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Combinatorial interpretation and positivity of Kerov's character polynomials

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Abstract Kerov's polynomials give irreducible character values in terms of the free cumulants of the associated Young diagram. We prove in this article a positivity result on their coefficients, which extends a conjecture of S. Kerov. Our method, through decomposition of maps, gives a description of the coefficients of the k-th Kerov's polynomial using permutations in $S(k)$. We also obtain explicit formulas or combinatorial interpretations for some coefficients. In particular, we are able to compute the subdominant term for character values on any fixed permutation (it was known for cycles).

Keywords Representations · Symmetric group · Maps

1 Introduction

1.1 Background

1.1.1 *Representations of the symmetric group*

Representation theory of the symmetric group $S(n)$ is an old research field in mathematics. Irreducible representations of $S(n)$ are indexed by partitions[1] λ of n, or equivalently by Young diagrams of size n. The associated character can be computed thanks to a combinatorial algorithm, but unfortunately it becomes quickly cumbersome when the size of the diagram is large and does not help to study asymptotic behaviors.

[1]Non-increasing sequences of non-negative integers of sum n.

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1.1.2 Free cumulants

To solve asymptotic problems in representation theory of the symmetric groups, P. Biane introduced in [2] the free cumulants $R_i(\lambda)$ (of the transition measure) of a Young diagram.[2] Asymptotically, the character value and the results of the classical operations on representations can be easily described with the free cumulants:

- Up to a good normalization, the $l+1$-th free cumulant is the leading term of the character value on the cycle $(1\dots l)$.
- Typical large Young diagrams (according to the Plancherel distribution) have, after rescaling, all free cumulants, except for the second one, very close to zero.
- Almost all diagrams appearing in a result of an elementary operation on irreducible representations (like restriction, tensor product) have free cumulants very close to specific values, which can be easily computed from the free cumulants of the original diagram(s).

So the free cumulants form a good way to encode the information contained in a Young diagram.

1.1.3 Kerov's polynomials

It is natural to wonder if there are exact expressions of the character value in terms of the free cumulants. Kerov's polynomials give a positive answer to this question for character values on cycles (they appeared first in a paper of P. Biane [3, Theorem 1.1] in 2003). Unfortunately, their coefficients remain very mysterious. A lot of work has been done to understand them: a general, but exploding in complexity, explicit formula [4, 8] and a combinatorial interpretation for linear terms in free cumulants [3] have been found. We also refer to [10, 13, 15] for a complete outline of the literature on the subject.

The positivity of the coefficients of Kerov's polynomial has been observed by numerical computations [3, 8] and was conjectured by S. Kerov. The main result of this paper is a positive answer to this conjecture.

1.1.4 Multirectangular Young diagrams

We use in this paper a new way to look at Young diagrams, initiated by R. Stanley in [16]. In this paper, he proved a nice combinatorial formula for character values, but only for Young diagrams of rectangular shape. To generalize it, we have to look at any Young diagram as a superposition of rectangles as in Fig. 1. With this description, Stanley's formula has been recently generalized (see [5, 17]).

The complexity of this general formula depends only on the size of the support of the permutation (and not of the size of the permutation!). As remarked in [6], it is useful to reformulate it with the notion of a bipartite graph associated to a pair of

[2] The transition measure of a Young diagram is a measure on the real line introduced by S. Kerov in [9]. Its free cumulants are a sequence of real numbers associated to this measure. The denomination comes from free probability theory, see [2] for more details.

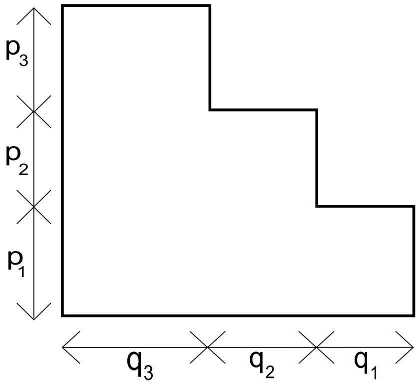


Fig. 1 Young diagram associated to sequences **p** and **q** (French convention)

permutations. This bipartite graph has in fact a canonical map structure,[3] which plays a key role here.

In this paper, we link these two recent developments. This gives a new combinatorial interpretation of the coefficients, proving Kerov's conjecture.

1.2 Normalized character

If σ is a permutation in $S(k)$, let $C(\sigma)$ be the partition of the set $[k] := \{1, \dots, k\}$ into orbits under the action of σ. The type of σ is, by definition, the partition μ of the integer k whose parts are the lengths of the cycles of σ. The conjugacy classes of $S(k)$ are exactly the sets of partitions of a given type.

By definition, for $\mu \vdash k$ and $\lambda \vdash n$ with $k \leq n$, the normalized character value is given by equation:

$$\Sigma_\mu(\lambda) := \frac{n(n-1)\dots(n-k+1)\chi^\lambda(\sigma)}{\chi^\lambda(\mathrm{Id}_n)}, \tag{1}$$

where σ is a permutation in $S(k)$ of type μ and χ^λ is the character value of the irreducible representation associated to λ (see [11]). Note that we have to identify σ with its image by the natural embedding of $S(k)$ in $S(n)$ to compute $\chi^\lambda(\sigma)$.

1.3 Minimal factorizations and non-crossing partitions

Non-crossing partitions and, in particular, their link with minimal factorizations of a cycle, are central in this work. This paragraph is devoted to definitions and known results in this domain. For more details, see P. Biane's paper [1].

Definition 1.3.1 A crossing of a partition π of the set $[j]$ is a quadruple $(a, b, c, d) \in [j]^4$ with $a < b < c < d$ such that

- a and c are in the same part of π;
- b and d are in the same part of π, different from the one containing a and c.

[3] For some pairs of permutations, this structure was introduced by I.P. Goulden and D.M. Jackson in [7].

A partition without crossings is called a non-crossing partition. The set of non-crossing partitions of $[j]$ is denoted $NC(j)$ and can be endowed with a partial order structure (by definition, $\pi \leq \pi'$ if every part of π is included in some part of π').

The partially ordered set (poset) $NC(j)$ appears in many domains: we will use its connection with the symmetric group.

Let us consider the following length on the symmetric group $S(j)$: denote by $l(\sigma)$ the minimal number h of factors needed to write σ as a product of transpositions $\sigma = \tau_1 \dots \tau_h$. One has:

$$l(\mathrm{Id}_j) = 0,$$
$$l(\sigma^{-1}) = l(\sigma),$$
$$l(\sigma \cdot \sigma') \leq l(\sigma) + l(\sigma').$$

We consider the associated partial order on $S(j)$: by definition, $\sigma \leq \sigma'$ if $l(\sigma') = l(\sigma) + l(\sigma^{-1}\sigma')$. It is easy to prove that

- Id_j is the smallest element;
- for any σ, one has $l(\sigma) = j - |C(\sigma)|$.

So, if we denote by $(1 \dots j)$ the cycle sending 1 onto 2, 2 onto 3, etc..., one has

$$\sigma \leq (1 \dots j) \iff |C(\sigma)| + |C(\sigma^{-1}(1 \dots j))| = j + 1.$$

If $\sigma \leq \sigma'$, let us consider the interval $[\sigma; \sigma']$ which is by definition the set $\{\tau \in S(k) \text{ s.t. } \sigma \leq \tau \leq \sigma'\}$. In his paper [1, Sect. 1.3], P. Biane gives a combinatorial description of these intervals:

Proposition 1.3.1 (Isomorphism with minimal factorizations) *The map*

$$\begin{aligned} [\mathrm{Id}_j; (1 \dots j)] &\longrightarrow NC(j) \\ \sigma &\mapsto C(\sigma) \end{aligned}$$

is a poset isomorphism.

Here is the inverse bijection: to a non-crossing partition τ of $[j]$, we associate the permutation $\sigma_\pi \in S(j)$, where $\sigma_\pi(i)$ is the next element in the same part of π as i for the cyclic order $(1, 2, \dots, j)$.

Since the order is invariant under conjugacy, every interval $[\mathrm{Id}_j; c]$, where c is a full cycle, is isomorphic as poset to a non-crossing partition set. More generally, if σ is a permutation in $S(j)$,

$$[\mathrm{Id}_j; \sigma] \simeq \prod_{i=1}^{|C(\sigma)|} NC(j_i),$$

where the j_i's are the numbers of elements of the cycles of σ. This result gives a description of all intervals of the symmetric group since, if $\sigma \leq \sigma'$, we have $[\sigma; \sigma'] \simeq [\mathrm{Id}; \sigma^{-1}\sigma']$.

1.4 Kerov's polynomials

We look for an expression of the normalized character value in terms of free cumulants. In the case when μ has only one part ($\mu = (k), \sigma = (1 \dots k)$), P. Biane shows[4] in [3] that:

Definition-Theorem 1.4.1 *For any $k \geq 1$, there exists a polynomial K_k, called k-th Kerov's polynomial, with integer coefficients, such that, for every Young diagram λ of size bigger than k, one has*:

$$\Sigma_k(\lambda) = K_k\big(R_2(\lambda), \dots, R_{k+1}(\lambda)\big). \tag{2}$$

Examples :

$$\Sigma_1 = R_2; \qquad \Sigma_4 = R_5 + 3R_3;$$

$$\Sigma_2 = R_3; \qquad \Sigma_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2;$$

$$\Sigma_3 = R_4 + R_2; \qquad \Sigma_6 = R_7 + 35R_5 + 35R_3 \cdot R_2 + 84R_3.$$

Our main result is the positivity of the coefficients of Kerov's polynomials. This result was conjectured by S. Kerov (according to P. Biane, see [3]).

Theorem 1.4.2 (Kerov's conjecture) *For any integer $k \geq 1$, the polynomial K_k has non-negative coefficients.*

Our proof gives a (complicated) combinatorial interpretation of the coefficients and allows us to compute some of them.

1.4.1 High graded degree terms

Theorem 1.4.3 *Let $j_1, \dots, j_t$ be non negative integers such that $\sum_i j_i = k - 1$. The coefficient of $\prod_i R_{j_i}$ in K_k is*

$$\frac{(k-1)k(k+1)}{24} |\operatorname{Perm}(\mathbf{j})| \prod_i (j_i - 1), \tag{3}$$

where $\operatorname{Perm}(\mathbf{j})$ *is the set of sequences equal to* $\mathbf{j}$ *up to a permutation* ($|\operatorname{Perm}(\mathbf{j})| = \frac{t!}{m_2! \dots m_{k-1}!}$ *is the multinomial coefficient of the m_l's, where m_l is the number of j_i equal to l*).

This theorem gives an explicit formula for the term of graded degree $k - 1$ in K_k, which is the subdominant term for character values on a cycle. It has already been proved in two different ways by I.P. Goulden and A. Rattan in [8] and by P. Śniady in [15]. The proof in this article is a new one and a consequence of our general combinatorial interpretation.

[4] P. Biane attributes this result to S. Kerov.

1.4.2 Low degree terms

Theorem 1.4.4 *The coefficient of the linear monomial R_d in K_k is the number of cycles $\tau \in S(k)$ such that $\tau^{-1}(12\dots k)$ has $d-1$ cycles.*

Let k, j, l be positive integers. Then the coefficient of $R_j R_l$ in K_k is the number (respectively half the number is $j = l$) of pairs (τ, φ) which fulfill the following conditions.

- *The first element τ is a permutation in $S(k)$ such that $|C(\tau)| = 2$. The second element φ is a bijection $C(\tau) \xrightarrow{\sim} \{1; 2\}$. So we count some permutations with numbered cycles.*
- *$C(\tau^{-1}\sigma)$ is a partition of $[k]$ in $j+l-2$ sets.*
- *Among these sets, at least j have an element in common with $\varphi^{-1}(1)$ and at least l with $\varphi^{-1}(2)$.*

The first part of this theorem was proved by R. Stanley and P. Biane [3]. The second part is a new result. As in our general combinatorial interpretation, these coefficients can be computed by counting permutations in $S(k)$. So, when the support of the permutations is quite small, we can compute quickly character values from free cumulants.

