Journal of Algebraic Combinatorics, 21, 331–349, 2005

The Combinatorial Quantum Cohomology Ring of G/B

AUGUSTIN-LIVIU MARE mareal@math.uregina.ca
Department of Mathematics and Statistics, University of Regina, College West 307.14, Regina, Saskatchewan, Canada S4S.0A2

Received March 5, 2003; Revised August 20, 2004; Accepted

Abstract. A purely combinatorial construction of the quantum cohomology ring of the generalized flag manifold is presented. We show that the ring we construct is commutative, associative and satisfies the usual grading condition. By using results of our previous papers [12, 13], we obtain a presentation of this ring in terms of generators and relations, and formulas for quantum Giambelli polynomials. We show that these polynomials satisfy a certain orthogonality property, which—for $G = SL_n(\mathbb{C})$—was proved previously in the paper [5].

Keywords: generalized flag manifolds, quantum cohomology, quantum Chevalley formula, quantum Giambelli problem

1. Introduction

Let us consider the complex flag manifold G/B, where G is a connected, simply connected, simple, complex Lie group and $B \subset G$ a Borel subgroup. Let $\mathfrak{t}$ be the Lie algebra of a maximal torus of a compact real form of G and $\Phi \subset \mathfrak{t}^*$ the corresponding set of roots. Consider an arbitrary W-invariant inner product $\langle , \rangle$ on $\mathfrak{t}$. The Weyl group W is the subgroup of $O(\mathfrak{t}, \langle , \rangle)$ generated by the reflections about the hyperplanes $\ker \alpha$, $\alpha \in \Phi^+$. To any root α corresponds the coroot

$$\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$

which is an element of $\mathfrak{t}$, by using the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$ induced by $\langle , \rangle$. If $\{\alpha_1, \ldots, \alpha_l\}$ is a system of simple roots then $\{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$ is a system of simple coroots. Consider $\{\lambda_1, \ldots, \lambda_l\} \subset \mathfrak{t}^*$ the corresponding system of fundamental weights, which are defined by $\lambda_i(\alpha_j^\vee) = \delta_{ij}$. It can be shown that the Weyl group W is actually generated by the *simple reflections* $s_1 = s_{\alpha_1}, \ldots, s_l = s_{\alpha_l}$ about the hyperplanes $\ker \alpha_1, \ldots, \ker \alpha_l$. The *length* $l(w)$ of w is the minimal number of factors in a decomposition of w as a product of simple reflections. We denote by w_0 the longest element of W.

Let $B^- \subset G$ denote the Borel subgroup opposite to B. To each $w \in W$ we assign the *Schubert variety* $X_w = \overline{B^-.w}$. The Poincaré dual of $[X_w]$ is an element of $H^{2l(w)}(G/B)$, which is called the *Schubert class*. The set $\{\sigma_w \mid w \in W\}$ is a basis of the cohomology[1] module $H^*(G/B)$. The Poincaré pairing $(,)$ on $H^*(G/B)$ is determined by:

$$(\sigma_u, \sigma_v) = \begin{cases} 1, & \text{if } u = w_0 v \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

According to a theorem of Borel [2], the ring homomorphism $S(\mathfrak{t}^*) \to H^*(G/B)$ defined by

$$\lambda_i \mapsto \sigma_{s_i}, \quad 1 \le i \le l$$

is surjective and it induces the ring isomorphism

$$H^*(G/B) \simeq \mathbb{R}[\{\lambda_i\}]/I_W, \tag{2}$$

where I_W is the ideal of $S(\mathfrak{t}^*) = \mathbb{R}[\lambda_1, \ldots, \lambda_l] = \mathbb{R}[\{\lambda_i\}]$ generated by the W-invariant polynomials of strictly positive degree. Recall that, by a result of Chevalley [4], there exist l homogeneous, functionally independent polynomials $u_1, \ldots, u_l \in S(\mathfrak{t}^*)$ which generate I_W. We identify $H^*(G/B)$ with Borel's presentation and denote them both by $\mathcal{H}$. So

$$\mathcal{H} = H^*(G/B) = \mathbb{R}[\{\lambda_i\}]/I_W.$$

In this way the homogeneous elements of $\mathcal{H}$ will be of the form

$$[f] = f \bmod I_W,$$

where $f \in \mathbb{R}[\{\lambda_i\}]$ is a homogeneous polynomial. In particular, the degree two Schubert classes will be $[\lambda_i]$, $1 \le i \le l$.

In fact we would like to see *all* Schubert classes as cosets of certain polynomials in the presentation (2). A construction of such polynomials was obtained by Bernstein et al. [1], as follows: To each positive root α we assign the *divided difference operator* Δ_α on the ring $\mathbb{R}[\{\lambda_i\}]$ (since the latter is just the symmetric algebra $S(\mathfrak{t}^*)$, it admits a natural action of the Weyl group W):

$$\Delta_\alpha(f) = \frac{f - s_\alpha f}{\alpha},$$

$f \in \mathbb{R}[\{\lambda_i\}]$. If w is an arbitrary element of W, take $w = s_{i_1} \ldots s_{i_k}$ a reduced expression and then set[2]

$$\Delta_w = \Delta_{\alpha_{i_1}} \circ \cdots \circ \Delta_{\alpha_{i_k}}.$$

The polynomial

$$c_{w_0} := \frac{1}{|W|} \prod_{\alpha \in \Phi^+} \alpha$$

is homogeneous, of degree $l(w_0)$ and has the property that $\Delta_{w_0}c_{w_0} = 1$. To any $w \in W$ we assign

$$c_w := \Delta_{w^{-1}w_0}c_{w_0}$$

which is a homogeneous polynomial of degree $l(w)$ satisfying

$$\Delta_v c_w = \begin{cases} c_{wv^{-1}}, & \text{if } l(wv^{-1}) = l(w) - l(v) \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

for any $v \in W$ (see for instance [9, Chapter 4]).

Theorem 1.1 ([1]) *By the identification* (2) *we have*

$$\sigma_w = [c_w],$$

for any $w \in W$.

The main goal of our paper is to construct in a purely combinatorial way a certain "quantum deformation" of the ring $\mathcal{H}$. This will depend on the "deformation parameters" $q_1, \ldots, q_l$, which are just some additional multiplicative variables. Let us begin with the following lemma, which was proved for instance in [12] (see also [15] or [3]). Recall first that if α is a positive root, then the *height* of the corresponding coroot $\alpha^\vee$ is by definition

$$\mathrm{ht}(\alpha^\vee) = m_1 + \cdots + m_l,$$

where the positive integers $m_1, \ldots, m_l$ are given by

$$\alpha^\vee = m_1\alpha_1^\vee + \cdots + m_l\alpha_l^\vee. \tag{4}$$

Lemma 1.2 *For any positive root* α *we have that* $l(s_\alpha) \leq 2\mathrm{ht}(\alpha^\vee) - 1$.

Denote by $\tilde{\Phi}^+$ the set of all positive roots α with the property that

$$l(s_\alpha) = 2\mathrm{ht}(\alpha^\vee) - 1.$$

We will obtain in Section 3 a complete description of the elements of $\tilde{\Phi}^+$ (see Lemma 3.1). One can easily deduce from this that if the root system of G is simply laced, then $\tilde{\Phi}^+ = \Phi^+$.

The following divided difference type operators on $\mathbb{R}[\{\lambda_i\}, \{q_i\}]$ have been considered by Peterson in [15]:

$$\Lambda_j = \lambda_j + \sum_{\alpha \in \tilde{\Phi}^+} \lambda_j(\alpha^\vee) q^{\alpha^\vee} \Delta_{s_\alpha}, \quad 1 \leq j \leq l \tag{5}$$

where we use the notation

$$q^{\alpha^\vee} = q_1^{m_1} \dots q_l^{m_l},$$

with $m_1 \dots, m_l$ given by (4). It is obvious that Λ_j leaves the ideal $I_W \otimes \mathbb{R}[\{q_i\}]$ of $\mathbb{R}[\{\lambda_i\}, \{q_i\}]$ invariant, hence it induces an operator on $\mathcal{H} \otimes \mathbb{R}[\{q_i\}]$.

