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A combinatorial proof of Klyachko's Theorem on Lie representations

L. G. Kovács · Ralph Stöhr

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Abstract Let L be a free Lie algebra of finite rank r over an arbitrary field K of characteristic 0, and let L_n denote the homogeneous component of degree n in L. Viewed as a module for the general linear group $GL(r, K)$, L_n is known to be semisimple with the isomorphism types of the simple summands indexed by partitions of n with at most r parts. Klyachko proved in 1974 that, for $n > 6$, almost all such partitions are needed here, the exceptions being the partition with just one part, and the partition in which all parts are equal to 1. This paper presents a combinatorial proof based on the Littlewood-Richardson rule. This proof also yields that if the composition multiplicity of a simple summand in L_n is greater than 1, then it is at least $\frac{n}{6} - 1$.

Keywords Free Lie algebra · General linear group · Littlewood-Richardson rule

Let V be a finite dimensional vector space over an arbitrary field of characteristic 0, and let T be the the tensor algebra of V, so $T = \bigoplus_{n\geq 0} T_n$ with $T_n = V^{\otimes n}$. Recall that the tensor powers T_n are semisimple $GL(V)$-modules, and the isomorphism types of the simple submodules of T_n correspond to the partitions of n into not more than $\dim V$ parts. The tensor product of any two such irreducibles is then also a direct sum of such irreducibles, and the relevant multiplicities are given by the Littlewood-Richardson rule.

Consider T a Lie algebra with respect to the Lie product $[x, y] = x \otimes y - y \otimes x$, and denote by L the Lie subalgebra generated by V $(= T_1)$. Then L is freely generated by any basis of V, $L = \bigoplus_{n\geq 1} L_n$ with $L_n = L \cap T_n$, and L_n is a submodule of T_n. In 1942, Thrall [12] asked for the composition multiplicities of these modules, and determined them for $n \leq 10$ (for $n = 10$, a correction was given in [1]). By 1949, Wever [14] gave a formula for these multiplicities in terms of characters of symmetric groups; nowadays, the highly

L. G. Kovács
Mathematics, Australian National University, Canberra ACT 0200, Australia
e-mail: kovacs@maths.anu.edu.au

R. Stöhr (✉)
School of Mathematics, University of Manchester, PO Box 88, Manchester M60 1QD, United Kingdom
ralph.stohr@manchester.ac.uk

illuminating Kraśkiewicz-Weyman Theorem [5] (see also Chapter 8 in [8]) may be invoked for an answer in terms of counting tableaux of certain kinds. Nevertheless, the work started by Thrall still continues today, with the scope of the problem having increased greatly; for further references and a recent overview, see [9].

A 1974 paper [4] of Alexander Klyachko provided a large impetus, and included the following remarkable result.

Theorem 1. [Klyachko [4]] *Let $n \geq 3$ and let ν be a partition of n. There is a simple submodule in L_n with isomorphism type corresponding to ν if and only if ν has no more than* $\dim V$ *parts and ν is not one of* (2^2), (2^3), (n), (1^n).

The notation used here is best explained by an example: $(2, 1^2)$ denotes the 3-part partition $2+1+1$ of 4. If $\nu = (2, 1^2)$, we shall write the corresponding module simply as $[\nu]$ or $[2, 1^2]$. The reader is expected to interpret everything that follows in the light of the convention that the $GL(V)$-module corresponding to a partition with more than $\dim V$ parts is 0, so then there is no such simple module.

The multiplicity formulas mentioned above are very useful when one wants to deal with one multiplicity at a time, but do not seem to help in proving global results like Klyachko's Theorem (or some others mentioned at the end of this note). As Schocker [10, p. 286] notes, "it seems to be rather difficult to give a combinatorial proof [...] by some analysis [...] and the Kraśkiewicz-Weyman Theorem only." There have been other proofs, perhaps the latest by Schocker [9, 10], built on the important developments which started with [4]. The aim of this note is to present a proof which does not do so, but relies only on the Littlewood-Richardson rule and on simple properties of free Lie algebras, and which has some further consequences. It is based on the following observation.

