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# On Modular Standard Modules of Association Schemes

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**Abstract.** We will determine the structure of the modular standard modules of association schemes of class two. In the process, we will give the theoretical interpretation for the $p$-rank theory for strongly regular graphs, and understand the $p$-rank as the dimension of a submodule of the modular standard module. Considering the modular standard module, we can obtain the detailed classification more than the $p$-rank and the parameters.

**Keywords:** association scheme, modular adjacency algebra, modular standard module, $p$-rank, strongly regular graph

## 1. Introduction

There are many examples of nonisomorphic association schemes such that adjacency algebras over the complex number field have the same structure constants (intersection numbers). The standard module over the complex number field is completely determined by the structure constants. So their standard modules over the complex number field are also isomorphic. In this case, their modular adjacency algebras are also isomorphic to each other. But there exist cases where their modular standard modules are nonisomorphic. Here, the modular standard module means the standard module over a positive characteristic field. Therefore the structure of the modular standard module can play a role for the structure theory of the association schemes.

In this paper, we will determine the structure of the modular standard modules of association schemes of class 2. We will use the $p$-rank theory of strongly regular graphs which are studied by Brouwer, van Eijl [5], and Peeters [8]. We will give the theoretical interpretation for the $p$-rank theory, and understand the $p$-rank as the dimension of a submodule of the standard module. Considering the standard module, we can obtain the detailed classification more than the $p$-rank and the parameters.

## 2. Definition and preliminaries

In this section, we assume $\mathfrak{X}$ to be commutative. Let $\mathfrak{X} = (X, \{R_i\}_{i=0,\dots,d})$ be an association scheme, and $A_i (i = 0, \dots, d)$ be its adjacency matrices. Put $n = |X|$ and $v_i$ the valency of $A_i$. Let us denote the adjacency algebra of $\mathfrak{X}$ over a field $L$ by $L\mathfrak{X}$ and the standard module by $LX$.

Let $(K, R, F)$ be a splitting $p$-modular system for the adjacency algebra, and let $(\pi)$ be the maximal ideal of $R$. We denote the image of the canonical epimorphism $R \to F$ by $*$ (For details about a $p$-modular system, see [7]).

Suppose that $\mathfrak{X}$ is commutative. Then

$$R\mathfrak{X}/\pi R\mathfrak{X} \cong F\mathfrak{X}$$

and $\pi R\mathfrak{X} \subseteq J(R\mathfrak{X})$ [7, Theorem I.14.1], so idempotents of $F\mathfrak{X}$ are liftable to idempotents of $R\mathfrak{X}$ [7, Theorem I.14.2]. Consider the primitive idempotents decomposition of $1_{F\mathfrak{X}}$ in $F\mathfrak{X}$:

$$1_{F\mathfrak{X}} = f_0 + \cdots + f_s \in F\mathfrak{X},$$

then we have the primitive idempotents decomposition of $1_{R\mathfrak{X}}$ in $R\mathfrak{X}$:

$$1_{R\mathfrak{X}} = e_{B_0} + \cdots e_{B_s} \in R\mathfrak{X}$$

such that $e^*_{B_i} = f_i$. These decompositions yield the decomposition of algebras. We call this $e_{B_i}(e^*_{B_i})$ a *block idempotent* of $R\mathfrak{X}(F\mathfrak{X})$, and we write $B_i = e_{B_i}R\mathfrak{X}$ and $B_i^* = e^*_{B_i}F\mathfrak{X}$.

Let $e_0, \dots, e_d$ be the set of primitive idempotents in $K\mathfrak{X}$. Then there is a partition $\{0, \dots, d\} = \bigcup_{j=0}^{s} T_j$ such that $e_{B_i} = \sum_{j \in T_i} e_j$. When $e_j \in T_i$, we say that $e_j$ belongs to the block $B_i$.

