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Base subsets of symplectic Grassmannians

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Abstract Let V and V' be $2n$-dimensional vector spaces over fields F and F'. Let also $\Omega\colon V \times V \to F$ and $\Omega'\colon V' \times V' \to F'$ be non-degenerate symplectic forms. Denote by Π and Π' the associated $(2n-1)$-dimensional projective spaces. The sets of k-dimensional totally isotropic subspaces of Π and Π' will be denoted by $\mathcal{G}_k$ and $\mathcal{G}'_k$, respectively. Apartments of the associated buildings intersect $\mathcal{G}_k$ and $\mathcal{G}'_k$ by so-called base subsets. We show that every mapping of $\mathcal{G}_k$ to $\mathcal{G}'_k$ sending base subsets to base subsets is induced by a symplectic embedding of Π to Π'.

Keywords Tits building · Symplectic Grassmannians · Base subsets

1 Introduction

An incidence geometry of rank n has the following ingredients: a set $\mathcal{G}$ whose elements are called *subspaces*, a symmetric *incidence relation* on $\mathcal{G}$, and a surjective *dimension function*

$$\dim\colon \mathcal{G} \to \{0, 1, \dots, n-1\}$$

such that the restriction of this function to every maximal flag is bijective (flags are sets of mutually incident subspaces).

A *Tits building* [12] is an incidence geometry together with a family of isomorphic subgeometries called *apartments* and satisfying a certain collection of axioms. One of these axioms says that for any two flags there is an apartment containing them.

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Let us consider an incidence geometry of rank n whose set of subspaces is denoted by $\mathcal{G}$. For every $k \in \{0, 1, \dots, n-1\}$ we denote by $\mathcal{G}_k$ the *Grassmannian* consisting of all k-dimensional subspaces. If this geometry is a building then the intersection of $\mathcal{G}_k$ with an apartment is called *the shadow* of this apartment in $\mathcal{G}_k$ [12]. In the projective and symplectic cases the intersections of apartments with Grassmannians are known as *base subsets* [8–10].

Let f be a bijective transformation of $\mathcal{G}_k$ preserving the class of the shadows of apartments. It is natural to ask: can f be extended to an automorphism of the corresponding geometry? This problem was solved in [8] for buildings of type A_n, in this case f is induced by a collineation of the associated projective space to itself or the dual projective space (the second possibility can be realized only for the case when $n = 2k + 1$). A more general result can be found in [9].

In the present paper we show that the extension is possible for symplectic buildings.

Note that apartment preserving transformations of the chamber set (the set of maximal flags) of a spherical building are induced by automorphisms of the corresponding complex; this follows from the results given in [1].

2 Symplectic geometry

Let V be a $2n$-dimensional vector space over a field F, $n \geq 2$. Let also

$$\Omega\colon V \times V \to F$$

be a non-degenerate symplectic form. Denote by $\Pi = (P, \mathcal{L})$ the $(2n-1)$-dimensional projective space associated with V (points are 1-dimensional subspaces of V and lines are defined by 2-dimensional subspaces).

We say that two points $p, q \in P$ are *orthogonal* and write $p \perp q$ if

$$p = \langle x \rangle, \quad q = \langle y \rangle \quad \text{and} \quad \Omega(x, y) = 0.$$

Similarly, two subspaces S and U of Π will be called *orthogonal* ($S \perp U$) if $p \perp q$ for any $p \in S$ and $q \in U$. The orthogonal complement to a subspace S (the maximal subspace orthogonal to S) will be denoted by $S^{\perp}$, if S is k-dimensional then the dimension of $S^{\perp}$ is equal to $2n - k - 2$ (throughout the paper the dimension is always assumed to be projective).

A base $\{p_1, \dots, p_{2n}\}$ of Π is said to be *symplectic* if for each $i \in \{1, \dots, 2n\}$ there exists unique $\sigma(i) \in \{1, \dots, 2n\}$ such that

$$p_i \not\perp p_{\sigma(i)}$$

(p_i and $p_{\sigma(i)}$ are non-orthogonal).

A subspace S of Π is called *totally isotropic* if any two points of S are orthogonal; in other words, $S \subset S^{\perp}$. The latter inclusion implies that the dimension of a totally isotropic subspace is not greater than $n - 1$.

Now consider the incidence geometry of totally isotropic subspaces. For every symplectic base B the subgeometry consisting of all totally isotropic subspaces spanned by points of B is *the symplectic apartment associated with* B. It is well-known that the incidence geometry of totally isotropic subspaces together with the family of all symplectic apartments is a building of type C_n.

For every $k \in \{0, 1, \dots, n-1\}$ we write $\mathcal{G}_k$ for the set of all k-dimensional totally isotropic subspaces, it is clear that $\mathcal{G}_0$ coincides with P. The set of all k-dimensional totally isotropic subspaces spanned by points of a symplectic base will be called the *base subset* of $\mathcal{G}_k$ associated with (defined by) this base.

Proposition 1. *Every base subset of $\mathcal{G}_k$ consists of*

$$2^{k+1}\binom{n}{k+1}$$

elements.