1.5 A combinatorial formula for character values

The main tool in this article is the following formula,[5] conjectured by R. Stanley in [17] and proved by the author in [5]. As noticed in paragraph 1.1, if we have two sequences $\mathbf{p}$ and $\mathbf{q}$ of non-negative integers with only finitely many non-zero terms, we consider the partition drawn in Fig. 1:

$$\lambda(\mathbf{p},\mathbf{q}) := \underbrace{\sum_{i\geq 1} q_i, \dots, \sum_{i\geq 1} q_i}_{p_1 \text{times}}, \underbrace{\sum_{i\geq 2} q_i, \dots, \sum_{i\geq 2} q_i}_{p_2 \text{times}}, \dots$$

With this notation, the $R_i(\lambda(\mathbf{p},\mathbf{q}))$ are homogeneous polynomials of degree i in $\mathbf{p}$ and $\mathbf{q}$.

Theorem 1.5.1 *Let $\mathbf{p}$ and $\mathbf{q}$ be two finite sequences, $\lambda(\mathbf{p},\mathbf{q}) \vdash n$ the associated Young diagram and $\mu \vdash k (k \leq n)$. If $\sigma \in S(k)$ is a permutation of type μ, the character value is given by the formula*:

$$\Sigma_\mu\big(\lambda(\mathbf{p},\mathbf{q})\big) = \sum_{\substack{\tau,\overline{\tau}\in S(k)\\ \tau\overline{\tau}=\sigma}} (-1)^{|C(\tau)|+l(\mu)} N^{\tau,\overline{\tau}}(\mathbf{p},\mathbf{q}), \tag{4}$$

where $l(\mu)$ is the number of parts of μ and $N^{\tau,\overline{\tau}}$ a homogeneous power series of degree $|C(\tau)|$ in $\mathbf{p}$ and $|C(\overline{\tau})|$ in $\mathbf{q}$ that will be defined in Sect. 2.

[5] The notations in this article are slightly different from the ones in the original papers.

This theorem gives a combinatorial interpretation of the coefficients of Σ_μ, expressed as a polynomial in variables $\mathbf{p}$ and $\mathbf{q}$. It is natural to wonder whether there exists such an expression for free cumulants. Since R_{l+1} is the term of graded degree $l+1$ of Σ_l (see [3, Theorem 1.3]), we have:[6]

$$\begin{aligned} R_{l+1}(\lambda(\mathbf{p},\mathbf{q})) &= \sum_{\substack{\tau,\overline{\tau}\in S(l)\\ \tau\overline{\tau}=(1\dots l)\\ |C(\tau)|+|C(\overline{\tau})|=l+1}} (-1)^{|C(\tau)|+1} N^{\tau,\overline{\tau}}(\mathbf{p},\mathbf{q}) \\ &= \sum_{\pi\in NC(l)} (-1)^{|\pi|+1} N^{\pi}(\mathbf{p},\mathbf{q}). \end{aligned} \tag{5}$$

The second equality comes from the fact that factorizations $\tau,\overline{\tau}$ of the long cycle $(1\dots l)$ such that $|C(\tau)|+|C(\overline{\tau})|=l+1$ are canonically in bijection with non-crossing partitions (see paragraph 1.3). Note that N^π is simply a short notation for $N^{\sigma_\pi,\sigma_\pi^{-1}(1\dots l)}$.

From now on, we consider Σ_k and R_l as power series in two infinite sets of variables $(\mathbf{p},\mathbf{q})$ and look at equality (2) in this algebra (equality as power series in $\mathbf{p}$ and $\mathbf{q}$ is equivalent to equality for all Young diagram λ, whose size is bigger than a given number). If we expand $K_k(R_2,\dots,R_{k+1})$, we obtain an algebraic sum of products of power series associated to minimal factorizations. In this article, we write each term of the right side of (4) as such a sum.

1.6 Generalized Kerov's polynomials

The theorems of paragraph 1.4 correspond to the case where μ has only one part. But, in fact, they have generalizations for any $\mu\vdash k$.

Firstly, there exist universal polynomials K_μ, called generalized Kerov's polynomials, such that:

$$\Sigma_\mu(\lambda) = K_\mu(R_2(\lambda),\dots,R_{k+1}(\lambda)). \tag{6}$$

$$\begin{aligned} \text{Examples: } \Sigma_{2,2} &= R_3^2 - 4R_4 - 2R_2^2 - 2R_2; \\ \Sigma_{3,2} &= R_3\cdot R_4 - 5R_2\cdot R_3 - 6R_5 - 18R_3; \\ \Sigma_{2,2,2} &= R_3^3 - 12R_3\cdot R_4 - 6R_3\cdot R_2^2 + 58R_3\cdot R_2 + 40R_5 + 80R_3. \end{aligned}$$

Secondly, although these polynomials do not have non-negative coefficients, the following generalization of Theorem 1.4.2 holds:

Theorem 1.6.1 *Let $\sigma\in S(k)$ be a permutation of type $\mu\vdash k$. Let us define*

$$\Sigma'_\mu := \sum_{\substack{\tau,\overline{\tau}\in S(k)\\ \tau\overline{\tau}=\sigma\\ \langle\tau,\overline{\tau}\rangle\,\text{trans.}}} (-1)^{|C(\tau)|+\mathbf{1}} N^{\tau,\overline{\tau}}, \tag{7}$$

[6] A. Rattan has also given a direct proof of this result in [12].

where $\langle\tau,\overline{\tau}\rangle$ trans. *means that the subgroup* $\langle\tau,\overline{\tau}\rangle$ *of* $S(k)$ *generated by* τ *and* $\overline{\tau}$ *acts transitively on the set* $[k]$. *Then there exists a polynomial* K'_μ *with non-negative integer coefficients such that, as power series*:

$$\Sigma'_\mu = K'_\mu(R_2, \dots, R_{k+1}). \tag{8}$$

$$\begin{aligned} \text{Examples: } \Sigma'_{2,2} &= 4R_4 + 2R_2^2 + 2R_2; \\ \Sigma'_{3,2} &= 6R_2 \cdot R_3 + 6R_5 + 18R_3; \\ \Sigma'_{2,2,2} &= 64R_3 \cdot R_2 + 40R_5 + 80R_3. \end{aligned}$$

Sections 2, 3 and 4 are devoted to the proof of this theorem.

The quantities Σ' are not only practical for the statement of this theorem, they also appear as disjoint cumulants [6, Proposition 22] in the study of asymptotics of character values in [14]. It is also easy to recover Σ from Σ' by looking, for each decomposition, at the set partition of $[k]$ into orbits under the action of $\langle\tau,\overline{\tau}\rangle$ (one has to be careful about the signs):

$$\Sigma_\mu = \sum_{\Pi \text{ partition of } [l(\mu)]} \left(\prod_{\{i_1,\dots,i_l\} \text{ part of } \Pi} (-1)^{l-1} \Sigma'_{\mu_{i_1},\dots,\mu_{i_l}} \right). \tag{9}$$

If we invert this formula with (usual) cumulants, then our positivity result on generalized Kerov's polynomials is exactly the one conjectured by A. Rattan and P. Śniady in [13].

1.6.1 Subdominant term for general μ

We can also compute some particular coefficients in this general context:

For low degree terms, the first part of Theorem 1.4.4 is still true (it has been proved in [13] in this general context) and the second is true with K'_μ instead of K_μ and with an additional condition in the second part that $<\tau,\tau^{-1}\sigma>$ acts transitively on $[k]$.

The highest graded degree in K'_μ is $|\mu|+2-l(\mu)$. In the case of $l(\mu)=2$, we can explicitly compute the corresponding term.

Theorem 1.6.2 *Let* $N(l_1,\dots,l_t;L)$ *be the number of solutions of the equation* $x_1+\dots+x_t=L$, *fulfilling the condition that, for each* i, x_i *is an integer between* 0 *and* l_i. *Then, the coefficient of a monomial* $\prod_{i=1}^t R_{j_i}$ *of graded degree* $r+s$ *in* $K'_{r,s}$ *is*:

$$\frac{r \cdot s}{t}\, |\mathrm{Perm}(\mathbf{j})|\; N(j_1-2,\dots,j_t-2;r-t). \tag{10}$$

This result gives the subdominant term for character values on any fixed permutation.

Corollary 1.6.3 *For any* $\mu = (k_1, \ldots, k_r) \vdash k$, *one has*:

$$\Sigma_\mu = \prod_{i=1}^{r} R_{k_i+1} + \sum_{i=1}^{r} \left[\left(\prod_{h \neq i} R_h \right) \left(\sum_{|\mathbf{j}|=i-1} \frac{(k-1)k(k+1)}{24} |\operatorname{Perm}(\mathbf{j})| \prod_{i}^{l(\mathbf{j})} (j_i - 1) R_{j_i} \right) \right] + \sum_{1 \le i_1 < i_2 \le r} \left[\left(\prod_{h \neq i_1, i_2} R_h \right) \times \left(\sum_{|\mathbf{j}|=i_1+i_2} \frac{i_1 \cdot i_2}{l(\mathbf{j})} |\operatorname{Perm}(\mathbf{j})| \, N(j_1 - 2, \ldots, j_t - 2; i_1 - t) \prod_{i=1}^{l(\mathbf{j})} R_{j_i} \right) \right] + \textit{lower degree terms}.$$

Proof In equation (9), the only summands which contain terms of degree $|\mu| + r - 2$ are the one indexed by the partition of $[l(\mu)]$ in singletons and those indexed by partitions in one pair and singletons. □

1.7 Organization of the article

In Sect. 2, we will associate a map to each pair of permutations. This will help us to define the associated power series N. In Sect. 3, for any map M, we write $N(M)$ as an algebraic sum of power series associated to minimal factorizations. The Sect. 4 is the end of the proof of Theorem 1.6.1. Then, in Sect. 5, we will compute some particular coefficients (proofs of Theorems 1.4.3, 1.4.4 and 1.6.2).

2 Maps and polynomials

In this section, we define the power series $N^{\tau,\overline{\tau}}$ as the composition of three functions:

$$S(k) \times S(k) \overset{\S\,2.1}{\longrightarrow} \text{bicolored labeled map} \overset{\text{Forget}}{\longrightarrow} \text{bicolored graph} \overset{\S\,2.2}{\longrightarrow} \mathbb{C}[[\mathbf{p}, \mathbf{q}]].$$

2.1 From permutations to maps

Let us give some definitions concerning graphs and maps.

Definition 2.1.1 (Graphs)

- A graph is given by:
  - a finite set of vertices V;
  - a set of half-edges H with a map *ext* from H to V (the image of a half-edge is called its extremity);

  - a partition of H into pairs (called edges, whose set is denoted E) and singletons (the external half-edges).
- A bicolored graph is a graph with a partition of V in two sets (the set of white vertices V_w and the set of black vertices V_b) such that, for each edge, among the extremities of its two half-edges, one is black and one is white.
- A labeled graph is a graph with a map ι from E in $\mathbb{N}^\star$. Moreover, we say that it is well labeled if ι is an injection with image $\{1, \dots, |E|\}$.
- An oriented edge e is an edge e with an order of its two half-edges.
- An oriented loop is a sequence of oriented edges $e_1, \dots, e_l$ such that:
  - For each i, the extremity v_i of the first half-edge of e_{i+1} is the same as the extremity of the second of e_i (with the convention $e_{l+1} = e_1$);
  - All the v_i's and the e_i's are different (an edge does not appear twice, even with different orientations).

  We identify sequences that differ only by a cyclic permutation of their oriented edges.
- The free abelian group on graphs has a natural ring structure: the product of two graphs is by definition their disjoint union.

Definition 2.1.2 (Maps)

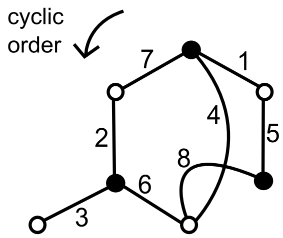
- A map is a graph supplied with, for each vertex v, a cyclic order on the set of all half-edges (including the external ones) with extremity v (*i.e.* $\mathrm{ext}^{-1}(v)$).
- Consider a half-edge h of a map M. Thanks to the map structure, there is a cyclic order on the set of half-edges having the same extremity as h. We call the element right after h in this order, the successor of h.
- Since a map is a graph with additional information, we have the notion of bicolored and/or (well-)labeled map.
- A face of a map is a sequence of oriented edges $e_1, \dots, e_k$ such that, for each i, the first half-edge of e_{i+1} ($e_{l+1} = e_1$) is the successor of the second half-edge of e_i. As for loops, we identify the sequences which differ by cyclic permutations of their oriented edges. Then each oriented edge is in exactly one face.
- If a face F of a map is labeled and bicolored, we denote by $E(F)$ the set of edges appearing in F with the white to black orientation. The word associated to a face is the word $w(F)$ of the labels of the elements of $E(F)$ (it is defined up to a cyclic permutation).
- A face that is also a loop (all vertices and edges of the face are distinct) and that does not contain an external half-edge, is called a polygon.

