathrm{I}, q, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d)$,
and r_1, r_2 are unique up to permutation.

(ii) a unique $r \in \mathbb{C}$ such that

$\psi = LS(\text{IA}, q, h^*, r, s, \theta_0, \theta_0^*, d)$.

(iii) $r_1, r_2 \in \mathbb{C}$ $(r_1 + r_2 = s + s^* + d + 1)$ such that
$\psi = LS(\text{II}, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d)$,
and r_1, r_2 are unique up to permutation.

(iv) a unique $r \in \mathbb{C}$ such that
$\psi = LS(\text{IIA}, h, r, s, s^*, \theta_0, \theta_0^*, d)$.

(v) a unique $r \in \mathbb{C}$ such that
$\psi = LS(\text{IIB}, h^*, r, s, s^*, \theta_0, \theta_0^*, d)$.

(vi) a unique $r \in \mathbb{C}$ such that
$\psi = LS(\text{IIC}, r, s, s^*, \theta_0, \theta_0^*, d)$.

(vii) $r_1, r_2 \in \mathbb{C}$ $(r_1 + r_2 = -s - s^* + d + 1)$ such that
$\psi = LS(\text{III}, h, h^*, r_1, r_2, s, s^*, \theta_0, \theta_0^*, d)$,
with r_1, r_2 unique if d is even, and
unique up to permutation if d is odd.

Furthermore, if $d \geq 3$ then exactly one of (18)–(24) occurs, and the parameters listed on that line are uniquely determined by ψ.

In order to define r_1, r_2 uniquely in Case I, II, and Case III (d odd), we will occasionally assume $r_1 \succeq r_2$, where $\succeq$ is from the end of the introduction.

Note 2.5. The data in Theorem 2.1 is from Bannai and Ito[3, p260], with the following translations.

Case	Notation of [3, p260]	Present notation
I	r_3	q^{-d-1}
IA	(r_1, r_2', s')	(q^{-d-1}, r, s)
II	r_3	$-d-1$
IIA	(r_1, r_2, s'^*)	$(-d-1, r, s^*)$
IIB	(r_1, r_2, s')	$(-d-1, r, s)$
IIC	(r_1, s', s'^*)	$(-d-1, s, s^*)$
III (d even)	(r_1, r_3)	$(-d-1, -r_1)$
III (d odd)	r_3	$d+1$

Note 2.6. (Bannai and Ito [3, p274]). The formulae in Case I($ss^* = 0$), IA, II, IIA, IIB, IIC, III of Theorem 2.1 can be obtained from the corresponding formulae in Case I ($ss^* \neq 0$) by taking limits. See the given reference for details.

3. The Subconstituent Algebra of an Association Scheme

In this section, we define the subconstituent algebra of an arbitrary commutative association scheme, and prove some general results.

Definition 3.1. Let D denote a non-negative integer. A *commutative, D-class association scheme* is a configuration $Y = (X, \{R_i\}_{0 \le i \le D})$, where X is a nonempty finite set and $R_0, R_1, \cdots, R_D$ are nonempty subsets of the Cartesian product $X \times X$, such that

(i) $(x,y) \in R_0$ if and only if $x = y \quad (x, y \in X)$.
(ii) $(x,y) \in R_i$ for exactly one $i \quad (0 \le i \le D)$, $(x, y \in X)$.
(iii) $R_i^t = R_{i'}$ for some $i' \in \{0, 1, \ldots, D\}$, where
$R_i^t = \{(y,x) \mid (x,y) \in R_i\} \quad (0 \le i \le D)$.
(iv) For all integers i, j, k $(0 \le i, j, k \le D)$, and all $x, y \in X$ with $(x,y) \in R_k$, the number p_{ij}^k of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is a constant that depends only on i, j, k.
(v) $p_{ij}^k = p_{ji}^k \quad (0 \le i, j, k \le D)$. (25)

The elements of X, the R_i, the constants p_{ij}^k, and the constant D are known as the *vertices*, the *associate classes*, the *intersection numbers*, and the *diameter*, of Y. For convenience, we will say *scheme* instead of *commutative association scheme*.