Proof: Let $B = \{p_1, \dots, p_{2n}\}$ be a symplectic base and $\mathcal{B}$ be the associated base subset of $\mathcal{G}_k$. By definition, $\mathcal{B}$ consists of all k-dimensional subspaces

$$\overline{\{p_{i_1}, \dots, p_{i_{k+1}}\}}$$

such that

$$\{i_1, \dots, i_{k+1}\} \cap \{\sigma(i_1), \dots, \sigma(i_{k+1})\} = \emptyset.$$

There are $2n$ possibilities to choose p_{i_1}, then p_{i_2} can be chosen in $2n-2$ ways, and so on. Since the order of the points must not be taken into account, we obtain that $\mathcal{B}$ has precisely

$$\frac{2n \cdot (2n-2) \dots (2n-2k)}{(k+1)!} = 2^{k+1}\binom{n}{k+1}$$

elements. □

Proposition 2. *For any two k-dimensional totally isotropic subspaces there is a base subset of $\mathcal{G}_k$ containing them.*

Proposition 2 can be obtained by an immediate verification or can be drawn from the fact that for any two flags there is an apartment containing them.

3 Results

From this moment we suppose that V and V' are $2n$-dimensional vector spaces over fields F and F' (respectively), $n \geq 2$, and

$$\Omega: V \times V \to F, \quad \Omega': V' \times V' \to F'$$

are non-degenerate symplectic forms. Let $\Pi = (P, \mathcal{L})$ and $\Pi' = (P', \mathcal{L}')$ be the $(2n-1)$-dimensional projective spaces associated with V and V', respectively.

An injection $f: P \to P'$ is called an *embedding* of Π to Π' if it maps lines to subsets of lines and for any line $L' \in \mathcal{L}'$ there is at most one line $L \in \mathcal{L}$ such that $f(L) \subset L'$. An embedding is said to be *strong* if it sends independent subsets to independent subsets. Every strong embedding of Π to Π' is induced by a semilinear injection of V to V' (with respect to a *monomorphism* of the underlying fields) preserving the linear independence [3–5].

Our projective spaces have the same dimension, and strong embeddings of Π to Π' (if they exist) map bases to bases. An example given in [6] shows that strong embeddings of Π to Π' cannot be characterized as mappings sending bases of Π to bases of Π'.

Theorem 1. *If a mapping $f: P \to P'$ transfers symplectic bases to symplectic bases then f is a strong embedding of Π to Π' and for any $p, q \in P$*

$$p \perp q \iff f(p) \perp f(q).$$

Since a surjective embedding is a collineation, we get the following.

Corollary. *Every surjection of P to P' sending symplectic bases to symplectic bases is a collineation of Π to Π' preserving the orthogonality relation.*

In what follows embeddings and collineations sending symplectic bases to symplectic bases will be called *symplectic*.

For every $k \in \{0, 1, \dots, n-1\}$ we denote by $\mathcal{G}_k$ and $\mathcal{G}'_k$ the sets of k-dimensional totally isotropic subspaces of Π and Π', respectively.

Let $f: P \to P'$ be a symplectic embedding of Π to Π'. For each $S \in \mathcal{G}_k$ the subspace spanned by $f(S)$ is an element of $\mathcal{G}'_k$. The mapping

$$\begin{aligned}(f)_k: \mathcal{G}_k &\to \mathcal{G}'_k\\ S &\to \overline{f(S)}\end{aligned}$$

is an injection sending base subsets to base subsets. If f is a collineation then every $(f)_k$ is bijective. Conversely, an easy verification shows that if $(f)_k$ is bijective for certain k then f is a collineation.

Theorem 2. *If a mapping of $\mathcal{G}_k$ to $\mathcal{G}'_k$ $(1 \le k \le n-1)$ transfers base subsets to base subsets then it is induced by a symplectic embedding of Π to Π'.*

Corollary. *Every surjection of $\mathcal{G}_k$ to $\mathcal{G}'_k$ $(1 \le k \le n-1)$ sending base subsets to base subsets is induced by a symplectic collineation of Π to Π'.*

For $k = n - 1 \geq 2$ Theorem 2 was established in [10]. In the present paper it will be proved for the general case.

Our proof of Theorem 2 is based on elementary properties of so-called inexact subsets (Section 5). If $k = n - 1$ then all maximal inexact subsets are of the same type. The case when $k < n - 1$ is more complicated: there are two different types of maximal inexact subsets.

Two elements of $\mathcal{G}_k$, $k \geq 1$ are called *adjacent* if their intersection belongs to $\mathcal{G}_{k-1}$. We say that two elements of $\mathcal{G}_k$ are *ortho-adjacent* if they are orthogonal and adjacent; this is possible only if $k < n - 1$. Using inexact subsets we characterize the adjacency and ortho-adjacency relations in terms of base subsets. This characterization shows that every mapping of $\mathcal{G}_k$ to $\mathcal{G}'_k$ sending base subsets to base subsets is adjacency and ortho-adjacency preserving (Section 7); after that arguments in the spirit of [2] give the claim (Section 8).

4 Proof of Theorem 1

A line of Π or Π' is said to be *hyperbolic* if it is not totally isotropic.

Lemma 1. *Let $p_1, p_2, p \in P$ be distinct points such that the line p_1p_2 is hyperbolic and $p \in p_1p_2$. Then for any symplectic base B containing p_1, p_2*

$$(B \setminus \{p_i\}) \cup \{p\} \quad i = 1, 2 \tag{1}$$

are symplectic bases.