Remark 1 A map, whose underlying graph is a tree, is a planar tree. It has exactly one face.

2.1.1 Map associated with a pair of permutations

The following construction is classical (it generalizes the work of I.P. Goulden and D.M. Jackson in [7]) but we recall it for completeness.

Fig. 2 Example of a bicolored labeled map, with exactly one face whose associated word is 12345678



Definition 2.1.3 To a well-labeled bicolored map M with k edges and no external half-edges, we associate the pair of permutations $(\tau, \overline{\tau}) \in S(k)^2$ defined as follows: if i is an integer in $[k]$, e the edge of M with label i and h its half-edge with a white (resp. black) extremity, then $\tau(i)$ (resp. $\overline{\tau}(i)$) is the label of the edge containing the successor of h.

It is easy to see that this defines a bijection between well-labeled bicolored maps and pairs of permutations in $S(k)$. Its inverse associates to a pair of permutations $(\tau, \overline{\tau})$ the following bicolored labeled map $M^{\tau,\overline{\tau}}$: the set of white vertices is $C(\tau)$, the one of black vertices is $C(\overline{\tau})$, the set of half-edges $\{1^w, 1^b, \dots, k^w, k^b\}$ is partitioned into edges $\{i^w, i^b\}$ and the cycle $(i_1, \dots, i_l)$ of τ (resp. $(j_1, \dots, j_l)$ of $\overline{\tau}$) is the extremity of the half-edges $i_1^w, \dots, i_l^w$ (resp. $j_1^b, \dots, j_l^b$) in this cyclic order.

The following property follows directly from the definition:

Proposition 2.1.1 *The words associated to the faces of $M^{\tau,\overline{\tau}}$ are exactly the cycles of the product $\tau\overline{\tau}$.*

Example 1 The map drawn in Fig. 2 is associated to the pair of permutations $\big((15)(27)(3)(486), (174)(236)(58)\big)$ with product (12345678). The word associated to its unique face is 12345678 as predicted by Proposition 2.1.1.

Note that the connected components of $M^{\tau,\overline{\tau}}$ are in bijection with the orbits of $[k]$ under the action of $\langle \tau, \overline{\tau} \rangle$. So, a factorization is transitive if and only if its map is connected. In particular, maps of minimal factorizations of the full cycle $(12 \dots k)$ are exactly the connected maps with $k+1$ vertices and k edges, that is to say the planar trees.

2.2 From graphs to polynomials

Definition 2.2.1 Let G be a bicolored graph and V its set of vertices, disjoint union of V_b and V_w. An evaluation $\psi : V \to \mathbb{N}^\star$ is said to be admissible if, for any edge between a white vertex w and a black one b, it satisfies $\psi(b) \geq \psi(w)$. The power series $N(G)$ in indeterminates $\mathbf{p}$ and $\mathbf{q}$ is defined by the formula:

$$N(G) = \sum_{\substack{\psi: V \to \mathbb{N} \\ \text{admissible}}} \prod_{w \in V_w} p_{\psi(w)} \prod_{b \in V_b} q_{\psi(b)}. \tag{11}$$

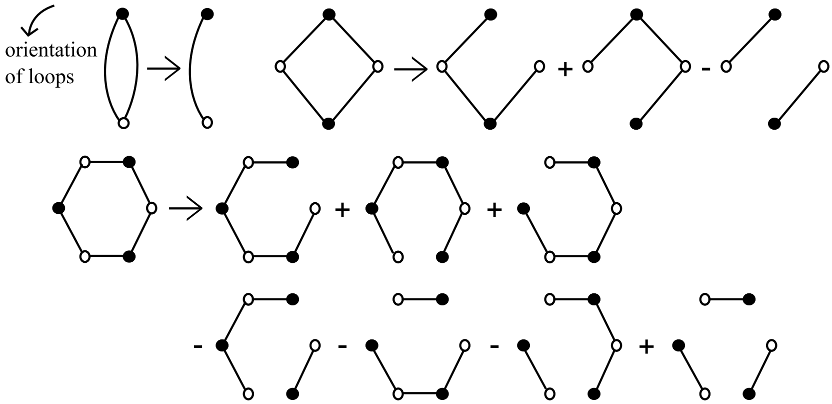


Fig. 3 Illustration of definition of transformation T_L

Note that N is extended to the ring $\mathbb{A}_{bg}$ of bicolored graphs by $\mathbb{Z}$-linearity. It is in fact a morphism of rings (the power series associated to a disjoint union of graphs is simply the product of the power series associated to these graphs).

If τ and $\overline{\tau}$ are two permutations in $S(k)$, we put:

$$N^{\tau,\overline{\tau}} := N(M^{\tau,\overline{\tau}}).$$

This definition is the one that appears in Theorem 1.5.1. The main step of our proof of Kerov's conjecture is to write the power series associated to any pair of permutations as an algebraic sum of power series associated to forests (*i.e.* products of power series associated to minimal factorizations).

Let G be a bicolored graph and L an oriented loop of G. We denote by $E(L)$ the set of edges that appear in the sequence L oriented from their white extremity to their black one. Let us define the following element of the $\mathbb{Z}$-module $\mathbb{A}_{bg}$:

$$T_L(G) = \sum_{\substack{E' \subset E(L) \\ E' \neq \emptyset}} (-1)^{|E'|-1} G \backslash E', \tag{12}$$

where $G\backslash E'$ denotes the graph obtained from G by erasing its edges belonging to E' (it is a subgraph of G with the same set of vertices). These elementary transformations are drawn in Fig. 3, where we have only drawn vertices and edges belonging to the loop L (so these schemes can be understood as local transformations).

An example of such a transformation is drawn in Fig. 4. G is the map of Fig. 2 (we forget the labels and the map structure) and L is the loop $7, 2, 6, 4$.

We have the following conservation property:

Proposition 2.2.1 *If G is a bicolored graph and L an oriented loop of G, then*

$$N\big(T_L(G)\big) = N(G). \tag{13}$$

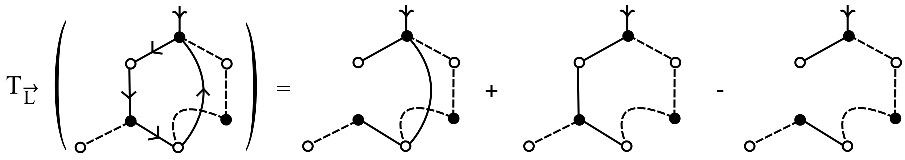


Fig. 4 Example of an elementary transformation

Proof Let G be a bicolored graph and V_w, V_b, E as in Definition 2.1.1. We write the series $N(G)$ as the following sum:

$$N(G) = \sum_{\psi_w : V_w \to \mathbb{N}^\star} \left[\sum_{\substack{\psi : V \to \mathbb{N}^\star \text{ admissible} \\ \psi_{/V_w} = \psi_w}} \prod_{w \in V_w} p_{\psi(w)} \prod_{b \in V_b} q_{\psi(b)} \right]$$
$$= \sum_{\psi_w : V_w \to \mathbb{N}^\star} N_{\psi_w}(G). \tag{14}$$

Since all the graphs in the equality (13) have the same set of white vertices V_w, it is enough to prove that, for every $\psi_w : V_w \to \mathbb{N}^\star$, one has:

$$N_{\psi_w}\big(T_L(G)\big) = N_{\psi_w}(G). \tag{15}$$

Let us fix a partial evaluation $\psi_w : V_w \to \mathbb{N}^\star$. If we choose a numbering $w_1, \dots, w_l$ (with respect to the loop order) of the white vertices of L, then there exists an index i such that $\psi_w(w_{i+1}) \geq \psi_w(w_i)$ (with the convention $w_{l+1} = w_1$). Denote by e the edge right after w_i in the loop L. It is an erasable edge. So we have a bijection:

$$\big\{E' \subset E(L), e \notin E'\big\} \xrightarrow{\sim} \big\{E'' \subset E(L), e \in E''\big\}$$
$$E' \mapsto E'' = E' \cup \{e\}.$$

But this bijection has the following property:

$$N_{\psi_w}(G \backslash E') = N_{\psi_w}(G \backslash (E' \cup \{e\}).$$

Indeed the admissible evaluations whose restriction to white vertices is ψ_w are the same for $G\backslash E'$ and $G\backslash(E' \cup \{e\})$. The only thing to prove is that, if such a ψ is admissible for $G\backslash(E' \cup \{e\})$, it also satisfies: $\psi(b_e) \geq \psi(w_i)$, where b_e is the black extremity of e. This is true because

$$\psi(b_e) \geq \psi(w_{i+1}) = \psi_w(w_{i+1}) \geq \psi_w(w_i) = \psi(w_i).$$

To conclude the proof, note that E' and $E' \cup \{e\}$ appear with different signs in $G - T_L(G)$. Their contributions to (15) cancel each other and the proof follows. □

Recall that N is a morphism of rings, so $(\mathbb{A}_{bg})_{/\operatorname{Ker} N}$ is a ring.

Corollary 2.2.2 *The ring* $(\mathbb{A}_{bg})_{/\operatorname{Ker} N}$ *is generated by trees.*

Proof It is enough to iterate the proposition by choosing any oriented loop until there is no loop left (if a graph is not a disjoint union of trees, there is always one). □

Remark 2 The idea of the proof of the main theorem is to write any series $N^{\tau,\overline{\tau}}$ as an algebraic sum of series associated to forests so that, one can write Σ'_μ in the form:

$$\Sigma'_\mu = \sum \pm N(F).$$

But free cumulants can be written in a similar form, so Kerov's polynomials give another formula for Σ'_μ as an algebraic sum of $N(F)$'s. If forests were linearly independent in $(\mathbb{A}_{bg})_{/\operatorname{Ker} N}$, we would know immediately that these two formulas are the same (up to a reordering of the terms).

Unfortunately, forests are not lineraly independent in $(\mathbb{A}_{bg})_{/\operatorname{Ker} N}$. So we will have to prove that our two formulas are indeed the same (this is the purpose of Sect. 4). This is not true for every decomposition of the $N^{\tau,\overline{\tau}}$'s as an algebraic sum of $N(F)$'s, so we will also have to construct a *good* one in Sect. 3.

3 Map decomposition

Given a graph G, by iterating Proposition 2.2.1 until there are only forests left, we obtain an algebraic sum of forests whose associated power series is $N(G)$. But there are many possible choices of oriented loops and they can give different sums of forests. In this section, we explain how, by restricting the choices, do we choose a particular one, which depends on the map structure and the labeling.

3.1 Elementary decomposition

To do coherent choices, it is convenient to add an external half-edge to our map. So, in this section, we deal with bicolored maps with exactly one external half-edge h. They generate a free $\mathbb{Z}$-module denoted $\mathbb{A}_{bm,1}$.

If M is such a map, let $\star$ be the extremity of its external half-edge. An (oriented) loop L is called admissible if:

- The vertex $\star$ is a vertex of the loop, that is to say that $\star$ is the extremity of the second half-edge $h_{i,2}$ of e_i and of the first half-edge $h_{i+1,1}$ of e_{i+1} for some i;
- The cyclic order at $\star$ restricted to the set $\{h, h_{i,2}, h_{i+1,1}\}$ is the cyclic order $(h, h_{i+1,1}, h_{i,2})$.

For example, the oriented loop L from Fig. 4 is admissible. If L satisfies the first condition, exactly one among the oriented loops L and L' is admissible (where L' is L with the opposite orientation).

Definition-Theorem 3.1.1 *There exists a unique linear operator*

$$D_1 : \mathbb{A}_{bm,1} \to \mathbb{A}_{bm,1}$$

such that:

- *The image of a given map M belongs to the vector space spanned by its submaps with the same set of vertices*;
- *If L is an admissible loop of M, then*

$$D_1(M) = D_1\big(T_L(M)\big). \tag{16}$$

  Note that this equality is meant as an equality between submaps of M, not just as abstract isomorphic maps;
- *If there is no admissible loops in M, then $D_1(M) = M$.*

Proof If M is a bicolored map, all graphs appearing in $T_L(M)$ have strictly fewer edges than M. So the uniqueness of D_1 is obvious.