Let $\chi_j$ be the (one-dimensional) irreducible representation of $K\mathfrak{X}$ corresponding to $e_j$. Then $e_j$ belongs to $B_i$ if and only if $\chi_j(e_{B_i}) = 1$. Since $B_i^*$ has the unique idempotent $e^*_{B_i}$, $B_i^*/J(B_i^*) \cong F$. If $\chi_i$ and $\chi_j$ belong to the same block, then $\chi_i^* = \chi_j^*$. So we have the following.

**Lemma 1** *Irreducible representations $\chi_i$ and $\chi_j$ of $K\mathfrak{X}$ belong to the same block if and only if $\chi_i(A_r) \equiv \chi_j(A_r) \pmod{(\pi)}$ for all $r = 0, \dots, d$.*

**Lemma 2** *The dimension of $B_i^*$ is equal to the number of $\chi_j$ belonging to $B_i$.*

**Proof:** We have $\dim_F B_i^* = \mathrm{rank}_R B_i = \dim_K e_{B_i}K\mathfrak{X}$. □

We consider a $(d+1) \times (d+1)$ matrix $P$ with the $(i, j)$-entry $\chi_i(A_j)$. We call $P$ the *character table* of $\mathfrak{X}$.

**Proposition 3** *The algebra $F\mathfrak{X}$ is semisimple if and only if* $\det P \not\equiv 0 \pmod{(\pi)}$.

**Proof:** If $F\mathfrak{X}$ is semisimple, then clearly we have $\det P^* \neq 0$. If $F\mathfrak{X}$ is not semisimple, the $P^*$ contains at least two same rows, so $\det P^* = 0$. $\square$

General sufficient criteria for adjacency algebras to be semisimple were obtained in [1] and [6].

Mapping each matrix $A_r$ to $v_r$ induces an algebra homomorphism from $K\mathfrak{X}$ to $K$. This representation is usually called the *trivial representation* of $K\mathfrak{X}$. We assume that $\chi_0$ is the trivial representation, and $\chi_0$ belongs to $B_0$. We call $B_0$ the *principal block* of $\mathfrak{X}$. Then we have the following.

**Proposition 4** *The dimension of the principal block $B_0^*$ is one if and only if $p \nmid n$. (This is also true for non-commutative association schemes.)*

**Proof:** Put $J = \sum_{i=0}^{d} A_i$. Since $JA_i = A_iJ = v_iJ$, $J^*$ generates a one-dimensional ideal of $F\mathfrak{X}$, and the corresponding representation is trivial. We have $J^2 = nJ$. If $p \nmid n$, then $n^{-1}J$ is a central primitive idempotent, thus $B_0^* = J^*F\mathfrak{X}$ is one-dimensional. If $p \mid n$, then $J^*$ is a central nilpotent element in $B_0^*$, so it is in the Jacobson radical. In this case, $\dim_F B_0^* > 1$. $\square$

## 3. Contragradient modules

Let $\mathfrak{X} = (X, \{R_i\}_{i=0,\ldots,d})$ be an association scheme (not necessary commutative), and $F$ a field. For a right $F\mathfrak{X}$-module $V$, we put $\hat{V} := \mathrm{Hom}_F(V, F)$. For $f \in \hat{V}$, $v \in V$, and $i \in \{0, \ldots, d\}$, we define the action of $F\mathfrak{X}$ to $\hat{V}$ by

$$(fA_i)(v) := f(vA_{i'}).$$

Here $A_{i'} = {}^tA_i$. Then

$$(f(A_iA_j))(v) = f(vA_{j'}A_{i'}) = ((fA_i)A_j)(v) \quad \text{for } i, j \in \{0, \ldots, d\}.$$

So $\hat{V}$ is a right $F\mathfrak{X}$-module. We call $\hat{V}$ the *contragradient* module of $V$.