For the rest of this section, let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote the scheme in Definition 3.1. We begin by summarizing some basic results from Bannai and Ito [3, pp 52–70].

For each integer i $(0 \le i \le D)$, the ith *associate matrix* $A_i \in \mathrm{Mat}_X(\mathbb{C})$ satisfies

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{if } (x,y) \notin R_i \end{cases} \qquad (x, y \in X). \tag{26}$$

Then Definition 3.1 implies

$$A_0 = I, \tag{27}$$

$$A_0 + A_1 + \cdots + A_D = J \qquad (J = \text{ all 1's matrix}), \tag{28}$$

$$A_i^t = A_{i'} \qquad (0 \le i \le D), \tag{29}$$

and

$$A_i A_j = \sum_{k=0}^{D} p_{ij}^k A_k \qquad (0 \le i, j \le D). \tag{30}$$

Setting $j = 0$ in (30), we find

$$p_{i0}^k = \delta_{ik} \qquad (0 \le i, k \le D). \tag{31}$$

The matrices $A_0, A_1, \cdots, A_D$ are certainly linearily independent, so they form a basis for a subspace M of $\mathrm{Mat}_X(\mathbb{C})$. Then M is a commutative semi-simple subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ by (25), (27), (29), (30), and is known as the *Bose-Mesner algebra* of Y. By [3, p59, p64], M has a second basis $E_0, E_1, \cdots, E_D$ such that

$$E_0 = |X|^{-1}J, \tag{32}$$

$$E_iE_j = \delta_{ij}E_i \qquad (0 \le i, j \le D), \tag{33}$$

$$E_0 + E_1 + \cdots + E_D = I, \tag{34}$$

$$E_i^t = \overline{E_i} \tag{35}$$

$$= E_{\hat{\imath}} \text{ for some } \hat{\imath} \in \{0, 1, \cdots, D\} \qquad (0 \le i \le D). \tag{36}$$

We refer to E_i as the ith *primitive idempotent* of $Y (0 \le i \le D)$.

Let $\circ$ denote entry-wise multiplication in $\mathrm{Mat}_X(\mathbb{C})$. Then

$$A_i \circ A_j = \delta_{ij}A_i \qquad (0 \le i, j \le D), \tag{37}$$

so M is closed under $\circ$. Thus there exists $q_{ij}^k \in \mathbb{C}$ $(0 \le i, j, k \le D)$ such that

$$E_i \circ E_j = |X|^{-1}\sum_{k=0}^{D} q_{ij}^k E_k \qquad (0 \le i, j \le D). \tag{38}$$

Taking the conjugate-transpose in (38), we find by (35) that $q_{ij}^k \in \mathbb{R}$ $(0 \le i, j, k \le D)$. Setting $j = 0$ in (38), we find by (32) that

$$q_{i0}^k = \delta_{ik} \qquad (0 \le i, k \le D). \tag{39}$$

The q_{ij}^k $(0 \le i, j, k \le D)$ are known as the *Krein parameters* of Y.

Since $A_0, A_1, \cdots, A_D$ and $E_0, E_1, \cdots, E_D$ are both bases for M, there exists $p_i(j), q_i(j) \in \mathbb{C}$ $(0 \le i, j \le D)$ such that

$$A_i = \sum_{j=0}^{D} p_i(j)E_j \qquad (0 \le i \le D), \tag{40}$$

$$E_i = |X|^{-1}\sum_{j=0}^{D} q_i(j)A_j \qquad (0 \le i \le D). \tag{41}$$

Taking the complex conjugate and transpose in (40), (41), we observe

$$\overline{p_i(j)} = p_{i'}(j) \qquad (0 \le i, j \le D), \tag{42}$$

$$= p_i(\hat{\jmath}) \qquad (0 \le i, j \le D), \tag{43}$$

and

$$\overline{q_i(j)} = q_{\hat{i}}(j) \qquad (0 \le i,j \le D), \tag{44}$$

$$= q_i(j') \qquad (0 \le i,j \le D). \tag{45}$$

Applying (33), (37) to (40), (41), respectively, we have

$$A_i E_j = p_i(j) E_j \qquad (0 \le i,j \le D), \tag{46}$$

$$E_i \circ A_j = |X|^{-1} q_i(j) A_j \qquad (0 \le i,j \le D). \tag{47}$$

The $p_i(j)$ (resp. $q_i(j)$) $(0 \le i,j \le D)$ are known as the *eigenvalues* (resp. *dual eigenvalues* of Y.