Lemma 1. *Let $n = k + l$ with $k > l > k/2$. The subspace $[L_k, L_l]$ of L_n spanned by the $[u, w]$ with $u \in L_k$, $w \in L_l$ is a submodule isomorphic to the tensor product $L_k \otimes L_l$.*

Proof: We shall argue in terms of a Hall basis $\mathcal{H}$ of L, but first we need to set the relevant conventions, for standard sources vary in their choices. We follow Marshall Hall's original paper [2]. There $\mathcal{H}$ consists of homogeneous elements of L (so $\mathcal{H} \cap L_m$ is always a basis of L_m) and is fully ordered by a relation $\leq$ which extends the partial order given by degrees. Every element of $\mathcal{H}$ of degree greater than 1 can be written uniquely in the form $[u, w]$ where $u, w \in \mathcal{H}$ and $u > w$. Finally, if $u, w \in \mathcal{H}$ and $u > w$, then $[u, w] \in \mathcal{H}$ if and only if either the degree of u is 1 or $u = [u', u'']$ with $u', u'' \in \mathcal{H}$ and $u' > u'' \leq w$.

Let us turn to the proof of the lemma itself. Since $GL(V)$ acts on L by Lie algebra automorphisms, the linear extension $L_k \otimes L_l \to L_n$ of $u \otimes w \mapsto [u, w]$ is in fact a module homomorphism. The set $\{ u \otimes w \mid u \in \mathcal{H} \cap L_k,\ w \in \mathcal{H} \cap L_l \}$ is a basis for $L_k \otimes L_l$; call it $\mathcal{B}$, say. We claim that the image $[u, w]$ of an element $u \otimes w$ of $\mathcal{B}$ is always in $\mathcal{H}$. The first part of this claim is that $u > w$: this holds because $k > l$. The second part is that if $u = [u', u'']$ with $u', u'' \in \mathcal{H}$, then $u'' \leq w$. In fact, $u'' < w$, because $u \in \mathcal{H}$ implies $u' > u''$ and so the degree of u'' is at most $k/2$ and hence strictly smaller than the degree l of w. It follows that our module homomorphism maps the basis $\mathcal{B}$ of its domain into a basis, $\mathcal{H} \cap L_n$, of its codomain. Moreover, its restriction to $\mathcal{B}$ is one-to-one, because the expression of an element of $\mathcal{H}$ in the form $[u, w]$ with $u, w \in \mathcal{H}$ and $u > w$ is unique. Consequently, the image of this homomorphism is isomorphic to its domain. □

The Littlewood-Richardson rule makes it possible to exploit this in an inductive argument. To start that off, we need to recall some of the information tabulated by Thrall [12] for the L_n with small n:

$$\begin{aligned} &L_1 \cong [1], \qquad L_2 \cong [1^2], \qquad L_3 \cong [2,1], \qquad L_4 \cong [3,1] \oplus [2,1^2], \\ &L_5 \cong [4,1] \oplus [3,2] \oplus [3,1^2] \oplus [2^2,1] \oplus [2,1^3], \\ &L_6 \cong [5,1] \oplus [4,2] \oplus [4,1^2]^{\oplus 2} \oplus [3^2] \oplus [3,2,1]^{\oplus 3} \oplus [3,1^2] \oplus [2^2,1^2]^{\oplus 2} \oplus [2,1^4]. \end{aligned} \tag{1}$$

Next we have to deal with the extreme cases. These will need only very special cases of the Littlewood-Richardson rule. One, that the simple modules which occur in $[\kappa] \otimes [\lambda]$ all correspond to partitions which are extensions of κ (and of λ, of course). Two, that if κ is a partition of k, then $[\kappa] \otimes [1]$ is the direct sum of one copy each of the $[\nu]$ as ν ranges over the partitions of $k+1$ which are extensions of κ.

Our notation for Lie products follows the left-normed convention: $[u, v, w]$ stands for $[[u, v], w]$, etc. Since L_n is spanned by the $[u, v]$ with $u \in L_{n-1}$, $v \in V$, the linear extension of $u \otimes v \mapsto [u, v]$ is a $GL(V)$-homomorphism of $L_{n-1} \otimes V$ onto L_n. We shall find this useful in proving the next result.

Lemma 2. *For $n \geq 3$, neither of the simple modules $[n]$ and $[1^n]$ can occur in L_n, and neither $[n-1, 1]$ nor $[2, 1^{n-2}]$ can occur with multiplicity greater than* 1.