**Proposition 5** *$\widehat{FX} \cong FX$ as right $F\mathfrak{X}$-modules.*

**Proof:** For $x \in X$, define $f_x \in \widehat{FX}$ by $f_x(y) = \delta_{xy}$. Put $\varphi : FX \to \widehat{FX}$ the $F$-linear map defined by $\varphi(x) = f_x$. Obviously $\varphi$ is an isomorphism of vector spaces. We show that

$\varphi$ is an $F\mathfrak{X}$-homomorphism. We have

$$\begin{aligned}
\varphi(xA_i)(y) &= \varphi\left(\sum_{(x,z)\in R_i} z\right)(y) = \sum_{(x,z)\in R_i} f_z(y) \\
&= \begin{cases} 1 & \text{if } (x,y)\in R_i, \\ 0 & \text{otherwise}, \end{cases} \\
(\varphi(x)A_i)(y) &= \varphi(x)(yA_{i'}) = f_x\left(\sum_{(y,z)\in R_{i'}} z\right) \\
&= \begin{cases} 1 & \text{if } (y,x)\in R_{i'}, \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}$$

Now $\varphi(xA_i) = \varphi(x)A_i$ and $\varphi$ is an $F\mathfrak{X}$-homomorphism. □

**Proposition 6** *We fix $x \in X$. Put $\iota : F\mathfrak{X} \to FX$ the $F$-linear map defined by $\iota(A_i) = xA_i$. Then $\iota$ is an $F\mathfrak{X}$-monomorphism.*

**Proof:** Obviously, $\iota$ is an $F\mathfrak{X}$-homomorphism. Assume $\iota(\sum_{i=0}^d a_i A_i) = 0$. Then

$$0 = \iota\left(\sum_{i=0}^d a_i A_i\right) = \sum_{i=0}^d a_i x A_i = \sum_{i=0}^d \sum_{(x,y)\in R_i} a_i y.$$

Now $a_i = 0$ for all $i \in \{0,\dots,d\}$, and so $\sum_{i=0}^d a_i A_i = 0$. □

We consider the structure of the contragradient module $\widehat{F\mathfrak{X}}$ of the regular $F\mathfrak{X}$-module. Define $\tau_i \in \widehat{F\mathfrak{X}}$ by $\tau_i(A_j) = \delta_{ij}$. Then $\{\tau_i \mid i \in \{0,\dots,d\}\}$ is a basis of $\widehat{F\mathfrak{X}}$. Now

$$(\tau_i A_j)(A_k) = \tau_i(A_k A_{j'}) = p^i_{kj'}.$$

So we have the following.

**Proposition 7** $\tau_i A_j = \sum_{k\in 0}^d p^i_{kj'} \tau_k$.

**Proposition 8** *Let $\iota : F\mathfrak{X} \to FX$ be as above. Put $\hat{\iota} : \widehat{FX} \to \widehat{F\mathfrak{X}}$ the map defined by $\hat{\iota}(f) = f \circ \iota$. Then $\hat{\iota}$ is a $F\mathfrak{X}$-epimorphism.*

**Proof:** For $i \in \{0,\dots,d\}$, we fix $y \in X$ such that $(x,y) \in R_i$. Then

$$\begin{aligned}
\hat{\iota}(f_y)(A_j) &= f_y \circ \iota(A_j) = f_y(xA_j) \\
&= \begin{cases} 1 & \text{if } (x,y)\in R_j, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned}$$

and so we have $\hat{\iota}(f_y) = \tau_i$. □

## 4. The structure of the adjacency algebras with $d = 2$

It is well-known that association schemes with at most five elements are commutative [12, Theorem 4.5.1(ii)]. We assume that $F$ is an algebraically closed field of characteristic $p$.

**Lemma 9** *The isomorphism classes of $F$-algebras of dimension 3 are $F \oplus F \oplus F$, $F \oplus F[x]/(x^2)$, $F[x]/(x^3)$, $F[x, y]/(x^2, xy, y^2)$, and $T_2(F) = \left(\begin{smallmatrix} F & 0 \\ F & F \end{smallmatrix}\right)$.*

Since $T_2(F)$ is non-commutative, $T_2(F)$ can not be an adjacency algebra of an association scheme. The algebra $F[x, y]/(x^2, xy, y^2)$ is not self injective, but it can be an adjacency algebra of an association scheme. The smallest such example is the adjacency algebra of the group association scheme of the symmetric group of degree 3 in characteristic 3.

