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Zonal polynomials for wreath products

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Abstract The pair of groups, symmetric group S_{2n} and hyperoctohedral group H_n, form a Gelfand pair. The characteristic map is a mapping from the graded algebra generated by the zonal spherical functions of (S_{2n}, H_n) into the ring of symmetric functions. The images of the zonal spherical functions under this map are called the zonal polynomials. A wreath product generalization of the Gelfand pair (S_{2n}, H_n) is discussed in this paper. Then a multi-partition version of the theory is constructed. The multi-partition versions of zonal polynomials are products of zonal polynomials and Schur functions and are obtained from a characteristic map from the graded Hecke algebra into a multipartition version of the ring of symmetric functions.

Keywords Zonal polynomial · Schur function · Gelfand pair · Hecke algebra · Hypergeometric function

1. Introduction

It is a well-known fact that the characteristic map ch gives an isomorphism between the character ring of the symmetric groups and the ring of symmetric functions Λ [7, I-7]. This mapping sends the irreducible characters to the Schur functions:

$$ch(\chi^{\lambda}) = S_{\lambda}(x),$$

where χ^{λ} is an irreducible character of a symmetric group indexed by a partition λ.

There are various similar results for other algebras. Below, we introduce two characteristic maps related to the symmetric groups.

Dedicated to Professor Eiichi Bannai on his 60th birthday.

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The first case is the character theory for the wreath products of a finite group with a symmetric group [7, I-Appendix B]. In this case the characteristic map sends the character ring of the wreath product into the multi-partition version of the ring of symmetric functions

$$\Lambda(G) = \mathbb{C}[p_r(C); C \text{ is a conjugacy class of } G,\ r \geq 1],$$

where $p_r(C)$'s are r-th power sum symmetric functions indexed by the conjugacy classes of G with variables $(x_{C1}, x_{C2}, \dots)$. Again, the characteristic map can be defined to be an isometry between these two rings. Then we see that the image of an irreducible character is a multi-partition version of Schur function.

Secondly, we consider the zonal spherical functions of the Gelfand pair (S_{2n}, H_n) (see [7, VII7-2]), where H_n is the centralizer of the element $(1,2)(3,4)\cdots(2n-1,2n)$. The precise definition of zonal spherical functions shall appear later (see Section 3). The zonal spherical functions of the Gelfand pair (S_{2n}, H_n) are indexed by the partitions of n. We define a graded ring $\mathcal{H}$ as a direct sum of Hecke algebras,

$$\mathcal{H} = \bigoplus_{n\geq 0} e_{H_n}\mathbb{C}S_{2n}e_{H_n}, \qquad \text{where} \quad e_{H_n} = \frac{1}{|H_n|}\sum_{h\in H_n} h.$$

In this case, the characteristic map Ch is an isomorphism between $\mathcal{H}$ and the ring of symmetric functions Λ. The images of the zonal spherical functions are the zonal polynomials (cf. [3, 12, 13]); the Jack symmetric functions $J_\lambda^\alpha(x)$ [7, 11] at the parameter $\alpha = 2$.

Several authors have (cf. [1, 2, 8–10]) written about Gelfand pairs of wreath products. In this paper, we generalize the theory of the Gelfand pair (S_{2n}, H_n) to wreath products. It might be expected that the images of the resulting zonal spherical function are products of zonal polynomials. This expectation is almost true but it runs out that we shall need the Schur functions as well as the zonal polynomials (see Theorem 11.2).

This paper is organized as follows. In Section 2 we establish notations. In Section 3 we recall the theory of Gelfand pairs of finite groups and in Section 4 we define the subgroup HG_n of $SG_{2n} = G \wr S_{2n}$. Section 5 analyzes the double cosets of HG_n in SG_{2n} and shows that the pair (SG_{2n}, HG_n) is a Gelfand pair. We recall the representation theory of wreath products in Section 6 and in Section 7 we determine the irreducible decomposition of the permutation representation $1_{HG_n}^{SG_{2n}}$. Here we compute two special types of zonal spherical functions. In Sections 8 and 9, we prepare algebraic setting for obtaining our main result. We construct a graded Hecke algebra and the multipartition version of the ring of symmetric functions. In Section 10 we define the characteristic map between these two algebras and in Section 11 we determine the images of zonal spherical functions of (SG_{2n}, HG_n) under the characteristic map (Theorem 11.2). In the last section, we apply our main theorem to discrete orthogonal polynomials of hypergeometric type.

2. Notation

Throughout this paper we use the following notation. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n. We write $|\lambda| = n$ or $\lambda \vdash n$. We denote by $m_r = m_r(\lambda) = |\{i; \lambda_i = r\}|$ the multiplicity of r in $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ and we write $\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \dots$. Let X be a (finite) set. If $\underline{\rho} = (\rho(x) | x \in X)$ is an $|X|$-tuple of partitions and $\sum_{x \in X} |\rho(x)| = n$ then we say that $\underline{\rho}$ is a ($|X|$-tuple of) partition(s) of n and write $|\underline{\rho}| = n$ or $\underline{\rho} \vdash n$. If $\rho(x) = 1^{m_1(x)} 2^{m_2(x)} 3^{m_3(x)} \dots$ $(x \in X)$ we put

$$\bigcup_{x \in X} \rho(x) = 1^{\sum_{x \in X} m_1(x)} 2^{\sum_{x \in X} m_2(x)} 3^{\sum_{x \in X} m_3(x)} \dots .$$

Let S_n be the symmetric group on n letters. Let $\rho = (\rho_1, \rho_2, \dots)$ be a partition of n. Define $[\rho] \in S_n$ by

$$[\rho] = (1, 2, \dots, \rho_1)(\rho_1 + 1, \rho_1 + 2, \dots, \rho_1 + \rho_2)(\rho_1 + \rho_2 + 1, \dots,) \dots .$$

Let G be a finite group. Let G^* be the set of irreducible characters of G and G_* the set of conjugacy classes of G. Let V_χ denote the irreducible G-module affording a character $\chi \in G^*$ and let $\chi(C)$ be the irreducible character χ evaluated at an element of the conjugacy class C. Let $\mathbb{C}G$ be the group ring of G. We always identify $\sum_{x \in G} f(x) x \in \mathbb{C}G$ with the corresponding function f on G. If H is a subgroup of G the Hecke algebra is

$$\mathcal{H}(G, H) = e_H \mathbb{C}G e_H, \qquad \text{where} \quad e_H = \frac{1}{|H|} \sum_{h \in H} h.$$

Viewed as functions, $\mathcal{H}(G, H)$ is the algebra of functions on G which are constant on each double coset in $H \backslash G / H$. Let

$$1_H^G = \mathrm{Ind}_H^G 1 \cong_G \mathbb{C}[G/H] \cong_G \mathbb{C}G e_H$$

denote the permutation representation of G on $\mathbb{C}[G/H]$. The scalar product on $\mathbb{C}G$ is defined by

$$\langle f, g \rangle_G = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{g(x)}.$$

The primitive idempotent corresponding to the V_χ-isotopic component of $\mathbb{C}G$ is written by e_χ or e_{V_χ}. The following proposition (cf. [7, I, (2.1)]) about double cosets for the Gelfand pair (S_{2n}, H_n) is frequently used in this paper.

Proposition 2.1. *A complete set of representatives of the double cosets $H_n \backslash S_{2n} / H_n$ is given by*

$$\{[2\rho]; \rho \vdash n\}.$$

3. Gelfand pair of finite groups and their zonal spherical functions

Throughout this section G is a finite group and H is a subgroup of G. We assume that $\mathbb{C}Ge_H$ is multiplicity free so that (G, H) is a *Gelfand pair*. With this assumption $\mathbb{C}Ge_H$ is a direct sum of non-isomorphic irreducible G-modules, say

$$1_H^G = \mathbb{C}Ge_H \cong \bigoplus_{i=1}^{s} V_i.$$

Proposition 3.1 [4, pp. 283 (11.27)]. $V_i = \mathbb{C}Ge_ie_H$ $(1 \le i \le s)$, *where* $e_i = e_{V_i}$.

It follows from Frobenius reciprocity that $\omega_i = \frac{\dim V_i}{|G|}e_ie_H$ is the unique H-invariant element in the V_i isotypic component of 1_H^G such that $\omega_i(1) = 1$. The functions ω_i $(1 \le i \le s)$ are the *zonal spherical functions* of the Gelfand pair (G, H). In terms of the inner product,

$$\omega_i(x) = \frac{\dim V_i}{|G|}e_ie_H(x) = \langle e_ie_H, xe_ie_H\rangle_G/\langle e_ie_H, e_ie_H\rangle_G, \ (1 \le i \le s, \ x \in G).$$

From this expression it is clear that the zonal spherical functions are constant on double cosets. See [7, VII, (1.4)] for other properties of the zonal spherical functions. The following proposition is useful for computing the zonal spherical functions.

Proposition 3.2 [7, VII, (1.3)]. *Suppose that W is a realization of V_i with a G-invariant Hermitian scalar product $\langle , \rangle$. Let F be a non-zero H-invariant element of W. Then the zonal spherical function ω_i is given by*

$$\omega_i(x) = \langle F, xF\rangle/\langle F, F\rangle.$$

4. The pair (SG_{2n}, HG_n)

Let S_{2n} be the group of permutations of $\{1, 2, \dots, 2n\}$ and let H_n be the subgroup of S_{2n} which is the centralizer of the involution $(1, 2)(3, 4)\dots(2n-1, 2n) \in S_{2n}$. We remark that H_n can be viewed as permutations of $\{\{2i-1, 2i\}; 1 \le i \le n\}$. Let G be a finite group. Let SG_{2n} be the wreath product of G with S_{2n} and let $\Delta G = \{(g, g) \mid g \in G\}$ be the diagonal subgroup of $G \times G$. Let θ be the action of S_{2n} on G^{2n} given by permuting the factors and let $\theta|_{H_n}$ be the restriction of θ to H_n. Then $(\Delta G)^n \subset G^{2n}$ is an invariant subset for $\theta|_{H_n}$. Now let $\tilde{\theta}$ be the action of H_n on $(\Delta G)^n$ induced by $\theta|_{Hn}$ and define a subgroup of SG_{2n} by

$$HG_n = (\Delta G)^n \rtimes_{\tilde{\theta}} H_n.$$

Note that HG_n is the normalizer of $(\Delta G)^n$ in SG_{2n}.

5. Description of double cosets

Let $x = (g_1, g_2, \ldots, g_{2n}; \sigma) \in SG_{2n}$. The *G-colored graph* $\Gamma_G(x) = \{V_G(x), E_G(x)\}$ is the graph with vertices

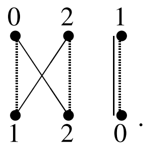
$$V_G(x) = \{g_1, g_2, \ldots, g_{2n}\}$$

and edges

$$E_G(x) = \Big\{ \{g_{2i-1}, g_{2i}\}, \big\{g_{\sigma(2j-1)}, g_{\sigma(2j)}\big\}; 1 \le i, j \le n \Big\}.$$

Here we call $\{g_{2i-1}, g_{2i}\} \in E_G(x)$ *broken* and $\big\{g_{\sigma(2i-1)}, g_{\sigma(2i)}\big\} \in E_G(x)$ *straight*.

