

Symmetric and quasi-symmetric functions associated to polymatroids

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Abstract To every subspace arrangement \mathbf{X} we will associate symmetric functions $\mathcal{P}[\mathbf{X}]$ and $\mathcal{H}[\mathbf{X}]$. These symmetric functions encode the Hilbert series and the minimal projective resolution of the product ideal associated to the subspace arrangement. They can be defined for discrete polymatroids as well. The invariant $\mathcal{H}[\mathbf{X}]$ specializes to the Tutte polynomial $\mathcal{T}[\mathbf{X}]$. Billera, Jia and Reiner recently introduced a quasi-symmetric function $\mathcal{F}[\mathbf{X}]$ (for matroids) which behaves valuably with respect to matroid base polytope decompositions. We will define a quasi-symmetric function $\mathcal{G}[\mathbf{X}]$ for polymatroids which has this property as well. Moreover, $\mathcal{G}[\mathbf{X}]$ specializes to $\mathcal{P}[\mathbf{X}]$, $\mathcal{H}[\mathbf{X}]$, $\mathcal{T}[\mathbf{X}]$ and $\mathcal{F}[\mathbf{X}]$.

Keywords Matroids · Polymatroids · Symmetric function · Quasi-symmetric function · Tutte polynomial · Subspace arrangement · Hyperplane arrangement

1 Introduction

1.1 Combinatorial invariants

Let X be a set with d elements. Suppose that $V_x, x \in X$ are subspaces of an n -dimensional vector space. Then $\mathcal{A} = \bigcup_{x \in X} V_x$ is called a *subspace arrangement*. Let $\text{Pow}(X)$ be the set of all subsets of X . The *rank function* $\text{rk} : \text{Pow}(X) \rightarrow \mathbb{N} := \{0, 1, 2, \dots\}$ is defined by

$$\text{rk}(A) = \dim V - \dim \bigcap_{i \in A} V_i$$

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for all subsets $A \subseteq X$.

Surprisingly, many topological invariants of the complement $V \setminus \mathcal{A}$ of subspace arrangements are *combinatorial*, i.e., they can be expressed in terms of $n := \dim V$ and the rank function. For example, Zaslavsky (see [40]) proved that number of regions in the complement of a real hyperplane arrangement is equal to

$$(-1)^n \chi(-1) = \sum_{A \subseteq X} (-1)^{\text{rk}(A)+|A|},$$

where $\chi(q)$ is the *characteristic polynomial* of the hyperplane arrangement defined by

$$\chi(q) = \sum_{A \subseteq X} q^{n-\text{rk}(A)} (-1)^{|A|}.$$

For *complex hyperplane arrangements*, the cohomology ring $H^*(V \setminus \mathcal{A})$ is isomorphic to the *Orlik-Solomon algebra* (see [29]), which is defined explicitly in terms of the rank function. For *arbitrary real subspace arrangements*, the topological Betti numbers of the complement $V \setminus \mathcal{A}$ are expressed in terms of the rank function using the *Goresky-MacPherson formula* (see [17]).

One may wonder whether various *algebraic* objects associated to a subspace arrangements are combinatorial invariants. Let K be a base field of characteristic 0, and denote the coordinate ring of V by $K[V]$. Terao defined the module of derivations $D(\mathcal{A})$ along a hyperplane arrangement \mathcal{A} (see [36]). An arrangement is called *free* if $D(\mathcal{A})$ is a free $K[V]$ -module. Terao has conjectured that “freeness” is a combinatorial property, i.e., whether $D(\mathcal{A})$ is free is determined by its rank function. Terao showed that free arrangements have the property that their characteristic polynomial factors into linear polynomials (see [36]). One should point out that for example the *Hilbert series* of the module $D(\mathcal{A})$ is *not* a combinatorial invariant.

In a recent paper, the author found an *algebraic* object which is a combinatorial invariant for subspace arrangements. Let $J_x \subseteq K[V]$ be the vanishing ideal of $V_x \subseteq V$ and let $J = \prod_{x \in X} J_x$ be the product ideal. The author showed in [12] that the Hilbert series $H(J, t)$ of J is a combinatorial invariant. For *hyperplane* arrangements the Hilbert series of J is always equal to $t^d/(1-t)^n$ and is therefore not an interesting invariant. Let W be an arbitrary vector space and denote its dual by W^* . We can tensor all the spaces with W^* . So let $J_x(W) \subseteq K[V \otimes W^*]$ be the vanishing ideal of the subspace $V_x \otimes W^*$ of $V \otimes W^*$ and $J(W) = \prod_{x \in X} J_x(W)$. Then the Hilbert series $H(J(W), t)$ is an interesting invariant, even for hyperplane arrangements. Moreover, since we have an action of $\text{GL}(W)$ on all the rings and ideals involved, we can define a $\text{GL}(W)$ -equivariant Hilbert series which is a more refined invariant for subspace arrangement.