Firstly, we consider the non-symmetric case, namely $A_{1'} = {}^tA_1 = A_2$. Put $k = v_1 = v_2 = (n-1)/2$, where $n = |X|$. Note that $k \equiv 3 \pmod 4$, in this case. Then we have

$$\begin{aligned} A_1^2 &= \frac{k-1}{2}A_1 + \frac{k+1}{2}A_2, \\ A_1A_2 &= kA_0 + \frac{k-1}{2}A_1 + \frac{k-1}{2}A_2, \\ A_2^2 &= \frac{k+1}{2}A_1 + \frac{k-1}{2}A_2. \end{aligned}$$

**Theorem 10** *Let $\mathfrak{X}$ be a non-symmetric association scheme with $d = 2$. Then we have the following.*
(1) *If $p \nmid n$, then $F\mathfrak{X} \cong F \oplus F \oplus F$.*
(2) *If $p \mid n$, then $F\mathfrak{X} \cong F[x]/(x^3)$.*

**Proof:** The character table is as follows:

| | $A_0$ | $A_1$ | $A_2$ |
|---|---|---|---|
| $\chi_0$ | 1 | $k$ | $k$ |
| $\chi_1$ | 1 | $\frac{-1+\sqrt{n}\,i}{2}$ | $\frac{-1-\sqrt{n}\,i}{2}$ |
| $\chi_2$ | 1 | $\frac{-1-\sqrt{n}\,i}{2}$ | $\frac{-1+\sqrt{n}\,i}{2}$ |

Obviously, (1) holds. (2) In this case, $(A_1^* - kA_0^*)^2 \neq 0$ and $(A_1^* - kA_0^*)^3 = 0$, so we have $F\mathfrak{X} \cong F[x]/(x^3)$. □

Now we consider symmetric association schemes with $d = 2$. They correspond to strongly regular graphs. If we consider a strongly regular graph with parameters $(n, k, \lambda, \mu)$ [4, p. 8], then we have

$$\begin{aligned} A_1^2 &= kA_0 + \lambda A_1 + \mu A_2, \\ A_1A_2 &= (k - \lambda - 1)A_1 + (k - \mu)A_2, \\ A_2^2 &= (n - k - 1)A_0 + (n - 2k + \lambda)A_1 + (n - 2k + \mu - 2)A_2. \end{aligned}$$

Now $A_1$ has three eigenvalues $k, r$, and $s$, here $r$ and $s$ are roots of the equation $x^2 + (\mu - \lambda)x + (\mu - k) = 0$. The character table is as follows.

| | $A_0$ | $A_1$ | $A_2$ |
|---|---|---|---|
| $\chi_0$ | 1 | $k$ | $n-k-1$ |
| $\chi_1$ | 1 | $r$ | $-r-1$ |
| $\chi_2$ | 1 | $s$ | $-s-1$ |

If the graph is not a conference graph, then $r$ and $s$ are rational integers.

**Theorem 11** *Let $\Gamma$ be a strongly regular graph with parameters $(n, k, \lambda, \mu)$ which is not a conference graph, and let $\mathfrak{X}$ be an association scheme defined by $\Gamma$. Then $A_1$ has eigenvalues $k, r$, and $s$ and the following holds.*
(1) *If $p \mid n$ and $r \not\equiv s \pmod p$, then $F\mathfrak{X} \cong F[x]/(x^2) \oplus F$ ($F[x]/(x^2)$ is the principal block),*
(2) *If $p \mid n, r \equiv s \pmod p$, and at least one of $k(k+1)$, $\lambda - 2k$ and $\mu$ are not zero (mod $p$), then $F\mathfrak{X} \cong F[x]/(x^3)$,*
(3) *If $p \mid n, r \equiv s \pmod p$, and $k(k+1) \equiv \lambda - 2k \equiv \mu \equiv 0 \pmod p$, then $F\mathfrak{X} \cong F[x, y]/(x^2, xy, y^2)$,*
(4) *If $p \nmid n$ and $r \not\equiv s \pmod p$, then $F\mathfrak{X} \cong F \oplus F \oplus F$,*
(5) *If $p \nmid n$ and $r \equiv s \pmod p$, then $F\mathfrak{X} \cong F \oplus F[x]/(x^2)$ ($F$ is the principal block).*