There are many equations relating the above constants. For example, set

$$m_i = \text{ rank } E_i \qquad (0 \le i \le D). \tag{48}$$

Then it is proved by Bannai and Ito [3, p62, p67] that

$$m_i = q_i(0) \quad (0 \le i \le D), \tag{49}$$

$$= q^0_{i\hat{i}} \qquad (0 \le i \le D),$$

and

$$m_j q^j_{\hat{i}k} = m_k q^k_{ij} \qquad (0 \le i,j,k \le D). \tag{50}$$

Now fix any $x \in X$. For each integer i $(0 \le i \le D)$, let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ satisfying

$$(E^*_i)_{yy} = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{if } (x,y) \notin R_i \end{cases} \qquad (y \in X). \tag{51}$$

Then immediately

$$E^*_i E^*_j = \delta_{ij} E^*_i \qquad (0 \le i,j \le D), \tag{52}$$

$$E^*_0 + E^*_1 + \cdots + E^*_D = I, \tag{53}$$

$$(E^*_i)^t = \overline{E^*_i} \tag{54}$$

$$= E^*_i \qquad (0 \le i \le D). \tag{55}$$

The matrices $E^*_0, E^*_1, \cdots, E^*_D$ are certainly linearily independent, so they form a basis for a subspace $M^* = M^*(x)$ of $\text{Mat}_X(\mathbb{C})$. Then M^* is a commutative semi-simple subalgebra of $\text{Mat}_X(\mathbb{C})$ by (52)–(54). We call M^* the *dual Bose-Mesner algebra of Y with respect to x*, and refer to E^*_i as the ith *dual idempotent of Y with respect to x* $(0 \le i \le D)$.

We now define the dual associate matrices in M^*. For each integer i $(0 \le i \le D)$, let $A^*_i = A^*_i(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ satisfying

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \qquad (y \in X). \tag{56}$$

Then from (40), (41) we obtain

$$A_i^* = \sum_{j=0}^{D} q_i(j) E_j^* \qquad (0 \le i \le D) \tag{57}$$

and

$$E_i^* = |X|^{-1} \sum_{j=0}^{D} p_i(j) A_j^* \qquad (0 \le i \le D). \tag{58}$$

In particular, $A_0^*, A_1^*, \cdots, A_D^*$ form a second basis for M^*. Multiplying (58) on the right by E_j^*, we find by (52) that

$$A_i^* E_j^* = q_i(j) E_j^* \qquad (0 \le i, j \le D). \tag{60}$$

Applying (56) to (32), (34)–(36), (38), we find

$$\begin{aligned} &A_0^* = I, \\ &A_0^* + A_1^* + \cdots + A_D^* = |X| E_0^*, \\ &\overline{A_i^*} = A_{\hat{i}}^*, \qquad (0 \le i \le D), \\ &A_i^* A_j^* = \sum_{k=0}^{D} q_{ij}^k A_k^* \qquad (0 \le i, j \le D). \end{aligned} \tag{61}$$

We call A_i^* the ith *dual associate matrix of* Y *with espect to* x $(0 \le i \le D)$.

Let $V, \langle , \rangle$ denote the standard module of $\mathrm{Mat}_X(\mathbb{C})$, defined near the end of Section 1. Then by (34), (35), (53), (54), we have the decompositions

$$V = E_0 V + E_1 V + \cdots + E_D V \quad \text{(orthogonal direct sum)}, \tag{62}$$

$$= E_0^* V + E_1^* V + \cdots + E_D^* V \quad \text{(orthogonal direct sum)}. \tag{63}$$

We call $E_i V$ and $E_i^* V$ the ith *eigenspace* and ith *subconstituent with respect to* x, respectively.