Proof: Direct verification. □

Lemma 2. *Let $p_1, p_2, p \in P$ be distinct points. If there exists a symplectic base B such that $p_1, p_2 \in B$ and* (1) *are symplectic bases then the line p_1p_2 is hyperbolic and $p \in p_1p_2$.*

Proof: Let $B = \{p_1, p_2, \ldots, p_{2n}\}$ be such symplectic base. Since $(B \setminus \{p_1\}) \cup \{p\}$ is a symplectic base, $p \not\perp p_{\sigma(1)}$. Similarly, $p \not\perp p_{\sigma(2)}$. Thus there is no symplectic base containing $p, p_{\sigma(1)}, p_{\sigma(2)}$; this implies that $\sigma(1) = 2$. Therefore, the line p_1p_2 is hyperbolic; moreover, it is the orthogonal complement to the subspace spanned by $p_3, \ldots, p_{2n}$. It is easy to see that $p \perp p_i$ for every $i \geq 3$. Thus p is a point on p_1p_2. □

Let $f : P \to P'$ be a mapping which sends symplectic bases of Π to symplectic bases of Π'. Since for any two points there is a symplectic base containing them, f is injective. By Lemmas 1 and 2, f transfers hyperbolic lines to subsets of hyperbolic lines; in particular,

$$p \not\perp q \implies f(p) \not\perp f(q).$$

We prove that

$$p \perp q \implies f(p) \perp f(q).$$

Let $p, q \in P$ be distinct orthogonal points and B be a symplectic base containing them. There is a unique point $p' \in B$ such that $p \not\perp p'$. Then $f(p) \not\perp f(p')$ and all other points of $f(B)$ are orthogonal to $f(p)$; in particular, we get $f(p) \perp f(q)$.

Now we can show that f maps totally isotropic lines to subsets of totally isotropic lines. Let L be a totally isotropic line of Π and p_1, p_2 be distinct points on this line. We can choose a symplectic base $\{p_1, p_2, \dots, p_{2n}\}$ such that L is the orthogonal complement to the subspace spanned by $p_1, p_2, \dots, p_{2n-2}$. Then $f(L)$ is contained in the line $f(p_1)f(p_2)$ which is the orthogonal complement to the subspace spanned by $f(p_1), f(p_2), \dots, f(p_{2n-2})$.

We have established that every line goes to a subset of a line. So we need to show that f maps every base of Π to a base of Π'. We will use the following fact.

Fact 1. [7] If $g: P \to P'$ is an injection transferring lines to subsets of lines then for every subset $X \subset P$

$$g(\overline{X}) \subset \overline{g(X)};$$

in particular, $\overline{g(X)}$ coincides with $\overline{g(\overline{X})}$.

Let B be a base of Π. Then

$$\overline{f(B)} = \overline{f(\overline{B})} = \overline{f(P)}$$

The f-image of any symplectic base is a symplectic base, hence $\overline{f(P)} = P'$ and Π' is spanned by $f(B)$. Since f is injective, $f(B)$ is a base of Π'.

5 Inexact subsets

Let $B = \{p_1, \dots, p_{2n}\}$ be a symplectic base of Π. Let also $1 \le k \le n-1$ and $\mathcal{B}$ be the base subset of $\mathcal{G}_k$ associated with B. It was noted in the proof of Proposition 1 that $\mathcal{B}$ consists of all k-dimensional subspaces

$$\overline{\{p_{i_1}, \dots, p_{i_{k+1}}\}}$$

such that

$$\{i_1, \dots, i_{k+1}\} \cap \{\sigma(i_1), \dots, \sigma(i_{k+1})\} = \emptyset.$$

If $k = n-1$ then every element of $\mathcal{B}$ contains precisely one of the points p_i or $p_{\sigma(i)}$ for each i.

We write $\mathcal{B}(+i)$ and $\mathcal{B}(-i)$ for the sets of all elements of $\mathcal{B}$ which contain p_i or do not contain p_i, respectively. For any $i_1, \dots, i_s$ and $j_1, \dots, j_u$ belonging to $\{1, \dots, 2n\}$ we define

$$\mathcal{B}(+i_1, \dots, +i_s, -j_1, \dots, -j_u) := \mathcal{B}(+i_1) \cap \dots \cap \mathcal{B}(+i_s) \cap \mathcal{B}(-j_1) \cap \dots \cap \mathcal{B}(-j_u).$$

The set of all elements of $\mathcal{B}$ incident with a subspace S will be denoted by $\mathcal{B}(S)$ (this set may be empty). Then $\mathcal{B}(-i)$ coincides with $\mathcal{B}(S)$, where S is the subspace spanned by $B \setminus \{p_i\}$. It is trivial that

$$\mathcal{B}(+i) = \mathcal{B}(+i, -\sigma(i))$$

and for the case when $k = n - 1$ we have

$$\mathcal{B}(-i) = \mathcal{B}(+\sigma(i)) = \mathcal{B}(+\sigma(i), -i).$$

Let $\mathcal{R} \subset \mathcal{B}$. We say that $\mathcal{R}$ is *exact* if there is only one base subset of $\mathcal{G}_k$ containing $\mathcal{R}$; otherwise, $\mathcal{R}$ will be called *inexact*. If $\mathcal{R} \cap \mathcal{B}(+i)$ is not empty then we define $S_i(\mathcal{R})$ as the intersection of all subspaces belonging to $\mathcal{R}$ and containing p_i, and we define $S_i(\mathcal{R}) := \emptyset$ if the intersection of $\mathcal{R}$ and $\mathcal{B}(+i)$ is empty. If

$$S_i(\mathcal{R}) = p_i$$

for all i then $\mathcal{R}$ is exact; the converse fails.