The existence of D_1 will be proved by induction. Denote, for every N, by $\mathbb{A}_{bm,1}^N$ the submodule of $\mathbb{A}_{bm,1}$ generated by graphs with at most N edges. We will prove that there exists, for every N, an operator $D_1^N : \mathbb{A}_{bm,1}^N \to \mathbb{A}_{bm,1}^N$, extending D_1^{N-1} if $N \geq 1$, and satisfying the conditions asked for D_1. The case $N = 0$ is very easy because $\mathbb{A}_{bm,1}^0$ is generated by graphs without admissible loops, so $D_1^0 = \mathrm{Id}$. If our statement is proved for any N, it implies the existence of D_1: it is the inductive limit of the D_1^N.

Let us fix $N \geq 1$ and suppose that D_1^{N-1} has been constructed. To prove the existence of D_1^N, we have to prove that, if M has admissible loops, then $D_1^{N-1}\big(T_L(M)\big)$ does not depend on the chosen admissible loop L.

To do this, let us denote by $M_\star$ the submap of M containing exactly all the edges of M which belong to some admissible loop of M. The maps M and $M_\star$ have exactly the same admissible loops. We define $H = |E(M_\star)| - |V(M_\star)| + 1$ (which might be understood as the number of independent loops in $M_\star$ since $M_\star$ is connected).

If $H = 0, 1$, the map M has at most one admissible loop, so there is nothing to prove:

- If M has exactly no admissible loop, then $D_1^N(M) = M$.
- If M has exactly one admissible loop L, then $D_1^N(M) = T_L(M)$.

If $H = 2$ and if there is a vertex of valence 4 in $M_\star$ different from $\star$, then there is at most one admissible loop. If $H = 2$ and if $\star$ is a vertex of valence 4, then there are two admissible loops L_1 and L_2 without any edges in common, so the transformations with respect to these loops commute, so

$$D^{N-1}\big(T_{L_1}(M)\big) = T_{L_2}\big(T_{L_1}(M)\big) = T_{L_1}\big(T_{L_2}(M)\big) = D^{N-1}\big(T_{L_2}(M)\big).$$

If $H = 2$ and if $\star$ and some other vertex v have valence 3, there are three admissible loops. In $M_\star$, there are three different paths c_0, c_1, c_2 going (without any repetition of vertices or edges) from $\star$ to v. We number them such that, if h_i is the first half-edge of the path c_i, the cyclic order at $\star$ is (h, h_0, h_1, h_2). Let us denote by E_i $(0 \leq i \leq 2)$ (resp. by $E_{\bar{i}}$) the set of edges appearing in c_i oriented from their black vertex to their white one (resp. from their white vertex to their black one). If $I = \{i_1, \dots, i_l\} \subset \{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$, we consider the following element of $\mathbb{A}_{bg,1}$:

$$M_I = \sum_{\emptyset \neq E'_1 \subset E_{i_1}, \dots, \emptyset \neq E'_l \subset E_{i_l}} (-1)^{|E'_1|-1} \dots (-1)^{|E'_l|-1} \left[M \backslash \left(E'_1 \cup \dots \cup E'_l \right) \right].$$

Let $L_1 = c_0 \cdot \overline{c_1}$, $L_2 = c_1 \cdot \overline{c_2}$ and $L_3 = c_0 \cdot \overline{c_2}$ be the three admissible loops of M. Their respective sets of erasable edges are $E_{\bar{0}} \cup E_1$, $E_{\bar{0}} \cup E_2$ and $E_{\bar{1}} \cup E_2$. So we have (the Fig. 5 shows this computation on an example, where all sets E_i are of cardinality 1):

$$\begin{aligned} T_{L_1}(M) &= \sum_{\substack{E' \subset E_1 \\ E' \neq \emptyset}} (-1)^{|E'|-1} M \backslash E' + \sum_{\substack{E' \subset E_{\bar{0}} \\ E' \neq \emptyset}} (-1)^{|E'|-1} M \backslash E' \\ &\quad + \sum_{\substack{E' \subset (E_1 \cup E_{\bar{0}}) \\ (E' \cap E_1) \neq \emptyset, (E' \cap E_{\bar{0}}) \neq \emptyset}} (-1)^{|E'|-1} M \backslash E'; \\ &= M_{\bar{0}} + M_1 - M_{1,\bar{0}}. \end{aligned}$$

For each graph appearing in $M_{\bar{0}}$, M_1 there is only one admissible loop so D_1^{N-1} is given by the corresponding elementary transform:

$$\begin{aligned} D_1^{N-1}(T_{L_1}(M)) &= M_{\bar{0},\bar{1}} + M_{2,\bar{0}} - M_{2,\bar{0},\bar{1}} + M_{1,\bar{0}} + M_{1,2} - M_{1,2,\bar{0}} - M_{1,\bar{0}}, \\ &= M_{\bar{0},\bar{1}} + M_{2,\bar{0}} - M_{2,\bar{0},\bar{1}} + M_{1,2} - M_{1,2,\bar{0}}. \end{aligned}$$

For the other admissible loops, we obtain:

$$\begin{aligned} D_1^{N-1}(T_{L_2}(M)) &= D_1^{N-1}(M_{\bar{1}} + M_2 - M_{2,\bar{1}}), \\ &= M_{\bar{0},\bar{1}} + M_{2,\bar{1}} - M_{2,\bar{0},\bar{1}} + M_{2,\bar{0}} + M_{1,2} - M_{1,2,\bar{0}} - M_{2,\bar{1}}, \\ &= M_{\bar{0},\bar{1}} - M_{2,\bar{0},\bar{1}} + M_{2,\bar{0}} + M_{1,2} - M_{1,2,\bar{0}}; \\ D_1^{N-1}(T_{L_3}(M)) &= D_1^{N-1}(M_{\bar{0}} + M_2 - M_{2,\bar{0}}), \\ &= M_{\bar{0},\bar{1}} + M_{2,\bar{0}} - M_{2,\bar{0},\bar{1}} + M_{2,\bar{0}} + M_{1,2} - M_{1,2,\bar{0}} - M_{2,\bar{0}}, \\ &= M_{\bar{0},\bar{1}} - M_{2,\bar{0},\bar{1}} + M_{2,\bar{0}} + M_{1,2} - M_{1,2,\bar{0}}. \end{aligned}$$

We observe, in our computation, that $D_1^{N-1}(T_L(M))$ does not depend on the admissible loop L.

If $H = 2$ and if there are two vertices v and v' of valence 3 distinct from $\star$, the proof is similar. We use the same notation, except that:

- The paths c_0, c_1 and c_2 go from v to v'.
- The vertex $\star$ is on c_0. The two others paths can eventually be switched.

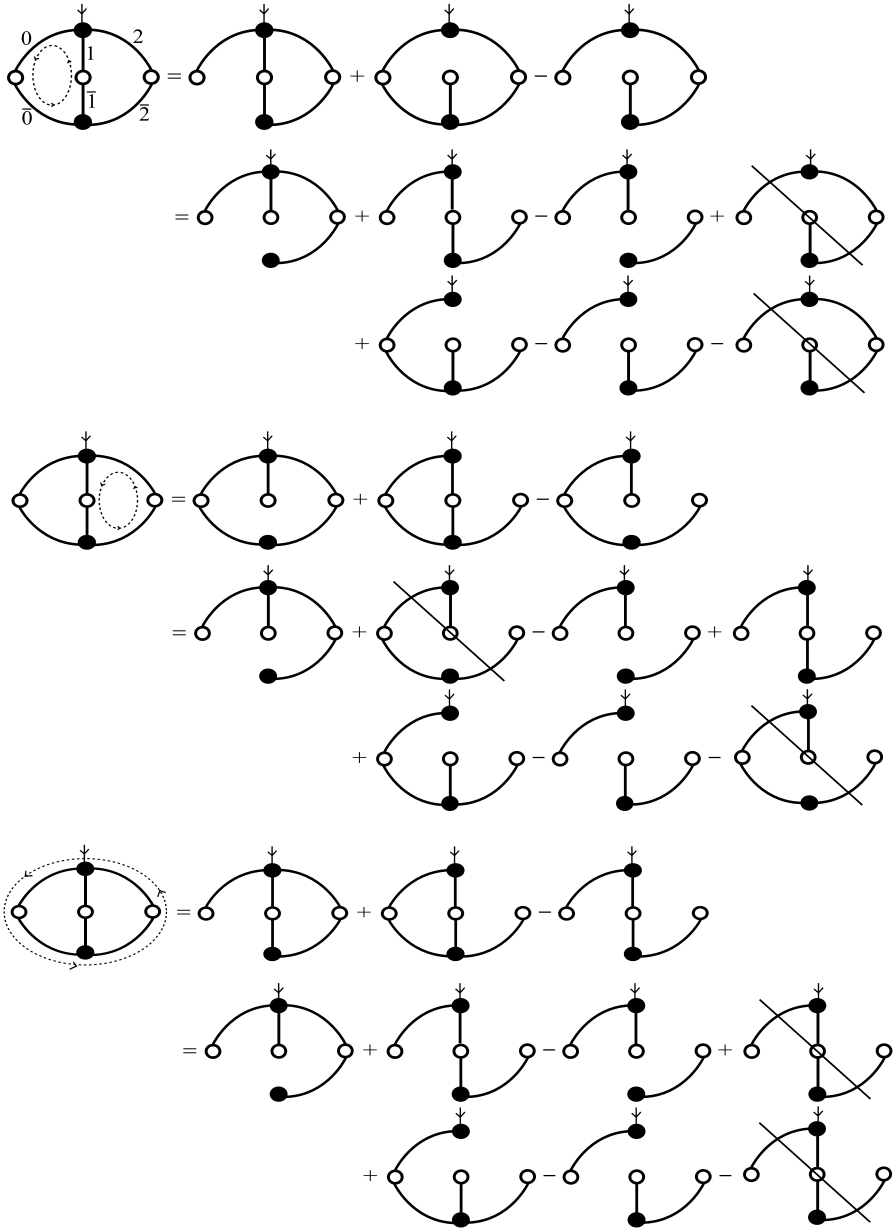


Fig. 5 One particular case of Definition-Theorem 3.1.1

- If the half-edge before (resp. after) $\star$ in c_0 is denoted by h_1 (resp. h_2), the cyclic order at $\star$ induces the order (h_1, h, h_2).

In this case, there are only two admissible loops L_1 and L_3 in M and a little computation proves the theorem:

$$\begin{aligned}
D_1^{N-1}(T_{L_1}(M)) &= D_1^{N-1}(M_{\bar{0}} + M_1 - M_{1,\bar{0}}),\\
&= M_{\bar{0}} + M_{1,\bar{0}} + M_{1,2} - M_{1,2,\bar{0}} - M_{1,\bar{0}},\\
&= M_{\bar{0}} + M_{1,2} - M_{1,2,\bar{0}};\\
D_1^{N-1}(T_{L_{\bar{0}}}(M)) &= D_1^{N-1}(M_{\bar{0}} + M_2 - M_{2,\bar{0}}),\\
&= M_{\bar{0}} + M_{2,\bar{0}} + M_{1,2} - M_{1,2,\bar{0}} - M_{2,\bar{0}},\\
&= M_{\bar{0}} + M_{1,2} - M_{1,2,\bar{0}}.
\end{aligned}$$

The proof is finished in the case $H = 2$.

The case $H \geq 3$ needs the following two lemmas:

Lemma 3.1.2 *Let L be an admissible loop of M and e an edge of $M \backslash L$. Then,*

$$D_1^{N-1}\big(T_L(M)\big) = D_1^{N-1}\big[D_1^{N-1}(M\backslash\{e\}) \cup \{e\}\big],$$

where, for a submap $M' \subset M$ with the same set of vertices which does not contain e, $M' \cup \{e\}$ is the map obtained by adding the edge e to M'.