Proof: The list (1) provides the inital step for an induction on n, so we proceed to the inductive step. The partition $(n+1)$ is not an extension of any partition of n except (n); by the inductive hypothesis, $[n]$ does not occur in L_n, so $[n+1]$ cannot occur in $L_n \otimes V$; as L_{n+1} is a homomorphic image of this tensor product, $[n+1]$ cannot occur in L_{n+1} either. The partition $(n, 1)$ is not an extension of any partition of n other than (n) and $(n-1, 1)$; by the inductive hypothesis, $[n]$ does not occur in L_n and $[n-1, 1]$ occurs at most once, so $[n, 1]$ cannot occur in $L_n \otimes V$ with multiplicity greater than 1; as L_{n+1} is a homomorphic image of this tensor product, the multiplicity of $[n, 1]$ in L_{n+1} cannot be larger either. The other cases are similar. □

Let $\circ$ and $\wedge$ denote symmetric and exterior products, respectively, and recall that $[n]$ is the symmetric power $V^{\circ n}$ while $[1^n]$ is the exterior power $V^{\wedge n}$.

Lemma 3. *For $n \geq 3$, the simple module $[n-1, 1]$ occurs in L_n provided* $\dim V \geq 2$, *and* $[2, 1^{n-2}]$ *also occurs if* $\dim V \geq n-1$.

Proof: There is a module homomorphism $L_n \to V \otimes V^{\circ(n-1)}$ such that

$$[v_1, v_2, v_3, \ldots, v_n] \mapsto v_1 \otimes (v_2 \circ v_3 \circ \cdots \circ v_n) - v_2 \otimes (v_1 \circ v_3 \circ \cdots \circ v_n)$$

(see [3, Theorem 3.1]), and this is clearly not zero when $\dim V \geq 2$. Since

$$V \otimes V^{\circ(n-1)} \cong [1] \otimes [n-1] \cong [n] \oplus [n-1, 1]$$

and we have seen that $[n]$ does not occur in L_n, it follows that $[n-1, 1]$ must occur in L_n.

It is also well known (see Levin [6] or Vaughan-Lee [13]) that

$$v_1 \otimes (v_2 \wedge \cdots \wedge v_n) \mapsto \sum_{\sigma} \operatorname{sgn}(\sigma)[v_1, v_{\sigma(2)}, \ldots, v_{\sigma(n)}],$$

where σ runs over all permutations of $\{2, \ldots, n\}$, extends to a nonzero homomorphism $V \otimes V^{\wedge(n-1)} \to L_n$ whenever $\dim V \geq n-1$. Since

$$V \otimes V^{\wedge(n-1)} \cong [1] \otimes [1^{n-1}] \cong [1^n] \oplus [2, 1^{n-2}]$$

and we have seen that $[1^n]$ does not occur in L_n, we conclude that $[2, 1^{n-2}]$ must occur in it. □

Our last lemma concerns only $GL(V)$-modules, not Lie algebras, and may be of some interest in itself. It does need the full generality of the Littlewood-Richardson rule, though not its full force: instead of counting precise multiplicities, it is sufficient to know that the relevant multiplicities are positive. For a complete statement and the terminology not explained here, see Macdonald's book [7, pp. 4–5, 68].

When we call a diagram or a skew-diagram a *rectangle*, we use the word in its everyday sense. A partition will be called *rectangular* if its diagram is a rectangle, that is, if it is of the form (r^s). Let U_n denote the direct sum of the $[\nu]$ as ν ranges through the *non-rectangular* partitions of n. We shall use that if Klyachko's Theorem holds for a particular value of n, then for this value L_n has a submodule isomorphic to U_n.

Lemma 4. *Let $n = k + l$ where $k, l \geq 3$. If ν is a partition of n other than (n), $(n-1, 1)$, $(2, 1^{n-2})$, (1^n), then $[\nu]$ does occur in $U_k \otimes U_l$.*

Proof: First we show that at least one of k and l has a partition κ such that $(*)$ κ is not rectangular, $\kappa \subset \nu$, and the skew diagram $\nu - \kappa$ is not a rectangle. If neither k nor l has a rectangular partition contained in ν, then any partition κ of k contained in ν will do: indeed, if $\nu - \kappa$ were an $r \times s$ rectangle, then (r^s) would be a rectangular partition of l contained in ν. Otherwise one of k and l, say k, has a rectangular partition, say (p^q), contained in ν. If $q = 1$, take $\kappa = (p-1, 1)$: the conditions on ν garantee that $\kappa \subset \nu$, and it is also easy to see that $\nu - \kappa$ is not a rectangle. Indeed, $\nu - \kappa$ contains the $1, p$ box and at least one of the $2, 2$ or $3, 1$ boxes, but not the $2, 1$ box, so $\nu - \kappa$ is not convex and therefore it cannot be a rectangle. Similarly, if $p = 1$, then we can take $\kappa = (2, 1^{q-2})$. Now suppose that $p, q \geq 2$. Then ν contains either the $1, p+1$ box or the $q+1, 1$ box. In the former case $\kappa = (p+1, p^{q-2}, p-1)$ and in the latter case $\kappa = (p^{q-1}, p-1, 1)$ will do.