*In (2) and (3), $k(k+1) \equiv \lambda - 2k \equiv \mu \equiv 0 \pmod p$ if and only if $(A_1^* - kA_0^*)^2 = 0$.*

**Proof:** We note that $r \equiv s \pmod{(\pi)}$ if and only if $r \equiv s \pmod p$ since $r$ and $s$ are rational integers.

Suppose that $p \mid n$. Then the principal block is not simple, so $k \equiv r \pmod p$ of $k \equiv s \pmod p$ by the character table. If only one of them holds, then (1) holds. Otherwise, $F\mathfrak{X}$ has only one irreducible representation, so it is a local algebra. In this case, we have $F\mathfrak{X} \cong F[x]/(x^3)$ or $F\mathfrak{X} \cong F[x, y]/(x^2, xy, y^2)$. If $(A_1^* - kA_0^*)^2 \neq 0$, then we have $F\mathfrak{X} \cong F[x]/(x^3)$. Suppose $(A_1^* - kA_0^*)^2 = 0$. Then, by a direct calculation, we have $k(k+1) \equiv \lambda - 2k \equiv \mu \equiv 0 \pmod p$ and $(\alpha A_1^* + \beta A_2^*)^2 = 0$ for any $\alpha, \beta \in F$. Thus $F\mathfrak{X} \cong F[x, y]/(x^2, xy, y^2)$.

Suppose that $p \nmid n$. Then the principal block is isomorphic to $F$. The result follows from Lemma 2. □

Note that there exist strongly regular graphs for every case in this theorem.

Now we consider conference graphs. Then the parameters are $(n, k, \lambda, \mu) = (4\mu + 1, 2\mu, \mu - 1, \mu)$, and the eigenvalues are $k, r = (-1 + \sqrt{n})/2$, and $s = (-1 - \sqrt{n})/2$.

**Theorem 12** *Let $\Gamma$ be a conference graph with parameters $(n, k, \lambda, \mu) = (4\mu+1, 2\mu, \mu-1, \mu)$, and let $\mathfrak{X}$ be an association scheme defined by $\Gamma$. Then the following holds.*
(1) *If $p \mid n$, then $F\mathfrak{X} \cong F[x]/(x^3)$.*
(2) *If $p \nmid n$, then $F\mathfrak{X} \cong F \oplus F \oplus F$.*

**Proof:** Suppose $p \mid n$. Then $k \equiv r \equiv s \pmod{(\pi)}$, so $F\mathfrak{X}$ is not semisimple. We can easily check that $(A_1^* - kA_0^*)^2 \neq 0$, thus $F\mathfrak{X} \cong F[x]/(x^3)$.

Suppose $p \nmid n$. Then $r - s = \sqrt{n} \not\equiv 0 \pmod{(\pi)}$. Thus $F\mathfrak{X}$ is semisimple. □

## 5. The structure of the standard module with $d = 2$

In the previous section, we showed that there exist the following five types for the structure of the adjacency algebras:

(1) $F\mathfrak{X} \cong F \oplus F \oplus F$,
(2) $F\mathfrak{X} \cong F \oplus F[x]/(x^2)$ (the principal block is $F$),
(3) $F\mathfrak{X} \cong F[x]/(x^2) \oplus F$ (the principal block is $F[x]/(x^2)$),
(4) $F\mathfrak{X} \cong F[x]/(x^3)$,
(5) $F\mathfrak{X} \cong F[x, y]/(x^2, xy, y^2)$.