The following result[3] was stated by Peterson [15] (for $G = SL(n, \mathbb{C})$, a proof can be found in [5]).

Lemma 1.3 *The operators $\Lambda_1, \dots, \Lambda_l$ on $\mathbb{R}[\{\lambda_i\}, \{q_i\}]$ commute.*

We will prove this lemma in Section 3 of our paper. The operator ψ defined in the next lemma will be an important object in our paper.

Lemma 1.4 *The map $\psi : \mathbb{R}[\{\lambda_i\}, \{q_i\}] \to \mathbb{R}[\{\lambda_i\}, \{q_i\}]$ given by*

$$\psi(f) = f(\{\Lambda_i\}, \{q_i\})(1),$$

$f \in \mathbb{R}[\{\lambda_i\}, \{q_i\}]$ is an isomorphism of $\mathbb{R}[\{q_i\}]$-modules. For $f \in \mathbb{R}[\{\lambda_i\}, \{q_i\}]$ of degree m with respect to $\lambda_1, \dots, \lambda_l$, we have

$$\begin{aligned}\psi^{-1}(f) &= \frac{I - (I - \psi)^m}{\psi}(f) \\ &= \binom{m}{1} f - \binom{m}{2}\psi(f) + \dots + (-1)^{m-2}\binom{m}{m-1}\psi^{m-2}(f) \\ &\quad + (-1)^{m-1}\psi^{m-1}(f),\end{aligned}$$

where $\binom{m}{1}, \dots, \binom{m}{m-1}$ are the binomial coefficients.

The proof follows in an elementary way from the fact that the degree of $f - \psi(f)$ with respect to $\lambda_1, \dots, \lambda_l$ is strictly less than the degree of f (the details can be found in [12, Lemma 3.4]).

Our aim is to investigate the ring defined as follows (note that for $G = SL(n, \mathbb{C})$ a similar object has been considered by Postnikov [16]).

Theorem-Definition 1.5 *The composition law $\star$ on the $\mathbb{R}[\{q_i\}]$-module $\mathcal{H} \otimes \mathbb{R}[\{q_i\}] = \mathbb{R}[\{\lambda_i\}, \{q_i\}]/(I_W \otimes \mathbb{R}[\{q_i\}])$ given by*

$$[f] \star [g] = [\psi(\psi^{-1}(f)\psi^{-1}(g))], \quad f, g \in \mathbb{R}[\{\lambda_i\}, \{q_i\}] \tag{6}$$

is well defined, commutative, associative, $\mathbb{R}[\{q_i\}]$-bilinear, and satisfies:

- $\deg(a \star b) = \deg a + \deg b$, *for any two homogeneous elements* a, b *of* $\mathcal{H} \otimes \mathbb{R}[\{q_i\}]$, *where we assign*

$$\deg[\lambda_i] = 2, \ \deg q_i = 4, \quad 1 \le i \le l.$$

- (*Frobenius property*) $(a \star b, c) = (a, b \star c)$, *for any* $a, b, c \in \mathcal{H}$, *where* $(,)$ *is the* $\mathbb{R}[\{q_i\}]$-*bilinear extension of the Poincaré pairing on* $\mathcal{H}$.

We will call $\star$ *the* combinatorial quantum product *on* $\mathcal{H} \otimes \mathbb{R}[\{q_i\}]$.

We will prove this theorem at the beginning of Section 2.

A complete knowledge of the combinatorial quantum cohomology $\mathbb{R}[\{q_i\}]$-algebra defined in the previous theorem can be achieved by finding the structure constants (which are in $\mathbb{R}[\{q_i\}]$) of the multiplication $\star$ with respect to the basis consisting of the Schubert classes $\sigma_w = [c_w]$, $w \in W$. Like in the classical situation (see the beginning of this section), we can obtain this information about $(\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \star)$ as follows:

(a) describe it in terms of generators and relations (i.e. find the quantum analogue of Borel's presentation (2))
(b) determine representatives of the Schubert classes in the quotient ring obtained at (a) (i.e. find the quantum analogue of the Bernstein-Gelfand-Gelfand polynomials, see Theorem 1.1).

The next two theorems give solutions to problems (a), respectively (b). The first theorem can be interpreted as the combinatorial version of B. Kim's theorem [11]. Our proof, which can be found in Section 2, is a direct application of a more general result obtained by us in [13].

Theorem 1.6 *Let* I_W^q *denote the ideal of* $\mathbb{R}[\{\lambda_i\}, \{q_i\}]$ *generated by* $F_k(\{\lambda_i\}, \{-\langle\alpha_i^\vee, \alpha_i^\vee\rangle q_i\})$, $1 \le k \le l$, *where* F_k *are polynomials in* $2l$ *variables which represent the integrals of motion of the Hamiltonian system of Toda lattice type associated to the coroot system of* G (*for more details, see Section* 2). *Then the map*

$$(\mathcal{H} \otimes \mathbb{R}[\{q_i\}] = \mathbb{R}[\{\lambda_i\}, \{q_i\}]/(I_W \otimes \mathbb{R}[q_i]), \star) \to \mathbb{R}[\{\lambda_i\}, \{q_i\}]/I_W^q,$$

given by

$$f \bmod I_W \ \mapsto \psi^{-1}(f) \bmod I_W^q,$$

$f \in \mathbb{R}[\{\lambda_i\}, \{q_i\}]$, *is an isomorphism of* $\mathbb{R}[\{q_i\}]$-*algebras.*

Alternatively, one can see that I_W^q is the ideal of $\mathbb{R}[\{\lambda_i\}, \{q_i\}]$ generated by the polynomials $\psi^{-1}(u_1), \ldots, \psi^{-1}(u_l)$, which is the same as $\psi^{-1}(I_W)$ (see Proposition 2.1).

What follows now is the combinatorial version of the main result of [12], where a quantum Giambelli formula for G/B has been obtained. In the context of our present paper, we obtain the same formula by a straightforward application of Theorem 1.6 and Lemma 1.4.

Corollary 1.7 *The isomorphism described by Theorem* 1.6 *maps the Schubert class* $\sigma_w = c_w \bmod I_W$ *to the class modulo* I_W^q *of the polynomial*

$$\psi^{-1}(c_w) = \frac{I-(I-\psi)^l}{\psi}(c_w)$$
$$= \binom{l}{1}c_w - \binom{l}{2}\psi(c_w) + \cdots + (-1)^{l-2}\binom{l}{l-1}\psi^{l-2}(c_w)$$
$$+ (-1)^{l-1}\psi^{l-1}(c_w),$$

where l *denotes* $l(w)$.

We will also show that the polynomials described by Corollary 1.7 satisfy a certain orthogonality condition (similar to (1)) with respect to the "quantum intersection pairing" (see Proposition 2.3).

Remarks 1 The actual quantum product $\circ$ on $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ is defined in terms of numbers of holomorphic curves which intersect "general" translates of three given Schubert varieties (for the precise definition, one can see [6] or [7]). The *quantum Chevalley formula* describes the multiplication of degree two Schubert classes by arbitrary Schubert classes. More precisely, in terms of the identification (2) (see also Theorem 1.1), it states that

$$[\lambda_i] \circ [c_w] = \Lambda_i([c_w]). \tag{7}$$

This formula was announced by Peterson in [15] and then proved by Fulton and Woodward [7]. In order to relate (7) to our product $\star$, we note that

$$\Lambda_i([c_w]) = [\Lambda_i(c_w)] = [\Lambda_i(\psi\psi^{-1}(c_w))] = [\psi(\lambda_i\psi^{-1}(c_w))] = [\lambda_i] \star [c_w] \tag{8}$$

where we have used that $\psi(\lambda_i) = \lambda_i$. We deduce that

$$[\lambda_i] \circ [c_w] = [\lambda_i] \star [c_w], \quad 1 \le i \le l,\, w \in W.$$

This implies that

$$[c_v] \circ [c_w] = [c_v] \star [c_w],$$

for any $v, w \in W$, because both $(\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \star)$ and $(\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \circ)$ are generated by $[\lambda_1], \ldots, [\lambda_l]$ as $\mathbb{R}[\{q_i\}]$-algebras. Now, since $\star = \circ$, the results about $\star$ which we prove in our paper hold for $\circ$ as well. In this way we are able to recover results about the actual quantum cohomology ring $QH^*(G/B) = (\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \circ)$ (see [11, 12] for the $\circ$-versions of Theorem 1.6 respectively Corollary 1.7).