1.2 Symmetric functions

The ring of symmetric functions is spanned by the Schur symmetric functions s_λ where λ runs over all partitions. Let $\mathbf{X} = (X, \text{rk})$ where rk is the rank function coming from a subspace arrangement $\bigcup_{x \in X} V_x \subseteq V$. In Section 2.3, we will define a

symmetric function $\mathcal{P}[\mathbf{X}]$ using a recursive formula (see Definition 2.3). We define another symmetric function $\mathcal{H}[\mathbf{X}] = \mathcal{H}[\mathbf{X}](q, t)$ with coefficients in $\mathbb{Z}[q, t]$ by

$$\mathcal{H}[\mathbf{X}](q, t) = \sum_{A \subseteq X} \mathcal{P}[\mathbf{X}|_A] q^{\text{rk}(A)} t^{|A|}. \tag{1}$$

Here $\mathbf{X}|_A = (A, \text{rk}|_A)$ can be viewed as the rank function of the sub-arrangement $\bigcup_{x \in A} V_x \subseteq V$. The definitions of $\mathcal{P}[\mathbf{X}]$ and $\mathcal{H}[\mathbf{X}](q, t)$ make sense even if the rank function rk does not come from a subspace arrangement. Therefore, these symmetric functions can also be defined for *polymatroids*. The symmetric function $\mathcal{H}[\mathbf{X}](q, t)$ essentially encodes Hilbert series of J and the $\text{GL}(W)$ -equivariant Hilbert series of $J(W)$. Also, the *minimal free resolutions* of J and $J(W)$ can be expressed in terms of $\mathcal{H}[\mathbf{X}](q, t)$. The symmetric functions behave nicely with respect to direct sums of polymatroids, namely

$$\mathcal{P}[\mathbf{X} \oplus \mathbf{Y}] = \mathcal{P}[\mathbf{X}] \cdot \mathcal{P}[\mathbf{Y}] \tag{2}$$

$$\mathcal{H}[\mathbf{X} \oplus \mathbf{Y}](q, t) = \mathcal{H}[\mathbf{X}](q, t) \cdot \mathcal{H}[\mathbf{Y}](q, t) \tag{3}$$

(see Proposition 2.6). The *Tutte polynomial* is defined by

$$\mathcal{T}[\mathbf{X}](x, y) = \sum_{A \subseteq X} (x - 1)^{\text{rk}(X) - \text{rk}(A)} (y - 1)^{|A| - \text{rk}(A)}. \tag{4}$$

The Tutte polynomial was introduced in [37] and generalized to matroids in [4] and [8]. It has the multiplicative property and it behaves well under matroid duality (see (5)). It specializes to the characteristic polynomial, namely

$$\chi(q) = q^{n - \text{rk}(X)} \mathcal{T}[\mathbf{X}](1 - q, 0).$$

The coefficients of $\mathcal{T}[\mathbf{X}](x, y)$ as a polynomial in x and y have combinatorial interpretations and are nonnegative. The invariant $\mathcal{H}[\mathbf{X}](q, t)$ specializes to the Tutte polynomial. The functions $\mathcal{P}[\mathbf{X}]$ and $\mathcal{H}[\mathbf{X}](q, t)$ do *not* seem to behave nicely under matroid duality. If the polymatroid \mathbf{X} is realizable as a subspace arrangement in characteristic 0, then the coefficients of $\mathcal{P}[\mathbf{X}]$, $\mathcal{H}[\mathbf{X}](q, t)$ and some of their specializations have homological interpretations. Therefore, the coefficients of these functions satisfy certain non-negativity conditions.

Brylawski defined a graph invariant in [5] which he called the *polychromate*. Sarmiento [31] proved that the polychromate is equivalent to the *U-polynomial* studied by Noble and Welch [28]. The polychromate and the U-polynomial specialize to Stanley’s *chromatic symmetric polynomial* [35]. There are graphs whose graphical matroids are the same, that can be distinguished by the Stanley symmetric function. This means that the Stanley symmetric function, the polychromatic, and the U-polynomial *cannot* be viewed as invariants of matroids.

Inspired by these graph invariants, Billera, Jia and Reiner defined a *quasi-symmetric* function which is an invariant for matroids (see [3]). This invariant will be discussed later.

1.3 Polarized Schur functions

Let us denote the Schur functor corresponding to the partition λ by S_λ . Suppose our base field K has characteristic 0, Z is a finite dimensional K -vector space, and $Z_1, \dots, Z_d \subseteq Z$ are subspaces. For a partition λ with $|\lambda| = d$ we will define a subspace

$$S_\lambda(Z_1, Z_2, \dots, Z_d) \subseteq S_\lambda(Z)$$

as the subspace spanned by the all $\pi(z_1 \otimes \dots \otimes z_d)$ where $z_i \in Z_i$ for all i and

$$\pi : \underbrace{Z \otimes Z \otimes \dots \otimes Z}_d \rightarrow S_\lambda(Z)$$

is a $GL(Z)$ -equivariant linear map.

The space $S_\lambda(Z_1, \dots, Z_d)$ has various interesting properties which will be discussed in Section 6. For example

$$S_\lambda(\underbrace{Z, Z, \dots, Z}_d) = S_\lambda(Z).$$

Also, permuting the spaces Z_1, \dots, Z_d does not change the subspace $S_\lambda(Z_1, \dots, Z_d)$. Let $V = Z^*$ be the dual space, and define $V_i = Z_i^\perp$ to be the subspace of V orthogonal to Z_i . Consider the subspace arrangement $\mathcal{A} = V_1 \cup \dots \cup V_d \subseteq V$. Then the dimension of $S_\lambda(Z_1, \dots, Z_d)$ can be expressed in terms of $\mathcal{H}[\mathcal{A}](q, t)$. This implies, that the dimension of $S_\lambda(Z_1, \dots, Z_d)$ is determined by the numbers