Proof The key point of the proof is the following: if M' is a submap of M which does not contain e and K an admissible loop of M', then K is also an admissible loop of $M' \cup \{e\}$ and

$$T_K(M' \cup \{e\}) = T_K(M') \cup \{e\}.$$

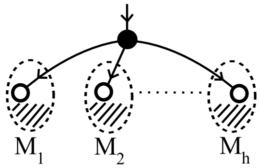
As $D_1^{N-1}(M\backslash\{e\}) = D_1^{N-1}\big(T_L(M\backslash\{e\})\big)$ can be obtained from $T_L(M\backslash\{e\})$ by iterating elementary transformations with respect to admissible loops, the formal expression $D_1^{N-1}(M\backslash\{e\}) \cup \{e\}$ can be obtained from $T_L(M\backslash\{e\}) \cup \{e\} = T_L(M)$ the same way. Therefore, they have the same image under D_1^{N-1}. □

Lemma 3.1.3 *If $H \geq 3$ and if L_1 and L_2 are two admissible loops with $L_1 \cup L_2 = M$, then there exists a third one L such that $L \cup L_1 \neq M$ and $L \cup L_2 \neq M$.*

Proof We choose a numbering of the oriented edges of the loops so that the first half-edge of e_1 has $\star$ for extremity. We suppose (by switching L_1 and L_2 if necessary) that the first half-edge of L_1 is between h_0 and the first half-edge of L_2 in the cyclic order of $\star$ (it might happen that they are equal, in which case we do not do anything). As $L_1 \cup L_2 = M$, the loops L_1 and L_2 have a vertex in common other than $\star$ (otherwise, M is a wedge of two cycles and $H = 2$). Let v be the first vertex of L_1 that is also in L_2 but such that the paths from $\star$ to v given by the beginnings of L_1 and L_2 are different. Let us consider the sequence L equal to the concatenation of the beginning of L_1 (from $\star$ to v) and the end of L_2 (from v to $\star$). With this definition:

- All vertices and edges appearing in L are distinct. Moreover, L is an admissible loop;

Fig. 6 General form of the connected component containing ⋆ of a map appearing in $D_1(M)$



- The edge before v in L_2 belongs neither to L_1 nor to L;
- As $H > 2$, the ends of L_1 and L_2 (from v to ⋆) are different. So there is an edge at the end of L_1 which belongs neither to L_2 nor to L. □

Lemma 3.1.2 implies that if L_1 and L_2 are admissible loops such that $L_1 \cup L_2 \neq M$, then we have:

$$D_1\big(T_{L_1}(M)\big) = D_1\big(T_{L_2}(M)\big).$$

Using Lemma 3.1.3, the equation above is still true without the assumption that $L_1 \cup L_2 \neq M$. So $D_1\big(T_L(M)\big)$ does not depend on the admissible loop L, which is exactly what we wanted to prove. □

Remark 3 (useful in paragraph 4.2) The definition of this operator does not really need the maps to be bicolored. It is enough to suppose that each edge has a privileged orientation. In this context, the erasable edges of an oriented loop are the ones which appear in the loop in their privileged orientation and we can define the operator T_L. A bicolored map can be seen this way if we choose as orientation of each edge the one from the white vertex to the black one.

3.2 Complete decomposition

It is immediate from the definition that every map M' appearing with a non-zero coefficient in $D_1(M)$ has no admissible loops. Thus they are of the following form (drawn in Fig. 6):

The vertex ⋆ is the extremity of half-edges h_i $(0 \leq i \leq l)$, including the external one h_0, numbered with respect to the cyclic order. For $i \geq 1$, h_i belongs to an edge e_i, whose other extremity is v_i. Each v_i is in a different connected component M_i (called leg) of $M \backslash \{h_1, \dots, h_l\}$. Note that we have only erased the half-edge h_i and not the whole edge e_i so that each M_i keeps an external half-edge.

If we have a family of submaps $M'_i = M_i \backslash \{E'_i\}$ of the M_i we consider the map $\phi_M(M'_1, \dots, M'_l) = M \backslash \bigcup \{E'_i\}$ obtained by replacing in M each M_i by M'_i.

The outcome of operator D_1 is an algebraic sums of maps that are much more complicated than planar forests. So, in order to write $N(M)$ as an algebraic sum of series associated to minimal factorizations, we have to iterate such operations.

We want to define decompositions of maps associated to pairs of permutations, that is, of well-labeled bicolored maps without external edges. But it is convenient to work in a bigger module: the ring $\mathbb{A}_{blm,\leq 1}$ of bicolored labeled maps with at most one external half-edge per connected component.

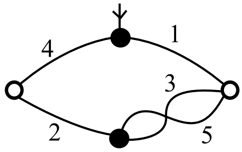


Fig. 7 Map $\overline{M}$

Definition-Proposition 3.2.1 *There exists a unique linear operator*

$$D : \mathbb{A}_{blm,\leq 1} \to \mathbb{A}_{blm,\leq 1}$$

such that:

(1) *If M has only one vertex*, *then $D(M) = M$*;
(2) *If M has more than one connected component $M = \prod M_i$, then one has $D(M) = \prod D(M_i)$*;
(3) *If M has only one connected component and no external half-edge*, *consider its edge e of smallest label. Let h be the half-edge of e with black extremity. We denote by $\overline{M}$ the map obtained by adding one external half-edge between h and its successor. Then $D(M) = D(\overline{M})$*;
(4) *If M has only one connected component with one half-edge but no admissible loops*, *we use the notation of the previous paragraph. As the M_i's are connected maps with an external half-edge*, *we can compute $D(M_i)$ (third or fifth case). Then $D(M)$ is given by the formula*:

$$D(M) = \phi_M\big(D(M_1), \dots, D(M_l)\big),$$

where ϕ_M is extended by multilinearity to algebraic sums of submaps of the M_i's.
(5) *Else*, *$D(M) = D(D_1(M))$.*

Existence and uniqueness of D are obvious. The image of a map M by D is in the subspace generated by its submaps with the same set of vertices, no isolated vertices and no loops, i.e. its covering forests without trivial trees. Note also that forests are fixed points for D (immediate induction).

Example 2 We will compute $D(M)$ where M is the map from Fig. 7 (without the external half-edge).

The map M belongs to the third kind, so we have to add an external half-edge as in the figure. Now, $\overline{M}$ is a map of the fifth type and we have to compute $D_1(\overline{M})$: this is very easy because the two transformations associated with admissible loops lead to the same sum of submaps that do not contain any admissible loop.

$$D_1(M) = M\backslash\{1\} + M\backslash\{2\} - M\backslash\{1,2\}.$$

$$\text{So } D(M) = D(M\backslash\{1\}) + D(M\backslash\{2\}) - D(M\backslash\{1,2\}).$$

The map $M\backslash\{1\}$ is a map of the fourth type with only one leg M_1, which is drawn in Fig. 8.

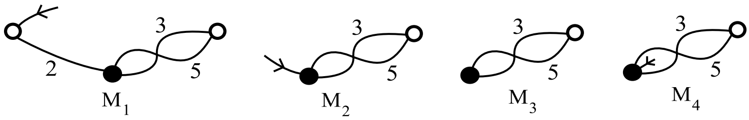


Fig. 8 Maps involved in the computation of the example

This map M_1 is again of the fourth type (with one leg: the map M_2 from Fig. 8) so we have to compute $D(M_2)$, which is simply $D_1(M_2) = M_2\backslash\{5\}$. This implies immediately that $D(M_1) = M_1\backslash\{5\}$ and:

$$D(M\backslash\{1\}) = M\backslash\{1,5\}.$$

$$\text{Similarly, } D(M\backslash\{2\}) = M\backslash\{2,3\}.$$

Now we look at the map $M\backslash\{1,2\}$. It has two connected components (we have to apply rule 2): one is a tree and has a trivial image by D, the other one M_3, has no external half-edge. We have to add one external half-edge to M_3 with the third rule and obtain M_4. Now, it is clear that $D_1(M_4) = M_4\backslash\{3\}$, so one has $D(M\backslash\{1,2\}) = M\backslash\{1,2,3\}$.

Finally

$$D(M) = M\backslash\{1,5\} + M\backslash\{2,3\} - M\backslash\{1,2,3\}.$$

As we can see from this example, when we replace M_i by its image under several elementary transformations in M, we obtain the image of M by the same transformations. So, by immediate induction, the operator D consists of applying to M an elementary transformation T_L (with restricted choices), then one to each map of the result that is not a forest, *etc.* until there are only forests left. An immediate consequence is the D-invariance of N.

Remark 4 Note that transformations indexed by loops which are in different connected components and/or in different legs of the map (fourth case) commute.

3.3 Signs

In this section, we study the signs of the coefficients in the expression $D(M)$. This is crucial in the proof of Theorem 1.6.1 because we will show that the coefficients of K'_μ can be written as sums of coefficients of $D(M)$, for some particular maps M.

Proposition 3.3.1 *Let $M' \subset M$ be two maps with the same set of vertices and respectively $t_{M'}$ and t_M connected components. The sign of the coefficient of M' in $(-1)^{t_M} D(M)$ is $(-1)^{t_{M'}}$.*

Proof Due to the inductive definition of D using D_1, it is enough to prove the result for operator D_1 in the case where M is a connected ($t_M = 1$) bicolored map with one external half-edge. We proceed by induction on the number of edges in $M\backslash M'$. If $M' = M$, the result is obvious. Note that if M' has a non-zero coefficient in $D_1(M)$, we have necessarily $M\backslash M' = \{e_1, \dots, e_l\}$ where each e_i belongs at least to one admissible loop.

First case: There exists an edge $e \in M\backslash M'$ such that $M\backslash\{e\}$ has at least one admissible loop. Let us define $M_1 = M\backslash\{e\}$ and apply the Lemma 3.1.2: $D_1(M) = D_1\big(D_1(M_1) \cup \{e\}\big)$. The submaps M'' of M_1 containing M' can be divided in two classes:

- Either $M'' \cup \{e\}$ has the same number t of connected components as M''. By induction hypothesis, the sign of the coefficient of $M'' \cup \{e\}$ in $D_1(M_1) \cup \{e\}$ is $(-1)^{t-1}$;
- Or $M'' \cup \{e\}$ has strictly less connected components than M''. In this case $\{e\}$ does not belong to any loops of $M'' \cup \{e\}$, so every graph appearing in $D_1(M'' \cup \{e\})$ does contain $\{e\}$. In particular, the coefficient of M' in $D_1(M'' \cup \{e\})$ is zero.

Finally, the coefficient of M' in $D_1(M)$ is the same as in the sum of $D_1(M'' \cup \{e\})$ for M'' of the first class. So the result follows from the induction hypothesis applied to $M' \subset M'' \cup \{e\}$ (which can be done because $M'' \cup \{e\}$ has strictly fewer edges than M).

Second case: Else, up to a new numbering of edges of $M\backslash M'$, the map M' has l connected components $M'_1, \dots, M'_l$ and, for each i, the two extremities of e_i belong to M'_i and M'_{i+1} (convention: $M'_{l+1} = M'_1$).

Choose any admissible loop L, it contains all the edges e_i. If we look at a map of the kind $M'' = M\backslash E'$, with $E' \subsetneq \{e_1, \dots, e_l\}$, all edges of $\{e_1, \dots, e_l\}\backslash E'$ do not belong to any loop of M'' and are never erased in the computation of $D_1(M'')$. So the only term in $T_L(M)$ which can contribute to the coefficient of M' is $(-1)^{l-1}M'$. □

4 Decompositions and cumulants

In Sect. 3, we have built an operator D on bicolored labeled maps which leaves N invariant and takes value in the ring spanned by forests. If we replace $N^{\tau,\overline{\tau}}$ by $N(D(M^{\tau,\overline{\tau}}))$ in the right hand side of equation (7), we obtain a decomposition of Σ'_μ as an algebraic sum of products of power series associated to minimal factorizations. In order to have something that looks like (8), we regroup some terms and make free cumulants appear via formula (5). To do this, it will be useful to encode these associations of terms in combinatorial objects that we will call cumulant maps.

4.1 Cumulant maps

Definition 4.1.1 A cumulant map $\mathcal{M}$ of size k is a triple $(M_{\mathcal{M}}, \mathbf{F}, \iota)$ where $M_{\mathcal{M}}$ is a bicolored map with $|E| - |V| = k$, $\mathbf{F} = (F_1, \dots, F_t)$ is a family of faces of $M_{\mathcal{M}}$ such that

- the faces $F_1, \dots, F_t$ are polygons (see Definition 2.1.2),
- every vertex of $M_{\mathcal{M}}$ belongs to exactly one face among $F_1, \dots, F_t$,

and ι is a function from $E\backslash\bigcup_i(E(F_i))$ (the set $E(F)$ was introduced in Definition 2.1.2) to $\mathbb{N}^\star$ (see Fig. 9 for an example). As in the case of classical maps, if ι is a bijection with image $[k]$, the cumulant map is called well-labeled.