2. We hope that a similar approach can be used by considering instead of the root system of G an arbitrary *affine* root system and obtain in this way a combinatorial model for the quantum cohomology ring of the infinite dimensional flag manifold LK/T, which is investigated in [14].

2. Definition and presentations of $(\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \star)$

Our first concern is to show that the combinatorial quantum product $\star$ described by Eq. (6) is well-defined.

Proof of Theorem 1.5: Let us note that in fact we can define the product $\star$ on $\mathbb{R}[\{\lambda_i\}, \{q_i\}]$, as follows:

$$f \star g := \psi(\psi^{-1}(f)\psi^{-1}(g)) = (\psi^{-1} f)(\{\Lambda_i\}, \{q_i\})(g), \tag{9}$$

$f, g \in \mathbb{R}[\{\lambda_i\}, \{q_i\}]$. If $g \in I_W \otimes \mathbb{R}[\{q_i\}]$, then the last expression in (9) is in $I_W \otimes \mathbb{R}[\{q_i\}]$ as well (since the latter is invariant under any Λ_j, $1 \le j \le l$). We deduce that $I_W \otimes \mathbb{R}[\{q_i\}]$ is an ideal of the ring $(\mathbb{R}[\{\lambda_i\}, \{q_i\}], \star)$. The quotient of the latter ring by the former ideal is just $(\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \star)$. It is commutative, associative and satisfies the grading condition $\deg(a \star b) = \deg a + \deg b$, because the ring $(\mathbb{R}[\{\lambda_i\}, \{q_i\}], \star)$ is commutative and associative, and the operator Λ_i defined by (5) satisfies

$$\deg \Lambda_i(f) = \deg f + 2,$$

for any homogeneous polynomial $f \in \mathbb{R}[\{\lambda_i\}, \{q_i\}]$ (provided that $\deg \lambda_i := 2$, $\deg q_i := 4$).

In order to prove the Frobenius property, we only have to check that

$$([\lambda_i] \star [c_v], [c_w]) = ([c_v], [\lambda_i] \star [c_w]) \tag{10}$$

for any $1 \le i \le l$, $v, w \in W$. In turn, (10) follows from the fact that

$$[\lambda_i] \star [c_w] = \Lambda_i([c_w])$$

(see Eq. (8) in the introduction), the definition (5) of Λ_i and the equation

$$(\Delta_{s_\alpha}[c_v], [c_w]) = ([c_v], \Delta_{s_\alpha}[c_w]),$$

$v, w \in W$, $\alpha \in \Phi^+$, which is a consequence of (1) and (3). □

We are interested now in obtaining a presentation of the ring $(\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \star)$ in terms of generators and relations. One way[4] of obtaining this is as follows:

Proposition 2.1 *Let I_W^q be the ideal of $\mathbb{R}[\{\lambda_i\}, \{q_i\}]$ generated by $\psi^{-1}(u_1), \ldots, \psi^{-1}(u_l)$. The map*

$$\psi^{-1} : (\mathcal{H} \otimes \mathbb{R}[\{q_i\}] = \mathbb{R}[\{\lambda_i\}, \{q_i\}]/(I_W \otimes \mathbb{R}[\{q_i\}]), \star) \to \mathbb{R}[\{\lambda_i\}, \{q_i\}]/I_W^q$$

given by

$$f \bmod I_W \otimes \mathbb{R}[\{q_i\}] \ \mapsto \ \psi^{-1}(f) \bmod I_W^q,$$

$f \in \mathbb{R}[\{\lambda_i\}, \{q_i\}]$ *is a ring isomorphism.*

Proof: From the definition (9) we can see that

$$\psi^{-1} : (\mathbb{R}[\{\lambda_i\}, \{q_i\}], \star) \to (\mathbb{R}[\{\lambda_i\}, \{q_i\}], \cdot) \tag{11}$$

is a ring isomorphism. As pointed out before (see the proof of Theorem 1.5), the combinatorial quantum cohomology ring $(\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \star)$ is the quotient of the ring $(\mathbb{R}[\{\lambda_i\}, \{q_i\}], \star)$ by its ideal $I_W \otimes \mathbb{R}[\{q_i\}]$. Note that the latter—regarded as an ideal of $(\mathbb{R}[\{\lambda_i\}, \{q_i\}], \star)$—is generated by the same fundamental W-invariant polynomials $u_1, \ldots, u_l$. This is because for any $f \in \mathbb{R}[\{\lambda_i\}, \{q_i\}]$ we have

$$f \star u_k = f \cdot u_k,$$

$k = 1, \ldots, l$. Consequently, the ring isomorphism (11) maps the quotient of $(\mathbb{R}[\{\lambda_i\}, \{q_i\}], \star)$ by the ideal generated by $u_1, \ldots, u_l$ isomorphically onto the quotient of $(\mathbb{R}[\{\lambda_i\}, \{q_i\}], \cdot)$ by the ideal generated by $\psi^{-1}(u_1), \ldots, \psi^{-1}(u_l)$. □

As pointed out out in the introduction, we are also able to deduce B. Kim's presentation [11] for the combinatorial quantum cohomology ring. In fact Theorem 1.6 is a straightforward consequence of the following result, which was proved in [13]:

Theorem 2.2 ([13]) *Let $\bullet$ be an $\mathbb{R}[\{q_i\}]$-bilinear product on $\mathcal{H} \otimes \mathbb{R}[\{q_i\}]$ with the following properties*:

(i) $\bullet$ *is commutative*
(ii) $\bullet$ *is associative*
(iii) $\bullet$ *is a deformation of the usual product, in the sense that if we formally replace all q_i by* 0, *we obtain the usual product on $\mathcal{H}$*
(iv) $(\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \bullet)$ *is a graded ring with respect to* $\deg[\lambda_i] = 2$ *and* $\deg q_i = 4$
(v) $[\lambda_i] \bullet [\lambda_j] = [\lambda_i][\lambda_j] + \delta_{ij} q_j$
(vi) $d_i([\lambda_j] \bullet a)_d = d_j([\lambda_i] \bullet a)_d$, *for any $a \in \mathcal{H}$, $1 \le i, j \le l$, and $d = (d_1, \ldots, d_l) \ge 0$ (here we use the notation $[\lambda_i] \bullet a = \sum_{d=(d_1,\ldots,d_l)\ge 0} ([\lambda_i] \bullet a)_d q_1^{d_1} \ldots q_l^{d_l}$, with $([\lambda_i] \bullet a)_d \in \mathcal{H}$).*

Then the following relation holds in the ring $(\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \bullet)$:

$$F_k\big(\{[\lambda_i]\bullet\}, \big\{ -\langle\alpha_i^\vee, \alpha_i^\vee\rangle q_i \big\}\big) = 0, \tag{12}$$

$1 \leq k \leq l$, *where* F_k *are the integrals of motion of the Toda lattice associated to the coroot system of* G (*see below*). *Moreover*, *the ring* $(\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \bullet)$ *is isomorphic to* $\mathbb{R}[\{\lambda_i\}, \{q_i\}]$ *modulo the ideal generated by* $F_k(\{\lambda_i\}, \{-\langle\alpha_i^\vee, \alpha_i^\vee\rangle q_i\})$, $1 \leq k \leq l$.