$$\dim \sum_{i \in A} Z_i, \quad A \subseteq \{1, 2, \dots, d\}.$$

1.4 Quasi-symmetric functions

Billera, Jia and Reiner defined a quasi-symmetric function $\mathcal{F}[\mathbf{X}]$ for any matroid \mathbf{X} in [3]. This invariant behaves nicely with respect to direct sums of matroids, matroid duality. There is also a very natural definition of this invariant in terms of the combinatorial Hopf algebras studied in [1] (see Section 7.4). In [3] it was proved that this quasi-symmetric function behaves valuatively with respect to matroid polytope decompositions, so it can be a useful tool for studying such decompositions. The quasi-symmetric $\mathcal{F}[\mathbf{X}]$ does not specialize to $\mathcal{H}[\mathbf{X}](q, t)$ because $\mathcal{F}[\mathbf{X}]$ cannot distinguish between a loop or an isthmus, and $\mathcal{H}[\mathbf{X}](q, t)$ can. We will show that $\mathcal{F}[\mathbf{X}]$ does specialize to $\mathcal{P}[\mathbf{X}]$. To prove this, we introduce another quasi-symmetric function $\mathcal{G}[\mathbf{X}]$ which should be of interest on its own right. First of all, we will choose a convenient basis $\{U_r\}$ of the ring of quasi-symmetric functions where r runs over all finite sequences of nonnegative integers. A complete chain is a sequence

$$\underline{X} : \emptyset = X_0 \subset X_1 \subset \dots \subset X_d = X$$

such that X_i has i elements for all i . The rank vector of this chain \underline{X} is defined by

$$r(\underline{X}) = (\text{rk}(X_1) - \text{rk}(X_0), \dots, \text{rk}(X_d) - \text{rk}(X_{d-1})).$$

Now we define

$$\mathcal{G}[\mathbf{X}] = \sum_{\underline{X}} U_{r(\underline{X})}$$

where \underline{X} runs over all $d!$ maximal chains in X . We will show that $\mathcal{G}[\mathbf{X}]$ behaves nicely with respect to direct sums and matroid duality. It defines a Hopf algebra homomorphism from the Hopf algebra of polymatroids to the Hopf algebra of quasi-symmetric functions. But unlike $\mathcal{F}[\mathbf{X}]$, it can distinguish between a loop and an isthmus. Moreover, $\mathcal{G}[\mathbf{X}]$ specializes to the Billera-Jia-Reiner quasi-symmetric function $\mathcal{F}[\mathbf{X}]$ as well as to $\mathcal{H}[\mathbf{X}](q, t)$. We will also show that $\mathcal{G}[\mathbf{X}]$ has the valuative property with respect to polymatroid polytope decompositions in Section 8. We question whether $\mathcal{G}[\mathbf{X}]$ might be universal with this property.

2 Symmetric functions associated to polymatroids

In this section we will define the invariants $\mathcal{H}[\mathbf{X}](q, t)$ and $\mathcal{P}[\mathbf{X}]$.

2.1 Discrete polymatroids

Definition 2.1 A (discrete) *polymatroid* is a pair $\mathbf{X} := (X, \text{rk})$ where X is a finite set, and $\text{rk} : \text{Pow}(X) \rightarrow \mathbb{N} = \{0, 1, 2, \dots\}$ is a function satisfying

1. $\text{rk}(\emptyset) = 0$;
2. $\text{rk}(A) \leq \text{rk}(B)$ if $A \subseteq B$ (nondecreasing);
3. $\text{rk}(A \cup B) + \text{rk}(A \cap B) \leq \text{rk}(A) + \text{rk}(B)$ (submodular).

If $\mathbf{X} = (X, \text{rk})$ is a polymatroid, and $A \subseteq X$ is a subset, then we restrict \mathbf{X} to A to get a polymatroid $\mathbf{X}|_A := (A, \text{rk}|_A)$. If $A^c = X \setminus A$ is the complement, then the *deletion* of A in \mathbf{X} is the polymatroid $\mathbf{X} \setminus A := \mathbf{X}|_{A^c} = (A^c, \text{rk}|_{A^c})$. The polymatroid $\mathbf{X}/A := (A^c, \text{rk}_{X/A})$ is defined by

$$\text{rk}_{X/A}(B) = \text{rk}(A \cup B) - \text{rk}(A)$$

for all $B \subseteq A^c$. We call \mathbf{X}/A the *contraction* of A in \mathbf{X} .

Two polymatroids $\mathbf{X} = (X, \text{rk}_X)$ and $\mathbf{Y} = (Y, \text{rk}_Y)$ are *isomorphic* if there exists a bijection $\varphi : X \rightarrow Y$ such that $\text{rk}_Y \circ \varphi = \text{rk}_X$. A polymatroid $\mathbf{X} = (X, \text{rk}_X)$ is a *matroid* if $\text{rk}_X(\{x\}) \in \{0, 1\}$ for all $x \in X$. For more on matroids, see [30, 38]. If $\mathbf{X} = (X, \text{rk}_X)$ is a matroid, then its dual is $\mathbf{X}^\vee := (X, \text{rk}_X^\vee)$ where rk_X^\vee is defined by

$$\text{rk}_X^\vee(A) := |A| - \text{rk}_X(X) + \text{rk}_X(X \setminus A)$$

for all $A \subseteq X$. The Tutte polynomial behaves nicely with respect to matroid duality:

$$\mathcal{T}[\mathbf{X}^\vee](x, y) = \mathcal{T}[\mathbf{X}](y, x). \tag{5}$$

There is also a formula expressing $\mathcal{F}[\mathbf{X}^\vee]$ in terms of $\mathcal{F}[\mathbf{X}]$ (see [3]).

Definition 2.2 If $\mathbf{X} = (X, \text{rk}_X)$ and $\mathbf{Y} = (Y, \text{rk}_Y)$ are polymatroids, then we define their direct sum by

$$\mathbf{X} \oplus \mathbf{Y} := (X \sqcup Y, \text{rk}_{X \sqcup Y})$$

where $X \sqcup Y$ is the disjoint union of X and Y and $\text{rk}_{X \sqcup Y} : X \sqcup Y \rightarrow \mathbb{N}$ is defined by

$$\text{rk}_{X \sqcup Y}(A \cup B) := \text{rk}_X(A) + \text{rk}_Y(B)$$

for all $A \subseteq X, B \subseteq Y$.