Example 5.1. If $G = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ and $x = (0, 1, 2, 2, 1, 0; (123)(56)) \in SG_6$ then we have

$\Gamma_G(x) =$ 

The following proposition shows that $\Gamma_G(x) = \Gamma_G(y_1 x y_2)$ for $y_1, y_2 \in HG_n$.

Proposition 5.2. *Let $x = (g_1, \ldots, g_{2n}; \sigma)$ be an element of SG_{2n}. The following conditions are equivalent.*
(1) $\{g_i, g_j\} \in E_G(x)$.
(2) $\{g_i, g_j\} \in E_G(y_1 x y_2)$*, where* $y_i = (1, \ldots, 1; h_i) \in HG_n$ $(i = 1, 2)$.

Proof: (2) ⇒ (1):

Case 1: $\{g_i, g_j\}$ is a broken edge of $\Gamma_G(x)$. In this case there is a number k' such that

$$\{g_i, g_j\} = \big\{g_{h_1^{-1}(2k'-1)}, g_{h_1^{-1}(2k')}\big\}$$

and the right hand side is a broken edge of $E_G(y_1 x y_2)$.
Case 2: $\{g_i, g_j\}$ is a straight edge of $\Gamma_G(x)$. By the definition of $\Gamma_G(x)$ we can put $i = \sigma(2k_1 - 1)$ and $j = \sigma(2k_1)$ for some k_1. Then there exists a number k_2 such that $\{h_2(2k_2 - 1), h_2(2k_2)\} = \{2k_1 - 1, 2k_1\}$. Therefore we have

$$\big\{g_{h_1^{-1}(h_1\sigma h_2(2k_2-1))}, g_{h_1^{-1}(h_1\sigma h_2(2k_2))}\big\} = \big\{g_{\sigma(2k_1-1)}, g_{\sigma(2k_1)}\big\}$$

and the left hand side is a straight edge of $\Gamma_G(y_1 x y_2)$. This establishes (1) ⇒ (2). The claim (2) ⇒ (1) is proved similarly. □

Proposition 5.3. *Let* $x = (g_1, g_2, \ldots, g_{2n}; \sigma) \in SG_{2n}$, $y_1 = (k_1, k_1, \ldots, k_n, k_n; 1) \in HG_n$ *and* $y_2 = (l_1, l_1, \ldots, l_n, l_n; 1) \in HG_n$.
(1) *If* $\{k_{i_1} g_{2i-1} l_{j_1}, k_{i_2} g_{2i} l_{j_2}\}$ *is a broken edge of* $\Gamma_G(y_1 x y_2)$ *then* $i_1 = i_2$.
(2) *If* $\{k_{i_1} g_{\sigma(2i-1)} l_{j_1}, k_{i_2} g_{\sigma(2i)} l_{j_2}\}$ *is a straight edge of* $\Gamma_G(y_1 x y_2)$ *then* $j_1 = j_2$.

Proof: Since the first claim is clear we have only to show the second claim. Put $l'_{2i-1} = l'_{2i} = l_i$. We calculate

$$xy_2 = \left(g_1 l'_{\sigma^{-1}(1)}, g_2 l'_{\sigma^{-1}(2)} \cdots g_{2n} l'_{\sigma^{-1}(2n)}; \sigma\right).$$

We have $g_{\sigma(2i-1)} l'_{\sigma^{-1}(\sigma(2i-1))} = g_{\sigma(2i-1)} l_i$ and $g_{\sigma(2i)} l'_{\sigma^{-1}(\sigma(2i))} = g_{\sigma(2i)} l_i$. □

Note that any element of HG_n can be written as

$$y = (1, \dots, 1; h)(k_1, k_1, \dots, k_n, k_n; 1), \quad \text{with} \quad h \in H_n \text{ and } k_i \in G.$$

With this in mind the following proposition follows from Propositions 5.2 and 5.3.

Proposition 5.4. *Let $x = (g_1, g_2, \dots, g_{2n}; \sigma) \in SG_{2n}$, $y_1 = (k_1, k_1, \dots, k_n, k_n; h_1) \in HG_n$, and $y_2 = (l_1, l_1, \dots, l_n, l_n; h_2) \in HG_n$. Suppose that $\{g_i, g_j\} \in E_G(x)$. Then*
(1) *$\{k_{n_1} g_i l_{m_1}, k_{n_2} g_j l_{m_2}\} \in E_G(y_1 x y_2)$ for some $1 \le n_1, m_1, n_2, m_2 \le n$,*
(2) *If $\{k_{n_1} g_i l_{m_1}, k_{n_2} g_j l_{m_2}\} \in E_G(y_1 x y_2)$ is a straight edge then $l_{m_1} = l_{m_2}$,*
(3) *If $\{k_{n_1} g_i l_{m_1}, k_{n_2} g_j l_{m_2}\} \in E_G(y_1 x y_2)$ is a broken edge then $k_{n_1} = k_{n_2}$.*

Fix an element $x = (g_1, g_2, \dots, g_{2n}; \sigma) \in SG_{2n}$. Let L be a cycle of $\Gamma_G(x)$. Assume that L has vertices $\{g_{i_j}; 1 \le j \le 2k\}$. Let $\{\{g_{i_{2j-1}}, g_{i_{2j}}\}; 1 \le j \le k\}$ be the broken edges of L and $\{\{g_{i_{2j}}, g_{i_{2j+1}}\}, \{g_{i_{2k}}, g_{i_1}\}; 1 \le j \le k-1\}$ the straight edges of L.

Definition 5.5. The *circuit product* of L is

$$p(L) = \prod_{j=1}^{k} g_{i_{2j-1}}^{-1} g_{i_{2j}}.$$

If L has $2k$ edges then $p(L)$ is a *circuit product of length k*.

Example 5.6. The circuit products of x for Example 5.1 are

$$-0 + 1 - 2 + 2 = 1 \quad \text{and} \quad -1 + 0 = 2.$$

Note that the circuit product $p(L)$ is not unique. Indeed there are two choices, the starting point and the orientation, clockwise or counterclockwise. Nonetheless, any circuit product $p(L)$ is an element of the set

$$\left\{xp(L)x^{-1}, xp(L)^{-1}x^{-1}; x \in SG_{2n}\right\}.$$

Now we define

$$G_{**} = \{R = C \cup C^{-1}; C \in G_*\}, \qquad \text{where} \quad C^{-1} = \{g^{-1}; g \in C\}.$$

A circuit product $p(L)$ determines a unique $R \in G_{**}$ such that

$$R = \{xp(L)x^{-1}, xp(L)^{-1}x^{-1}; x \in SG_{2n}\}.$$

We call a conjugacy class *real* (resp. *complex*) when $C = C^{-1}$(resp. $C \neq C^{-1}$).

Definition 5.7. Put

$$m_k(R) = |\{L; L \text{ is a } 2k\text{-cycle of } \Gamma_G(x) \text{ and } p(L) \in R\}|.$$

Define a tuple of partitions

$$\underline{\rho}(x) = (\rho(R); R \in G_{**}),$$

where $\rho(R) = (1^{m_1(R)}, 2^{m_2(R)}, \dots, n^{m_n(R)})$. This tuple of partitions $\underline{\rho}(x)$ is called the *circuit type of* x.

Example 5.8. If $G = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$, then $G_{**} = \{R_0 = \{0\}, R_1 = \{1, 2\}\}$. For Example 5.1,

$$\underline{\rho}(x) = (\rho(R_0), \rho(R_1)) = ((\emptyset), (2, 1))$$

is the circuit type of x.

The following proposition is a consequence of Proposition 5.4.

Proposition 5.9. *Put $x \in SG_{2n}$ and $y_1, y_2 \in HG_n$. Then*

$$\underline{\rho}(x) = \underline{\rho}(y_1 x y_2).$$

We will show that the converse of Proposition 5.9 holds. If x and y are elements of the same double coset write $x \sim_d y$. We may assume that $\sigma = h_1[2\rho]h_2 \in S_{2n}$ ($\rho \vdash n$), where h_1 and h_2 are elements of H_n. Then

$$\begin{aligned} x &= (g_1, g_2, \dots, g_{2n}; \sigma) \\ &= (g_1, g_2, \dots, g_{2n}; h_1[2\rho]h_2) \\ &\sim_d (g_{h_1(1)}, g_{h_1(2)}, \dots, g_{h_1(2n)}; [2\rho]) \\ &\sim_d (\underbrace{1, \dots, 1, c_1}_{\rho_1}, \underbrace{1, \dots, 1, c_2}_{\rho_2}, \dots, \underbrace{1, \dots, 1, c_\ell}_{\rho_\ell}; [2\rho]) = x'. \end{aligned}$$

In the last relation above, can be any element of G which is conjugate to

$$\prod_{j=1}^{\rho_i} g^{-1}_{h_1(2u_i+2j-1)} g_{h_1(2u_i+2j)} \qquad \left(u_1 = 0 \text{ and } u_i = \sum_{k=1}^{i-1} \rho_k\right)$$

which is obtained by solving

$$(k_1, k_1, \dots, k_n, k_n; 1)(g_{h_1(1)}, g_{h_1(2)}, \dots, g_{h_1(2n)}; [2\rho])(l_1, l_1, \dots, l_n, l_n; 1) = x'.$$

Fix an element $y \in SG_{2n}$ with the same circuit type as x and let y' be such that

$$y' = (\underbrace{1, \dots, 1, c'_1}_{\rho_1}, \underbrace{1, \dots, 1, c'_2}_{\rho_2}, \dots, \underbrace{1, \dots, 1, c'_\ell}_{\rho_\ell}; \sigma_\rho) \sim_d y.$$

Because of our assumptions we can choose an element $c'_i \in K'$ which satisfies c'_i or $c'^{-1}_i \in K$, where $K = \{c_i; \rho_i = k\}$ and $K' = \{c'_i; \rho_i = k\}$. Let $(K')^{-1} = \{c^{-1} \mid c \in K'\}$. Then the converse of Proposition 5.9 will be established by showing that $K' \cup (K')^{-1}$ is an HG_n double coset. Consider the following four operations.

(OP1) If $\rho_i = \rho_{i'}$ then multiply y' by

$$\left(1, \dots, 1; \prod_{j=0}^{\rho_i-1} (2(u_i + j) + 1, 2(t_{i'} + j) + 1)(2(u_i + j) + 2, 2(t_{i'} + j) + 2)\right) \in HG_n$$

on both sides. Here $u_i = \sum_{k=0}^{i-1} \rho_k$ and $t_{i'} = \sum_{k=0}^{i'-1} \rho_k$ for $1 \le k, k' \le \ell$.