The Toda lattice we are referring to in the theorem is the Hamiltonian system whose phase space is $(\mathbb{R}^{2l}, \sum_{i=1}^{l} dr_i \wedge ds_i)$ and Hamiltonian function

$$E = \sum_{i,j=1}^{l} \langle\alpha_i^\vee, \alpha_j^\vee\rangle r_i r_j + \sum_{i=1}^{l} e^{2s_i}.$$

It turns out (see for instance [8]) that this system admits l independent integrals of motion $E = F_1, F_2, \ldots, F_l$, which are all polynomial functions in variables $r_1, \ldots, r_l, e^{2s_1}, \ldots, e^{2s_l}$ and satisfy the condition

$$F_k(\lambda_1, \ldots, \lambda_l, 0, \ldots, 0) = u_k(\lambda_1, \ldots, \lambda_l), \tag{13}$$

where $u_1, \ldots, u_l$ are the fundamental W-invariant polynomials (see Section 1). According to Theorem 2.2, the ring $(\mathcal{H} \otimes \mathbb{R}[\{q_i\}], \bullet)$ is generated by $[\lambda_1], \ldots, [\lambda_l], q_1, \ldots, q_l$, and the relations are obtained by taking all polynomials F_k and for each of them making the replacements

$$r_i \mapsto [\lambda_i]\bullet, \quad e^{2s_i} \mapsto -\langle\alpha_i^\vee, \alpha_i^\vee\rangle q_i, \quad 1 \leq i \leq l.$$

It is easy to see that the combinatorial quantum product $\star$ satisfies the hypotheses (i)–(iv) of Theorem 2.2. We prove condition (v) as follows:

$$[\lambda_i] \star [\lambda_j] = [\psi(\lambda_i)] \star [\psi(\lambda_j)] = [\psi(\lambda_i\lambda_j)] = [\Lambda_i(\lambda_j)] = [\lambda_i\lambda_j + \delta_{ij} q_j],$$

$1 \leq i, j \leq l$. In order to prove (vi), we note that the coefficient of $q^{\alpha^\vee}$ in

$$[\lambda_j] \star a = \Lambda_j(a)$$

is $\lambda_i(\alpha^\vee)\Delta_{s_\alpha}(a)$; thus for the multi-index $d = \alpha^\vee = \lambda_1(\alpha^\vee)\alpha_1^\vee + \cdots + \lambda_l(\alpha^\vee)\alpha_l^\vee$ we have

$$d_i([\lambda_j] \star a)_d = \lambda_i(\alpha^\vee)\lambda_j(\alpha^\vee)\Delta_{s_\alpha}(a),$$

which is symmetric in i and j.

Our next goal is to show that the "quantum BGG-polynomials" (see Theorem 1.1) $\psi^{-1}(c_w)$, $w \in W$, satisfy a certain orthogonality property, which can be thought of as the quantum version of (1). For any $f \in \mathbb{R}[\{\lambda_i\}, \{q_i\}]$ we denote by $[f]_q$ its class modulo

I_W^q. By Theorem 1.6, the set $\{[\psi^{-1}(c_w)]_q \mid w \in W\}$ is a basis of $\mathbb{R}[\{\lambda_i\}, \{q_i\}]/I_W^q$ as an $\mathbb{R}[\{q_i\}]$-module. Define

$$\langle\!\langle [f]_q \rangle\!\rangle = \alpha_{w_0}$$

where the elements α_w of $\mathbb{R}[\{q_i\}]$ are defined by

$$[f]_q = \sum_{w \in W} \alpha_w [\psi^{-1}(c_w)]_q.$$

Consider the pairing $\langle\!\langle\, , \,\rangle\!\rangle$ on $\mathbb{R}[\{\lambda_i\}, \{q_i\}]/I_W^q$ given by

$$\langle\!\langle [f]_q, [g]_q \rangle\!\rangle = \langle\!\langle [fg]_q \rangle\!\rangle.$$

Proposition 2.3 *We have that*

$$\langle\!\langle [\psi^{-1}(c_u)]_q, [\psi^{-1}(c_v)]_q \rangle\!\rangle = \begin{cases} 1, & \text{if } u = w_0 v \\ 0, & \text{otherwise} \end{cases}$$

Proof: Write

$$[\psi^{-1}(c_u)\psi^{-1}(c_v)]_q = \sum_{w \in W} \alpha_w [\psi^{-1}(c_w)]_q,$$

which means that the polynomial

$$\psi^{-1}(c_u)\psi^{-1}(c_v) - \sum_{w \in W} \alpha_w \psi^{-1}(c_w) \tag{14}$$

is in I_W^q. Consider ψ of the expression (14), take into account that $\psi^{-1}(c_w)(\{[\lambda_i]\star\}, \{q_i\}) = [c_w]$ and that $\psi(I_W^q) = I_W \otimes \mathbb{R}[\{q_i\}]$ (see Proposition 2.1) and obtain in this way the following equality in $\mathcal{H} \otimes \mathbb{R}[\{q_i\}]$:

$$[c_u] \star [c_v] = \sum_{w \in W} \alpha_w [c_w]$$

If $(\, , \,)$ denotes the usual Poincaré pairing[5] on $\mathcal{H} \otimes \mathbb{R}[\{q_i\}]$, we deduce that

$$\alpha_{w_0} = ([c_u] \star [c_v], 1) = ([c_u], [c_v])$$

where we have used the Frobenius property of $\star$. The orthogonality relation stated in the lemma is a direct consequence of Eq. (1). ☐

3. Commutativity of the operators $\Lambda_1, \ldots, \Lambda_l$

The goal of this section is to provide a proof of Lemma 1.3. Let us start with the following recursive construction of the elements of $\tilde{\Phi}^+$ (the latter has been defined immediately after Lemma 1.2).

Proposition 3.1 *A positive root α is in $\tilde{\Phi}^+$ if and only if it is simple, or else there exist $k \geq 2$ and $i_1, \ldots, i_k \in \{1, \ldots, l\}$ such that*

$$\alpha = s_{i_k} \ldots s_{i_2}(\alpha_{i_1})$$

and

$$\alpha_{i_{j+1}}\big(s_{i_j} \ldots s_{i_2}(\alpha_{i_1})^\vee\big) = -1,$$

for all $1 \leq j \leq k-1$. When this is true, the expression

$$s_\alpha = s_{i_k} \ldots s_{i_2} s_{i_1} s_{i_2} \ldots s_{i_k}$$

is reduced and we have

$$\alpha^\vee = \alpha_{i_1}^\vee + \cdots + \alpha_{i_k}^\vee,$$

hence $\mathrm{ht}(\alpha^\vee) = k$. All roots $s_{i_j} \ldots s_{i_2}(\alpha_{i_1})$, $1 \leq j \leq k$, are in $\tilde{\Phi}^+$.

Proof: First we use induction on $k \geq 1$ to prove that any root of the form described in the lemma is in $\tilde{\Phi}^+$. Since any simple root is in $\tilde{\Phi}^+$, we only have to perform the induction step. Assume that $k \geq 2$. The root

$$\beta := s_{i_{k-1}} \ldots s_{i_2}(\alpha_{i_1})$$

satisfies the hypotheses of the lemma, hence it is in $\tilde{\Phi}^+$. Moreover, we have $\alpha_{i_k}(\beta^\vee) = -1$, hence

$$\alpha^\vee = s_{i_k}(\beta^\vee) = \beta^\vee + \alpha_{i_k}^\vee,$$

which implies that

$$\mathrm{ht}(\alpha^\vee) = \mathrm{ht}(\beta^\vee) + 1. \tag{15}$$

In particular, α is not a simple root. Also because $\alpha_{i_k}(\alpha^\vee) = 1$, we deduce that the roots

$$s_\alpha(\alpha_{i_k}) = \alpha_{i_k} - \alpha_{i_k}(\alpha^\vee)\alpha \text{ and } s_{i_k} s_\alpha(\alpha_{i_k}) = \big(\alpha(\alpha_{i_k}^\vee)\alpha_{i_k}(\alpha^\vee) - 1\big)\alpha_{i_k} - \alpha_{i_k}(\alpha^\vee)\alpha$$

are both negative. Consequently we have

$$l(s_\alpha) = l\big(s_{i_k} s_\alpha s_{i_k}\big) + 2 = l(s_\beta) + 2 = 2\mathrm{ht}(\beta^\vee) - 1 + 2 = 2\mathrm{ht}(\alpha^\vee) - 1,$$

where we have used (15). Hence $\alpha \in \tilde{\Phi}^+$.