The Tutte polynomial satisfies the multiplicative property

$$\mathcal{T}[\mathbf{X} \oplus \mathbf{Y}] = \mathcal{T}[\mathbf{X}] \cdot \mathcal{T}[\mathbf{Y}]. \tag{6}$$

2.2 The ring of symmetric functions

Let

$$\text{Sym} := \mathbb{Z}[e_1, e_2, e_3, \dots] \subset \mathbb{Z}[x_1, x_2, x_3, \dots]$$

be the ring of symmetric functions in infinitely many variables, where

$$e_k := \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k}$$

is the k -th elementary symmetric function. The monomials in e_1, e_2, \dots form a \mathbb{Z} -basis of Sym . A partition of n is a tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of positive integers with $\lambda_1 \geq \dots \geq \lambda_r \geq 1$ and $|\lambda| := \lambda_1 + \dots + \lambda_r$ equal to n . Another basis of Sym is given by the Schur symmetric functions s_λ where λ runs over all partitions. For standard results on symmetric functions, we refer to the book [22]. The natural grading of $\mathbb{Z}[x_1, x_2, x_3, \dots]$ induces a grading on Sym . In this grading e_k has degree k and s_λ has degree $|\lambda|$. Let

$$\overline{\text{Sym}} = \mathbb{Z}[[e_1, e_2, e_3, \dots]]$$

be the set of power series in e_1, e_2, \dots . Define

$$\sigma = 1 + s_1 + s_2 + s_3 + \dots \in \overline{\text{Sym}}.$$

The inverse is given by

$$\sigma^{-1} = 1 - e_1 + e_2 - e_3 + \dots = 1 - s_1 + s_{11} - s_{111} + \dots. \tag{7}$$

2.3 The definitions of $\mathcal{P}[\mathbf{X}]$ and $\mathcal{H}[\mathbf{X}](q, t)$

Definition 2.3 For every polymatroid $\mathbf{X} = (X, \text{rk})$ we define a symmetric polynomial $\mathcal{P}[\mathbf{X}] \in \text{Sym}$ by induction as follows. If $X = \emptyset$, then $\mathcal{P}[\mathbf{X}] = 1$. If $X \neq \emptyset$, then we may assume that $\mathcal{P}[\mathbf{X}|_A]$ has been defined for all *proper* subsets $A \subset X$. We define

$$\mathcal{P}[\mathbf{X}] = u_0 + u_1 + \dots + u_{|X|-1} \tag{8}$$

where $u_i \in \text{Sym}$ is homogeneous of degree i for all i such that

$$\sum_{i=0}^{\infty} u_i = - \sum_{A \subset X} \mathcal{P}[\mathbf{X} | A] \sigma^{\text{rk}(X) - \text{rk}(A)} (-1)^{|X| - |A|}. \tag{9}$$

Here A runs over all proper subsets of X .

Definition 2.4 For every polymatroid $\mathbf{X} = (X, \text{rk})$ we define a symmetric polynomial

$$\mathcal{H}[\mathbf{X}](q, t) \in \text{Sym}[q, t] = \mathbb{Z}[q, t] \otimes_{\mathbb{Z}} \text{Sym}$$

by

$$\mathcal{H}[\mathbf{X}](q, t) = \sum_{A \subset X} \mathcal{P}[\mathbf{X} | A] q^{\text{rk}(A)} t^{|A|}. \tag{10}$$

The coefficient of $t^{|X|}$ in $\mathcal{H}[\mathbf{X}](q, t)$ is $q^{\text{rk}(X)} \mathcal{P}[\mathbf{X}]$.

Remark 2.5 If we evaluate (10) at $q = \sigma^{-1}$ and $t = -1$, then we obtain

$$\mathcal{H}[\mathbf{X}](\sigma^{-1}, -1) = \sum_{A \subset X} \mathcal{P}[\mathbf{X} | A] \sigma^{-\text{rk}(A)} (-1)^{|A|} \in \overline{\text{Sym}}.$$

From (8) and (9) it follows that $\mathcal{H}[\mathbf{X}](\sigma^{-1}, -1)$ vanishes in degree $< d = |X|$.

Proposition 2.6 (multiplicative property) *For polymatroids $\mathbf{X} = (X, \text{rk}_X)$ and $\mathbf{Y} = (Y, \text{rk}_Y)$ we have*

$$\mathcal{P}[\mathbf{X} \oplus \mathbf{Y}] = \mathcal{P}[\mathbf{X}] \cdot \mathcal{P}[\mathbf{Y}] \tag{11}$$

and

$$\mathcal{H}[\mathbf{X} \oplus \mathbf{Y}](q, t) = \mathcal{H}[\mathbf{X}](q, t) \cdot \mathcal{H}[\mathbf{Y}](q, t). \tag{12}$$

Proof We prove the proposition by induction on $|X| + |Y|$. The case where $X = Y = \emptyset$ is clear. So let us assume that $|X| + |Y| > 0$. We may assume that

$$\mathcal{P}[\mathbf{X} | A \oplus \mathbf{Y} | B] = \mathcal{P}[\mathbf{X} | A] \cdot \mathcal{P}[\mathbf{Y} | B]$$

for all subsets $A \subseteq X$ and $B \subseteq Y$ such that $A \neq X$ or $B \neq Y$.