Lemma 3. *Let $\mathcal{R} \subset \mathcal{B}$. Suppose that there exist distinct i, j such that*

$$p_j \in S_i(\mathcal{R}) \ \ \textit{and} \ \ p_{\sigma(i)} \in S_{\sigma(j)}(\mathcal{R}).$$

Then $\mathcal{R}$ is inexact.

Proof: On the line $p_i p_j$ we choose a point p'_i different from p_i and p_j. The line $p_{\sigma(i)} p_{\sigma(j)}$ contains a unique point orthogonal to p'_i; we denote this point by $p'_{\sigma(j)}$. Then

$$\left(B \setminus \left\{p_i, p_{\sigma(j)}\right\}\right) \cup \left\{p'_i, p'_{\sigma(j)}\right\}$$

is a symplectic base. The associated base subset of $\mathcal{G}_k$ contains $\mathcal{R}$ and we get the claim. □

Proposition 3. *The subset $\mathcal{B}(-i)$ is inexact; moreover, if $k < n - 1$ then this is a maximal inexact subset. In the case when $k = n - 1$, the inexact subset $\mathcal{B}(-i)$ is not maximal.*

Proof: Let us take a point p'_i on the line $p_i p_{\sigma(i)}$ different from p_i and $p_{\sigma(i)}$. Then

$$(B \setminus \{p_i\}) \cup \{p'_i\}$$

is a symplectic base and the associated base subset of $\mathcal{G}_k$ contains $\mathcal{B}(-i)$. Hence this subset is inexact.

Let $k < n-1$. For any $j \neq i$ we can choose distinct

$$i_1, \dots, i_k \in \{1, \dots, 2n\} \setminus \{i, j, \sigma(i), \sigma(j)\}$$

such that

$$\{i_1, \dots, i_k\} \cap \{\sigma(i_1), \dots, \sigma(i_k)\} = \emptyset.$$

The subspaces spanned by

$$p_{i_1}, \dots, p_{i_k}, p_j \quad \text{and} \quad p_{\sigma(i_1)}, \dots, p_{\sigma(i_k)}, p_j$$

belong to $\mathcal{B}(-i)$. Since the intersection of these subspaces is p_j, we have

$$S_j(\mathcal{B}(-i)) = p_j \quad \text{if} \quad j \neq i. \tag{2}$$

Let U be an arbitrarily taken element of

$$\mathcal{B} \setminus \mathcal{B}(-i) = \mathcal{B}(+i).$$

This subspace is spanned by p_i and some $p_{i_1}, \dots, p_{i_k}$. Since p_i is the unique point of U orthogonal to $p_{\sigma(i_1)}, \dots, p_{\sigma(i_k)}$, (2) shows that the subset

$$\mathcal{B}(-i) \cup \{U\} \tag{3}$$

is exact. This implies that the inexact subset $\mathcal{B}(-i)$ is maximal.

Now let $k = n-1$. We take an arbitrary element $U \in \mathcal{B}(+i)$. There exists j such that $p_{\sigma(j)}$ does not belong to U. Then p_j is a point of the subspace

$$S_i(\mathcal{B}(-i) \cup \{U\}) = U.$$

Since $p_{\sigma(i)}$ belongs to every element of $\mathcal{B}(-i)$ and $p_{\sigma(j)}$ does not belong to U,

$$S_{\sigma(j)}(\mathcal{B}(-i)) = S_{\sigma(j)}(\mathcal{B}(-i) \cup \{U\})$$

contains $p_{\sigma(i)}$. By Lemma 3, the subset (3) is inexact and the inexact subset $\mathcal{B}(-i)$ is not maximal. □

Proposition 4. *If $j \neq i, \sigma(i)$ then*

$$\mathcal{R}_{ij} := \mathcal{B}(+i, +j) \cup \mathcal{B}(+\sigma(i), +\sigma(j)) \cup \mathcal{B}(-i, -\sigma(j))$$

is a maximal inexact subset.

We remark that

$$\mathcal{R}_{ij} = \mathcal{B}(+i, +j) \cup \mathcal{B}(-i)$$

if $k = n - 1$.

Proof: Since

$$S_i(\mathcal{R}_{ij}) = p_i p_j \quad \text{and} \quad S_{\sigma(j)}(\mathcal{R}_{ij}) = p_{\sigma(j)} p_{\sigma(i)},$$

Lemma 3 shows that $\mathcal{R}_{ij}$ is inexact. We want to show that

$$S_l(\mathcal{R}_{ij}) = p_l \quad \text{if} \quad l \neq i, \sigma(j). \tag{4}$$

Let $l \neq i, j, \sigma(i), \sigma(j)$. If $k \geq 2$ then there exist

$$i_1, \ldots, i_{k-2} \in \{1, \ldots, n\} \setminus \{i, j, \sigma(i), \sigma(j), l, \sigma(l)\}$$

such that

$$\{i_1, \ldots, i_k\} \cap \{\sigma(i_1), \ldots, \sigma(i_k)\} = \emptyset;$$

the subspaces spanned by

$$p_{i_1}, \ldots, p_{i_{k-2}}, p_l, p_i, p_j \quad \text{and} \quad p_{\sigma(i_1)}, \ldots, p_{\sigma(i_{k-2})}, p_l, p_{\sigma(i)}, p_{\sigma(j)}$$

are elements of $\mathcal{R}_{ij}$ intersecting in the point p_l. If $k = 1$ then the lines $p_l p_{\sigma(i)}$ and $p_l p_j$ are as required.