Throughout this paper, we adopt the convention that $E_i = 0, E_i^* = 0$ for any integer i such that $i < 0$ or $i > D$.

We now find some relations between M, M^*.

LEMMA 3.2. *Let the scheme* $Y = (X, \{R_i\}_{0 \le i \le D})$ *be as in Definition 3.1, pick any* $x \in X$, *and write* $E_i^* = E_i^*(x), A_i^* = A_i^*(x)$ $(0 \le i \le D)$. *Then*

$$E_i^* A_j E_k^* = 0 \quad \text{if and only if} \quad p_{ij}^k = 0 \qquad (0 \le i, j, k \le D), \tag{64}$$

$$E_i A_j^* E_k = 0 \quad \text{if and only if} \quad q_{ij}^k = 0 \qquad (0 \le i, j, k \le D). \tag{65}$$

The key result (65) is due to Cameron, Goethals, and Seidel [17].

Proof. First consider (64). By (26), (51), the y, z entry of $E_i^* A_j E_k^*$ is nonzero exactly when $(x, y) \in R_i$, $(y, z) \in R_j$, and $(x, z) \in R_k$ $(y, z \in X)$. By (iv) of Definition 3.1, such y, z exist if and only if $p_{ij}^k \neq 0$, so (64) holds. To see (65), recall trace $(\alpha\beta) =$ trace $(\beta\alpha)$. Now the sum of the squares of the norms of the entries of $E_i A_j^* E_k$ is equal to

$$
\begin{aligned}
& \operatorname{trace}\left((E_i A_j^* E_k)\overline{(E_i A_j^* E_k)^t}\right) \\
&= \operatorname{trace}\left(E_i A_j^* E_k \overline{A_j^*}\right) \\
&= \sum_{y,z \in X} (E_i)_{yz}(A_j^*)_{zz}(E_k)_{zy}(A_{\hat{j}}^*)_{yy} \\
&= |X|^2 \sum_{y,z \in X} (E_{\hat{i}})_{zy}(E_j)_{xz}(E_k)_{zy}(E_j)_{yx} \\
&= |X|^2 (E_j(E_{\hat{i}} \circ E_k)E_j)_{xx} \\
&= |X| q_{\hat{i}k}^j (E_j)_{xx} \\
&= m_j q_{\hat{i}k}^j, \\
&= m_k q_{ij}^k.
\end{aligned}
$$

But $m_k \neq 0$ $(0 \leq k \leq D)$ by (48), so (65) holds. This proves Lemma 3.2. □

Definition 3.3. Let the scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be as in Definition 3.1, pick any $x \in X$, and let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by the Bose-Mesner algebra M and the dual Bose-Mesner algebra $M^*(x)$. We call T the *subconstituent algebra of* Y *with respect to* x.

Before proceeding, let us emphasise some facts about $T(x)$.

LEMMA 3.4. *Let the scheme* $Y = (X, \{R_i\}_{0 \leq i \leq D})$ *be as in Definition 3.1, pick any* $x \in X$, *and write* $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$, $T = T(x)$. *Then*

- (i) *T is closed under the conjugate-transpose map. In particular, T is semisimple.*
- (ii) *The standard module V decomposes into an orthogonal direct sum of irreducible T-modules.*
- (iii) *Each irreducible T-module W is the orthogonal direct sum of the nonvanishing* $E_i W$ $(0 \leq i \leq D)$, *and the orthogonal direct sum of the nonvanishing* $E_i^* W$ $(0 \leq i \leq D)$.

Proof. T is closed under conjugate-transpose by (35), (54). The result (ii) follows from the discussion at the end of Section 1. Result (iii) follows from (62), (63). □

We will mainly be interested in the following modules for the subconstituent algebras.