In this section, we will determine the structure of the standard module for each type. We assume that $A_1$ has eigenvalues $k, r, s$ with multiplicities $1, m_r, m_s$.

If $\mathfrak{X}$ is commutative, $(FX)M$ is an $F\mathfrak{X}$-module for any $M \in F\mathfrak{X}$. Then it follows that $\dim(FX)M = rk_F(M)$. Especially, if $\mathfrak{X}$ is class 2, the $p$-rank of $M$ such that $M \in \mathrm{rad}F\mathfrak{X}$ is very important. Since many of them have the relevant $p$-rank, the structure of $FX$ plays a role for the structure theory of the association scheme (See [5]).

### 5.1. *Type (1)* $F\mathfrak{X} \cong F \oplus F \oplus F$

Let $B_0$, $B_1$ and $B_2$ be the corresponding blocks to $k, r$ and $s$, respectively. Let $S_i$ be the simple module such that the corresponding representation belongs to $B_i$ for $i = 0, 1, 2$. In this case, it is known that

$$FX \cong S_0 \oplus m_r S_1 \oplus m_s S_2.$$

In this case, the structure of $FX$ is completely determined by the parameters.

In the case of type (2) or type (3), we set that $F\mathfrak{X} = B_0 \oplus B_1$ is the block decomposition of the adjacency algebra, where $B_0$ is the principal block. Let $S_i$ be the simple module such that the corresponding representation belongs to $B_i$ for $i = 0, 1$. In these cases, it is enough that we determine $\dim_F \mathrm{rad}FX$.

### 5.2. *Type (2)* $F\mathfrak{X} \cong F \oplus F[x]/(x^2)$

Let $P$ be the projective cover of $S_1$ and $e_1$ the block idempotent of $B_1$. We set that $M = e_1(A_1 - sA_0)$. Then $\mathrm{rad}F\mathfrak{X} = FM^*$. Since the coefficient of $A_i^*$ in $M^*$ is in $\mathbb{F}_p$ for each $i$, $rk_F(M) = rk_p(M)$. We set that $t = rk_p(M)$. Then it follows that

$$FX \cong S_0 \oplus tP \oplus (n - 1 - 2t)S_1.$$

Interesting examples are the association schemes defined by (26,10,3,4)-strongly regular graphs. In this case, there are 10 non-isomorphic association schemes. The adjacency algebras over a field of characteristic 5 are type (2). Then the structures of the modular standard modules are as follows:

| $FX$ | The number of non-isomorphic association schemes |
|---|---|
| $S_0 \oplus 9P \oplus 7S_1$ | 1 |
| $S_0 \oplus 11P \oplus 3S_1$ | 2 |
| $S_0 \oplus 12P \oplus S_1$ | 7 |

### 5.3. *Type (3)* $F\mathfrak{X} \cong F[x]/(x^2) \oplus F$

Let $P$ be the projective cover of $S_0$. In this case, we have either $k \equiv r$ or $k \equiv s$. Without loss of generality, we assume that $k \equiv r$. Since $\mathrm{rad}F\mathfrak{X}$ is $FJ^*$, $\mathrm{rad}FX$ is one-dimensional. Therefore it follows that

$$FX \cong P \oplus (n - m_s - 2)S_0 \oplus m_s S_1.$$

In this case, the structure of $FX$ is completely determined by the parameters.

### 5.4. *Type (4)* $F\mathfrak{X} \cong F[x]/(x^3)$

We set that $M_i \cong F\mathfrak{X}/(\mathrm{rad}F\mathfrak{X})^i$ as $F\mathfrak{X}$-modules for $i = 1, 2, 3$. We set $M = A_1 - kA_0$. Then $\mathrm{rad}^2 F\mathfrak{X} = FJ^*$ and $\mathrm{rad}F\mathfrak{X} = FJ^* + FM^*$.