Now we choose distinct

$$i_1, \ldots, i_{k-1} \in \{1, \ldots, n\} \setminus \{i, j, \sigma(i), \sigma(j)\}$$

such that

$$\{i_1, \ldots, i_{k-1}\} \cap \{\sigma(i_1), \ldots, \sigma(i_{k-1})\} = \emptyset$$

and consider the subspace spanned by

$$p_{i_1}, \ldots, p_{i_{k-2}}, p_j, p_{\sigma(i)}.$$

This subspace intersects the subspaces spanned by

$$p_{i_1}, \ldots, p_{i_{k-1}}, p_j, p_i \quad \text{and} \quad p_{i_1}, \ldots, p_{i_{k-1}}, p_{\sigma(i)}, p_{\sigma(j)}$$

precisely in the points p_j and $p_{\sigma(i)}$, respectively. Since all these subspaces are elements of $\mathcal{R}_{ij}$, we get (4) for $l = j, \sigma(i)$.

A direct verification shows that

$$\mathcal{B} \setminus \mathcal{R}_{ij} = \mathcal{B}(+i, -j) \cup \mathcal{B}(+\sigma(j), -\sigma(i)).$$

Thus for every $U \in \mathcal{B} \setminus \mathcal{R}_{ij}$ one of the following possibilities is realized:

(1) $U \in \mathcal{B}(+i, -j)$ intersects $S_i(\mathcal{R}_{ij}) = p_i p_j$ by p_i,
(2) $U \in \mathcal{B}(+\sigma(j), -\sigma(i))$ intersects $S_{\sigma(j)}(\mathcal{R}_{ij}) = p_{\sigma(j)}p_{\sigma(i)}$ by $p_{\sigma(j)}$.

Since $p_{\sigma(j)}$ is the unique point of the line $p_{\sigma(j)}p_{\sigma(i)}$ orthogonal to p_i and p_i is the unique point on $p_i p_j$ orthogonal to $p_{\sigma(j)}$, the subset

$$\mathcal{R}_{ij} \cup \{U\}$$

is exact for each U belonging to $\mathcal{B} \setminus \mathcal{R}_{ij}$. Thus the inexact subset $\mathcal{R}_{ij}$ is maximal. □

The maximal inexact subsets considered in Propositions 3 and 4 will be called of *first* and *second* type, respectively.

Proposition 5. *Every maximal inexact subset is of first or second type. In particular, if* $k = n - 1$ *then each maximal inexact subset is of second type.*

Proof: Let $\mathcal{R}$ be a maximal inexect subset of $\mathcal{B}$, and let B' be another symplectic base of Π such that the associated base subset of $\mathcal{G}_k$ contains $\mathcal{R}$. If certain $S_i(\mathcal{R})$ is empty then $\mathcal{R} \subset \mathcal{B}(-i)$. In the case when $k = n - 1$, this is impossible (the inexact subset $\mathcal{B}(-i)$ is not maximal). If $k < n - 1$ then the inverse inclusion holds (since our inexact subset is maximal).

Now suppose that each $S_i(\mathcal{R})$ is not empty. Denote by I the set of all i such that the dimension of $S_i(\mathcal{R})$ is non-zero. Since $\mathcal{R}$ is inexact, I is non-empty. Suppose that for certain $l \in I$ the subspace $S_l(\mathcal{R})$ is spanned by $p_l, p_{j_1}, \dots, p_{j_u}$ and

$$M_1 := S_{\sigma(j_1)}(\mathcal{R}), \dots, M_u := S_{\sigma(j_u)}(\mathcal{R})$$

do not contain $p_{\sigma(l)}$. Then p_l belongs to $M_1^{\perp}, \dots, M_u^{\perp}$; on the other hand,

$$p_{j_1} \notin M_1^{\perp}, \dots, p_{j_u} \notin M_u^{\perp}$$

and we have

$$M_1^{\perp} \cap \dots \cap M_u^{\perp} \cap S_l(\mathcal{R}) = p_l.$$

Since $S_1(\mathcal{R}), \dots, S_{2n}(\mathcal{R})$ and their orthogonal complements are spanned by points of the base B', the point p_l belongs to B'. The fact that $B \neq B'$ implies the existence of $i \in I$ and $j \neq i, \sigma(i)$ such that

$$p_j \in S_i(\mathcal{R}) \text{ and } p_{\sigma(i)} \in S_{\sigma(j)}(\mathcal{R}).$$

Then $\mathcal{R} = \mathcal{R}_{ij}$. □

Maximal inexact subsets of the same type have the same cardinality. These cardinalities will be denoted by $c_1(k)$ and $c_2(k)$, respectively. An immediate verification shows that each of the following possibilities

$$c_1(k) = c_2(k),\ c_1(k) < c_2(k),\ c_1(k) > c_2(k)$$

is realized for suitable k.