$$\begin{aligned} \mathcal{H}[\mathbf{X} \oplus \mathbf{Y}](q, t) &= \sum_{C \subseteq X \sqcup Y} \mathcal{P}[(\mathbf{X} \oplus \mathbf{Y}) | C] q^{\text{rk}_{X \sqcup Y}(C)} t^{|C|} \\ &= \sum_{A \subseteq X} \sum_{B \subseteq Y} \mathcal{P}[\mathbf{X} | A \oplus \mathbf{Y} | B] q^{\text{rk}_X(A) + \text{rk}_Y(B)} t^{|A| + |B|} \\ &= \sum_{A \subseteq X} \mathcal{P}[\mathbf{X} | A] q^{\text{rk}_X(A)} t^{|A|} \cdot \sum_{B \subseteq Y} \mathcal{P}[\mathbf{Y} | B] q^{\text{rk}_Y(B)} t^{|B|} \end{aligned}$$

$$\begin{aligned}
& + (\mathcal{P}[\mathbf{X} \oplus \mathbf{Y}] - \mathcal{P}[\mathbf{X}]\mathcal{P}[\mathbf{Y}])q^{\text{rk}_X(X)+\text{rk}_Y(Y)}t^{|X|+|Y|} \\
& = \mathcal{H}[\mathbf{X}](q, t) \cdot \mathcal{H}[\mathbf{Y}](q, t) \\
& + (\mathcal{P}[\mathbf{X} \oplus \mathbf{Y}] - \mathcal{P}[\mathbf{X}]\mathcal{P}[\mathbf{Y}])q^{\text{rk}_X(X)+\text{rk}_Y(Y)}t^{|X|+|Y|}. \quad (13)
\end{aligned}$$

If we substitute $q = \sigma^{-1}$ and $t = -1$ we get

$$\begin{aligned}
& \mathcal{H}[\mathbf{X} \oplus \mathbf{Y}](\sigma^{-1}, -1) - \mathcal{H}[\mathbf{X}](\sigma^{-1}, -1) \cdot \mathcal{H}[\mathbf{Y}](\sigma^{-1}, -1) \\
& = (-1)^{|X|+|Y|} (\mathcal{P}[\mathbf{X} \oplus \mathbf{Y}] - \mathcal{P}[\mathbf{X}] \cdot \mathcal{P}[\mathbf{Y}])\sigma^{-\text{rk}_X(X)-\text{rk}_Y(Y)}.
\end{aligned}$$

The left-hand side has no terms in degree $< |X| + |Y|$ by Remark 2.5 and

$$\mathcal{P}[\mathbf{X} \oplus \mathbf{Y}] - \mathcal{P}[\mathbf{X}] - \mathcal{P}[\mathbf{Y}]$$

is a symmetric polynomial of degree $< |X| + |Y|$. It follows that

$$\mathcal{P}[\mathbf{X} \oplus \mathbf{Y}] = \mathcal{P}[\mathbf{X}] \cdot \mathcal{P}[\mathbf{Y}].$$

From (13) it follows that

$$\mathcal{H}[\mathbf{X} \oplus \mathbf{Y}](q, t) = \mathcal{H}[\mathbf{X}](q, t) \cdot \mathcal{H}[\mathbf{Y}](q, t). \quad \square$$

The Tutte polynomial is closely related to the *rank generating function*

$$\mathcal{R}[\mathbf{X}](q, t) = \sum_{A \subseteq X} q^{\text{rk}(A)} t^{|A|}.$$

We have

$$(x-1)^{\text{rk}(X)} \mathcal{R}[\mathbf{X}]((y-1)^{-1}(x-1)^{-1}, (y-1)) = \mathcal{T}[\mathbf{X}](x, y),$$

so the Tutte polynomial is completely determined by the rank generating function and vice versa. The rank generating function makes sense for polymatroids, not just matroids. The Tutte invariant may not be a polynomial for polymatroids, because we could have $\text{rk}(A) > |A|$ for some subset $A \subseteq X$. Define

$$\Theta : \text{Sym} \rightarrow \mathbb{Q}$$

by

$$\Theta(s_\lambda) = \begin{cases} 1 & \text{if } \lambda = (); \\ 0 & \text{otherwise.} \end{cases}$$

Using base extension, we also get a $\mathbb{Q}(q, t)$ -linear map

$$\text{Sym} \otimes_{\mathbb{Q}} \mathbb{Q}(q, t) \rightarrow \mathbb{Q}(q, t)$$

which we also will denote by Θ . It is straightforward to prove by induction on $|X|$ that $\Theta(\mathcal{P}[\mathbf{X}]) = 1$.

Corollary 2.7 *We have*

$$\Theta(\mathcal{H}[\mathbf{X}](q, t)) = \sum_{A \subseteq X} q^{\text{rk}(A)} t^{|A|} = \mathcal{R}[\mathbf{X}](q, t).$$

So $\mathcal{H}[\mathbf{X}](q, t)$ specializes to the rank generating function and the Tutte polynomial.

3 Examples

Example 3.1 Let $\mathbf{0} = (\{v\}, \text{rk}_0)$ be the loop matroid, and $\mathbf{1} = (\{v\}, \text{rk}_1)$ be the co-loop matroid defined by

$$\text{rk}_0(v) = 0 \text{ and } \text{rk}_1(v) = 1.$$

Then we have $P[\mathbf{0}] = P[\mathbf{1}] = 1$, $\mathcal{H}[\mathbf{0}] = 1 + t$, $\mathcal{H}[\mathbf{1}] = 1 + qt$, $\mathcal{G}[\mathbf{0}] = U_{(0)}$ and $\mathcal{G}[\mathbf{1}] = U_{(1)}$.