Now we will use induction on $l(s_\alpha)$ in order to prove that any element of $\tilde{\Phi}^+$ can be realized in this way. If $l(s_\alpha) = 1$, then α is simple, hence it is of the type indicated in the lemma. Assume now that $\alpha \in \tilde{\Phi}^+$ is not simple. There exists a simple root α_i such that $\alpha(\alpha_i^\vee) > 0$ (otherwise we would be led to $\alpha(\alpha^\vee) \leq 0$). Also $\alpha_i(\alpha^\vee)$ must be strictly positive, hence the roots

$$s_\alpha(\alpha_i) = \alpha_i - \alpha_i(\alpha^\vee)\alpha \quad \text{and} \quad s_i s_\alpha(\alpha_i) = \big(\alpha\big(\alpha_i^\vee\big)\alpha_i(\alpha^\vee) - 1\big)\alpha_i - \alpha_i(\alpha^\vee)\alpha$$

are both negative. We deduce that $l(s_i s_\alpha s_i) = l(s_\alpha) - 2$. From

$$s_i(\alpha)^\vee = s_i(\alpha^\vee) = \alpha^\vee - \alpha_i(\alpha^\vee)\alpha_i^\vee$$

it follows that $s_i(\alpha)$ is a positive root which satisfies $\mathrm{ht}(s_i(\alpha)^\vee) = \mathrm{ht}(\alpha^\vee) - \alpha_i(\alpha^\vee)$. By Lemma 1.2, we have that:

$$\begin{aligned} l(s_\alpha) &= l(s_i s_\alpha s_i) + 2 \leq 2\mathrm{ht}(s_i(\alpha)^\vee) - 1 + 2 \\ &= 2\mathrm{ht}(\alpha^\vee) - 1 + 2(1 - \alpha_i(\alpha^\vee)) \leq 2\mathrm{ht}(\alpha^\vee) - 1. \end{aligned}$$

Since $\alpha \in \tilde{\Phi}^+$, the two inequalities from the last equation must be equalities. In other words, $s_i\alpha \in \tilde{\Phi}^+$ and $\alpha_i(\alpha^\vee) = 1$, the latter being equivalent to $\alpha_i((s_i\alpha)^\vee) = -1$. We use the induction hypothesis for $s_i\alpha$, which has the property that $l(s_{s_i\alpha}) = l(s_i s_\alpha s_i) = l(s_\alpha) - 2$ and the induction step is accomplished. □

The following property of $\tilde{\Phi}^+$ will be needed later.

Lemma 3.2 *If $\alpha, \beta \in \tilde{\Phi}^+$ are such that*

$$l(s_\alpha s_\beta) = l(s_\alpha) + l(s_\beta)$$

and $s_\alpha s_\beta \neq s_\beta s_\alpha$, then $\alpha(\beta^\vee) < 0$.

Proof: We use induction on $l(s_\beta)$. If β is simple, the condition $l(s_\alpha s_\beta) = l(s_\alpha) + 1$ is equivalent to the fact that the root $s_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha$ is positive, which implies $\beta(\alpha^\vee) \leq 0$, and then $\alpha(\beta^\vee) \leq 0$. We cannot have $\alpha(\beta^\vee) = 0$, since otherwise s_α and s_β would commute.

The induction step will follow now. Let us assume first that the root system involved here is not of type G_2. Consider $\alpha, \beta \in \tilde{\Phi}^+$ both non-simple; by Proposition 3.1, β is of the form $\beta = s_i(\tilde{\beta})$, where $\tilde{\beta} \in \tilde{\Phi}^+$ and $\alpha_i(\tilde{\beta}^\vee) = -1$. Suppose that $\alpha(\beta^\vee) \geq 1$. Since $\alpha_i(\beta^\vee) = 1$, the root $s_\beta(\alpha_i) = \alpha_i - \beta$ is negative, hence $l(s_\beta s_i) = l(s_\beta) - 1$. From $l(s_\alpha s_\beta) = l(s_\alpha) + l(s_\beta)$, we deduce now that $l(s_i s_\alpha) = l(s_\alpha) + 1$, hence the root $s_\alpha(\alpha_i) = \alpha_i - \alpha_i(\alpha^\vee)\alpha$ is positive, which implies $\alpha_i(\alpha^\vee) \leq 0$.

Claim. $\alpha_i(\alpha^\vee) \neq 0$.

Because otherwise s_i and s_α commute, hence

$$\begin{aligned} l(s_{\tilde{\beta}} s_\alpha) &= l(s_{\tilde{\beta}} s_i s_\alpha s_i) \\ &= l(s_{\tilde{\beta}} s_i s_\alpha) - 1 \\ &= l(s_i s_{\tilde{\beta}} s_i s_\alpha) - 2 \\ &= l(s_\beta s_\alpha) - 2 \\ &= l(s_\beta) - 2 + l(s_\alpha) \\ &= l(s_{\tilde{\beta}}) + l(s_\alpha) \end{aligned}$$

where the second equality holds since $l(s_{\tilde{\beta}} s_i s_\alpha) = l(s_{\tilde{\beta}}) + l(s_\alpha) + 1 > l(s_{\tilde{\beta}} s_\alpha)$. By the induction hypothesis, we must have $\tilde{\beta}(\alpha^\vee) \leq 0$. On the other hand we have

$$\tilde{\beta}(\alpha^\vee) = s_i \beta(\alpha^\vee) = \beta(s_i \alpha^\vee) = \beta(\alpha^\vee)$$

the last number being strictly positive. This contradiction concludes the claim.

From the claim we deduce that

$$\alpha(\tilde{\beta}^\vee) = \alpha(\beta^\vee) - \alpha\left(\alpha_i^\vee\right) \geq 2. \tag{16}$$

Since the root system is not of type G_2, we must have equality in (16), hence

$$\alpha\left(\alpha_i^\vee\right) = -1. \tag{17}$$

We distinguish the following two possibilities:

(i) $\alpha \neq \tilde{\beta}$. From (16) we deduce that $||\tilde{\beta}|| < ||\alpha||$. Since $||\tilde{\beta}|| = ||s_i \tilde{\beta}|| = ||\beta||$, we have that $||\beta|| < ||\alpha||$, hence $\alpha(\beta^\vee) \geq 2$. Consequently,

$$\alpha(\tilde{\beta}^\vee) = \alpha(\beta^\vee) - \alpha\left(\alpha_i^\vee\right) \geq 3, \tag{18}$$

which cannot happen as long as the root system is not of type G_2.

(ii) $\alpha = \tilde{\beta}$. This means that $\beta = s_i(\alpha)$,

$$\alpha_i(\alpha^\vee) = -1, \tag{19}$$

and $\beta^\vee = \alpha^\vee + \alpha_i^\vee$. From (17) and (19) we deduce that

$$s_\alpha s_i s_\alpha(\alpha_i) = -\alpha,$$

which is a negative root, hence

$$\begin{aligned} l(s_\alpha s_\beta) &= l(s_\alpha s_i s_\alpha s_i) = l(s_\alpha s_i s_\alpha) - 1 \le l(s_\alpha) + l(s_i s_\alpha) - 1 \\ &= l(s_\alpha) + l(s_i s_\alpha s_i) - 2 = l(s_\alpha) + l(s_\beta) - 2. \end{aligned}$$

This is a contradiction.