6 Complement subsets

Let $\mathcal{B}$ be as in the previous section. We say that $\mathcal{R} \subset \mathcal{B}$ is a *complement subset* if $\mathcal{B} \setminus \mathcal{R}$ is a maximal inexact subset. A complement subset is said to be of *first* or *second* type if the corresponding maximal inexact subset is of first or second type, respectively. The complement subsets for the maximal inexact subsets from Propositions 3 and 4 are

$$\mathcal{B}(+i) \quad \text{and} \quad \mathcal{B}(+i, -j) \cup \mathcal{B}(+\sigma(j), -\sigma(i)).$$

If $k = n - 1$ then the second subset coincides with

$$\mathcal{B}(+i, +\sigma(j)) = \mathcal{B}(+i, +\sigma(j), -j, -\sigma(i)).$$

In the case when $k = n - 1 = 1$, a complement subset has one element only.

Lemma 4. *Let $k = n - 1 \geq 2$. Then $S, U \in \mathcal{B}$ are adjacent if and only if there are precisely $\binom{n-1}{2}$ distinct complement subsets of $\mathcal{B}$ containing both S and U.*

Proof: Denote by m the dimension of $S \cap U$. The complement subset $\mathcal{B}(+i, +j)$ contains our subspaces if and only if p_i, p_j belong to $S \cap U$. Thus there are precisely $\binom{m+1}{2}$ distinct complement subsets of $\mathcal{B}$ containing S and U. □

Lemma 5. *Let $k < n - 1$ and $\mathcal{R}$ be a complement subset of $\mathcal{B}$. If $\mathcal{R}$ is of first type then there are precisely $4n - 3$ distinct complement subsets of $\mathcal{B}$ which do not intersect $\mathcal{R}$. If $\mathcal{R}$ is of second type then there are precisely 4 distinct complement subsets of $\mathcal{B}$ which do not intersect $\mathcal{R}$.*

To prove Lemma 11 we use the following.

Lemma 6. *Let $k < n - 1$ and i, i', j, j' be elements of $\{1, \ldots, 2n\}$ such that $i \neq j$ and $i' \neq j'$. If the intersection of*

$$\mathcal{B}(+i, -j) \quad \textit{and} \quad \mathcal{B}(+i', -j')$$

is empty then one of the following possibilities is realized: $i' = \sigma(i)$, $i' = j$, $j' = i$.

Proof: Direct verification. □

Proof of Lemma 5. Let us fix $l \in \{1, \dots, 2n\}$ and consider the complement subset $\mathcal{B}(+l)$. If $\mathcal{B}(+i)$ is disjoint with $\mathcal{B}(+l)$ then $i = \sigma(l)$. If for some $i, j \in \{1, \dots, 2n\}$ the complement subset

$$\mathcal{B}(+i, -j) \cup \mathcal{B}(+\sigma(j), -\sigma(i))$$

does not intersect $\mathcal{B}(+l)$ then one of the following possibilities is realized:

(1) $i = \sigma(l)$, the condition $j \neq i, \sigma(i)$ shows that there are precisely $2n - 2$ possibilities for j;
(2) $j = l$ and there are precisely $2n - 2$ possibilities for i (since $i \neq j, \sigma(j)$).

Now fix $i, j \in \{1, \dots, 2n\}$ such that $j \neq i, \sigma(i)$ and consider the associated complement subset

$$\mathcal{B}(+i, -j) \cup \mathcal{B}(+\sigma(j), -\sigma(i)). \tag{5}$$

There are only two complement subsets of the first type disjoint with (5):

$$\mathcal{B}(+\sigma(i)) \quad \text{and} \quad \mathcal{B}(+j).$$

If

$$\mathcal{B}(+i', -j') \cup \mathcal{B}(+\sigma(j'), -\sigma(i'))$$

does not intersect (5) then one of the following two possibilities is realized:

$$i' = j,\ \ j' = i \quad \text{or} \quad i' = \sigma(i),\ j' = \sigma(j)$$

(see Lemma 12). □

7 Main lemma

Let $f: \mathcal{G}_k \to \mathcal{G}'_k$ $(1 \leq k \leq n-1)$ be a mapping which sends base subsets to base subsets. Since for any two elements of $\mathcal{G}_k$ there exists a base subset containing them (Proposition 2) and the restriction of f to every base subset of $\mathcal{G}_k$ is a bijection to a base subset of $\mathcal{G}'_k$, the mapping f is injective.

Throughout the section we suppose that $n \geq 3$. In this section the following statement will be proved.

Lemma 7 (Main Lemma). *Let $S, U \in \mathcal{G}_k$. Then S and U are adjacent if and only if $f(S)$ and $f(U)$ are adjacent. Moreover, for the case when $k < n-1$, the subspaces S and U are ortho-adjacent if and only if the same holds for $f(S)$ and $f(U)$.*

Let $\mathcal{B}$ be a base subset of $\mathcal{G}_k$ containing S and U. Then $\mathcal{B}' := f(\mathcal{B})$ is a base subset of $\mathcal{G}_k(\Omega')$ and the restriction $f|_{\mathcal{B}}$ is a bijection to $\mathcal{B}'$.