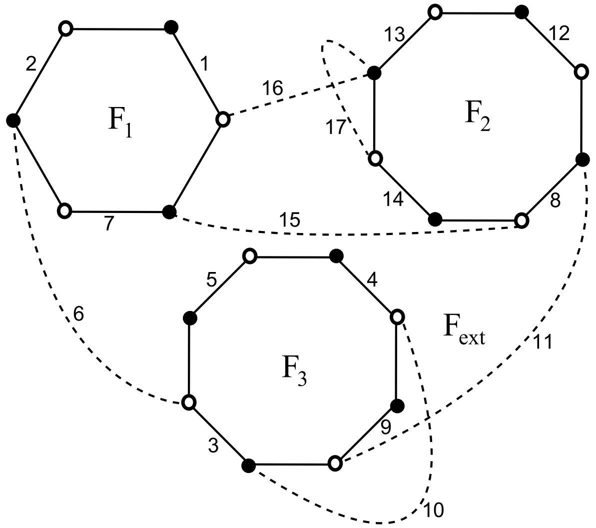


Fig. 9 Example of a well-labeled cumulant map of resultant $(1\dots17)$

By definition, the number of connected components of $\mathcal{M}$ is the one of $M_{\mathcal{M}}$, and its resultant $\sigma_{\mathcal{M}}$ is the product of the cycles associated to the faces of $M_{\mathcal{M}}$ different from $F_1, \dots, F_t$.

4.1.1 Non-crossing partitions as compressions of a polygon

Consider a polygon with $2j$ vertices, colored alternatively black and white. We choose an orientation: begin at a black vertex and label the edges $1', 1, 2', 2, \dots, j', j$. Given a non-crossing partition $\pi \in NC(j)$, we glue, for each i, the edge i with the edge $\sigma_\pi(i)'$ (σ_π is the permutation of $[\mathrm{Id}_j; (1\dots j)]$ canonically associated to π by Proposition 1.3.1) so that their black extremities are glued together and also their white ones. In each of these gluings we only keep the label without prime. The result is the labeled bicolored planar tree associated to the pair $\big(\sigma_\pi, \sigma_\pi^{-1}(1\dots j)\big)$.

This construction defines a bijection between $NC(j)$ and the different ways to compress a polygon with $2j$ vertices (with labeled edges) in a bicolored labeled planar tree with j edges. So we reformulate (5):

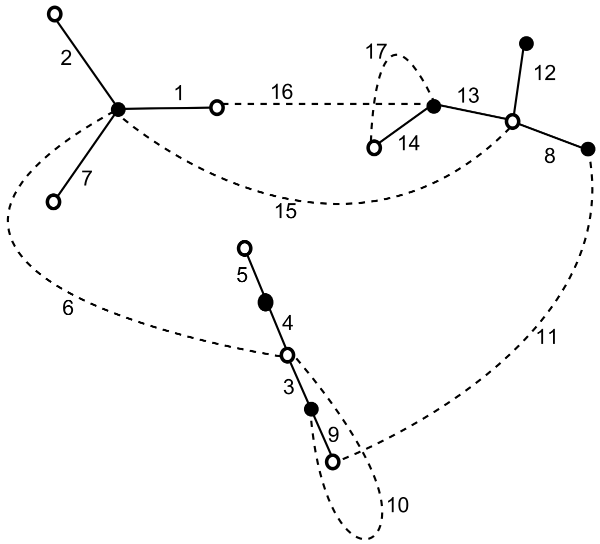
$$R_{j+1} = \sum_{\substack{\text{tree } T \text{ obtained by compression}\\ \text{of a polygon with } 2j \text{ vertices}}} (-1)^{|V_w(T)|+1} N(T), \qquad (17)$$

as power series in $\mathbf{p}$ and $\mathbf{q}$ (where $|V_w(T)|$ is the number of white vertices of T). If we consider a polygon without the labels $1', 1, \dots, j', j$, the bijection between $NC(j)$ and the different ways to compress it as a tree is only defined up to a rotation of the polygon but this formula still holds.

4.1.2 Compressions of a cumulant map

Given a cumulant map $\mathcal{M}$, consider all maps M obtained from $M_{\mathcal{M}}$ by compressing each F_i into a tree (we do not touch the edges—dotted in our example—that do not

Fig. 10 Example of a map obtained by compressing the polygons of the cumulant map from Figure 9



belong to any face F_i). Such maps M have the same number of connected components as $\mathcal{M}$ and are maps of pairs of permutations whose product is the resultant of $\mathcal{M}$. The disjoint union of the trees obtained by compression of the face F_i is a covering forest of M with no trivial trees (i.e. with only one vertex), which is denoted F_M.

Example 3 The map M from Fig. 10 can be obtained from the cumulant map from Fig. 9 by compressing each polygon into a tree in a certain way. The corresponding forest F_M can be seen in the figure by erasing the dotted edges.

Let $\mathcal{M}$ be a cumulant map of resultant σ. Consider the function

$$N_{\mathcal{M}} : \left\{(\tau, \overline{\tau}) \in S(k) \times S(k) \text{ s.t. } \tau\overline{\tau} = \sigma\right\} \to \mathbb{C}[[\mathbf{p}, \mathbf{q}]],$$

defined by:

- If the map $M^{\tau,\overline{\tau}}$ is obtained from $M_{\mathcal{M}}$ by compressing in a certain way (necessarily unique) the faces $F_1, \dots, F_t$, we put:

$$N_{\mathcal{M}}(\tau, \overline{\tau}) = N\left(F_{M^{\tau,\overline{\tau}}}\right).$$

- Else $N_{\mathcal{M}}(\tau, \overline{\tau}) = 0$.

This function satisfies:

$$\sum_{\substack{\tau,\overline{\tau} \in S(k) \\ \tau\overline{\tau} = \sigma_{\mathcal{M}}}} (-1)^{|C(\tau)| + t_{\mathcal{M}}} N_{\mathcal{M}}(\tau, \overline{\tau}) = \prod_{i=1}^{t_{\mathcal{M}}} R_{j_i+1}. \tag{18}$$

To see that, it is enough to use formula (17) on the right hand side and to expand it: the non-zero terms of the two sides of equality are exactly the same (with same signs because and M and F_M always have the same number of white vertices).

Thanks to this property, functions of this type are a good tool to put series associated to forests together in order to make the product of free cumulants appear.

Remark 5 Let $\mathcal{M}$ be a cumulant map of resultant σ. The sets

$$\left\{\tau \in S(k) \text{ such that } N_{\mathcal{M}}(\tau, \tau^{-1}\sigma) \neq 0\right\}$$
$$\text{and } \left\{\overline{\tau} \in S(k) \text{ such that } N_{\mathcal{M}}(\sigma\overline{\tau}^{-1}, \overline{\tau}) \neq 0\right\}$$

are intervals $I_{\mathcal{M}}$ and $\overline{I_{\mathcal{M}}}$ of the symmetric group. So they are isomorphic as posets to products of non-crossing partition sets (for the order described in paragraph 1.3). The power series $N_{\mathcal{M}}(\tau, \tau^{-1}\sigma)$ is simply the one associated to the image of τ by this isomorphism (this image is defined up to the action of the full cycle on non-crossing partitions, so the associated power series is well-defined) and equation (5) is a consequence of this fact.

4.2 Multiplicities

As for classical maps in paragraph 3.2, we define a decomposition operator for cumulant maps. Denote by $\mathbb{A}_{cm,\leq 1}$ the ring generated as $\mathbb{Z}$-module by the cumulant maps with at most one external half-edge per connected component. If $\mathcal{M}$ is a cumulant map, denote by $M'_{\mathcal{M}}$ the map obtained by replacing, for each i, the face F_i by a vertex (this map is not bicolored but each edge has a privileged orientation: the former white to black orientation).

Definition-Proposition 4.2.1 *There exists a unique linear operator*

$$\mathcal{D} : \mathbb{A}_{cm,\leq 1} \to \mathbb{A}_{cm,\leq 1}$$

such that:

- *If $M'_{\mathcal{M}}$ has only one vertex, then $\mathcal{D}(\mathcal{M}) = \mathcal{M}$;*
- *If $\mathcal{M}$ has more than one connected component ($\mathcal{M} = \prod_i \mathcal{M}^i$), then one has $\mathcal{D}(\mathcal{M}) = \prod \mathcal{D}(\mathcal{M}^i)$;*
- *If $\mathcal{M}$ has only one connected component and no external half-edge, let h be the half-edge of black extremity of its edge with the smallest label. We denote by $\overline{\mathcal{M}}$ the cumulant map obtained by adding one external half-edge between h and its successor (as some edges have no labels, the half-edge is never in one of the faces F_i). Then $\mathcal{D}(\mathcal{M}) = \mathcal{D}(\overline{\mathcal{M}})$.*
- *If $M'_{\mathcal{M}}$ has only one connected component with one half-edge but no admissible loops, denote by $e_1, \dots, e_l$ the edges leaving the same face F_{i_0} as the external half-edge. The map $M_{\mathcal{M}} \backslash F_{i_0}$ has l connected components $M_1, \dots, M_l$, each with an external half-edge (at the place where e_i leaves M_i). These maps have a cumulant map structure $M_i = M_{\mathcal{M}_i}$. Then $\mathcal{D}(\mathcal{M})$ is given by the formula:*

  $$\mathcal{D}(\mathcal{M}) = \phi_{\mathcal{M}}\big(\mathcal{D}(\mathcal{M}_1), \dots, \mathcal{D}(\mathcal{M}_l)\big),$$

  where $\phi_{\mathcal{M}}$ is the multilinear operator on algebraic sums of sub-cumulant maps of the $\mathcal{M}_i$'s defined as ϕ_M in Sect. 3.2.

- *Else, consider $D_1(M'_{\mathcal{M}})$ thanks to Remark 3. In each map of the result, replace the vertices by faces F_i and denote the resulting sum of the cumulant map by $CM(D_1(M'_{\mathcal{M}}))$. Then,*

$$\mathcal{D}(M) = \mathcal{D}\left[CM\big(D_1(M'_{\mathcal{M}})\big)\right].$$

Definition 4.2.2 The multiplicity $c(\mathcal{M})$ of a cumulant map $\mathcal{M}$ is the coefficient of the disjoint union of the faces F_i in the decomposition $\mathcal{D}(\mathcal{M})$ multiplied by $(-1)^{t_{\mathcal{M}}-1}$ (it can be zero!).

Proposition 3.3.1 also holds for cumulant maps and $\mathcal{D}$. So $c(\mathcal{M})$ is non-negative if $\mathcal{M}$ is connected.

If M is a map and F_M is a covering forest without trivial trees of M, denote by $\mathcal{M}_{M,F_M}$ the cumulant map obtained by replacing in M each tree of F_M by a polygon. The corresponding map M'_{M,F_M} is obtained from M by replacing all trees of F_M by a vertex. So the edges of $M\backslash F_M$ are in bijection with those of M'_{M,F_M}.

Lemma 4.2.1 *For any bicolored labeled map M, one has*

$$D(M) = \sum_{F_M \subset M} (-1)^{t_{F_M}-1} c(\mathcal{M}_{M,F_M}) F_M,$$

where the sum runs over covering forests of M with no trivial trees.

Proof Let $F_M \subset M$ be a covering forest with no trivial trees of a bicolored labeled map. The operator D applied to M consists of making transformations of type T_L with restricted choices until there are only forests left. Thanks to Remark 4, we choose loops containing a vertex of $T_\star$ (the tree of F_M containing the external half-edge) as long as possible. As we are interested in the coefficient of F_M, we can forget at each step all maps that do not contain F_M. Now we notice that doing an elementary transformation with respect to L and keeping only maps containing F_M is equivalent to applying formula (12) with $E(L) \cap (M\backslash F_M)$ instead of $E(L)$.