(OP2) Multiply y' by

$$(\underbrace{1, \dots, 1, c_i^{-1}, c_i^{-1}}_{2\rho_1+\cdots+2\rho_i}, 1, \dots, 1; 1) \in HG_n$$

on the left.

(OP3) Multiply y' by

$$(1, \dots, 1; (2u_i + 1, 2u_i + 3, \dots, 2u_i + 2\rho_i - 1)^{-1}(2u_i + 2, 2u_i + 4, \dots, 2u_i + 2\rho_i)^{-1}) \in HG_n$$

on the right, where $u_i = \sum_{j=0}^{i-1} \rho_j$.

(OP4) Multiply y' by

$$\left(1, \dots, 1; \left(\prod_{j=1}^{u_i-1} (2u_i + j, 2u_i + 2\rho_i - j - 1)\right) (2(u_i + \rho_i) - 1, 2(u_i + \rho_i))\right) \in HG_n$$

on both sides, where $u_i = \sum_{j=0}^{i-1} \rho_j$.

The following example illustrates these operations.

Example 5.10. We take an element

$$x = (g_1, \ldots, g_6, g_7, \ldots, g_{12}; (1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12)).$$

(OP1) Take $a = (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)$ and compute

$$axa = x = (g_7, \ldots, g_{12}, g_1, \ldots, g_6; (1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12)).$$

We take

$$y' = (1, 1, 1, 1, 1, 1, 1, c'_i; (1, 2, 3, 4, 5, 6, 7, 8)).$$

(OP2) Take $b = (1, 1, 1, 1, c_i'^{-1}, {c'}_i^{-1}; 1)$ and compute

$$by' = (1, 1, 1, 1, 1, 1, c_i'^{-1}, 1; (1, 2, 3, 4, 5, 6, 7, 8))$$

(OP3) Take c = (1, 1, 1, 1, 1, 1; (7, 5, 3, 1)(8, 6, 4, 2)) and compute

$$by'c = (1, 1, 1, 1, 1, 1, c_i'^{-1}, 1; (8, 7, 6, 5, 4, 3, 2, 1)).$$

(OP4) Take $d = (1, 1, 1, 1, 1, 1, 1, 1; (1, 6)(2, 5)(3, 4)(7, 8))$ and compute

$$dby'cd = (1, 1, 1, 1, 1, 1, 1, c_i'^{-1}; (1, 2, 3, 4, 5, 6, 7, 8)).$$

The role of (OP1) is to interchange any two elements of K'. Operations (OP2), (OP3) and (OP4) make it possible to change ${c'}_i$ to ${c'}_i^{-1}$, namely

$$\big(\ldots 1, \ldots, 1, c'_i \ldots; [2\rho]\big) \sim_d (\ldots 1, \ldots, 1, c_i'^{-1}, \ldots; [2\rho]\big).$$

This establishes the following proposition.

Proposition 5.11. *If* $x, y \in SG_{2n}$ *and* x *and* y *have the same circuit type then* $x \sim_d y$.

Moreover, using the operations (OP2) and (OP3) gives the following proposition.

Proposition 5.12. $x \sim_d x^{-1}$, *for all* $x \in SG_{2n}$.

Remark 5.13. A consequence of Proposition 5.12 is that the pair (SG_{2n}, HG_n) is a Gelfand pair (cf. [7, VII, (1.2)]).

Consequently we have the following theorem.

Theorem 5.14. *Let* $x, y \in SG_{2n}$. *Then*
(1) $x \sim_d y \Leftrightarrow \underline{\rho}(x) = \underline{\rho}(y)$.
(2) $\underline{\rho}(x) = \underline{\rho}(x^{-1})$.

We know that there is a one-to-one correspondence between the double cosets $HG_n \backslash SG_{2n}/HG_n$ and the set of $|G_{**}|$-tuples of partitions of n;

$$HG_n \backslash SG_{2n}/HG_n \overset{1:1}{\leftrightarrow} \{\underline{\rho} = (\rho(R)|R \in G_{**}); |\underline{\rho}| = n\}.$$

The remainder of this section is devoted to computation of the cardinality of the double coset indexed by $\underline{\rho} = (\rho(R); R \in G_{**}) = \rho(R) = (1^{m_1(R)}, 2^{m_2(R)}, \dots, n^{m_n(R)})$. First, we recall a proposition about the double cosets in $H_n \backslash S_{2n}/H_n$.

Proposition 5.15 [7, VII, (2.3)]. *Let* $\sigma = h_1[2\rho]h_2 \in S_{2n}$ $(h_1, h_2 \in H_n)$ *then* $|H_n \sigma H_n| = \frac{|H_n|^2}{z_{2\rho}}$, *where* $z_\rho = 1^{m_1} 2^{m_2} \dots m_1! m_2! \dots$ *for* $\rho = 1^{m_1} 2^{m_2} \dots$.

Suppose that the circuit type $x \in SG_{2n}$ is $\underline{\rho} = (\rho(R)|R \in G_{**})$ and let $\rho = \bigcup_{R \in G_{**}} \rho(R)$. Then the multiplicity of r in ρ is

$$m_r = \sum_{R \in G_{**}} m_r(R)$$

and an easy computation gives

$$\frac{|H_n|^2}{z_{2\rho}} = \frac{|H_n|^2}{\prod_{R \in G_{**}} z_{2\rho(R)}} \left(\frac{m_k!}{\prod_{R \in G_{**}} m_k(R)!} \right)^{-1}.$$

This is the number of elements in $S_{2n} x S_{2n}$. Then, for each element of S_{2n} as in Proposition 5.15, there are

$$m_k! \Big/ \prod_{R \in G_{**}} m_k(R)!$$

ways of distributing m_k cycles of length $2k$ in the G-colored graph and for each cycle there are

$$(|G|^{2k-1}|R|)^{m_k(R)}$$

corresponding elements of G which correspond to the same distribution. Hence,

$$\begin{aligned}|HG_nxHG_n| &= \frac{|H_n|^2}{z_{2\rho}}\prod_{k=1}^{n}\prod_{R\in G_{**}}(|G|^{2k-1}|R|)^{m_k(R)}\times\prod_{k=1}^{n}\frac{m_k!}{\prod_{R\in G_{**}}m_k(R)!}\\ &= \frac{|H_n|^2\prod_{k=1}^{n}\prod_{R\in G_{**}}(|G|^{2k-1}|R|)^{m_k(R)}}{\prod_{R\in G_{**}}2^{\ell(\rho(R))}1^{m_1(R)}2^{m_2(R)}\dots n^{m_n(R)}\prod_{R\in G_{**}}m_r(R)!}\\ &= \frac{|H_n|^2\prod_{k=1}^{n}\prod_{R\in G_{**}}(|G|^{2k-1}|R|)^{m_k(R)}}{\prod_{R\in G_{**}}z_{2\rho(R)}}\\ &= \frac{|H_n|^2|G|^{2n}}{\prod_{R\in G_{**}}z_{2\rho(R)}}\times\frac{\prod_{R\in G_{**}}|R|^{\ell(\rho(R_i))}}{|G|^{\ell(\rho)}}.\end{aligned}$$

Proposition 5.16. *Suppose $x\in SG_{2n}$ has circuit type $\underline{\rho}(x)=(\rho(R)|R\in G_{**})$, where $\rho(R)=(1^{m_1(R)},2^{m_2(R)},\dots,n^{m_n(R)})$. Let $\zeta_C=\frac{|G|}{|C|}$ for $C\in G_*$. Then*

$$\begin{aligned}|HG_nxHG_n| &= \frac{|H_n|^2|G|^{2n}}{\prod_{R\in G_{**}}z_{2\rho(R)}}\times\frac{\prod_{R\in G_{**}}|R|^{\ell(\rho(R))}}{|G|^{\ell(\rho)}}\\ &= |H_n|^2|G|^{2n}\prod_{\substack{R=C\in G_{**}\\ C=C^{-1}}}\frac{1}{z_{2\rho(R)}\zeta_C^{\ell(\rho(R))}}\times\prod_{\substack{R=C\cup C^{-1}\in G_{**}\\ C\neq C^{-1}}}\frac{1}{z_{\rho(R)}\zeta_C^{\ell(\rho(R))}}.\end{aligned}$$

In Section 8 we will use this result to properly normalize the inner product on the multi-partition version of the ring of symmetric functions.

Definition 5.17.

$$z_{\underline{\rho}}=\prod_{\substack{R=C\in G_{**}\\ C=C^{-1}}}z_{2\rho(R)}\zeta_C^{\ell(\rho(R))}\times\prod_{\substack{R=C\cup C^{-1}\in G_{**}\\ C\neq C^{-1}}}z_{\rho(R)}\zeta_C^{\ell(\rho(R))},\quad\text{where}\quad\underline{\rho}=(\rho(R)|R\in G_{**}).$$

6. Representation theory of wreath products

In this section we recall a method of constructing the irreducible representations of a wreath product $SG_n=G\wr S_n$ (cf. [6, 14]). Let c be the cardinality of G^*.

Let

$$\mathcal{C}_n=\left\{\underline{n}=(n_\chi;\chi\in G^*);\sum_{\chi\in G^*}n_\chi=n,\ n_\chi\geq 0\right\}$$

be the set of c-compositions of n. For $\underline{n}\in\mathcal{C}_n$ let

$$\mathcal{P}(\underline{n})=\{(\lambda^\chi|\chi\in G^*);\lambda^\chi\vdash n_\chi\}.$$

The elements of $\mathcal{P}(\underline{n})$ are c-tuples of partitions. The product $SG(\underline{n}) = \prod_{\chi \in G^*} SG_{n_\chi}$ is an *inertia group* of SG_n. Define two $SG(\underline{n})$-modules for $\underline{n} \in \mathcal{C}_n$ and $\underline{\lambda} = (\lambda(\chi)|\chi \in G^*) \in \mathcal{P}(\underline{n})$,

$$R(\underline{n}) = \bigotimes_{\chi \in G^*} {V_\chi}^{\otimes n_\chi} \quad \text{and} \quad S(\underline{\lambda}) = \bigotimes_{\chi \in G^*} S^{\lambda(\chi)},$$

where S^λ denotes the irreducible module of the symmetric group indexed by the partition λ. The action of $SG(\underline{n})$ is defined by

$$\begin{cases} (g_1, \dots, g_n; \sigma) v_1 \otimes \cdots \otimes v_n = g_1 v_{\sigma^{-1}(1)} \otimes \cdots \otimes g_n v_{\sigma^{-1}(n)} \text{ on } R(\underline{n}), \\ (g_1, \dots, g_n; \sigma) v = \sigma v \text{ on } S(\underline{\lambda}). \end{cases}$$

For an irreducible representation $\mathcal{S}(\underline{\lambda}) = R(\underline{n}) \otimes S(\underline{\lambda})$ of $SG(\underline{n})$ let $\mathfrak{S}(\underline{\lambda}) = \mathcal{S}(\underline{\lambda}) \uparrow_{SG(\underline{n})}^{SG_n}$.