Lemma 8. *A subset $\mathcal{R} \subset \mathcal{B}$ is inexact if and only if $f(\mathcal{R})$ is inexact; moreover, $\mathcal{R}$ is a maximal inexact subset if and only if the same holds for $f(\mathcal{R})$.*

Proof: If $\mathcal{R}$ is inexact then there are two distinct base subsets of $\mathcal{G}_k$ containing $\mathcal{R}$ and their f-images are distinct base subsets of $\mathcal{G}'_k$ containing $f(\mathcal{R})$, hence $f(\mathcal{R})$ is inexact. The base subsets $\mathcal{B}$ and $\mathcal{B}'$ have the same number of inexact subsets and the first part of our statement is proved. Since $\mathcal{B}$ and $\mathcal{B}'$ have the same number of maximal inexact subsets, every maximal inexact subset of $\mathcal{B}'$ is the image of a maximal inexact subset of $\mathcal{B}$. □

Lemma 9. *$\mathcal{R} \subset \mathcal{B}$ is a complement subset if and only if $f(\mathcal{R})$ is a complement subset of $\mathcal{B}'$.*

Proof: This is a simple consequence of the previous lemma. □

If $k = n - 1$ then Main Lemma (Lemma 7) can be drawn directly from Lemmas 10 and 15. In [10] this statement was proved in a more complicated way.

Lemma 10. *If $k < n - 1$ then the mapping $f|_{\mathcal{B}}$ together with the inverse mapping preserve types of maximal inexact and complement subsets.*

Proof: This statement is trivial if $c_1(k) \neq c_2(k)$. In the general case it follows from Lemma 5. □

We write $\mathcal{X}_i$ and $\mathcal{X}'_i$ for the sets of all i-dimensional subspaces spanned by points of the symplectic bases associated with $\mathcal{B}$ and $\mathcal{B}'$, respectively. We denote by $\mathcal{B}(N)$ and $\mathcal{B}'(N')$ the sets of all elements of $\mathcal{B}$ and $\mathcal{B}'$ incident with subspaces N and N', respectively.

Lemma 11. *Let $k < n - 1$. There exists a bijection $g\colon \mathcal{X}_{k+1} \to \mathcal{X}'_{k+1}$ such that*

$$f(\mathcal{B}(N)) = \mathcal{B}'(g(N))$$

for every $N \in \mathcal{X}_{k+1}$.

Proof: Lemma 10 guarantees that $f|_{\mathcal{B}}$ and the inverse mapping send maximal inexact subsets of first type to maximal inexact subsets of first type. This implies the existence of a bijection $h\colon \mathcal{X}_{2n-2} \to \mathcal{X}'_{2n-2}$ such that

$$f(\mathcal{B}(M)) = \mathcal{B}'(h(M))$$

for all $M \in \mathcal{X}_{2n-2}$. Each $N \in \mathcal{X}_{k+1}$ can be presented as the intersection of

$$M_1, \dots, M_{2n-k-2} \in \mathcal{X}_{2n-2}.$$

Then

$$g(N) := \bigcap_{i=1}^{2n-k-2} h(M_i)$$

is as required. □

Now we prove Lemma 7 for $k < n - 1$. Two subspaces $S, U \in \mathcal{B}$ are adjacent if and only if they belong to $\mathcal{B}(T)$ for certain $T \in \mathcal{X}_{k+1}$; moreover, S and U are ortho-adjacent if and only if $\mathcal{B}(T)$ consists of $k + 2$ elements (in other words, T is totally isotropic). The required statement follows from Lemma 11.

8 Proof of Theorem 2 for $n \geq 3$

Let M, N be a pair of incident subspaces of Π such that $\dim M < k < \dim N$. We denote by $[M, N]_k$ the set of k-dimensional subspaces of Π incident with both M and N; in the case when $M = \emptyset$ or $N = P$, we write $(N]_k$ or $[M)_k$, respectively.

We say that $\mathcal{X} \subset \mathcal{G}_k$ is an A-*subset* if any two distinct elements of $\mathcal{X}$ are adjacent.

Example 1. If $k < n - 1$ and N is an element of $\mathcal{G}_{k+1}$ then $(N]_k$ is a maximal A-subset of $\mathcal{G}_k$. Subsets of such type will be called *tops*. Any two distinct elements of a top are ortho-adjacent.

Example 2. If M belongs to $\mathcal{G}_{k-1}$ then

$$[M, M^{\perp}]_k = [M)_k \cap \mathcal{G}_k$$

is a maximal A-subset of $\mathcal{G}_k$. Such maximal A-subsets are known as *stars*, they contain non-orthogonal elements.

Fact 2 ([2, 11]). Each A-subset is contained in a maximal A-subset. Every maximal A-subset of $\mathcal{G}_{n-1}$ is a star. If $k < n - 1$ then every maximal A-subset of $\mathcal{G}_k$ is a top or a star.