We set $t = rk_p(M) = rk_F(M)$. Since $(M^*)^2 = \mu J^* \neq 0$, $(FX)J^* \subset (FX)M^*$. Therefore it follows that

$$FX \cong M_3 \oplus (t-2)M_2 \oplus (n - 2t + 1)M_1.$$

Interesting examples are the association schemes defined by the (25,12,5,6)-strongly regular graphs. Their adjacency algebras over a field of characteristic 5 are type (4). There are 15 strongly regular graphs with parameters (25,12,5,6), but the only one of them is self-complementary, so there are 8 non-isomorphic association schemes. Then the structures of the modular standard modules are as follows:

| $FX$ | The number of non-isomorphic association schemes |
|---|---|
| $M_3 \oplus 10M_2 \oplus 2M_1$ | 5 |
| $M_3 \oplus 9M_2 \oplus 4M_1$ | 2 |
| $M_3 \oplus 7M_2 \oplus 8M_1$ | 1 |

*5.5. Type (5)* $F\mathfrak{X} \cong F[x, y]/(x^2, xy, y^2)$

In this case, $p \mid v_1$ and $p \mid v_2 + 1$, or $p \mid v_1 + 1$ and $p \mid v_2$. Let $\alpha$ and $\beta$ be $p \mid v_\alpha$ and $p \mid v_\beta + 1$. In this case, we can define an isomorphism $F\mathfrak{X} \to F[x, y]/(x^2, xy, y^2)$ by $A_\alpha \mapsto x$ and $A_0 + A_\alpha + A_\beta \mapsto y$. From here, we identify them.

Now we consider $\widehat{F\mathfrak{X}}$. Consider a basis $\{\tau_0, \tau_\alpha - \tau_0, \tau_0 - \tau_\beta\}$ of $\widehat{F\mathfrak{X}}$. Then the matrix representation of $F\mathfrak{X}$ on $\widehat{F\mathfrak{X}}$ with respect to this basis is as follows:

$$x \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we consider the structure of $FX$. Let

$$FX = M_1 \oplus M_2 \oplus \cdots \oplus M_r$$

be an indecomposable decomposition of $FX$. Since $\dim_F FXy = 1$, there is the unique $M_i$ such that $M_i y \neq 0$. We assume $M_1 y \neq 0$.

Since $F\mathfrak{X}$ has tame representation type, we can classify all indecomposable $F\mathfrak{X}$-modules [3, Section 4.3]. We have the following.

**Proposition 13** *Let $M$ be an indecomposable $F\mathfrak{X}$-module such that* $\dim_F My = 0$. *Then the corresponding representation is one of the following*:

$$\Lambda_1 : x \mapsto (0), \quad y \mapsto (0) \text{ (simple module)},$$
$$\Lambda_2 : x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

**Proposition 14** *Let $M$ be an indecomposable $F\mathfrak{X}$-module such that* $\dim_F My = 1$. *Then the corresponding representation is one of the following*:

$$\Gamma_1(\lambda) : x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} (\lambda \neq 0),$$
$$\Gamma_2 : x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$\Gamma_3 : x \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_4 : x \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_5 : x \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We set $t = rk_p(A_\alpha) = rk_F(A_\alpha)$. From Propositions 5 and 13, $M_1$ must be self-contragradient. It follows that $FX \cong \Gamma_5 \oplus (t-2)\Lambda_2 \oplus (n-2t)\Lambda_1$, $\Gamma_1(\lambda) \oplus (t-1)\Lambda_2 \oplus (n-2t)\Lambda_1$, or $\Gamma_2 \oplus t\Lambda_2 \oplus (n-2t-2)\Lambda_1$.