Theorem 6.1 ([6]). $\{\mathfrak{S}(\underline{\lambda}); \underline{n} \in \mathcal{C}_n,\ \underline{\lambda} \in \mathcal{P}(\underline{n})\}$ *is a complete system of irreducible representations of* SG_n.

7. Gelfand pair (SG_{2n}, HG_n)

The following theorem is a result of the analysis in Section 5 (cf. Remark 5.13).

Theorem 7.1. (SG_{2n}, HG_n) *is a Gelfand pair.*

Since (SG_{2n}, HG_n) is a Gelfand pair $1_{HG_n}^{SG_{2n}}$ is multiplicity free as SG_{2n}-module.

Definition 7.2. A character $\chi \in G^*$ is *real* if $\chi = \overline{\chi}$, and *complex* if $\chi \neq \overline{\chi}$. Let G_R^* be the set of real characters and G_C^* the set of complex characters. Define a relation $\sim$ on G_C^* by

$$\chi \sim \chi' \Leftrightarrow \overline{\chi} = \chi' \text{ or } \chi = \chi', \quad \text{and put} \quad G^{**} = G_R^* \cup G_C^*/_\sim.$$

Throughout this paper we view G^{**} as a subset of G^* by fixing representatives of the equivalence classes in $G_C^*/_\sim$.

Let us record the following basic results (cf. [7, VII-2]).

Proposition 7.3. (S_{2n}, H_n) *and* $(G \times G, \Delta G)$ *are Gelfand pairs.*

Proposition 7.4. $1_{H_n}^{S_{2n}} = \bigoplus_{\lambda \vdash n} S^{2\lambda}$ *and* $1_{\Delta G}^{G \times G} = \bigoplus_{\chi \in G^*} V_\chi \otimes V_{\overline{\chi}}$.

In particular, $(S_n \times S_n, \Delta S_n)$ is a Gelfand pair and $1_{\Delta S_n}^{S_n \times S_n} = \bigoplus_{\lambda \vdash n} S^\lambda \otimes S^\lambda$. Using notations as in Section 6 define a subset of $\mathcal{C}_{2n}$ by

$$\mathcal{C}_n^{**} = \{(n_\chi | n_\chi = 2m_\chi(\chi \in G_R^*, m_\chi \in \mathbb{Z}_{\geq 0}),\ n_\chi = n_{\overline{\chi}}(\chi \in G_C^*)) \in \mathcal{C}_{2n}\}.$$

Example 7.5. Let $\eta_i (0 \leq i \leq n-1)$ be the irreducible characters of the cyclic group $\mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}$ given by $\eta_i(j) = \exp(\frac{ij2\pi\sqrt{-1}}{n})$.

If $G = \mathbb{Z}/2\mathbb{Z}$ then $\mathcal{C}_n^{**} = \{(n_{\eta_0}, n_{\eta_1}) = (2n-2k, 2k); 0 \leq k \leq n\}$ and if $G = \mathbb{Z}/3\mathbb{Z}$ then $\mathcal{C}_n^{**} = \{(n_{\eta_0}, n_{\eta_1}, n_{\eta_2}) = (2n-2k, k, k); 0 \leq k \leq n\}$.

Define a subset of $\mathcal{P}(\underline{n})$ $(\underline{n} \in \mathcal{C}_n^{**})$ by

$$\mathcal{P}^{**}(\underline{n}) = \{(\lambda^\chi | \chi \in G^*) | \lambda^\chi = 2\mu^\chi(\chi \in G_R^*),\ \ \lambda^\chi = \lambda^{\overline{\chi}}(\chi \in G_C^*)\}.$$

Put $\mathcal{P}_n^{**} = \bigcup_{\underline{n} \in \mathcal{C}_n^{**}} \mathcal{P}^{**}(\underline{n})$.

Example 7.6. If $G = \mathbb{Z}/2\mathbb{Z}$ then $\mathcal{P}_n^{**} = \{(\lambda^{\eta_0}, \lambda^{\eta_1}) = (2\lambda, 2\mu); |\lambda| + |\mu| = n\}$ and if $G = \mathbb{Z}/3\mathbb{Z}$ then $\mathcal{P}_n^{**} = \{(\lambda^{\eta_0}, \lambda^{\eta_1}, \lambda^{\eta_2}) = (2\lambda, \mu, \mu); |\lambda| + |\mu| = n\}$.

We shall decompose $1_{HG_n}^{SG_{2n}}$ in terms of $\mathcal{P}_n^{**}$. For irreducible representations of SG_n $S(\lambda(\chi)) = V_\chi^{\otimes n} \otimes S^\lambda$ and $\underline{\lambda} = (\lambda^\chi | \chi \in G^*) \in \mathcal{P}^{**}(\underline{n})$,

$$\mathcal{S}(\underline{\lambda}) = \bigotimes_{\chi \in G^*} S(\lambda^\chi(\chi))$$

are irreducible representations $\mathcal{S}(\underline{\lambda})$ of an inertia group of SG_n (see Section 6).

Now we consider two special types of representations: The irreducible SG_{2n}-modules

$$S(2\lambda(\chi)), \qquad \text{for } \chi \in G_R^*,$$

and the $SG_n \times SG_n$ modules

$$S(\mu(\chi, \overline{\chi})) = (V_\chi \otimes V_{\overline{\chi}})^{\otimes n} \otimes (S^\mu \otimes S^\mu) \cong S(\mu(\chi)) \otimes S(\mu(\overline{\chi})), \qquad \text{for } \chi \in G_C^*.$$

Then

$$S(\chi(2\lambda))^{HG_n} = ((V_\chi \otimes V_\chi)^{\Delta G})^{\otimes n} \otimes (S^{2\lambda})^{H_n} \quad \text{is 1-dimensional,}$$

and

$$S(\mu(\chi, \overline{\chi}))^{\Delta SG_n} = ((V_\chi \otimes V_{\overline{\chi}})^{\Delta G})^{\otimes n} \otimes (S^\mu \otimes S^\mu)^{\Delta SG_n} \quad \text{is nonzero.}$$

Define idempotents

$$e_{2\lambda(\chi)} = \frac{\deg 2\lambda(\chi)}{|SG_{2n}|} \sum_{x \in SG_{2n}} \overline{2\lambda(\chi)(x)}x \quad \text{and}$$
$$e_{\mu(\chi,\overline{\chi})} = \frac{\deg \mu(\chi,\overline{\chi})}{|SG_n|^2} \sum_{x \in SG_n \times SG_n} \overline{\mu(\chi,\overline{\chi})(x)}x,$$

where $2\lambda(\chi)$ (resp. $\mu(\chi,\overline{\chi})$) is the character of $S(2\lambda(\chi))$ (resp. $S(\mu(\chi,\overline{\chi}))$).

Definition 7.7. For $\underline{\lambda} = (\lambda^\chi; \chi \in G^*) \in \mathcal{P}_n^{**}$ put

$$e(\underline{\lambda}) = \bigotimes_{\chi \in G_R^*} e_{\lambda^\chi(\chi)} \otimes \bigotimes_{\chi \in G_C^*/\sim} e_{\lambda^\chi(\chi,\overline{\chi})}.$$

For $\underline{n} \in \mathcal{C}_n^{**}$ define a subgroup $HG(\underline{n})$ of $SG(\underline{n})$ by

$$HG(\underline{n}) = HG_n \cap SG(\underline{n}) \cong \prod_{\chi \in G_R^*} HG_{n^\chi} \times \prod_{\chi \in G_C^*/\sim} \Delta SG_{n^\chi}.$$

By Proposition 7.3,

$$(SG(\underline{n}), HG(\underline{n})) \quad \text{is a Gelfand pair}$$

and thus, by Proposition 3.1,

$$\mathbb{C}SG_{2n}e(\underline{\lambda})e_{HG(\underline{n})} \cong \mathbb{C}SG_{2n} \otimes_{\mathbb{C}SG(\underline{n})} \mathbb{C}SG(\underline{n})e(\underline{\lambda})e_{HG(\underline{n})} \quad (\underline{\lambda} \in \mathcal{P}_n^{**})$$

is an irreducible SG_{2n}-module. Consequently,

$$\mathbb{C}SG_{2n}e(\underline{\lambda})e_{HG(\underline{n})}$$

are irreducible representations of SG_{2n} indexed by $|G^{**}|$-tuples of partitions $\lambda \in \mathcal{P}_n^{**}$. We recall a lemma of Brauer (cf. [5, Chapter 6 (6.32)]).

Proposition 7.8. $|G^{**}| = |G_{**}|$.

This proposition induces the following proposition.

Proposition 7.9. $|HG_n \backslash SG_{2n}/HG_n| = |\mathcal{P}_n^{**}|$.

Therefore, if we can prove that $\mathbb{C}SG_{2n}e(\underline{\lambda})e_{HG(\underline{n})}$ has a non-zero HG_n-invariant then we have determined the irreducible decomposition of $1_{HG_n}^{SG_{2n}}$. If $\tilde{\Omega}^{\underline{\lambda}}$ is the function on SG_{2n} given by

$$\tilde{\Omega}^{\underline{\lambda}}(x) = \langle e_{HG_n} e(\underline{\lambda}), x e_{HG_n} e(\underline{\lambda}) \rangle_{SG_n}$$

then

$$\begin{aligned}
\tilde{\Omega}^{\underline{\lambda}}(x) &= \langle e_{HG_n} e(\underline{\lambda}), x e_{HG_n} e(\underline{\lambda}) \rangle_{SG_n} = \langle e_{HG_n} x^{-1} e_{HG_n} e(\underline{\lambda}), e(\underline{\lambda}) \rangle_G \\
&= \frac{1}{|SG_{2n}|} \sum_{g \in SG_{2n}} e_{HG_n} x^{-1} e_{HG_n} e(\underline{\lambda})(g) \overline{e(\underline{\lambda})(g)} \\
&= \frac{1}{|SG_{2n}|} \sum_{g \in SG(\underline{n})} e_{HG_n} x^{-1} e_{HG_n} e(\underline{\lambda})(g) e(\underline{\lambda})(g^{-1}) \\
&= \frac{1}{|SG_{2n}|} e_{HG_n} x^{-1} e_{HG_n} e(\underline{\lambda}) e(\underline{\lambda})(1) \\
&= \frac{1}{|SG_{2n}|} e_{HG_n} x^{-1} e_{HG_n} e(\underline{\lambda})(1) = \frac{1}{|SG_{2n}|} e_{HG_n} e(\underline{\lambda}) e_{HG_n}(x).
\end{aligned}$$

When $x = 1$,

$$\begin{aligned}
\tilde{\Omega}^{\underline{\lambda}}(1) &= \langle e_{HG_n} e(\underline{\lambda}), e_{HG_n} e(\underline{\lambda}) \rangle_G = \langle e_{HG_n} e(\underline{\lambda}), e(\underline{\lambda}) \rangle_G \\
&= \frac{1}{|SG_{2n}|} \sum_{g \in SG_{2n}} e_{HG_n} e(\underline{\lambda})(g) \overline{e(\underline{\lambda})(g)} \\
&= \frac{1}{|SG_{2n}|} \sum_{g \in SG(\underline{n})} e_{HG_n} e(\underline{\lambda})(g) e(\underline{\lambda})(g^{-1}) \\
&= \frac{1}{|SG_{2n}|} e_{HG_n} e(\underline{\lambda}) e(\underline{\lambda})(1) = \frac{1}{|SG_{2n}|} e(\underline{\lambda}) e_{HG_n}(1) \\
&= \frac{1}{|SG_{2n}|} \frac{|HG(\underline{n})|}{|HG_n|} e(\underline{\lambda}) e_{HG_n \cap SG(\underline{n})}(1) = \frac{\dim S(\underline{\lambda})}{|SG_{2n}||SG(\underline{n})|} \frac{|HG(\underline{n})|}{|HG_n|}.
\end{aligned}$$

In particular, $\tilde{\Omega}^{\underline{\lambda}} \neq 0$ and so $\mathbb{C}SG_{2n} e(\underline{\lambda}) e_{HG(\underline{n})}$ has an HG_n-invariant. This proves the following theorem.