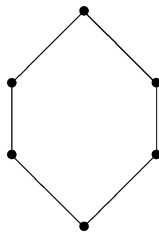
An important class of matroids is the class of graphical matroids. Suppose that $\Gamma = (Y, X, \phi)$ where Y is the set of vertices, X is the set of edges, and $\phi : X \rightarrow \text{Pow}(Y)$ is a map such that $\phi(e)$ is the set of endpoints of the edge e . So $\phi(e)$ has 1 or 2 elements for all $e \in X$. Let $V = K^n$, and denote the coordinate functions by x_1, \dots, x_n . To each vertex $e \in X$, with $\phi(e) = \{i, j\}$ we can associate a subspace $V_e \subseteq V$ defined by $x_i = x_j$. So V_e is a hyperplane unless e is a loop (i.e., $i = j$), in which case $V_e = V$. For $A \subseteq X$, we define $V_A = \bigcap_{a \in A} V_a$. We define a rank function by

$$\text{rk}(A) = \dim V - \dim V_A, \quad A \subseteq X.$$

Now $\mathbf{X} = (X, \text{rk})$ is a matroid.

Example 3.2 Suppose (Y, X, ϕ) is an m -gon.

$m = 6 :$



Then we have

$$\begin{aligned} \mathcal{T}[\mathbf{X}](x, y) &= y + x + x^2 + \dots + x^{m-1}, \\ \mathcal{P}[\mathbf{X}] &= 1 - s_1 + s_{11} - \dots + (-1)^{m-1} s_{1^{m-1}}, \\ \mathcal{H}[\mathbf{X}](q, t) &= (1 + qt)^m - (qt)^m + q^{m-1} t^m \mathcal{P}[\mathbf{X}], \end{aligned}$$

$$\mathcal{G}[\mathbf{X}] = m!U_{(1,1,\dots,1,0)}.$$

Example 3.3 Suppose that (Y, X, ϕ) is the graph with 2 vertices and m edges between them.

$$m = 5 : \begin{array}{c} \circ \\ \text{---} \\ \circ \end{array}$$

Then we have

$$\mathcal{T}[\mathbf{X}](x, y) = x + y + y^2 + \cdots + y^{m-1},$$

$$\mathcal{P}[\mathbf{X}] = 1 - \binom{m-1}{1}s_1 + \binom{m-1}{2}s_2 - \cdots + (-1)^{m-1} \binom{m-1}{m-1}s_{m-1}, \quad (14)$$

$$\mathcal{H}[\mathbf{X}](q, t) = 1 + q \sum_{i=1}^m \binom{m}{i} t^i \left(\sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} s_j \right). \quad (15)$$

Here, we use the convention $s_0 = 1$. To prove the formulas (14) and (15) it suffices to show that the right-hand side of (15) vanishes in degree $< m$ if we substitute $q = \sigma^{-1}$ and $t = -1$. If we make these substitutions, we get (using the combinatorial identity [23, §1.2.6, (33)])

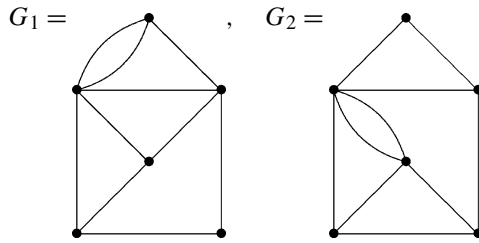
$$\begin{aligned} & 1 + \sigma^{-1} \sum_{i=1}^m \binom{m}{i} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} s_j \right) \\ &= 1 + \sigma^{-1} \sum_{j=0}^{m-1} s_j \sum_{i=j+1}^m (-1)^{i+j} \binom{m}{i} \binom{i-1}{j} \\ &= 1 + \sigma^{-1} \sum_{j=0}^{m-1} s_j \left((-1)^{j+1} \binom{-1}{j} + \sum_{i=0}^m (-1)^{i+j} \binom{m}{i} \binom{i-1}{j} \right) \\ &= 1 - \sigma^{-1} \sum_{j=0}^{m-1} s_j. \end{aligned} \quad (16)$$

This vanishes in degree $< m$ because $\sigma = 1 + s_1 + s_2 + \cdots$. We also have

$$\mathcal{G}[\mathbf{X}] = m!U_{(1,0,0,\dots,0)}.$$

The following example appeared in [5], and was pointed out to the author by Nathan Reading.

Example 3.4 The Gray graphs



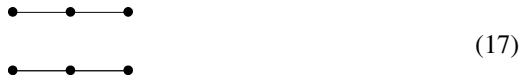
have the same Tutte polynomial, namely

$$\begin{aligned} T[G_1](x, y) = T[G_2](x, y) = & y^5 + 4y^4 + xy^4 + x^2y^3 + 6xy^3 + 7y^3 \\ & + x^3y^2 + 6y^2 + 6x^2y^2 + 13xy^2 + 10xy + x^4y + 13x^2y + 6x^3y \\ & + 2y + 2x + 7x^3 + x^5 + 4x^4 + 6x^2. \end{aligned}$$

However, the coefficients of $s_{2,2,2}$ in $\mathcal{P}[G_1]$ and $\mathcal{P}[G_2]$ are 56 and 55 respectively.

The examples below appeared in the survey of Brylawski and Oxley in [39, pp. 197], and were also featured in [3].

Example 3.5 Consider 6 points in $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$ according to the diagram below



Here 3 or more points are collinear if and only if they lie on a line segment in the diagram. Dualizing gives us 6 projective lines in \mathbb{P}^2 which can be viewed as 6 hyperplanes in \mathbb{C}^3 .

Denote the matroid associated with this arrangement by \mathbf{X} . Consider 6 points in \mathbb{P}^2 according to the diagram below



Again, dualizing gives a hyperplane arrangement in \mathbb{C}^3 . Denote the matroid associated with this arrangement by \mathbf{Y} .