Definition 3.5. Let the scheme $Y = (X, \{R_i\}_{0 \le i \le D})$ be as in Definition 3.1, pick any $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, $T = T(x)$. Let W denote an irreducible T-module, and define

$$W_s := \{i \mid 0 \le i \le D, E_i W \ne 0\}.$$

We call W_s the *support* of W. The *diameter* of W is defined to equal $|W_s| - 1$. Now define

$$W_\sigma := \{i \mid 0 \le i \le D, E_i^* W \ne 0\}.$$

We call W_σ the *dual-support* of W. The *dual-diameter* of W is defined to equal $|W_\sigma| - 1$. W is said to be *thin* whenever

$$\dim E_i^* W \le 1 \qquad \text{for all } i \qquad (0 \le i \le D). \tag{66}$$

W is said to be *dual-thin* whenever

$$\dim E_i W \le 1 \qquad \text{for all } i \qquad (0 \le i \le D). \tag{67}$$

Y is said to be *thin* (resp. *dual-thin*) *with respect to* x if each irreducible $T(x)$-module is thin (resp. dual-thin). Y is said to be *thin* (resp. *dual-thin*) if Y is thin (resp. dual-thin) with respect to each vertex in X.
We now show each subconstituent algebra possesses at least one irreducible module that is both thin and dual-thin.

LEMMA 3.6. *Let the scheme $Y = (X, \{R_i\}_{0 \le i \le D})$ be as in Definition 3.1. Pick any $x \in X$, and write $E_i^* = E_i^*(x)$, $A_i^* = A_i^*(x)$ $(0 \le i \le D)$, $M^* = M^*(x)$, $T = T(x)$. Then*

$$A_i \hat{x} = E_i^* \delta \qquad (0 \le i \le D), \tag{68}$$

$$A_i^* \delta = |X| E_i \hat{x} \qquad (0 \le i \le D), \tag{69}$$

where δ denotes the all 1's *vector in the standard module. In particular, $M\hat{x} = M^*\delta$ is a thin, dual-thin irreducible T-module of dimension $D + 1$.*

Proof. To obtain (68), evaluate both sides using (26), (51). To obtain (69), compare the two sides using (56). Now $M\hat{x} = M^*\delta$ by the definition of M, M^*. Now $M\hat{x}$ is a T-module, since $M\hat{x}$ is M-invariant, and $M^*\delta$ is M^*-invariant. $M\hat{x}$ is irreducible, since by part (ii) of Lemma 3.4, there exists an irreducible T-module W that is not orthogonal to $\hat{x}$. But then $\hat{x} \in E_0^* W \subseteq W$, forcing $M\hat{x} \subseteq W$ and then $M\hat{x} = W$ by the irreducibility of W. That $M\hat{x}$ is thin and dual-thin with dimension $D + 1$ is a consequence of (68), (69). This proves Lemma 3.6. □

For an arbitrary commutative scheme, it seems difficult to describe the remaining irreducible modules for the subconstituent algebras. Therefore we focus on a special class of schemes called the *P- and Q-polynomial schemes* . These schemes have many vanishing intersection numbers and Krein parameters, so the relations (64), (65) should give us a lot of information.

Definition 3.7. The scheme $Y = (X, \{R_i\}_{0 \le i \le D})$ in Definition 3.1 is said to be *P-polynomial* (with respect to the given ordering $A_0, A_1, \cdots, A_D$ of the associate matrices), if for all integers i, j, k $(0 \le i, j, k \le D)$, $p^k_{ij} = 0$ (resp $p^k_{ij} \ne 0$) whenever one of i, j, k is greater than (resp. equal to) the sum of the other two.

LEMMA 3.8. [3, p190]. *Assume the scheme $Y = (X, \{R_i\}_{0 \le i \le D})$ in Definition 3.1 is P-polynomial with respect to the ordering $A_0, A_1, \cdots, A_D$ of the associate matrices. Then the following* (i)–(iii) *hold.*

(i) The Bose-Mesner algebra M is symmetric; equivalently

$$i' = i, \qquad \hat{\imath} = i \qquad (0 \le i \le D). \tag{70}$$

(ii) There exists polynomials $\varphi_i \in \mathbb{R}[\lambda]$ $(0 \le i \le D)$ such that

$$\deg \varphi_i = i \qquad (0 \le i \le D),$$

$$A_i = \varphi_i(A) \qquad (0 \le i \le D). \tag{71}$$

In particular, A generates M.
(iii) The eigenvalues $p_1(j)$ $(0 \le j \le D)$ are mutually distinct real numbers.