Theorem 7.10.

$$1_{HG_n}^{SG_{2n}} = \bigoplus_{\underline{\lambda} \in \mathcal{P}_n^{**}} \mathfrak{S}(\underline{\lambda})$$

Example 7.11. Let η_0, η_1 and η_2 be defined as in Example 7.5. If $G = \mathbb{Z}/2\mathbb{Z}$ then

$$1_{HG_n}^{SG_{2n}} = \bigoplus_{|\lambda|+|\mu|=n} \mathfrak{S}(\underbrace{2\lambda}_{\eta_0}, \underbrace{2\mu}_{\eta_1}),$$

and if $G = \mathbb{Z}/3\mathbb{Z}$ then

$$1_{HG_n}^{SG_{2n}} = \bigoplus_{|\lambda|+|\mu|=n} \mathfrak{S}(\underbrace{2\lambda}_{\eta_0}, \underbrace{\mu}_{\eta_1}, \underbrace{\mu}_{\eta_2}).$$

At the same time we have proved that

Theorem 7.12. *The complete set of the zonal spherical functions for the pair (SG_{2n}, HG_n) is*

$$\left\{\Omega^{\underline{\lambda}} = \frac{\tilde{\Omega}^{\underline{\lambda}}}{\tilde{\Omega}^{\underline{\lambda}}(1)} = \frac{|SG(\underline{n})||HG_n|}{\dim S(\underline{\lambda})|HG_n|} e_{HG_n} e(\underline{\lambda}) e_{HG_n}; \underline{\lambda} \in \mathcal{P}_n^{**}\right\}.$$

Let $\Omega^{\lambda(\chi)}$ be the zonal spherical function corresponding to $S(2\lambda)\uparrow^{SG_{2n}}$ and let $\Omega^{\mu(\chi,\overline{\chi})}$ be the zonal spherical function corresponding to $S(\mu(\chi,\overline{\chi})))\uparrow^{SG_{2n}}$. Then

$$\Omega^{\underline{\lambda}} = M(\underline{n}) e_{HG_n} \bigotimes_{\chi \in G_R^*} \Omega^{\lambda(\chi)} \otimes \bigotimes_{\chi \in G_C^*/\sim} \Omega^{\mu(\chi,\overline{\chi})} e_{HG_n}$$

for $\underline{\lambda} \in \mathcal{P}_n^{**}$, where

$$M(\underline{n}) = \frac{|HG_n|}{\prod_{\chi \in G_R^*} |HG_{|\lambda(\chi)|}| \times \prod_{\chi \in G_C^*/\sim} |HG_{|\mu(\chi,\overline{\chi})|}|}.$$

We shall compute value of two zonal spherical functions, $\Omega^{\lambda(\chi)}$ and $\Omega^{\mu(\chi,\overline{\chi})}$, on each double coset.

First assume $\chi = \overline{\chi}$. Then

$$\begin{aligned}\Omega^{\lambda(\chi)}(x) &= \frac{|SG_{2n}|}{\dim {V_\chi}^{2n} \dim S^{2\lambda}} e_{HG_n} e_{S(2\lambda(\chi))} e_{HG_n}(x) \\ &= \frac{|G|^{2n} 2n!}{\dim {V_\chi}^{2n} \dim S^{2\lambda}} e_{HG_n} e_{V_\chi^{\otimes 2n}} e_{HG_n}(x) e_{HG_n} e_{S^{2\lambda}} e_{HG_n}(x).\end{aligned}$$

Here we regard $S^{2\lambda}$ as the irreducible representation $V_1^{\otimes 2n} \otimes S^{2\lambda}$ where V_1 is the trivial representation of G. Assuming that $x \in SG_{2n}$ has a circuit type $\underline{\rho} = (\rho(R)|R \in G_{**})$ and writing $\rho = \bigcup_{R \in G_{**}} \rho(R)$,

$$e_{HG_n} e_{S^{2\lambda}} e_{HG_n}(x) = \frac{\dim S^{2\lambda}}{2n!} \omega_\rho^\lambda,$$

where ω_ρ^λ is the zonal spherical function of (S_{2n}, H_n) evaluated at the double coset indexed by ρ.

We remark that V_χ is a unitary representation. To compute $e_{HG_n}e_{V_\chi^{\otimes 2n}}e_{HG_n}(x)$ we consider a G-invariant scalar product $\langle,\rangle_{V_\chi}$ on V_χ. If $\{v_1, v_2, \dots, v_d\}$ of V_χ is an orthonormal basis of V_χ the corresponding matrix representation is given by

$$A_\chi(x)v_j = \sum_{i=1}^{d} \chi_{ij}(x)v_i.$$

Let

$$f = \sum_{1\le ij\le d} a_{ij} v_i \otimes v_j$$

be a ΔG-invariant element in $V_\chi \otimes V_\chi$. If $A = (a_{ij})$ then

$$(g,g)f = \sum_{1\le ij\le d} a_{ij} g v_i \otimes g v_j = \sum_{1\le ij\le d} \left\{ \sum_{1\le lk\le d} \chi_{ik}(x)a_{kl}\chi_{lj}(x^{-1}) \right\} v_i \otimes v_j = f$$

gives $A_\chi A = AA_\chi$. So A is a scalar matrix. In the following we put

$$f = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} v_i \otimes v_i.$$

Define an inner product on $(V_\chi)^{\otimes 2n}$ by

$$\langle v_1 \otimes \cdots \otimes v_{2n}, w_1 \otimes \cdots \otimes w_{2n}\rangle = \prod_{i=1}^{2n} \langle v_i, w_i\rangle_{V_\chi}.$$

Then $f^{\otimes n} \in (V_\chi \otimes V_\chi)^{\otimes n}$ is an HG_n-invariant and $\langle f^{\otimes n}, f^{\otimes n}\rangle = 1$. If $x = (x_1, y_1, \dots, x_n, y_n; (12\dots 2n)) \in SG_n$ then the zonal spherical function is determined by

$$\begin{aligned}\langle f^{\otimes n}, xf^{\otimes n}\rangle &= \frac{1}{d^n} tr A_\chi\left(x_1^{-1}y_1x_2^{-1}y_2\dots x_n^{-1}y_n\right)\\ &= \frac{1}{d^n}\chi\left(x_1^{-1}y_1x_2^{-1}y_2\dots x_n^{-1}y_n\right).\end{aligned}$$

Repeating the same computation with a general $x \in SG_{2n}$ of circuit type $\underline{\rho} = (\rho(R)|R \in G_{**})$ gives

$$\frac{|G|^{2n}}{\dim V_\chi^{2n}} e_{HG_n}e_{V_\chi^{\otimes 2n}}e_{HG_n}(x) = \frac{1}{\dim {V_\chi}^n} \prod_{\substack{R\in G_{**}\\ R=C\cup C^{-1}}} \chi(C)^{\ell(\rho(R))}.$$

This computation establishes the following proposition.

Proposition 7.13. *If $x \in GS_{2n}$ has a circuit type $\underline{\rho} = (\rho(R)|R \in G_{**})$ then*

$$\Omega^{\lambda(\chi)}(x) = \frac{1}{(\dim V_\chi)^n} \prod_{\substack{R \in G_{**} \\ R = C \cup C^{-1}}} \chi(C)^{\ell(\rho(R))} \omega^\lambda_\rho.$$

Now assume $\chi \neq \overline{\chi}$ and consider the zonal spherical function;

$$\Omega^{\mu(\chi,\overline{\chi})} = \frac{2^n |G|^{2n} n!^2}{(\dim V_\chi)^{2n} (\dim S^\lambda)^2} e_{HG_n} e_{\mu(\chi,\overline{\chi})} e_{HG_n}.$$

For convenience write $SG_n \times SG_n$ for the inertia group of SG_{2n} given by

$$\{(g_1, g_2, \dots, g_{2n}; \sigma_o\sigma_e)|\sigma_o(2i) = 2i, \sigma_e(2i-1) = 2i-1 \ (1 \le i \le n)\} \cong SG_n \times SG_n.$$

Define

$$C_n = \langle (1, \dots, 1; (2i-1, 2i)); 1 \le i \le n \rangle$$

and put $z_0 = (1, \dots, 1; (12)(34)\dots(2n-1, 2n)) \in C_n$. Note that

$$\omega_{\mu(\chi,\overline{\chi})} = \frac{|G|^{2n} n!^2}{(\dim V_\chi)^{2n} (\dim S^\lambda)^2} e_{\Delta SG_n} e_{\mu(\chi,\overline{\chi})} e_{\Delta SG_n}$$

are the zonal spherical functions of the Gelfand pair $(SG_n \times SG_n, \Delta SG_n)$. We have

$$\Omega^\mu = \frac{2^n |G|^{2n} n!^2}{(\dim V_\chi)^{2n} (\dim S^\lambda)^2} e_{C_n} e_{\Delta SG_n} e_{\mu(\chi,\overline{\chi})} e_{\Delta SG_n} e_{C_n} = 2^n e_{C_n} \omega_{\mu(\chi,\overline{\chi})} e_{C_n}$$

and

$$e_{C_n} \omega_{\mu(\chi,\overline{\chi})} e_{C_n}(x) = \frac{1}{2^{2n}} \sum_{\substack{\varepsilon, \epsilon \in C^n \\ \varepsilon x \epsilon \in SG_n \times SG_n}} \omega_{\mu(\chi,\overline{\chi})}(\varepsilon x \epsilon).$$