As edges of $M\backslash F_M$ are in bijection with edges of M'_{M,F_M}, this new set of erasable edges is a set of edges of M'_{M,F_M}. With our choice of order of loops, this set of edges of M'_{M,F_M} is always the set of erasable edges of an admissible transformation. So, computing $D(F_M)$ and keeping only the submap containing F_M is the same thing as computing $\mathcal{D}(\mathcal{M}_{M,F_M})$, except that we have trees instead of the polygonal faces. This shows that the coefficient of F_M in $D(M)$ is the same as the one of the union of the faces F_i in $\mathcal{D}(\mathcal{M}_{M,F_M})$. The lemma is now obvious from the definition of the multiplicity of cumulant maps. □

With the notation of the previous paragraph, the lemma implies:

$$N(D(M^{\tau,\overline{\tau}})) = \sum_{\substack{\mathcal{M}-\text{cumulant map}\\ \text{of resultant } \sigma}} (-1)^{t_{\mathcal{M}}-1} c(\mathcal{M}) N_{\mathcal{M}}(\tau,\overline{\tau}). \tag{19}$$

Remark 6 By Remark 5 and Lemma 4.2.1, for every $\sigma \in S(k)$, the family of intervals $I_{\mathcal{M}}$, where $\mathcal{M}$ describes the set of cumulant maps of resultant σ with multiplicities $(-1)^{t_{\mathcal{M}}-1}c(\mathcal{M})$, is a signed covering (the sum of multiplicities of intervals containing a given permutation is 1) of the symmetric group by intervals $[\pi, \pi']$ such that

- the quantity $|C(\tau)| + |C(\tau^{-1}\sigma)|$ is constant on these intervals;
- the intervals are centered: $|C(\pi^{-1}\sigma)| = |C(\pi')|$.

Note that the power series N does not appear in this result but is central to our construction. This interpretation of Kerov's polynomials' coefficients was conjecturally suggested by P. Biane in [3].

4.3 End of the proof of the main theorem

We use the D-invariance of N to write Σ'_μ as an algebraic sum of power series associated to minimal factorizations:

$$\begin{aligned}\Sigma'_\mu &= \sum_{\substack{\tau,\overline{\tau}\in S(k)\\ \tau\overline{\tau}=\sigma\\ \langle\tau,\overline{\tau}\rangle \text{ trans.}}} (-1)^{|C(\tau)|+1} N(D(M^{\tau,\overline{\tau}}))\\ &= \sum_{\substack{\tau,\overline{\tau}\in S(k)\\ \tau\overline{\tau}=\sigma\\ \langle\tau,\overline{\tau}\rangle \text{ trans.}}} (-1)^{|C(\tau)|+1} \left[\sum_{\substack{\mathcal{M}-\text{cumulant map}\\ \text{of resultant } \sigma}} (-1)^{t_{\mathcal{M}}-1} c(\mathcal{M}) N_{\mathcal{M}}(\tau,\overline{\tau})\right].\end{aligned}$$

The second equality is just equation (19). Now, we change the order of summation (note that transitive factorizations have connected maps, so they appear only as compressions of connected cumulant maps) and use (18):

$$\begin{aligned}\Sigma'_\mu &= \sum_{\substack{\mathcal{M}-\text{connected}\\ \text{cumulant map of}\\ \text{resultant } \sigma}} c(\mathcal{M}) \left[\sum_{\substack{\tau,\overline{\tau}\in S(k)\\ \tau\overline{\tau}=\sigma}} (-1)^{|C(\tau)|+t_{\mathcal{M}}} N_{\mathcal{M}}(\tau,\overline{\tau})\right]\\ &= \sum_{\substack{\mathcal{M}-\text{connected}\\ \text{cumulant map of}\\ \text{resultant } \sigma}} c(\mathcal{M}) \left[\prod_{i=1}^{t_{\mathcal{M}}} R_{j_i(\mathcal{M})+1}\right].\end{aligned} \tag{20}$$

This finishes the proof of Theorem 1.6.1 because:

- the multiplicity of a connected cumulant map is non negative;
- the monomials in the R_i's are linearly independent as power series in $\mathbf{p}$ and $\mathbf{q}$.

5 Computation of some particular coefficients

5.1 How to compute coefficients?

In the proof of the main theorem, we have observed that the coefficient of the monomial $\prod_{i=1}^{t} R_{j_i+1}$ in K'_μ is the sum of $c(\mathcal{M})$ over all connected cumulant maps $\mathcal{M}$ of resultant σ, with t polygons of respective sizes $2j_1, \ldots, 2j_t$.

But it is easier to look, instead of the connected cumulant map $\mathcal{M}$, at the map M_0 obtained from $M_{\mathcal{M}}$ by compressing each polygon in a tree with only one black vertex. Recall that, in this context, F_M is the disjoint union of these trees. Thanks to Lemma 4.2.1, the coefficient of F_M in $D(M)$ is, up to a sign, equal to $c(\mathcal{M})$. Note that each pair (M, F_M), where M is the map of a transitive decomposition of σ, and F_M is a covering forest whose trees have exactly one black vertex and at least a white one, can be obtained in this way from one cumulant map $\mathcal{M}$.

This remark leads to the following proposition, which will be used for explicit computations in the next paragraphs:

Proposition 5.1.1 *The coefficient of monomial $\prod_{i=1}^{t} R_{j_i+1}$ in K'_μ is the coefficient of the disjoint union of t trees, each with one black and respectively $j_1, \dots, j_t$ white vertices in*

$$(-1)^{t-1} \sum_{\substack{\tau,\overline{\tau}\in S(k)\\ \tau\overline{\tau}=\sigma,\langle\tau,\overline{\tau}\rangle \text{ trans.}\\ |C(\overline{\tau})|=t}} D(M^{\tau,\overline{\tau}}).$$

As remarked before, for coefficients of monomials of low degree, all the coefficients can be computed by counting some statistics on permutations in $S(k)$ (which can be much smaller than the symmetric group whose character values we are looking for).

5.2 Low degrees in R

5.2.1 Linear coefficients

A direct consequence of Proposition 5.1.1 is the (well-known) combinatorial interpretation of coefficients of linear monomials in the free cumulants: the coefficient of R_{l+1} in K'_μ (or equivalently in K_μ) is the number of permutations $\tau \in S(k)$ with l cycles whose complementary permutation $\overline{\tau} = \tau^{-1}\sigma$ is a full cycle, that is exactly the number of factorizations of σ, whose map has exactly one black vertex and l whites. Indeed, if M is a map with one black vertex, it is connected and has only loops of length 2. So transformations with respect to these loops only consist of erasing an edge, and $D(M)$ is a tree with one black vertex and as many white vertices as in M.

5.2.2 Quadratic coefficients

We have to compute $D(M)$, where M is a connected map with two black vertices. Denote by $w_0, \dots, w_u$ the white vertices of M linked to both black vertices. The first step is the computation of $D_1(\tilde{M})$, where $\tilde{M}$ is M with an external half-edge h (see Definition 3.2.1).

We begin by transformations with respect to all loops of length 2 going through the extremity $\star$ of h. So we suppose that every w_i is linked by only one edge e_i to $\star$, but there can be more than one edge between w_i and the other black vertex v, so we denote by $\mathbf{f_i}$ the family of these edges. Let h_i, h'_i be the two half-edges of e_i, where

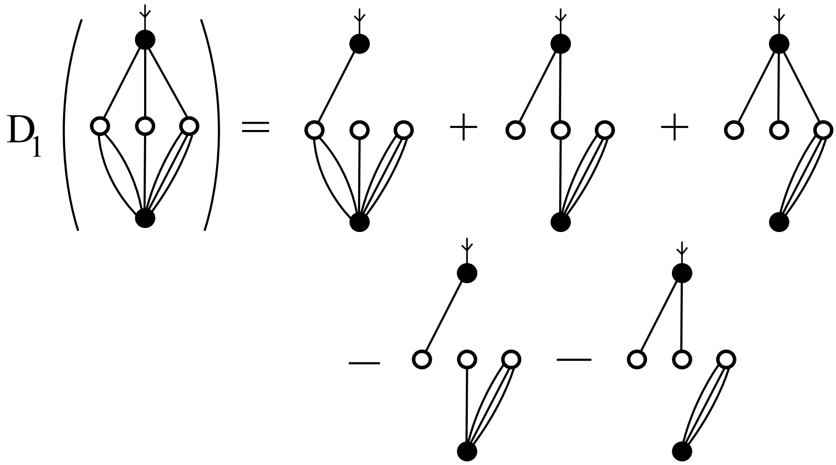


Fig. 11 Elementary decomposition of a map with two black vertices

the extremity of h_i is $\star$. With a good choice of numbering for the w_i, the cyclic order at $\star$ induces the order $h, h_0, \dots, h_u$.

Lemma 5.2.1 *With these notations, we have*:

$$D_1(\tilde{M}) = \sum_{i=0}^{u} \tilde{M}\backslash\{\mathbf{f_0}, \dots, \mathbf{f_{i-1}}, e_{i+1}, \dots, e_u\} - \sum_{i=1}^{u} \tilde{M}\backslash\{\mathbf{f_0}, \dots, \mathbf{f_{i-1}}, e_i, \dots, e_u\}. \tag{21}$$

An example for $u = 3$ is drawn in Fig. 11.

Proof If $u = 0$, there is no admissible loop and this result is $D_1(\tilde{M}) = \tilde{M}$. The case $u = 1$ is left to the reader (it is an easy induction on the number of edges in $\mathbf{f_0}$, the case where $\mathbf{f_0}$ has two elements is contained in the case $H = 2$ in the proof of Definition-Theorem 3.1.1). Then we proceed by induction on u by using the formula:

$$D_1(\tilde{M}) = D_1\big(D_1(\tilde{M}\backslash\{e_u\}) \cup \{e_u\}\big).$$

Suppose that the lemma holds for $u - 1$:

$$D_1(\tilde{M}\backslash\{e_u\}) \cup \{e_u\} = \sum_{i=0}^{u-1} \tilde{M}\backslash\{\mathbf{f_0}, \dots, \mathbf{f_{i-1}}, e_{i+1}, \dots, e_{u-1}\}$$
$$- \sum_{i=1}^{u-1} \tilde{M}\backslash\{\mathbf{f_0}, \dots, \mathbf{f_{i-1}}, e_i, \dots, e_{u-1}\}. \tag{22}$$

The graphs of the first line still have admissible loops. To compute their image under D_1, we have to compute the image of the submaps whose set of edges is

$\{e_i, \mathbf{f_i}, e_u, \mathbf{f_u}\}$, since all other edges do not belong to any admissible loops. This is an application of the case $u = 1$:

$$\begin{aligned} D_1(\tilde{M}\backslash\{\mathbf{f_0}, \dots, \mathbf{f_{i-1}}, e_{i+1}, \dots, e_{u-1}\}) = {} & \tilde{M}\backslash\{\mathbf{f_0}, \dots, \mathbf{f_{i-1}}, \mathbf{f_i}, e_{i+1}, \dots, e_{u-1}\} \\ & + \tilde{M}\backslash\{\mathbf{f_0}, \dots, \mathbf{f_{i-1}}, e_{i+1}, \dots, e_{u-1}, e_u\} \\ & - \tilde{M}\backslash\{\mathbf{f_0}, \dots, \mathbf{f_{i-1}}, \mathbf{f_i}, e_{i+1}, \dots, e_{u-1}, e_u\}. \end{aligned}$$

Using this formula for each i, the first summand balances with the negative term in (22) (except for $i = u - 1$) and the two other summands are exactly the ones in (21). So the lemma is proved by induction. □

Now, in all maps appearing in $D_1(\tilde{M})$, there are only loops of length 2, so the end of the decomposition algorithm consists of erasing some edges without changing the number of connected components.