Now let us consider the case when the root system is of type G_2. Let α_1, α_2 be the standard basis of the root system G_2, with $||\alpha_1|| > ||\alpha_2||$. By Proposition 3.1 we can see that $\tilde{\Phi}^+$ consists of α_1, α_2, $s_2(\alpha_1) = \alpha_1 + 3\alpha_2$, and $s_1 s_2(\alpha_1) = 2\alpha_1 + 3\alpha_2$. Since $\alpha(\beta^\vee) \ge 1$ and none of α and β is simple, we can only have $\alpha = s_1 s_2(\alpha_1)$ and $\beta = s_2(\alpha_1)$, which implies $s_\alpha = s_1 s_2 s_1 s_2 s_1$ and $s_\beta = s_2 s_1 s_2$, hence $s_\alpha s_\beta = s_1 s_2 s_1 s_2 s_1 s_2 s_1 s_2 = (s_1 s_2)^4$; but the latter is the same as $(s_2 s_1)^2$, having length 4, which is strictly less than $l(s_\alpha) + l(s_\beta) = 5 + 3 = 8$. The contradiction shows that also in this case we must have $\alpha(\beta^\vee) < 0$. □

Lemma 3.3 *If $\alpha, \beta \in \tilde{\Phi}^+$ with $l(s_\alpha s_\beta) = l(s_\alpha) + l(s_\beta)$ and $s_\alpha s_\beta \ne s_\beta s_\alpha$, then there exists $\gamma \in \tilde{\Phi}^+$ such that*

$$\alpha^\vee + \beta^\vee = \gamma^\vee.$$

Proof: By Lemma 3.2, one of the numbers $\alpha(\beta^\vee)$ and $\beta(\alpha^\vee)$ is -1. We will actually prove that if $\beta(\alpha^\vee) = -1$ then $s_\beta(\alpha) \in \tilde{\Phi}^+$ (it is obvious that $s_\beta(\alpha)^\vee = s_\beta(\alpha^\vee) = \alpha^\vee + \beta^\vee$). We will use induction on $l(s_\beta)$. If β is simple, the result follows immediately from Proposition 3.1. Consider now the case when $\beta \in \tilde{\Phi}^+$ is non-simple; by Proposition 3.1, β is of the form $\beta = s_i(\tilde{\beta})$, where $\tilde{\beta} \in \tilde{\Phi}^+$ and $\alpha_i(\tilde{\beta}^\vee) = -1$. From $l(s_\beta s_i) = l(s_\beta) - 1$ and $l(s_\alpha s_\beta) = l(s_\alpha) + l(s_\beta)$ it follows that $l(s_\alpha s_i) = l(s_\alpha) + 1$, hence $s_\alpha(\alpha_i) = \alpha_i - \alpha_i(\alpha^\vee)\alpha$ is positive, which means $\alpha_i(\alpha^\vee) \le 0$. We show that the only possible values for $\alpha_i(\alpha^\vee)$ are -1 and 0. Otherwise, the root system is *not* simply laced and the roots α and α_i are short, respectively long; on the other hand, $\alpha_i(\beta^\vee) = 1$, so $||\alpha_i|| \le ||\beta||$ and $\beta(\alpha^\vee) = -1$, so $||\beta|| \le ||\alpha||$, which gives a contradiction.

Case 1. $\alpha_i(\alpha^\vee) = 0$. This implies $s_i(\alpha) = \alpha$, hence

$$-1 = \beta(\alpha^\vee) = s_i(\tilde{\beta})(\alpha^\vee) = \tilde{\beta}(s_i(\alpha)^\vee) = \tilde{\beta}(\alpha^\vee).$$

From the induction hypothesis, $s_{\tilde{\beta}}(\alpha) = s_i s_\beta(\alpha) := \gamma$ is in $\tilde{\Phi}^+$. We also have that

$$\alpha_i(\gamma^\vee) = \alpha_i(s_i s_\beta(\alpha^\vee)) = -\alpha_i(s_\beta(\alpha^\vee)) = -\alpha_i(\alpha^\vee + \beta^\vee) = -1.$$

By Proposition 3.1, the root $s_i(\gamma) = s_\beta(\alpha)$ is in $\tilde{\Phi}^+$.

Case 2. $\alpha_i(\alpha^\vee) = -1$. We have again that

$$-1 = \beta(\alpha^\vee) = s_i(\tilde{\beta})(\alpha^\vee) = \tilde{\beta}(s_i(\alpha)^\vee).$$

By Proposition 3.1, the root $s_i(\alpha)$ is in $\tilde{\Phi}^+$. A simple calculation shows that

$$s_{\tilde{\beta}} s_i s_\alpha(\alpha_i) = \alpha - \big(1 + \alpha\big(\alpha_i^\vee\big)\big)\alpha_i - (1 + \alpha(\beta^\vee))\tilde{\beta}$$

which is a positive root. Consequently we have

$$l\big(s_{\tilde{\beta}} s_{s_i(\alpha)}\big) = l(s_{\tilde{\beta}} s_i s_\alpha s_i) = l(s_{\tilde{\beta}} s_i s_\alpha) + 1 = l(s_{\tilde{\beta}}) + l(s_i s_\alpha s_i) = l(s_{\tilde{\beta}}) + l\big(s_{s_i(\alpha)}\big).$$

From the induction hypothesis we deduce that $s_{\tilde{\beta}}(s_i(\alpha)) = s_i s_\beta(\alpha) := \gamma$ is also in $\tilde{\Phi}^+$. But as before,

$$\alpha_i(\gamma^\vee) = -\alpha_i(\alpha^\vee + \beta^\vee),$$

the right hand side being now 0. It follows that $\gamma = s_i(\gamma) = s_\beta(\alpha)$. □

We are now able to prove Lemma 1.3:

Proof of Lemma 1.3: Denote by λ_i^* the operator of multiplication by λ_i on $\mathbb{R}[\{\lambda_1, \ldots, \lambda_l\}]$, $1 \leq i \leq l$. The following formula can be found for instance in [9, Chapter 4, Section 3]:

$$\Delta_w \lambda_i^* - w\lambda_i^* w^{-1} \Delta_w = \sum_{\beta \in \Phi^+, l(ws_\beta) = l(w) - 1} \lambda_i(\beta^\vee) \Delta_{ws_\beta}, \tag{20}$$

where $w \in W$. Put $w = s_\alpha$ in (20) and obtain that:

$$\Delta_{s_\alpha} \lambda_i^* = (\lambda_i^* - \lambda_i(\alpha^\vee)\alpha^*)\Delta_{s_\alpha} + \sum_{\gamma \in \Phi^+, l(s_\alpha s_\gamma) = l(s_\alpha) - 1} \lambda_i(\gamma^\vee)\Delta_{s_\alpha s_\gamma}.$$

We deduce that:

$$\begin{aligned} \Lambda_j \Lambda_i = {} & (\lambda_j \lambda_i)^* + \sum_{\alpha \in \tilde{\Phi}^+} \lambda_i(\alpha^\vee) q^{\alpha^\vee} \lambda_j^* \Delta_{s_\alpha} + \sum_{\alpha \in \tilde{\Phi}^+} \lambda_j(\alpha^\vee) q^{\alpha^\vee} \lambda_i^* \Delta_{s_\alpha} \\ & - \sum_{\alpha \in \tilde{\Phi}^+} \lambda_j(\alpha^\vee)\lambda_i(\alpha^\vee) q^{\alpha^\vee} \alpha^* \Delta_{s_\alpha} \\ & + \sum_{\alpha \in \tilde{\Phi}^+, \gamma \in \Phi^+, l(s_\alpha s_\gamma) = l(s_\alpha) - 1} \lambda_j(\alpha^\vee)\lambda_i(\gamma^\vee) q^{\alpha^\vee} \Delta_{s_\alpha s_\gamma} \\ & + \sum_{\beta, \delta \in \tilde{\Phi}^+, l(s_\beta s_\delta) = l(s_\beta) + l(s_\delta)} \lambda_j(\beta^\vee)\lambda_i(\delta^\vee) q^{\beta^\vee + \delta^\vee} \Delta_{s_\beta s_\delta}. \end{aligned}$$

Denote by $\Sigma_1, \ldots, \Sigma_5$ the five consecutive sums in the right hand side. It is obvious that $(\lambda_i \lambda_j)^*$, $\Sigma_1 + \Sigma_2$ and Σ_3 are symmetric in i and j. We split $\Sigma_4 = \Sigma_4' + \Sigma_4''$, where Σ_4' contains only terms corresponding to α simple (consequently $\gamma = \alpha$ and $\Delta_{s_\alpha s_\gamma}$ is the identity operator) and Σ_4'' consists of the remaining terms. We also split $\Sigma_5 = \Sigma_5' + \Sigma_5''$, where Σ_5' contains only terms corresponding to β, δ with $s_\beta s_\delta = s_\delta s_\beta$, and Σ_5'' consists of the remaining terms.