If $x_0 = (x_1, y_1, \dots, x_n, y_n; (1, 2, 3, 4, \dots 2n-1, 2n)) \in SG_{2n}$ then

$$\varepsilon x_0 \epsilon \in SG_n \times SG_n \Leftrightarrow \varepsilon = 1, \epsilon = z_0 \text{ or } \varepsilon = z_0, \epsilon = 1.$$

Recalling the zonal spherical function of $(SG_n \times SG_n, \Delta SG_n)$ [7, VII, Ex. 9] and the characters of the wreath product [7, I-Appendix B] and making the identification

$$x_0 z_0 \leftrightarrow ((x_1, x_2, \dots, x_n; [n]), (y_1, \dots, y_n; 1)) \in SG_n \times SG_n,$$

$$x_0 z_0 \leftrightarrow ((y_1, \dots, y_n; 1), (x_1, x_2, \dots, x_n; [n])) \in SG_n \times SG_n,$$

we compute

$$\begin{aligned}\omega_{\mu(\chi,\overline{\chi})}(x_0z_0) &= \chi\left(x_2^{-1}y_2x_3^{-1}y_3x_4^{-1}\dots x_n^{-1}y_nx_1^{-1}y_1\right)\chi^{\mu}_{(n)}\\ &= \chi\left(x_1^{-1}y_1x_2^{-1}y_2x_3^{-1}\dots x_{n-1}^{-1}y_{n-1}x_n^{-1}y_n\right)\chi^{\mu}_{(n)},\\ \omega_{\mu(\chi,\overline{\chi})}(z_0x_0) &= \chi\left(y_n^{-1}x_ny_{n-1}^{-1}x_{n-1}y_{n-2}^{-1}\dots y_2^{-1}x_2y_1^{-1}x_1\right)\chi^{\mu}_{(n)}\\ &= \overline{\chi\left(x_1^{-1}y_1x_2^{-1}y_2x_3^{-1}\dots x_{n-1}^{-1}y_{n-1}x_n^{-1}y_n\right)}\chi^{\mu}_{(n)}.\end{aligned}$$

We remark that $x_1^{-1}y_1x_2^{-1}y_2x_3^{-1}\dots x_{n-1}^{-1}y_{n-1}x_n^{-1}y_n$ is the circuit product of x_0. If $x=(x_1,y_1,\dots,x_n,y_n;[2\rho])\in SG_{2n}$ and $\varepsilon x\epsilon\in SG_n\times SG_n$ then ρ is a cycle type of the S_{2n}-part of $\varepsilon x\epsilon$. Repeating these computations with x in place of x_0 establishes the following proposition.

Proposition 7.14. *If $x\in GS_{2n}$ has circuit type $\underline{\rho}=(\rho(R)|R\in G_{**})$ then*

$$\Omega^{\mu}(x)=\frac{1}{(2\dim V_\chi)^n}\prod_{\substack{R\in G_{**}\\ R=C\cup C^{-1}}}(\chi(C)+\overline{\chi}(C))^{\ell(\rho(R))}\frac{\chi^{\lambda}_{\rho}}{\dim S^{\lambda}}.$$

8. The ring $\tilde{\Lambda}(G)$

In this section we define the multi-partition version of the ring of symmetric functions.

For $R\in G_{**}$ let $p_r(R)(r\geq 1)$ be the power sum symmetric function in the variables $(x_{R1},x_{R2},x_{R3},\dots)$. If $\underline{\rho}=(\rho(R);R\in G_{**})$ is a $|G_{**}|$-tuple of partitions put

$$P_{\underline{\rho}}(G_{**})=\prod_{R\in G_{**}}p_{\rho(R)}(R).$$

The multi-partition version of the ring of symmetric functions is

$$\tilde{\Lambda}(G)=\mathbb{C}[p_r(R);R\in G_{**}]$$

with scalar product given by

$$\langle P_{\underline{\rho}}(G_{**}),P_{\underline{\sigma}}(G_{**})\rangle_{\tilde{\Lambda}}=\delta_{\underline{\rho}\underline{\sigma}}z_{\underline{\rho}}.$$

Here $z_{\underline{\rho}}$ is as in Definition 5.17. Change variables by setting

$$p_r(\chi)=\sum_{\substack{R=C\cup C^{-1}\in G_{**}\\ C=C^{-1}}}\frac{\chi(C)}{\zeta_C}p_r(R)+\sum_{\substack{R=C\cup C^{-1}\in G_{**}\\ C\neq C^{-1}}}\frac{\chi(C)+\overline{\chi(C)}}{\zeta_C}p_r(R),$$

for $\chi(C)=\chi(x)$ $(x\in C)$. For a tuple of partitions $\underline{\lambda}=(\lambda^{\chi};\chi\in G^{**})$ put

$$P_{\underline{\lambda}}(G^{**})=\prod_{\chi\in G^{**}}p_{\lambda^{\chi}}(\chi)$$

and note that $p_r(\chi)=p_r(\overline{\chi})$. The second orthogonality relation gives

$$p_r(R)=\sum_{\chi\in G^{*}}\chi(C)p_r(\chi)\quad\text{if } R=C\cup C^{-1}.$$

Therefore $\tilde{\Lambda}(G)=\mathbb{C}[p_r(\chi);\chi\in G^{**}]$. Let $\hat{Z}_{\lambda}(\chi)$ be the zonal polynomial in the $p_{\rho}(\chi)$'s (see [7, VII, (2.13)]). Set

$$\ddot{\chi}(R)=\begin{cases}\dfrac{\chi(C)}{\zeta_C} & C=C^{-1},\ R=C\cup C^{-1},\\ \frac{\chi(C)+\overline{\chi(C)}}{\zeta_C} & C\neq C^{-1},\ R=C\cup C^{-1}\end{cases}.$$

and expand $\hat{Z}_{\lambda}(\chi)$ in the basis $\{P_{\underline{\rho}}(G_{**})\}$,

$$\begin{aligned}
\frac{1}{|H_n|}\hat{Z}_{\lambda}(\chi)&=\sum_{\rho\vdash n}z_{2\rho}^{-1}\omega_{\rho}^{\lambda}p_{\rho}(\chi)=\sum_{\rho\vdash n}\frac{\omega_{\rho}^{\lambda}}{\prod_{r=1}^{n}m_r(\rho)!(2r)^{m_r(\rho)}}\prod_{r=1}^{n}p_r(\chi)^{m_r(\rho)}\\
&=\sum_{\rho\vdash n}\frac{\omega_{\rho}^{\lambda}}{\prod_{r=1}^{n}m_r(\rho)!(2r)^{m_r(\rho)}}\prod_{r=1}^{n}\left\{\sum_{R\in G_{**}}\ddot{\chi}(R)p_r(R)\right\}^{m_r(\rho)}\\
&=\sum_{\rho\vdash n}\frac{\omega_{\rho}^{\lambda}}{\prod_{r=1}^{n}m_r(\rho)!(2r)^{m_r(\rho)}}\\
&\quad\times\prod_{r=1}^{n}\left\{\sum_{\sum_R m_r(\rho(R))=m_r(\rho)}\frac{m_r(\rho)!\prod_{R\in G_{**}}(\ddot{\chi}(R)p_r(R))^{m_r(\rho(R))}}{\prod_{R\in G_{**}}m_r(\rho(R))!}\right\}\\
&=\sum\frac{1}{\prod_{R\in G_{**}}z_{2\rho(R)}}\omega_{\rho}^{\lambda}\prod_{R\in G_{**}}(\ddot{\chi}(R))^{\ell(\rho(R))}\prod_{R\in G_{**}}p_{\rho(R)}(R)\\
&=\sum_{\underline{\rho}=(\rho(R)|R\in G_{**})\vdash n}\frac{1}{z_{\underline{\rho}}}\omega_{\rho}^{\lambda}\prod_{\substack{R\in G_{**}\\ R=C\cup C^{-1}}}\left(\frac{\chi(C)+\overline{\chi}(C)}{2}\right)^{\ell(\rho(R))}\prod_{R\in G_{**}}p_{\rho(R)}(R)\\
&=\sum_{\underline{\rho}=(\rho(R)|R\in G_{**})\vdash n}\frac{1}{z_{\underline{\rho}}}\omega_{\rho}^{\lambda}\prod_{\substack{R\in G_{**}\\ R=C\cup C^{-1}}}\chi(C)^{\ell(\rho(R))}\prod_{R\in G_{**}}p_{\rho(R)}(R)\\
&=\frac{(\dim V_{\chi})^n}{|HG_n|^2}\sum_{\underline{\rho}=(\rho(R)|R\in G_{**})\vdash n}|D_{\underline{\rho}}|\Omega_{\underline{\rho}}^{\lambda(\chi)}P_{\underline{\rho}}(G_{**}),
\end{aligned}$$

where, for the last equality we use Proposition 7.13. As in the classical setting the zonal polynomials are considered as generating functions of the $\Omega^{\lambda(\chi)}$'s.

Next we analyze the case of $\Omega^{\lambda(\chi,\overline{\chi})}$. Here the Schur functions play the role of the zonal polynomials. Define $\hat{S}_\lambda(\chi)$ to be a Schur function in the $p_r(\chi)$'s and expand $\hat{S}_\lambda(\chi)$ in the basis $\{P_{\underline{\rho}}(G_{**})\}$,

$$\begin{aligned}
\hat{S}_\lambda(\chi) &= \sum_{\rho\vdash n} z_\rho^{-1}\chi_\rho^\lambda p_\rho(\chi) = \sum_{\rho\vdash n} \frac{\chi_\rho^\lambda}{\prod_{r=1}^n m_r(\rho)! r^{m_r(\rho)}} \prod_{r=1}^n p_r(\chi)^{m_r(\rho)} \\
&= \sum_{\rho\vdash n} \frac{\chi_\rho^\lambda}{\prod_{r=1}^n m_r(\rho)! r^{m_r(\rho)}} \prod_{r=1}^n \left\{ \sum_{R\in G_{**}} \ddot{\chi}(R) p_r(R) \right\}^{m_r(\rho)} \\
&= \sum_{\rho\vdash n} \frac{\chi_\rho^\lambda}{\prod_{r=1}^n m_r(\rho)! r^{m_r(\rho)}} \\
&\quad \times \prod_{r=1}^n \left\{ \sum_{\sum_R m_r(\rho(R))=m_r(\rho)} \frac{m_r(\rho)! \prod_{R\in G_{**}} (\ddot{\chi}(R) p_r(R))^{m_r(\rho(R))}}{\prod_{R\in G_{**}} m_r(\rho(R))!} \right\} \\
&= \sum \frac{1}{\prod_{R\in G_{**}} z_{\rho(R)}} \chi_\rho^\lambda \prod_{R\in G_{**}} (\ddot{\chi}(R))^{\ell(\rho(R)} \prod_{R\in G_{**}} p_{\rho(R)}(R) \\
&= \sum_{\underline{\rho}=(\rho(R)|R\in G_{**})\vdash n} \frac{1}{z_{\underline{\rho}}} \chi_\rho^\lambda \prod_{\substack{R\in G_{**} \\ R=C\cup C^{-1}}} (\chi(C)+\overline{\chi}(C))^{\ell(\rho(R)} \prod_{R\in G_{**}} p_{\rho(R)}(R) \\
&= \frac{(2\dim V_\chi)^n \dim S^\lambda}{|HG_n|^2} \sum_{\underline{\rho}=(\rho(R)|R\in G_{**})\vdash n} |D_{\underline{\rho}}|\, \Omega_{\underline{\rho}}^{\lambda(\chi,\overline{\chi})} P_{\underline{\rho}}(G_{**})
\end{aligned}$$

The *augmented Schur functions* are defined by $\tilde{S}_\lambda = h(\lambda)\hat{S}_\lambda$, where $h(\lambda)$ is the product of the hook lengths of λ. The augmented Schur functions are the Jack symmetric functions at the parameter $\alpha = 1$. These computations establish the following proposition.