Let us denote the column space of $A_\alpha$ by $\langle A_\alpha \rangle$ and the all-one column vector by $\mathbf{1}$. Then $(FX)J^* \subset (FX)A_\alpha$ if and only if $\mathbf{1} \in \langle A_\alpha \rangle$. We assume that $\mathbf{1} \in \langle A_\alpha \rangle$. Let us denote the $i$-th column vector of $A_\alpha$ by $A_\alpha(i)$. We set $\mathbf{1} = \sum_{i=1}^{n} a_i A_\alpha(i)$, where $a_i \in F$ for all $i$. Then it follows that

$$FX \cong \begin{cases} \Gamma_5 \oplus (t-2)\Lambda_2 \oplus (n-2t)\Lambda_1 & \text{if } \mathbf{1} \in \langle A_\alpha \rangle \text{ and } \displaystyle\sum_{i=1}^{n} a_i \equiv 0 \pmod{p}, \\ \Gamma_1(\lambda) \oplus (t-1)\Lambda_2 \oplus (n-2t)\Lambda_1 & \text{if } \mathbf{1} \in \langle A_\alpha \rangle \text{ and } \displaystyle\sum_{i=1}^{n} a_i \not\equiv 0 \pmod{p}, \\ \Gamma_2 \oplus t\Lambda_2 \oplus (n-2t-2)\Lambda_1 & \text{if } \mathbf{1} \notin \langle A_\alpha \rangle. \end{cases}$$

The example whose modular standard module contains $\Gamma_5$ is the association scheme defined by the (16,5,0,2)-strongly regular gaph. The adjacency algebra over a field of characteristic 2 is type (5) and $FX \cong \Gamma_5 \oplus 4\Lambda_2 \oplus 4\Lambda_1$.

The example whose modular standard module contains $\Gamma_2$ is the association scheme defined by the triangular graph $T(8)$. The adjacency algebra over a field of characteristic 2 is type (5) and $FX \cong \Gamma_2 \oplus 6\Lambda_2 \oplus 14\Lambda_1$. There are 4 strongly regular graphs with parameters (28, 12, 6, 4), namely $T(8)$ and three Chang graphs. We know that $T(8)$ is characterized by the 2-rank $rk_2(A_1)$ (See [5]), but $T(8)$ is also characterized by the structure of the modular standard module. Because it follows that $FX \cong \Gamma_5 \oplus 6\Lambda_2 \oplus 12\Lambda_1$ for the association scheme defined by a Chang graph.

The example whose modular standard module contains $\Gamma_1(\lambda)$ is the association scheme defined by the (36,14,4,6)-strongly regular graphs. The adjacency algebras over a field of characteristic 3 are type (5). There are 180 strongly regular graphs with parameters (36,14,4,6). Two of them have the structure of the modular standard module $FX \cong \Gamma_1(\lambda) \oplus 12\Lambda_2 \oplus 10\Lambda_1$.

*Notes* In the cases type (1) and (3), the structure of the modular standard module is determined completely by the multipicities (if the association scheme is defined by a strongly regular graph, they are determined by the parameters). Therefore we are interested in type

(2), (4) and (5). In the cases type (2) and (4), we can determine the structure of the modular standard module by the $p$-rank of the corresponding matrix and the parameters of the corresponding strongly regular graph.

Here we focus on the case type (5). Then there is the possibility that we can obtain more information than the $p$-rank. Namely, they have the same $p$-rank of $A_\alpha$, but their structures of the modular standard module are non-isomorphic. Such an example is (36, 15, 6, 6)-strongly regular graph. There are 227 strongly regular graphs with the parameter (36, 15, 6, 6) in the list by Spence [10], [11]. There are 60 strongly regular graphs such that $rk_2(A_2) = 14$. Then three strongly regular graphs of them have $FX \cong \Gamma_2 \oplus 14\Lambda_2 \oplus 6\Lambda_1$, and the others $FX \cong \Gamma_5 \oplus 12\Lambda_2 \oplus 8\Lambda_1$. We do not know the examples that there are all of three cases with the same $p$-rank.

In general, it is not so easy to calculate the value $\lambda$ for $\Gamma_1(\lambda)$. We do not know the value $\lambda$ and its range.

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