Then \mathbf{X} and \mathbf{Y} give nonisomorphic matroids, but they have the same Tutte polynomial and the same Billera-Jia-Reiner quasi-symmetric function (see [3]). Moreover,

$$\begin{aligned} \mathcal{P}[\mathbf{X}] = \mathcal{P}[\mathbf{Y}] = & 1 - 3s_1 + 3s_2 + 6s_{1,1} - s_3 - 8s_{2,1} - 8s_{1,1,1} \\ & + 3s_{3,1} + 6s_{2,2} + 11s_{2,1,1} - 3s_{3,2} - 4s_{3,1,1} - 3s_{2,2,1}, \end{aligned}$$

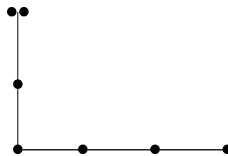
$$\mathcal{H}[\mathbf{X}](q, t) = \mathcal{H}[\mathbf{Y}](q, t),$$

and

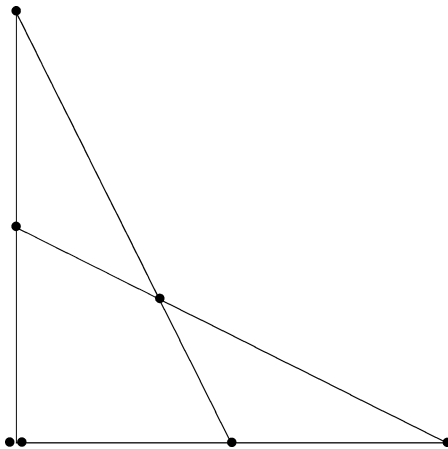
$$\mathcal{G}[\mathbf{X}] = \mathcal{G}[\mathbf{Y}] = 72U_{(1,1,0,1,0,0)} + 648U_{(1,1,1,0,0,0)}.$$

The last equation can easily be computed by hand as follows. There are $6!$ ways of labeling the points in diagram (17) by $p_1, p_2, p_3, p_4, p_5, p_6$. If p_1, p_2, p_3 are not colinear, then the labeling gives the rank sequence $(1, 1, 0, 1, 0, 0)$, because p_1 spans a subspace of dimension 1 in \mathbb{C}^3 , p_1 and p_2 span a subspace of dimension $1 + 1$, p_1, p_2, p_3 span a subspace of dimension $1 + 1 + 0$, p_1, p_2, p_3, p_4 span a subspace of dimension $1 + 1 + 0 + 1$, etc. There are $2 \cdot 3!^2 = 72$ ways of choosing a labeling such that p_1, p_2, p_3 are colinear. All other $720 - 72 = 648$ labelings, give the rank sequence $(1, 1, 1, 0, 0, 0)$. A similar reasoning can be used to compute $\mathcal{G}[\mathbf{Y}]$.

Example 3.6 Let \mathbf{X} be the matroid corresponding to the hyperplane arrangement dual to the point arrangement of the following diagram



Let \mathbf{Y} be the matroid corresponding to the hyperplane arrangement dual to the point arrangement of the following diagram



The Tutte polynomial is the same for \mathbf{X} and \mathbf{Y} . The Billera-Jia-Reiner quasi-symmetric function *does* distinguish the arrangements. We have

$$\begin{aligned} \mathcal{P}[\mathbf{X}] = & 1 - 4s_1 + 6s_2 + 9s_{1,1} - 4s_3 - 17s_{2,1} - 10s_{1,1,1} \\ & + s_4 + 12s_{3,1} + 13s_{2,2} + 17s_{2,1,1} - 3s_{4,1} - 10s_{3,2} - 10s_{3,1,1} - 8s_{2,2,1} \\ & + 2s_{4,2} + 2s_{4,1,1} + 2s_{3,3} + 3s_{3,2,1} + s_{2,2,2} \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}[\mathbf{Y}] = & 1 - 4s_1 + 6s_2 + 9s_{1,1} - 4s_3 - 17s_{2,1} - 10s_{1,1,1} \\ & + s_4 + 12s_{3,1} + 14s_{2,2} + 17s_{2,1,1} - 3s_{4,1} - 12s_{3,2} - 10s_{3,1,1} - 10s_{2,2,1} \\ & + 3s_{4,2} + 2s_{4,1,1} + 2s_{3,3} + 4s_{3,2,1} + s_{2,2,2}. \end{aligned}$$

We also have

$$\begin{aligned} \mathcal{G}[\mathbf{X}] = & 3456U_{(1,1,1,0,0,0,0)} + 1080U_{(1,1,0,1,0,0,0)} + 264U_{(1,1,0,0,1,0,0)} \\ & + 216U_{(1,0,1,1,0,0,0)} + 24U_{(1,0,1,0,1,0,0)} \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}[\mathbf{Y}] = & 3456U_{(1,1,1,0,0,0,0)} + 1104U_{(1,1,0,1,0,0,0)} + 240U_{(1,1,0,0,1,0,0)} \\ & + 192U_{(1,0,1,1,0,0,0)} + 48U_{(1,0,1,0,1,0,0)}. \end{aligned}$$

So the invariants \mathcal{H} , \mathcal{P} and \mathcal{G} distinguish these two matroids as well.