Let $n \geq 3$ and f be as in the previous section. The first part of Lemma 7 says that f transfers A-subsets to A-subsets. The second part of Lemma 7 guarantees that stars go to subsets of stars. In other words, for any $M \in \mathcal{G}_{k-1}$ there exists $M' \in \mathcal{G}'_{k-1}$ such that

$$f([M, M^{\perp}]_k) \subset [M', M'^{\perp}]_k. \tag{6}$$

Suppose that

$$f([M, M^{\perp}]_k) \subset [M'', M''^{\perp}]_k$$

for $M'' \in \mathcal{G}'_{k-1}$ other than M'. Then $f([M, M^{\perp}]_k)$ is contained in the intersection of $[M', M'^{\perp}]_k$ and $[M'', M''^{\perp}]_k$. This intersection is not empty only if $M' = M''$ or M' and M'' are ortho-adjacent; but in the second case our intersection consists of one element only. Thus there is unique $M' \in \mathcal{G}'_{k-1}$ satisfying (6). We have established the existence of a mapping

$$g\colon \mathcal{G}_{k-1} \to \mathcal{G}'_{k-1}$$

such that

$$f([M, M^{\perp}]_k) \subset [g(M), g(M)^{\perp}]_k$$

for every $M \in \mathcal{G}_{k-1}$. It is easy to see that

$$g((N]_{k-1}) \subset (f(N)]_{k-1} \qquad \forall\, N \in \mathcal{G}_k. \tag{7}$$

Now we show that g *sends base subsets to base subsets.*

Proof: Let $\mathcal{B}_{k-1}$ be a base subset of $\mathcal{G}_{k-1}$ and B be the associated symplectic base. This base defines a base subset $\mathcal{B} \subset \mathcal{G}_k$. Now let B' be the symplectic base associated with the base subset $\mathcal{B}' := f(\mathcal{B})$ and $\mathcal{B}'_{k-1}$ be the base subset of $\mathcal{G}'_{k-1}$ defined by B'. If $S \in \mathcal{B}_{k-1}$ then we take $U_1, U_2 \in \mathcal{B}$ such that $S = U_1 \cap U_2$, and (7) shows that

$$g(S) = f(U_1) \cap f(U_2) \in \mathcal{B}'_{k-1}.$$

Thus $g(\mathcal{B}_{k-1})$ is contained in $\mathcal{B}'_{k-1}$. Suppose that $g(\mathcal{B}_{k-1})$ is a proper subset of $\mathcal{B}'_{k-1}$. Then $g(S) = g(U)$ for some $S, U \in \mathcal{B}_{k-1}$. The f-image of

$$\mathcal{B}(S) = \mathcal{B} \cap [S, S^{\perp}]_k$$

is contained in

$$\mathcal{B}'(g(S)) = \mathcal{B}' \cap [g(S), g(S)^{\perp}]_k.$$

Since these sets have the same cardinality,

$$f(\mathcal{B}(S)) = \mathcal{B}'(g(S)).$$

Similarly,

$$f(\mathcal{B}(U)) = \mathcal{B}'(g(U)).$$

The equality $f(\mathcal{B}(S)) = f(\mathcal{B}(U))$ contradicts the injectivity of f. Hence $g(\mathcal{B}_{k-1})$ coincides with $\mathcal{B}'_{k-1}$. □

If $k = 1$ then the mapping $g: P \to P'$ sends symplectic bases to symplectic bases. By Theorem 1, g is a symplectic embedding of Π to Π', and we have $f = (g)_1$.

Now suppose that $k > 1$ and g is induced by a symplectic embedding h of Π to Π'. Let us consider an arbitrary element $S \in \mathcal{G}_k$ and take ortho-adjacent $M, N \in \mathcal{G}_{k-1}$ such that $S = \overline{M \cup N}$. Then

$$\{S\} = [M, M^{\perp}]_k \cap [N, N^{\perp}]_k$$

and $f(S)$ belongs to the intersection of $[g(M), g(M)^{\perp}]_k$ and $[g(N), g(N)^{\perp}]_k$. Since

$$g(M) = \overline{h(M)} \quad \text{and} \quad g(N) = \overline{h(N)}$$

are ortho-adjacent, the intersection of $[g(M), g(M)^{\perp}]_k$ and $[g(N), g(N)^{\perp}]_k$ consists of one element and we have

$$f(S) = \overline{\overline{h(M)} \cup \overline{h(N)}} = \overline{h(S)}.$$

This means that f is induced by h. Therefore, Theorem 2 can be proved by induction.

9 Proof of Theorem 2 for $n = 2$

Let $k = n - 1 = 1$. In this case, a base subset consists of 4 elements.

For $S, U \in \mathcal{G}_k$ we denote by $\mathcal{X}(S, U)$ the set of all $M \in \mathcal{G}_k \setminus \{S, U\}$ such that there is a base subset containing S, U, M.

Since $k = 1$, two distinct elements of $\mathcal{G}_k$ are adjacent or non-intersecting.

Lemma 12. *Two distinct $S, U \in \mathcal{G}_k$ are non-intersecting if and only if for any distinct $M, N \in \mathcal{X}(S, U)$*

$$\{S, U, M, N\}$$

is a base subset.

Proof: Direct verification. □

This is a partial case of a more general property of generalized polygons [13].

Let f be as in Sections 7 and 8. Since

$$f(\mathcal{X}(S, U)) \subset \mathcal{X}(f(S), f(U)),$$

Lemma 12 shows that f maps adjacent elements of $\mathcal{G}_k$ to adjacent elements of $\mathcal{G}'_k$. Then stars go to subsets of stars and f induces a mapping $g: P \to P'$. As above, we establish that g sends bases of Π to bases of Π' and Theorem 1 gives the claim.

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