We only need to show that

$$\Sigma_4'' + \Sigma_5'' = \sum_{\alpha\in\tilde{\Phi}^+,\gamma\in\Phi^+,l(s_\alpha s_\gamma)=l(s_\alpha)-1\geq 1} \lambda_j(\alpha^\vee)\lambda_i(\gamma^\vee)q^{\alpha^\vee}\Delta_{s_\alpha s_\gamma} + \sum_{\beta,\delta\in\tilde{\Phi}^+,l(s_\beta s_\delta)=l(s_\beta)+l(s_\delta),s_\beta s_\delta\neq s_\delta s_\beta} \lambda_j(\beta^\vee)\lambda_i(\delta^\vee)q^{\beta^\vee+\delta^\vee}\Delta_{s_\beta s_\delta}$$

is symmetric in i and j. To this end, let us take first two arbitrary elements β, δ of $\tilde{\Phi}^+$ with $l(s_\beta s_\delta) = l(s_\beta) + l(s_\delta)$ and $s_\beta s_\delta \neq s_\delta s_\beta$; by Lemmas 3.2 and 3.3, there exists $\alpha \in \tilde{\Phi}^+$ such that $\alpha^\vee = \beta^\vee + \delta^\vee$; we will show that:

- there exists a unique $\gamma \in \Phi^+$ with $s_\alpha s_\gamma = s_\beta s_\delta$ and $l(s_\alpha s_\gamma) = l(s_\alpha) - 1$,
- for γ determined above, the sum

$$\lambda_j(\alpha^\vee)\lambda_i(\gamma^\vee)\Delta_{s_\alpha s_\gamma} + \lambda_j(\beta^\vee)\lambda_i(\delta^\vee)\Delta_{s_\beta s_\delta} = (\lambda_j(\alpha^\vee)\lambda_i(\gamma^\vee) + \lambda_j(\beta^\vee)\lambda_i(\delta^\vee))\Delta_{s_\beta s_\delta} := S_{ij}^{\beta,\delta}\Delta_{s_\beta s_\delta}$$

is symmetric in i and j.

By Lemma 3.2, we distinguish the following two cases:

Case 1. $\beta(\delta^\vee) = -1$, which implies $\alpha = s_\beta(\delta)$, so the condition $s_\alpha s_\gamma = s_\beta s_\delta$ is equivalent to $\gamma = \beta$. Note that

$$l(s_\alpha) = 2\mathrm{ht}(\alpha^\vee) - 1 = 2(\mathrm{ht}(\beta^\vee) + \mathrm{ht}(\delta^\vee)) - 1 = l(s_\beta s_\delta) + 1 = l(s_\alpha s_\gamma) + 1.$$

We deduce that

$$S_{ij}^{\beta,\delta} = \lambda_j(\alpha^\vee)\lambda_i(\beta^\vee) + \lambda_j(\beta^\vee)\lambda_i(\delta^\vee) = \lambda_j(\beta^\vee)\lambda_i(\beta^\vee) + \lambda_j(\delta^\vee)\lambda_i(\beta^\vee) + \lambda_j(\beta^\vee)\lambda_i(\delta^\vee)$$

which is obviously symmetric in i and j.

Case 2. $\delta(\beta^\vee) = -1$, which implies that $\alpha = s_\delta(\beta)$, so this time the condition $s_\alpha s_\gamma = s_\beta s_\delta$ is equivalent to $\gamma = \pm s_\alpha(\delta)$. Because $\delta(\alpha^\vee) = 1$, the number $\alpha(\delta^\vee)$ is strictly positive, hence the root $s_\alpha(\delta)^\vee = s_\alpha(\delta^\vee) = \delta^\vee - \alpha(\delta^\vee)\alpha^\vee = \delta^\vee - \alpha(\delta^\vee)(\beta^\vee + \delta^\vee)$ is negative, so we must have $\gamma = -s_\alpha(\delta)$. We have again that

$$l(s_\alpha) = 2\mathrm{ht}(\alpha^\vee) - 1 = 2(\mathrm{ht}(\beta^\vee) + \mathrm{ht}(\delta^\vee)) - 1 = l(s_\beta s_\delta) + 1 = l(s_\alpha s_\gamma) + 1.$$

This time we can express $S_{ij}^{\beta,\delta}$ as follows:

$$\begin{aligned}S_{ij}^{\beta,\delta} &= -\lambda_j(\alpha^\vee)\lambda_i(s_\alpha(\delta^\vee)) + \lambda_j(\beta^\vee)\lambda_i(\delta^\vee)\\ &= -\lambda_j(\alpha^\vee)(\lambda_i(\delta^\vee) - \lambda_i(\alpha^\vee)\alpha(\delta^\vee)) + \lambda_j(\beta^\vee)\lambda_i(\delta^\vee)\\ &= -\lambda_j(\delta^\vee)\lambda_i(\delta^\vee) + \lambda_j(\alpha^\vee)\lambda_i(\alpha^\vee)\alpha(\delta^\vee),\end{aligned}$$

which is again symmetric in i and j.

In order to complete the proof, we must show that the map

$$\{(\beta,\delta) \in \tilde{\Phi}^+ \times \tilde{\Phi}^+ : l(s_\beta s_\delta) = l(s_\beta) + l(s_\delta),\ s_\beta s_\delta \neq s_\delta s_\beta\}\\ \to \{(\alpha,\gamma) \in \tilde{\Phi}^+ \times \Phi^+ : l(s_\alpha s_\gamma) = l(s_\alpha) - 1\}$$

given by $\alpha^\vee = \beta^\vee + \delta^\vee$ and $s_\beta s_\delta = s_\alpha s_\gamma$, is bijective.

Injectivity. Suppose that there exist two different pairs (β_1, δ_1), (β_2, δ_2) which are mapped to a given (α, γ). By looking at Cases 1 and 2 from above, we see that we must have $\beta_1(\delta_1^\vee) = -1$, $\delta_2(\beta_2^\vee) = -1$ and correspondingly

$$\delta_1 = s_\gamma(\alpha),\ \beta_1 = \gamma \text{ and } \delta_2 = -s_\alpha(\gamma),\ \beta_2 = s_{\delta_2}(\alpha).$$

We deduce that

$$-1 = \beta_1(\delta_1^\vee) = \gamma(s_\gamma(\alpha)^\vee) = \gamma(\alpha^\vee - \gamma(\alpha^\vee)\gamma^\vee) = -\gamma(\alpha^\vee).$$

If $\alpha(\gamma^\vee) = 1$, then $\delta_1 = \alpha - \gamma = \delta_2$, hence also $\beta_1 = \beta_2$, which is a contradiction. If $\alpha(\gamma^\vee) \geq 2$ then the roots $\delta_1^\vee = s_\gamma(\alpha^\vee) = \alpha^\vee - \gamma^\vee$ and

$$\beta_2^\vee = s_{\delta_2}(\alpha^\vee) = \gamma^\vee + (1 - \alpha(\gamma^\vee))\alpha^\vee = -[\alpha^\vee - \gamma^\vee + (\alpha(\gamma^\vee) - 2)\alpha^\vee]$$

cannot be simultaneously positive, which is again a contradiction.