Proposition 8.1. *If $\chi = \overline{\chi}$ then*

$$\left(\frac{|G|}{\dim V_\chi}\right)^n \hat{Z}_\lambda(\chi) = \frac{1}{|HG_n|} \sum_{\underline{\rho}=(\rho(R)|R\in G_{**})\vdash n} |D_{\underline{\rho}}| \Omega_{\underline{\rho}}^{\lambda(\chi)} P_{\underline{\rho}}(G_{**}).$$

If $\chi \neq \overline{\chi}$ then

$$\left(\frac{|G|}{\dim V_\chi}\right)^n h(\lambda)\hat{S}_\lambda(\chi) = \frac{1}{|HG_n|} \sum_{\underline{\rho}=(\rho(R)|R\in G_{**})\vdash n} |D_{\underline{\rho}}|\, \Omega_{\underline{\rho}}^{\lambda(\chi,\overline{\chi})} P_{\underline{\rho}}(G_{**}).$$

Definition 8.2. Put

$$Z_\lambda(\chi) = \left(\frac{|G|}{\dim V_\chi}\right)^n \hat{Z}_\lambda(\chi) \text{ and } \tilde{S}_\lambda(\chi) = \left(\frac{|G|}{\dim V_\chi}\right)^n h(\lambda)\hat{S}_\lambda(\chi).$$

9. The graded algebra $\mathcal{H}(G)$

Let $\mathcal{H}_n = e_{HG_n}\mathbb{C}SG_{2n}e_{HG_n} (n \geq 1), \mathcal{H}_0 = \mathbb{C}$, and let $\mathcal{H}(G)$ be the graded vector space

$$\mathcal{H}(G) = \bigoplus_{n\geq 0} \mathcal{H}_n.$$

If $f = \sum_{n\geq 0} f_n$ and $g = \sum_{n\geq 0} g_n$ with $f_n, g_n \in \mathcal{H}_n$, define

$$\langle f, g\rangle_{\mathcal{H}} = \sum_{n\geq 0} \langle f_n, g_n\rangle_{\mathcal{H}_n}, \qquad \text{where} \qquad \langle f_n, g_n\rangle_{\mathcal{H}_n} = \sum_{x\in SG_{2n}} f_n(x)\overline{g_n(x)}.$$

Since the zonal spherical functions of (SG_{2n}, HG_n) form an orthogonal basis of $\mathcal{H}_n$, they also form a basis of $\mathcal{H}(G)$. The multiplication of $\mathcal{H}(G)$ is defined by

$$u * v = e_{HG_{n+m}}(u \times v)e_{HG_{n+m}},$$

where we view $u \times v$ as a function on the parabolic subalgebra $\mathbb{C}S_{2n} \times \mathbb{C}S_{2m}$ of $\mathbb{C}S_{2n+2m}$. In this way $\mathcal{H}(G)$ has the structure of a graded algebra.

10. The characteristic map

Define a linear map $CH : \mathcal{H}(G) \longrightarrow \tilde{\Lambda}(G)$ by

$$CH\left(\sum_{g\in SG_{2n}} f(g)g\right) = \sum_{g\in SG_{2n}} f(g)P_{\underline{\rho}(g)}(G_{**}),$$

where $\underline{\rho}(g)$ is the circuit type of $g \in SG_{2n}$. Let $w_{\underline{\rho}}$ be an element of SG_{2n} whose circuit type is $\underline{\rho}$ and define

$$\Psi_{\underline{\rho}} = e_{HG_n}w_{\underline{\rho}}e_{HG_n}.$$

Let $D_{\underline{\rho}}$ be the double coset containing $w_{\underline{\rho}}$. Then

$$\Psi_{\underline{\rho}}(x) = \begin{cases} |D_{\underline{\rho}}|^{-1}, & \text{if } x \text{ has circuit type } \underline{\rho}, \\ 0, & \text{otherwise.} \end{cases}$$

The $\Psi_{\underline{\rho}}$ form a basis of $\mathcal{H}(SG_{2n}, HG_n)$ and the image of $\Psi_{\underline{\rho}}$ under CH is

$$CH(\Psi_{\underline{\rho}}) = P_{\underline{\rho}}(G_{**}).$$

Suppose that $x \in SG_{2n}$ and $y \in SG_{2m}$ have circuit type $\underline{\rho}$ and $\underline{\sigma}$ respectively. The G-colored graph

$$\Gamma_G(x \times y) = \Gamma_G(x) \cup \Gamma_G(y),$$

where $x \times y \in SG_{2n} \times SG_{2m} \subset SG_{2n+2m}$. Therefore the circuit type of $x \times y$ is $\underline{\rho} \cup \underline{\sigma} = (\rho(R) \cup \sigma(R); R \in G_{**})$. Since

$$\Psi_{\underline{\rho}} * \Psi_{\underline{\sigma}} = e_{HG_{n+m}}(e_{HG_n} w_{\underline{\rho}} e_{HG_n}) \times (e_{HG_m} w_{\underline{\sigma}} e_{HG_m}) e_{HG_{n+m}}$$
$$= e_{HG_{n+m}} w_{\underline{\rho}} \times w_{\underline{\sigma}} e_{HG_{n+m}} = \Psi_{\underline{\rho} \cup \underline{\sigma}}$$

it follows that

$$CH(\Psi_{\underline{\rho}} * \Psi_{\underline{\sigma}}) = P_{\underline{\rho} \cup \underline{\sigma}}(G_{**}) = P_{\underline{\rho}}(G_{**}) P_{\underline{\sigma}}(G_{**}),$$

and hence CH is an isomorphism of graded $\mathbb{C}$-algebras. Together with the fact that

$$\langle \Psi_{\underline{\rho}}, \Psi_{\underline{\sigma}} \rangle_{\mathcal{H}} = \delta_{\underline{\rho},\underline{\sigma}} |D_{\underline{\rho}}|^{-1} = \delta_{\underline{\rho},\underline{\sigma}} z_{\underline{\rho}} |HG_n|^{-2} = \left\langle \frac{P_{\underline{\rho}}(G_{**})}{|HG_n|}, \frac{P_{\underline{\sigma}}(G_{**})}{|HG_n|} \right\rangle_{\tilde{\Lambda}}.$$

this establishes the following theorem.

Theorem 10.1.

$$|HG_n|^{-1} CH : f \mapsto |HG_n|^{-1} CH(f)$$

is an isometry of $\mathcal{H}(G)$ *onto* $\tilde{\Lambda}(G)$.

11. Zonal polynomials for wreath products

In this section we compute the images of the zonal spherical functions under the map CH. The following proposition follows from Proposition 8.1.

Proposition 11.1. $|HG_n|^{-1} CH(\Omega^{2\lambda(\chi)}) = Z_\lambda(\chi)$ *and* $|HG_n|^{-1} CH(\Omega^{\mu(\chi,\overline{\chi})}) = S_\mu(\chi)$.

Let $\underline{\lambda} = (2\lambda(\chi), \mu(\chi', \overline{\chi'}) | \chi \in G_R^{**}, \chi' \in G_C^{**})) \in \mathcal{P}_n^{**}$. Then

$$\Omega^{\underline{\lambda}} = M(\underline{n}) e_{HG_n} \bigotimes_{\chi \in G_R^*} \Omega^{\lambda(\chi)} \otimes \bigotimes_{\chi \in G_C^*/\sim} \Omega^{\mu(\chi,\overline{\chi})} e_{HG_n}$$

(see Section 7, formula after Theorem 7.12). Viewing $\bigotimes_{\chi\in G_R^*}\Omega^{\lambda(\chi)}\otimes\bigotimes_{\chi\in G_C^*/\sim}\Omega^{\mu(\chi,\overline{\chi})}$ as an element of a parabolic subalgebra of $\mathcal{H}(G)$,

$$\begin{aligned}CH(\Omega^{\underline{\lambda}}) &= M(\underline{n})CH\left(e_{HG_n}\bigotimes_{\chi\in G_R^*}\Omega^{\lambda(\chi)}\otimes\bigotimes_{\chi\in G_C^*/\sim}\Omega^{\mu(\chi,\overline{\chi})}e_{HG_n}\right)\\ &= M(\underline{n})\prod_{\chi\in G_R^*}CH\left(\Omega^{\lambda(\chi)}\right)\times\prod_{\chi\in G_C^*/\sim}CH\left(\Omega^{\mu(\chi,\overline{\chi})}\right)\\ &= |HG_n|\prod_{\chi\in G_R^*}Z_{\lambda(\chi)}(\chi)\times\prod_{\chi\in G_C^*/\sim}\tilde{S}_{\mu(\chi,\overline{\chi})}(\chi).\end{aligned}$$

This establishes the main theorem of this paper.

Theorem 11.2. *Let* $\underline{\lambda}=(2\lambda(\chi),\mu(\chi',\overline{\chi'})|\chi\in G_R^{**},\chi'\in G_C^{**}))\in\mathcal{P}_n^{**}$. *Then we have*

$$|HG_n|^{-1}CH(\Omega^{\underline{\lambda}})=\prod_{\chi\in G_R^*}Z_{\lambda(\chi)}(\chi)\times\prod_{\chi\in G_C^*/\sim}\tilde{S}_{\mu(\chi,\overline{\chi})}(\chi).$$

Define

$$\mathcal{Z}_{\underline{\lambda}}=\prod_{\chi\in G_R^*}Z_{\lambda(\chi)}(\chi)\times\prod_{\chi\in G_C^*/\sim}\tilde{S}_{\mu(\chi,\overline{\chi})}(\chi),\qquad\text{for }\underline{\lambda}\in\mathcal{P}_n^{**}.$$

Since $|HG_n|^{-1}CH$ is an isometry we have

Corollary 11.3.

$$\langle\mathcal{Z}_{\underline{\lambda}},\mathcal{Z}_{\underline{\mu}}\rangle_{\tilde{\Lambda}}=\langle\Omega^{\underline{\lambda}},\Omega^{\underline{\mu}}\rangle_{\mathcal{H}}=\delta_{\underline{\lambda},\underline{\mu}}\frac{|SG_{2n}|}{\dim\mathfrak{S}(\underline{\lambda})}.$$

Via *CH* we obtain

Corollary 11.4.

$$\mathcal{Z}_{\underline{\lambda}}=|HG_n|\sum_{\underline{\rho}}z_{\underline{\rho}}^{-1}\Omega_{\underline{\rho}}^{\underline{\lambda}}P_{\underline{\rho}}(G_{**}),$$

where $\underline{\rho}$ *runs over* $\{\underline{\rho}=(\rho(R)|R\in G_{**});|\underline{\rho}|=n\}$.