Since the lemma is symmetric in k and l, we may assume that κ is a partition of k satisfying $(*)$. We claim that then there exists a non-rectangular partition λ of $l = n - k$ such that $[\nu]$ occurs in $[\kappa] \otimes [\lambda]$. To see this, consider the tableau T obtained by putting consecutive numbers $1, 2, \ldots$ down each column of the skew diagram $\nu - \kappa$. It is easily seen that in this way we get a tableau for which $w(\mathsf{T})$ is a lattice permutation. Let the weight of T be λ. Then the Littlewood-Richardson rule implies that $[\nu]$ occurs in $[\kappa] \otimes [\lambda]$. If λ is not rectangular, we are done, so it remains to deal with the case where λ is rectangular. This can only happen if $\nu - \kappa$ consists of columns of equal length, and there must be at least two columns since $\nu - \kappa$ is not a rectangle. Moreover, for the same reason we can find at least one column whose last box is strictly lower than the last box of the rightmost column. Take the last of those columns, and modify T and λ by adding 1 to the entry in its last box.

This ensures that λ is not rectangular, while the word $w(\mathsf{T})$ is still a lattice permutation. Again, the Littlewood-Richardson rule implies that $[\nu]$ occurs in $[\kappa] \otimes [\lambda]$, and hence in $U_k \otimes U_l$. □

Proof of Klyachko's Theorem: In view of the list (1), the theorem is valid for $n \leq 6$. For a proof by induction on n, we may therefore assume that $n \geq 7$. Then one can write n as a sum $n = k + l$ with $k > l > k/2$ and $k, l \geq 3$. By Lemma 1, L_n has a submodule isomorphic to the tensor product $L_k \otimes L_l$; by the inductive hypothesis, the theorem holds for L_k and L_l, so $L_k \otimes L_l$ contains $U_k \otimes U_l$; therefore, by Lemma 4, every $[\nu]$ occurs in L_n except perhaps $[n]$, $[n-1, 1]$, $[2, 1^{n-2}]$ and $[1^n]$. Finally, $[n-1, 1]$ and $[2, 1^{n-2}]$ do occur by Lemma 3, while $[n]$ and $[1^n]$ do not, by Lemma 2. This completes the inductive step. □

Remark. We have proved more than Klyachko's Theorem, namely (see Lemmas 2 and 3) that the multiplicities of $[n-1, 1]$ and $[2, 1^{n-2}]$ in L_n are 0 or 1, depending only on $\dim V$. This was proved by Zhuravlev [15, § 4] and by Schocker [9, 10]. In [9, 10], it was also shown that no other multiplicity is 1 when $n > 8$. Instead of pursuing that here, we note that it is easy to modify the proof of Lemma 1 to show that in L_n

$$\sum_{n/2<k<2n/3} [L_k, L_{n-k}] \cong \bigoplus_{n/2<k<2n/3} (L_k \otimes L_{n-k}).$$

(Indeed, if $\mathcal{H}$ is a Hall basis of L, then the right hand side has as basis the disjoint union

$$\bigcup_{n/2<k<2n/3} \{u \otimes w \mid u \in \mathcal{H} \cap L_k,\ w \in \mathcal{H} \cap L_{n-k}\}$$

and this basis is mapped by $u \otimes w \mapsto [u, w]$ one-to-one into $\mathcal{H}$, with the image spanning the left hand side.) These sums have at least $\frac{n}{6} - 1$ summands. If ν is a partition of n other than (n), $(n-1, 1)$, $(2, 1^{n-2})$, (1^n), then $[\nu]$ does occur in each of these summands. It follows that *the multiplicity of such a* $[\nu]$ *in* L_n *is at least* $\frac{n}{6} - 1$ (provided of course that ν has no more than $\dim V$ parts). In particular, *if a multiplicity in* L_n *is larger than* 1*, then it is at least* $\frac{n}{6} - 1$.

Added in proof (March 8, 2006). Since this paper was submitted, Marianne Johnson at the University of Manchester has been able to deduce Klyachko's Theorem directly from the Kraśkiewicz-Weyman Theorem ('Standard tableaux and Klyachko's Theorem on Lie representations', J. Combin. Theory Ser. A, to appear).

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