Proof of (i). Setting $j = i'$ in (30), we have $p^0_{ii'} \ne 0$ $(0 \le i \le D)$ since $A_i A_{i'} = A_i A_i^t$ has non-zero trace. Now $0, i, i'$ must satisfy the triangle inequality by Definition 3.7, so $i = i'$ $(0 \le i \le D)$. Now M is symmetric by (29), and $\hat{\imath} = i$ $(0 \le i \le D)$ by (35), (36). □

Proof of (ii). The existence of $\varphi_0, \varphi_1, \cdots, \varphi_D$ follows directly from (30) and Definition 3.7. □

Proof of (iii). The eigenvalues $p_1(j)$ $(0 \le j \le D)$ are distinct because A generates M. The eigenvalues are real numbers by (42), (70). □

LEMMA 3.9. *Assume the scheme $Y = (X, \{R_i\}_{0 \le i \le D})$ in Definition 3.1 is P-polynomial with respect to the ordering $A_0, A_1, \cdots, A_D$ of the associate matrices. Pick any $x \in X$, and write $E^*_i = E^*_i(x)$ $(0 \le i \le D)$, $T = T(x)$. Let W denote an irreducible T-module, with diameter d and dual-diameter d^*. Set*

$$\nu := \min\{i \mid 0 \le i \le D,\ E^*_i W \ne 0\}. \tag{72}$$

We call ν the *dual-endpoint* of W (with respect to the given ordering the associate matrices). Now the following (i)–(v) hold.

(i) $AE_j^*W \subseteq E_{j-1}^*W + E_j^*W + E_{j+1}^*W \qquad (0 \leq j \leq D).$ (73)

(ii) The dual-support $W_\sigma = \{\nu, \nu+1, \cdots, \nu + d^*\}$.

(iii) $E_i^*AE_j^*W \neq 0$ if $|i-j| = 1 \qquad (\nu \leq i, j \leq \nu + d^*)$.

(iv) Suppose W is thin. Then

$$E_\nu^*W + E_{\nu+1}^*W + \cdots + E_{\nu+i}^*W$$
$$= E_\nu^*W + AE_\nu^*W + \cdots + A^iE_\nu^*W \qquad (0 \leq i \leq d^*). \tag{74}$$

(v) Suppose W is thin. Then

$$E_jW = E_jE_\nu^*W \qquad (0 \leq j \leq D). \tag{75}$$

In particular, W is dual-thin, forcing $d = d^*$.

Proof of (i). We have $p_{i1}^j = 0$ whenever $|i-j| > 1$ by Definition 3.7, so by (53), (63),

$$AE_j^*W = \sum_{i=0}^{D} E_i^*AE_j^*W$$
$$= E_{j-1}^*AE_j^*W + E_j^*AE_j^*W + E_{j+1}^*AE_j^*W$$
$$\subseteq E_{j-1}^*W + E_j^*W + E_{j+1}^*W \qquad (0 \leq j \leq D).$$

This proves (i). □

Now set

$$W[i,j] = E_i^*W + E_{i+1}^*W + \cdots + E_j^*W \qquad (0 \leq i \leq j \leq D),$$

and assume for the moment that $W[i,j]$ is A-invariant for some integers i, j $(0 \leq i, j \leq D)$. Then $W[i,j]$ is M-invariant by Lemma 3.8, and M^*-invariant by construction, so $W[i,j]$ is a T-module. But then $W[i,j] = W$ by the irreducibility of W. Now consider the assertions (ii)–(v).