The orthogonality relation of the zonal spherical functions gives the following Frobenius formula.

Corollary 11.5.

$$P_{\underline{\rho}}(G_{**}) = |HG_n| \sum_{\underline{\lambda} \in \mathcal{P}_n^{**}} \frac{\dim \mathfrak{S}(\underline{\lambda})}{|SG_{2n}|} \Omega_{\underline{\rho}}^{\underline{\lambda}} \mathcal{Z}_{\underline{\lambda}}.$$

12. Orthogonal polynomials of hypergeometric type arising from frobenius formula

In this section we apply our main theorem to discrete orthogonal polynomials. Multivariate orthogonal polynomials of hypergeometric type are considered in [1, 8, 9]. These papers are devoted to multivariate orthogonal polynomials which are expressed in terms of the $(m+1, n+1)$-hypergeometric functions

$$F(\alpha, \beta; \gamma, X) = \sum_{(a_{ij}) \in M_{n,m-n-1}(\mathbb{N}_0)} \frac{\prod_{i=1}^{n} (\alpha_i)_{\sum_{j=1}^{n} a_{ij}} \prod_{i=1}^{m-n-1} (\beta_i)_{\sum_{j=1}^{m-n-1} a_{ji}}}{(\gamma)_{\sum_{i,j} a_{ij}}} \frac{\prod x_{ij}^{a_{ij}}}{\prod a_{ij}!},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, $\beta = (\beta_1, \dots, \beta_{m-n-1}) \in \mathbb{C}^{m-n-1}$ and $X = (x_{ij})_{\substack{1 \le i \le n, \\ 1 \le j \le m-n-1}}$. If (G, H) is a Gelfand pair then the zonal spherical functions of the Gelfand pair $(G \wr S_n, H \wr S_n)$ can be written in terms of $(m+1, n+1)$-hypergeometric functions as follows.

Proposition 12.1 [9, Remark 2.1]. *If the zonal spherical function values for (G, H) are given by the matrix $\Omega = (\omega_{ij})_{0 \le ij \le r-1}$ then the zonal spherical function values for $(G \wr S_n, H \wr S_n)$ are given by the matrix*

$$(F(-\underline{m}, -\underline{k}; -n | J - \Omega))_{\underline{m}, \underline{k}}.$$

where $\underline{m} = (m_0, m_1, \dots, m_{r-1}) \in \mathbb{Z}_{\ge 0}^r$ and $\underline{k} = (k_0, k_1, \dots, k_{r-1}) \in \mathbb{Z}_{\ge 0}^n$ runs over $k_0 + \cdots + k_{r-1} = m_0 + \cdots + m_{r-1} = n$ and J is $r \times r$-all-one matrix.

The following generating function can be considered as an integral expression of hypergeometric functions.

Proposition 12.2 [8, Proposition 4.1].

$$\prod_{i=0}^{r-1} \left(\sum_{j=0}^{r-1} x_{ji} t_j \right)^{m_i} = \sum_{\underline{k}} \binom{n}{k_0, \dots, k_{r-1}} F(-\underline{k}, -\underline{m}; -n | J - X) t_0^{k_0} \dots t_{r-1}^{k_{r-1}},$$

where $\underline{k} = (k_0, k_1, \dots, k_{r-1}) \in \mathbb{Z}_{\ge 0}^r$ runs over $k_0 + \cdots + k_{r-1} = n$, and $X = (x_{ij})_{0 \le ij \le r-1}$ with $x_{0j} = x_{i0} = 1$.

Let $\rho = (1^{m_R} | R \in G_{**})$. Then the left hand side of formula in Corollary 11.5 can be written as

$$\prod_{R=C\cup C^{-1}\in G_{**}} \left(\sum_{\chi\in G^*} \chi(C) p_1(\chi) \right)^{m_R}.$$

Put $r = |G_{**}|$, $t_\chi = p_1(\chi)$ and let d_χ be the degree of χ. Note that $t_\chi = t_{\overline{\chi}}$. Proposition 12.2 gives

$$\prod_{R=C\cup C^{-1}\in G_{**}} \left(\sum_{\chi\in G^*} \chi(C) t_\chi \right)^{m_R} = \prod_{R=C\cup C^{-1}\in G_{**}} \left(\sum_{\chi\in G^*} \frac{\chi(C)}{d_\chi} d_\chi t_\chi \right)^{m_R}$$
$$= \sum_{\underline{k}} \frac{n!}{\prod_{\chi\in G^{**}} k_\chi !} F(-\underline{k}, -\underline{m}; -n | J - \mathcal{X}) \prod_{\chi\in G_R^*} (d_\chi t_\chi)^{k_\chi} \prod_{\chi\in G_C^*/\sim} (2 d_\chi t_\chi)^{k_\chi},$$

where $\mathcal{X} = (\frac{\chi(C)+\overline{\chi(C)}}{2d_\chi})_{\chi\in G^{**}, R=C\cup C^{-1}\in G_{**}}$ and $\underline{k} = (k_\chi | \chi \in G^{**}) \in \mathbb{Z}_{\geq 0}^{|G^{**}|}$ runs over $\sum_{\chi\in G^{**}} k_\chi = n$. We have

$$\left(\frac{|G|}{d_\chi} t_\chi \right)^{k_\chi} = |H_{k_\chi}| \sum_{\lambda\vdash k_\chi} h(2\lambda)^{-1} Z_\lambda(\chi), \qquad \text{for } \chi \in G_R^*, \text{ and}$$

$$\left(\frac{|G|}{d_\chi} t_\chi \right)^{k_\chi} = |S_{k_\chi}| \sum_{\lambda\vdash k_\chi} h(\lambda)^{-2} \tilde{S}_\lambda(\chi), \qquad \text{for } \chi \in G_C^*.$$

Compute

$$\frac{n!}{\prod_{\chi\in G^{**}} k_\chi !} \prod_{\chi\in G_R^*} (d_\chi t_\chi)^{k_\chi} \prod_{\chi\in G_C^*/\sim} (2 d_\chi t_\chi)^{k_\chi}$$
$$= \binom{n}{\underline{k}} \prod_{\chi\in G_R^*} \left(\frac{d_\chi^2}{|G|} \right)^{k_\chi} \sum_{\lambda\vdash k_\chi} \frac{|H_{k_\chi}|}{h(2\lambda)} Z_\lambda(\chi) \prod_{\chi\in G_C^*/\sim} \left(\frac{2d_\chi^2}{|G|} \right)^{k_\chi} \sum_{\lambda\vdash k_\chi} \frac{|S_{k_\chi}|}{h(\lambda)^2} \tilde{S}_\lambda(\chi)$$
$$= \frac{n!}{2n!} \left(\frac{2}{|G|} \right)^n \sum_{\underline{\lambda}} \dim \mathcal{S}(\underline{\lambda}) \mathcal{Z}_{\underline{\lambda}},$$

where $\underline{\lambda} = (\lambda^\chi | \chi \in G^{**})$ runs over $\lambda^\chi \vdash k^\chi (\chi \in G^{**})$. By comparing coefficients on each side of the Frobenius formula, we obtain the following theorem.

Theorem 12.3. *Let* $\underline{k} = (2k_\chi, k_{\chi'} | \chi \in G_R^*,\ \chi' \in G_C^*) \in \mathcal{P}_k^{**}$, $\underline{\lambda} \in \mathcal{P}(\underline{k})$ *and put* $\underline{k}^{**} = (k_\chi | \chi \in G^{**})$ *and* $\underline{m} = (m_R | R \in G^{**})$. *Then*

$$\Omega^{\underline{\lambda}}_{(1^{m_R} | R\in G_{**})} = F(-\underline{m}, -\underline{k}^{**}; -n | J - \mathcal{X}).$$

Remark 12.4. Since the table of zonal spherical functions of a Gelfand pair (SG_2, HG_1) is given by $\mathcal{X}$, the same orthogonal polynomials are obtained from a Gelfand pair of $(SG_2 \wr S_n, HG_1 \wr S_n)$ (cf. [9, Remark 2.1]).

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References

1. H. Akazawa and H. Mizukawa, "Orthogonal polynomials arising from the wreath products of dihedral group," *J. Combin. Theory. Ser. A* **104** (2003), 371–380.
2. T. Ceccherini-Silberstein, F. Scarabotti, and F. Tolli, "Trees, wreath products and finite Gelfand pairs," *Advances in Math*. in press.
3. N. Bergeron and A. Garsia, "Zonal polynomials and domino tableaux," *Discrete Math.* **99** (1992), 3–15.
4. C. Curtis and I. Reiner, *Methods of Representations Theory*, vol. 1 A Wiley-Interscience, 1981.
5. I.M. Issacs, "Character theory of finite groups," *Dover Publications Inc*, 1994.
6. G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, *Encyclopedia of Mathematics and its Applications* **16** (1981).
7. I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd. ed., Oxford, Clarendon Press, 1995.
8. H. Mizukawa, Zonal spherical functions on the complex reflection groups and $(m+1, n+1)$-hypergeometric functions, *Advances in Math.* **184** (2004), 1–17.
9. H. Mizukawa and H Tanaka, "$(n+1, m+1)$-hypergeometric functions associated to character algebras," *Proc AMS* **132**(9) (2004), 2613–2618.
10. D. Rockmore, "Fast fourier transforms for wreath products," *Appl. Comput. Harmon. Anal.* **2** (1995), 279–292.
11. R. Stanley "Some combinatorial properties of Jack symmetric functions," *Advances in Math.* **77** (1989), 76–115.
12. J. Stembridge, "On Schur's Q-functions and the primitive idempotents of a commutative Hecke algebra," *J. Alg. Comb.* **1** (1992), 71–95.
13. A. Takemura, *Zonal Polynomials*, Inst. of Math. Statistics Lecture Notes,-Monograph Series vol. 4. Hayward, California, 1984.
14. A. Zelevinsky, *Representations of finite classical groups. A Hopf Algebra Approach*, Lecture Notes in Mathematics, 869. Springer-Verlag, Berlin-New York, 1981.