Surjectivity. We take $\alpha \in \tilde{\Phi}^+$ non-simple, $\gamma \in \Phi^+$ with $l(s_\alpha s_\gamma) = l(s_\alpha) - 1$ and show that there exists $\beta, \delta \in \tilde{\Phi}^+$ with $\beta^\vee + \delta^\vee = \alpha^\vee$, $s_\alpha s_\gamma = s_\beta s_\delta$ and $s_\beta s_\delta \neq s_\delta s_\beta$. Consider the reduced decomposition $s_\alpha = s_{i_k} \dots s_{i_2} s_{i_1} s_{i_2} \dots s_{i_k}$ given by Proposition 3.1. By the "strong exchange condition" (see for instance [10, Section 5.8]) we distinguish the following two cases:

Case A. $s_\gamma = s_{i_k} \dots s_{i_{j+1}} s_{i_j} s_{i_{j+1}} \dots s_{i_k}$ for some j between 2 and k. We deduce that $\gamma = s_{i_k} \dots s_{i_{j+1}}(\alpha_{i_j})$, the latter being a positive root since the expression $s_{i_k} \dots s_{i_{j+1}} s_{i_j}$ is reduced. We notice that

$$\gamma(\alpha^\vee) = s_{i_k} \dots s_{i_{j+1}}(\alpha_{i_j})\big(s_{i_k} \dots s_{i_2}(\alpha_{i_1}^\vee)\big) = \alpha_{i_j}\big(s_{i_j} \dots s_{i_2}(\alpha_{i_1}^\vee)\big) = 1,$$

where we have used Proposition 3.1. The root

$$s_\gamma(\alpha) = s_{i_k} \dots \hat{s}_{i_j} \dots s_{i_2}(\alpha_{i_1})$$

is positive, because the expression

$$s_\alpha s_\gamma = s_{i_k} \dots s_{i_2} s_{i_1} s_{i_2} \dots \hat{s}_{i_j} \dots s_{i_k}$$

is reduced, which implies that $s_{i_k} \dots \hat{s}_{i_j} \dots s_{i_1}$ is reduced as well. We set $\beta = \gamma$ and $\delta = s_\gamma(\alpha)$, so that $\delta^\vee = s_\gamma(\alpha^\vee) = \alpha^\vee - \gamma^\vee$, which implies $\alpha^\vee = \beta^\vee + \delta^\vee$. We obviously have $s_\beta s_\delta = s_\alpha s_\gamma$, hence

$$l(s_\beta s_\delta) = 2\mathrm{ht}(\alpha^\vee) - 2 = 2\mathrm{ht}(\beta^\vee) - 1 + 2\mathrm{ht}(\gamma^\vee) - 1.$$

From Lemma 1.2 we deduce that β and δ are both in $\tilde{\Phi}^+$ and $l(s_\beta s_\delta) = l(s_\beta) + l(s_\delta)$. If we had $s_\beta s_\delta = s_\delta s_\beta$, then $s_\alpha s_\gamma = s_\gamma s_\alpha$, hence $s_\gamma(\alpha) = \alpha$; this is not true, as $\alpha(\gamma^\vee) > 0$.

Case B.

$$\begin{aligned} s_\gamma &= s_{i_k} \dots s_{i_2} s_{i_1} s_{i_2} \dots s_{i_{j-1}} s_{i_j} s_{i_{j-1}} \dots s_{i_2} s_{i_1} s_{i_2} \dots s_{i_k} \\ &= s_\alpha s_{i_k} \dots s_{i_{j+1}} s_{i_j} s_{i_{j+1}} \dots s_{i_k} s_\alpha \end{aligned}$$

which implies that

$$\gamma = -s_\alpha\big(s_{i_k} \dots s_{i_{j+1}}(\alpha_{i_j})\big) = s_{i_k} \dots s_{i_2} s_{i_1} s_{i_2} \dots s_{i_{j-1}}(\alpha_{i_j}).$$

A straightforward calculation shows that $\gamma(\alpha^\vee) = 1$. We set $\delta = -s_\alpha(\gamma)$, and $\beta = -s_\alpha s_\gamma(\alpha)$ (it is not difficult to see that both $s_\alpha(\gamma)$ and $s_\alpha s_\gamma(\alpha)$ are negative roots). We have that $\delta^\vee = -\gamma^\vee + \alpha(\gamma^\vee)\alpha^\vee$ and $\beta^\vee = \gamma^\vee - (\alpha(\gamma^\vee) - 1)\alpha^\vee$, which implies that $\beta^\vee + \delta^\vee = \alpha^\vee$. We can easily check that $s_\beta s_\delta = s_\alpha s_\gamma$. As in the previous situation, we show that β and δ are both in in $\tilde{\Phi}^+$ and we have $l(s_\beta s_\delta) = l(s_\beta) + l(s_\delta)$. If we had $s_\beta s_\delta = s_\delta s_\beta$, then $s_\alpha s_\gamma = s_\gamma s_\alpha$, hence $-s_\alpha(\gamma) = \gamma$; this implies $\alpha = 2\gamma$, which is not true. □

Acknowledgments

I would like to thank Lisa Jeffrey for a careful reading of the manuscript and several helpful comments. I am also thankful to the referee for several valuable suggestions.

Notes

1. Only cohomology with coefficients in $\mathbb{R}$ will be considered in this paper.
2. One can show (see for instance [9, Chapter 4]) that the definition does not depend on the choice of the reduced expression.
3. The proof of this result given in [12] relies essentially on the associativity of the ring $QH^*(G/B)$, which is a highly nontrivial fact; the proof of Lemma 1.3 we are going to give here is entirely in the realm of root systems.
4. I am grateful to the referee for suggesting me this idea.
5. Actually its $\mathbb{R}[\{q_i\}]$-linear extension.

References

1. I.N. Bernstein, I.M. Gelfand, and S.I. Gelfand, "Schubert cells and cohomology of the space G/P," *Russian Math. Surveys* **28** (1973), 1–26.
2. A. Borel, "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts," *Ann. of Math.* (2) **57** (1953), 115–207.
3. F. Brenti, S. Fomin, and A. Postnikov, "Mixed Bruhat operators and Yang-Baxter equations for Weyl groups," *IMRN* **8** (1999), 420–441.
4. C. Chevalley, "Invariants of finite groups generated by reflections," *Amer. J. Math.* **77** (1955), 778–782.
5. S. Fomin, S. Gelfand, and A. Postnikov, "Quantum Schubert polynomials," *J. Amer. Math. Soc.* **10** (1997), 565–596.
6. W. Fulton and R. Pandharipande, "Notes on stable maps and quantum cohomology," in Algebraic geometry—Santa Cruz 1995, *Proc. Sympos. Pure Math.* **62**, (Part 2), J. Kollar, R. Lazarsfeld and D.R. Morrison (eds.), 1997, pp. 45–96.
7. W. Fulton and C. Woodward, "On the quantum product of Schubert classes," preprint `math.AG/0112183`.
8. R. Goodman and N.R. Wallach, "Classical and quantum-mechanical systems of Toda lattice type, I," *Comm. Math. Phys.* **83** (1982), 355–386.
9. H. Hiller, *Geometry of Coxeter Groups*, Pitman Advanced Publishing Program, 1982.
10. J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, 1990.
11. B. Kim, "Quantum cohomology of flag manifolds G/B and quantum Toda lattices," *Ann. of Math.* **149** (1999), 129–148.
12. A.-L. Mare "Polynomial representatives of Schubert classes in $QH^*(G/B)$," *Math. Res. Lett.* **9** (2002), 757–770.
13. A.-L. Mare "Relations in the quantum cohomology ring of G/B," *Math. Res. Lett.* **11** (2004), 35–48.
14. A.-L. Mare "Quantum cohomology of the infinite dimensional generalized flag manifolds," *Adv. Math.* **185** (2004), 347–369.
15. D. Peterson, "Lectures on quantum cohomology of G/P," M.I.T., 1996.
16. A. Postnikov, *Enumeration in Algebra and Geometry*, Ph.D. thesis, M.I.T., 1997.