As explained in Proposition 5.1.1, we have to look at the sizes of trees in the two-tree forests (these forests come from the second sum of the right member of (21)). If, in M, there are h^1_M white vertices linked to $\star$ (including the w_i) and h^2_M to v, we obtain pairs of trees with h^1 and h^2 vertices, where h^1 and h^2 attain all integer values satisfying the conditions:

$$\begin{cases} h^1 - 1 < h^1_M; \\ h^2 - 1 < h^2_M; \\ h^1 + h^2 = |V_w(M)|. \end{cases}$$

So any permutation with two black vertices contributes to coefficients of $R_{h^1} R_{h^2}$, where h^1 and h^2 satisfy the condition above. If $j \neq l$, a permutation may contribute twice to the coefficient of $R_j R_l$ if the conditions above are satisfied for $j = h^1, l = h^2$ and for $l = h^1, j = h^2$. Finally, one has:

$$[R_j R_l] K_k = \begin{Bmatrix} 1 & \text{if } j \neq l \\ 1/2 & \text{if } j = l \end{Bmatrix}.$$

$$\sum_{\substack{\tau, \overline{\tau} \in S(k) \\ \tau\overline{\tau} = \sigma, \langle \tau, \overline{\tau} \rangle \text{trans.} \\ |C(\overline{\tau})| = 2}} \delta_{j \le h^1_{M^{\tau,\overline{\tau}}}} \delta_{l \le h^2_{M^{\tau,\overline{\tau}}}} + \delta_{l \le h^1_{M^{\tau,\overline{\tau}}}} \delta_{j \le h^2_{M^{\tau,\overline{\tau}}}},$$

which is exactly the second part of Theorem 1.4.4 (the second δ in the equation above disappears if we consider permutations with numbered cycles).

5.3 High degrees in p, q

If the graded degree in **p** and **q** is high, the maps we are dealing with have few loops. Therefore, it is easier to compute their image under D and to count them.

Proof of Theorem 1.6.2 Let $r, s, t, j_1, \dots, j_t$ be integers such that $\sum j_i = r + s$. As in the whole paper $\sigma \in S(k)$ is a permutation of type μ (here r, s). We can suppose that 1 is in the support of the cycle c_1 of σ of size s.

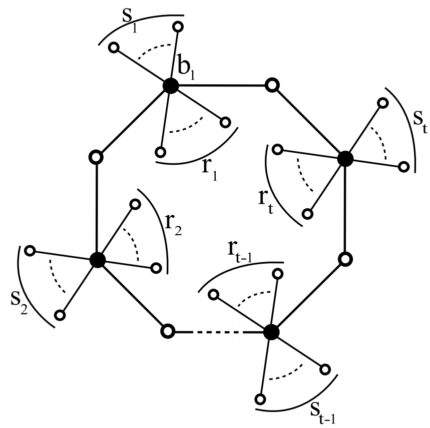


Fig. 12 Maps contributing to terms of graded degree $r+s$ in $K'_{r,s}$

We have to count connected maps with $r+s$ edges and $r+s$ vertices, that is to say, up to a change of orientation, one loop L. So, eventually by replacing L by L' (if 1 is in the word associated to the external face, L must be going counterclockwise), $D(M) = T_L(M)$. Only maps M such that, in $D(M)$, there is (at least) a forest with one black vertex per tree, contribute to coefficients of Kerov's polynomials. In such maps, all vertices of $M\backslash L$ are white and only the forest $M\backslash E(L)$ (see formula (12)) satisfies the condition above.

Let us consider such a map M. We can choose arbitrarily a first black vertex b_1 of M (M will be called marked) and number all its black vertices $b_1, \dots, b_t$ in the order of L. Suppose that there are w_i white vertices of $M\backslash L$ linked to b_i. Then M contributes only to the coefficient of $\prod R_{w_i+2}$ in K'_{f_1,f_2} (where $2f_1$ and $2f_2$ are the lengths of the two faces of M) with coefficient 1.

We count the number of marked labeled maps M contributing to the coefficient of $\prod_{i=1}^{t} R_{j_i}$ in $K'_{r,s}$. They are of the form from Fig. 12 with:

- the word $(r_1+s_1, r_2+s_2, \dots, r_t+s_t)$ equal up to a permutation to $(j_1-2, \dots, j_t-2)$, and
- the length $r_1+r_2+\dots+r_t$ of the face F_r that is on the left side of L, is equal to r.

Such a map can be labeled in $r \cdot s$ different ways such that its faces are the cycles of σ. Indeed, if we fix one element in the support of each cycle of σ, such a labeling is determined by the edges labeled by these elements. We have r (resp. s) choices for the first (resp. second) one: the r (resp. s) edges whose labels are in the word associated to the face F_r (resp. F_s). As we deal for the moment with maps with a marked black vertex, all the numberings give a different map.

If we choose a permutation $\mathbf{j}'-2$ of the word $(j_1-2, \dots, j_t-2)$, non-negative integers $r_1, s_1, \dots, r_t, s_t$ such that $\sum_i r_i = r-t$, $\sum_i s_i = s-t$ and $\mathbf{r}+\mathbf{s} = \mathbf{j}'-2$, and labels on the corresponding map, we obtain a marked map M contributing to the coefficient of $\prod_{i=1}^{t} R_{j_i}$ in $K'_{r,s}$. To obtain the number of such non-marked maps, we have to divide by t (thanks to the labels, there is no problem of symmetry).

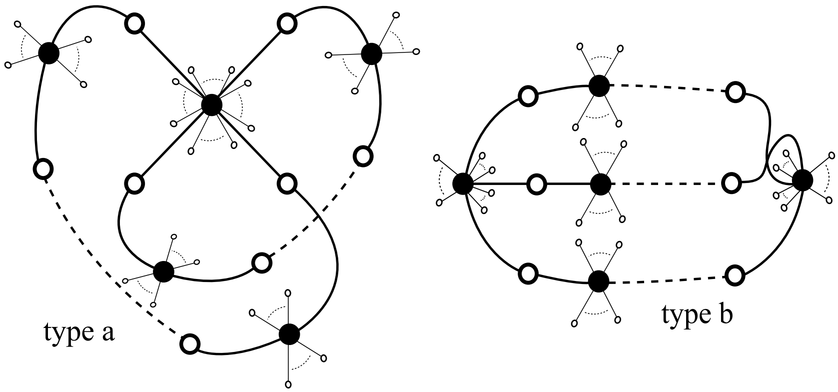


Fig. 13 Maps contributing to terms of degree $k-1$ in K_k

So the coefficient of $\prod_{i=1}^{t} R_{j_i}$ in $K'_{r,s}$ is

$$\frac{r \cdot s}{t} \text{ Perm}(\mathbf{j}) \left|\left\{(r_1, s_1, \dots, r_t, s_t)\right\}\right|,$$

where $r_1, s_1, \dots, r_t, s_t$ describe the set of non-negative integers satisfying the equations

$$\begin{cases} r_1 + s_1 = j_1 - 2; \\ \vdots \\ r_t + s_t = j_t - 2; \\ r_1 + \ldots + r_t = r - t. \end{cases}$$

But, in the system of equations satisfied by the r_i's and the s_i's, we can forget the s_i's and only keep an inequality on each r_i ($r_i \le j_i - 2$), which corresponds to the positivity of s_i. So the cardinality of the set in the formula above is exactly $N(j_1 - 2, \dots, j_t - 2; r - t)$. □

We use the same ideas for subdominant term in the case $l(\mu) = 1$.

Proof of Theorem 1.4.3 To compute the coefficients of a monomial of degree $k-1$ in K_k, we have to count the contributions of labeled maps with k edges, $k-1$ vertices and one face. As in the previous proof, if a map has a non-zero contribution, all vertices which do not belong to any loop are white. Such maps can be sorted in five classes: see Fig. 13 for types a and b, type c (resp. d) is type b with one black and one white (resp. two white) vertices at the extremities, and type e is type a with a white central vertex of valence 4 instead of a black one.

Thanks to the case $H = 2$ in the proof of Definition-Theorem 3.1.1, the decomposition of these maps is easy to compute:

Types a and e: the two loops have no edges in common and their associated transformations commute;
Types b, c and d: we obtain a result close to the one from Fig. 5.

Here is the description of the forests with t trees for each type (it is quite surprising that it does not depend on the labels).

Type a: in $D(M)$, there is one forest F with one black star per tree: in addition to those which do not belong to loops, there are two white vertices linked to the central black vertex and one to each other black vertex.

Type b: in $D(M)$, there are two forests F_1 and F_2 with one black star per tree: in F_1 (resp. in F_2), in addition to those which do not belong to loops, there are two white vertices linked to the vertex at the left (resp. right) extremity and one to each other black vertex (including the right (resp. left) extremity).

Type c: in $D(M)$, there is one forest F with one black vertex per tree: in addition to those which do not belong to loops, there is one white vertex linked to each black vertex.

Types d and e: in $D(M)$, there is no forest F with one black vertex per tree.

Now we compute the coefficient of $\prod_{i=1}^{t} R_{j_i}$ in K_k. We give all the details only for the contributions of maps of type a.

Let us count the number of maps of type a (contributing to this monomial) with a marked half-edge of extremity v_0, the central black vertex of the map (we will have to divide this number by 4 at the end to find the number of maps of type a). We number the black vertices of such a map M in the following order: those of the loop containing the marked half-edge, the central one and those of the other loop. Such maps are entirely determined by:

- A permutation $\mathbf{j}'$ of the word $(j_1, \dots, j_t)$ (j'_i is the number of vertices of the tree of F of black vertex b_i).
- The length of the first loop, *i.e.* the label $p \in [t]$ of the central black vertex.
- For each black vertex different from the central one, we have to link $j'_i - 2$ white vertices that do not belong to loops. We have to fix the number of these vertices which are on a given side of the loop: there are $j'_i - 1$ possibilities.
- For the central black vertex, we have $j'_p - 3$ white vertices to place on 4 sides, so $\binom{j'_p}{3}$ possibilities.
- The labels of such a map are determined by the choice of one edge which has label 1, so k possibilities.

Finally the contribution of type a maps to the coefficient of $\prod_{i=1}^{t} R_{j_i}$ in K_k is

$$C_a = \frac{k}{4} \sum_{\mathbf{j}'} \left[\sum_{p=1}^{t} \frac{j'_p(j'_p - 2)}{6} \prod_{i=1}^{t} (j'_i - 1) \right].$$

The expression in the bracket is symmetric in $\mathbf{j}'$, so equal to its value for $\mathbf{j}$:

$$C_a = \frac{k}{4} |\operatorname{Perm}(\mathbf{j})| \prod_{i=1}^{t} (j_i - 1) \sum_{p=1}^{t} \frac{j_p(j_p - 2)}{6}.$$

We can find similar arguments for types b and c:

- In type b, p_1 and p_2 are the labels of the black vertices at the extremities if we numbered by following the face beginning right after an extremity (6 possibilities to choose where to begin);
- In type c, p_1 is the label of the black extremity and p_2 of the black vertex preceding the white extremity if we begin just after the white extremity (3 possibilities to choose where to begin), note also that in this type we have to symmetrize our expression in $\mathbf{j}'$.

We obtain:

$$C_b = \frac{k}{6}|\operatorname{Perm}(\mathbf{j})|\prod_{i=1}^{t}(j_i-1)\sum_{1\le p_1<p_2\le t}\frac{j_{p_1}(j_{p_2}-2)}{4}+\frac{j_{p_2}(j_{p_1}-2)}{4};$$

$$C_c = \frac{k}{3}|\operatorname{Perm}(\mathbf{j})|\prod_{i=1}^{t}(j_i-1)\sum_{1\le p_1\le p_2\le t}\frac{1}{2}\left(\frac{j_{p_1}}{2}+\frac{j_{p_2}}{2}\right).$$

Finally, if we note that

$$A = \frac{k}{24}|\operatorname{Perm}(\mathbf{j})|\prod_{i=1}^{t}(j_i-1),$$

and split the summation in C_c into the cases $j_{p_1} < j_{p_2}$ and $j_{p_1} = j_{p_2}$, the coefficient we are looking for is:

$$\begin{aligned}
C_a + C_b + C_c &= A\Biggl(\sum_{p=1}^{t} j_p(j_p-2) + \sum_{1\le p_1\le t} 4j_{p_1} \\
&\quad + \sum_{1\le p_1<p_2\le t}\bigl(j_{p_1}(j_{p_2}-2)+j_{p_2}(j_{p_1}-2)+2j_{p_1}+2j_{p_2}\bigr)\Biggr); \\
&= A\Biggl(2\sum_{p=1}^{t} j_p + \sum_{p=1}^{t} j_p^2 + \sum_{1\le p_1<p_2\le t}\bigl(j_{p_1}j_{p_2}+j_{p_2}j_{p_1}\bigr)\Biggr); \\
&= A\Biggl[\Biggl(\sum_{p=1}^{t} j_p\Biggr)^2 + 2\sum_{p=1}^{t} j_p\Biggr]; \\
&= A\bigl((k-1)^2+2(k-1)\bigr) = A(k-1)(k+1),
\end{aligned}$$

which is exactly the expression claimed in Theorem 1.4.3. □

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