Proof of (ii). Suppose $E_j^*W = 0$ for some integer j $(\nu \leq j \leq \nu + d^*)$. Then $W[\nu, j-1]$ is A-invariant by (73), and hence equals W by our preliminary remarks. But this contradicts the definition of d^*, so $E_j^*W \neq 0$ for all j $(\nu \leq j \leq \nu + d^*)$. Now (ii) holds by the definition of d^*. □

Proof of (iii). Certainly

$$E_{j+1}^*AE_j^*W \neq 0 \qquad (\nu \leq j < \nu + d^*), \tag{76}$$

for if (76) failed for some j $(\nu \leq j < \nu + d^*)$, then $AE_j^*W \subseteq E_{j-1}^*W + E_j^*W$ by (73), making $W[\nu, j]$ A-invariant, and contradicting our preliminary remarks. Similarly,

$$E_{j-1}^*AE_j^*W \neq 0 \qquad (\nu < j \leq \nu + d^*),$$

for otherwise $W[j, \nu + d^*]$ is A-invariant, contradicting our preliminary remarks. □

Proof of (iv). Immediate from (i)–(iii). □

Proof of (v). Setting $i = d^*$ in (74), we find

$$W = ME_\nu^*W. \tag{77}$$

Now (75) is obtained by applying E_j to both sides of (77).

Definition 3.10. The scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$ in Definition 3.1 is said to be *Q-polynomial* (with respect to the given ordering $E_0, E_1, \cdots, E_D$ of the primitive idempotents), if for all integers i, j, k $(0 \leq i, j, k \leq D), q_{ij}^k = 0$ (resp. $q_{ij}^k \neq 0$) whenever one of i, j, k is greater than (resp. equal to) the sum of the other two. We abbreviate $A^*(x) = A_1^*(x)$ whenever Y is Q-polynomial.

LEMMA 3.11. *Assume the scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$ in Definition 3.1 is Q-polynomial with respect to the ordering $E_0, E_1, \cdots, E_D$ of the primitive idempotents. Then the following* (i)–(iii) *hold.*

(i) *The Bose-Mesner algebra M is symmetric.*
(ii) *There exists polynomials $\varphi_i^* \in \mathbb{R}[\lambda]$ $(0 \leq i \leq D)$ such that*

$$\deg \varphi_i^* = i \qquad (0 \leq i \leq D),$$

$$A_i^*(x) = \varphi_i^*(A^*(x)) \qquad (0 \leq i \leq D, x \in X). \tag{78}$$

In particular, $A^*(x)$ generates the dual Bose-Mesner algebra $M^*(x)$ for all $x \in X$.
(iii) *The dual eigenvalues $q_1(j)$ $(0 \leq j \leq D)$ are mutually distinct real numbers.*

Proof. Similar to Lemma 3.8. □

LEMMA 3.12. *Assume the scheme $Y = (X, \{R_i\}_{0 \leq i \leq D})$ in Definition 3.1 is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. Pick any $x \in X$, and write $E_i^* = E_i^*(x), (0 \leq i \leq D), A^* = A_1^*(x), T = T(x)$. Let W denote an irreducible T-module, with diameter d and dual-diameter d^*. Set*

$$\mu := \min \{i \mid 0 \leq i \leq D, E_iW \neq 0\}. \tag{79}$$

We call μ the *endpoint of* W *(with respect to the given ordering of the primitive idempotents). Now the following* (i)-(v) *hold.*

(i) $A^* E_j W \subseteq E_{j-1}W + E_j W + E_{j+1}W \qquad (0 \le j \le D).$ (80)

(ii) *The support* $W_s = \{\mu, \mu+1, \ldots, \mu+d\}$.

(iii) $E_i A^* E_j W \neq 0$ *if* $|i-j| = 1 \qquad (\mu \le i, j \le \mu + d).$

(iv) *Suppose* W *is dual-thin. Then*

$$E_\mu W + E_{\mu+1}W + \cdots + E_{\mu+i}W = E_\mu W + A^* E_\mu W + \cdots + (A^*)^i E_\mu W \qquad (0 \le i \le d). \tag{81}$$

(v) *Suppose* W *is dual-thin. Then*

$$E_j^* W = E_j^* E_\mu W \qquad (0 \le j \le D).$$

In particular, W *is thin, forcing* $d = d^*$.

Proof. Similar to the proof of Lemma 3.9. □

In [11, p239], Brouwer, Cohen, and Neumaier show how to determine if a given P-polynomial scheme is Q-polynomial.

Sections 4, 5, 6, 7 will appear in subsequent issues.

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