4 Ideals and regularity

4.1 Equivariant free resolutions

Let K be a field, and V be an n -dimensional K -vector space. For any partition λ , S_λ denotes its corresponding Schur functor. In particular, $S_d V$ is the d -th symmetric power of V , and $S_{1^d} V = S_{1,\dots,1} V$ is the d -th exterior power. Let $R = K[V]$ be the ring of polynomial functions on V . The space R_d of polynomial functions of degree d can be identified with $S_d(Z)$, where $Z = V^*$ is the dual space of V . Also, the ring $R = \bigoplus_{d=0}^\infty R_d$ can be identified with the symmetric algebra $S(Z) := \bigoplus_{d=0}^\infty S_d(Z)$ on $Z = V^*$. By choosing a basis in V and a dual basis $\{x_1, \dots, x_n\}$ in V^* we may identify R with the polynomial ring $K[x_1, \dots, x_n]$. Let $\mathfrak{m} = \bigoplus_{d=1}^\infty R_d = (x_1, \dots, x_n)$ be the maximal homogeneous ideal of R .

Suppose that M is a finitely generated graded R -module. Its minimal resolution can be constructed as follows. First define $D_0 := M$ and $E_0 = D_0/\mathfrak{m}D_0$. Then E_0 is a finite dimensional, graded vector space. The homogeneous quotient map $\psi_0 : D_0 \rightarrow E_0$ has a homogeneous linear section $\phi_0 : E_0 \rightarrow D_0$ (which does not need to be an R -module homomorphism) such that $\psi_0 \circ \phi_0 = \text{id}$. We can extend ϕ_0 to a R -module homomorphism $\phi_0 : R \otimes_K E_0 \rightarrow D_0$ in a unique way. The tensor product $R \otimes_K E_0$ has a natural grading as a tensor product of two graded vector spaces, and ϕ_0 is homogeneous with respect to this grading. We inductively define D_i, E_i, ψ_i, ϕ_i as follows. Define D_i as the kernel of $\phi_{i-1} : R \otimes E_{i-1} \rightarrow D_{i-1}$. We set $E_i = D_i/\mathfrak{m}D_i$. Let $\phi_i : E_i \rightarrow D_i$ be a homogeneous linear section to the homogeneous quotient map $\psi_i : D_i \rightarrow E_i$. We can extend ϕ_i to an R -module homomorphism $\phi_i : R \otimes E_i \rightarrow D_i$. By Hilbert’s Syzygy theorem (see [21] and [13, Corollary 19.7]), we get that $D_i = 0$ for $i > n$. We end up with the minimal free resolution

$$0 \rightarrow R \otimes E_n \rightarrow R \otimes E_{n-1} \rightarrow \dots \rightarrow R \otimes E_0 \rightarrow M \rightarrow 0.$$

Here E_i can be naturally identified with $\text{Tor}_j(M, K)$.

For a group G and sets X and Y on which G acts, we say that a map $\phi : X \rightarrow Y$ is G -equivariant if it respects the action, i.e., $\phi(g \cdot x) = g \cdot \phi(x)$ for all $x \in X$ and $g \in G$. Suppose that G is a linearly reductive linear algebraic group and V is a representation of G . Assume that G also acts on the finitely generated graded R -module $M = \bigoplus_d M_d$ such the multiplication $R \times M \rightarrow M$ is G -equivariant, and M_d is a representation of G for every d . By the definition of linear reductivity, we can choose the sections $\phi_i : E_i \rightarrow K_i$ to be G -equivariant. So by induction we see that G acts regularly on $D_0, E_0, D_1, E_1, D_2, E_2, \dots$. Also, by induction one can show that the structure of D_i as a G -equivariant graded R -module, and E_i as graded representation of G do not depend on the choices of the G -equivariant sections ϕ_i . We conclude that $E_i \cong \text{Tor}_i(M, K)$ has a well-defined structure as a graded G -module.

4.2 Castelnuovo-Mumford regularity

For a finite dimensional graded K -vector space $W = \bigoplus_{d \in \mathbb{Z}} W_d$ we define

$$\text{deg}(W) := \max\{i \mid W_i \neq 0\}.$$

If $W = \{0\}$ then we define $\text{deg}(W) = -\infty$. A finitely generated graded R -module M is called s -regular if $\text{deg}(\text{Tor}^i(M, K)) \leq s + i$ for all i . The *Castelnuovo-Mumford regularity* $\text{reg}(M)$ of M is the smallest integer s such that M is s -regular. See [13, §20.5] for more on Castelnuovo-Mumford regularity.

4.3 Product ideals and regularity bounds

Suppose that $V_x, x \in X$ are subspaces of V for some finite set X with d elements. Assume that $X = \{1, 2, \dots, d\}$. Let $J_x \subseteq K[V] = S(Z)$ be the vanishing ideal of V_x . The ideal J_x is generated by the subspace $Z_x = V_x^\perp \subseteq Z = V^*$ of all linear functions vanishing on V_x . For every subset $A \subseteq X$, we define $J_A := \prod_{x \in X} J_x$, and let $J = J_X$. A crucial result we need is:

Theorem 4.1 (Conca and Herzog [7]) *The Castelnuovo-Mumford regularity of J is equal to d .*

We define

$$C_k = \bigoplus_{|A|=k} J_A. \tag{19}$$

Following [33, Chapter IV] we construct a complex

$$0 \rightarrow C_d \rightarrow C_{d-1} \rightarrow \dots \rightarrow C_0 \rightarrow 0. \tag{20}$$

The map $\partial_k : C_k \rightarrow C_{k-1}$ can be written as $\partial_k = \sum_{A,B} \partial_k^{A,B}$, where

$$\partial_k^{A,B} : J_A \rightarrow J_B.$$

Suppose that $A = \{i_1, i_2, \dots, i_k\}$ with $i_1 < i_2 < \dots < i_k$, then we define

$$\partial_k^{A,B} := \begin{cases} 0 & \text{if } B \not\subseteq A; \\ (-1)^r \text{id} & \text{if } B = \{i_1, \dots, i_{r-1}, i_{r+1}, \dots, i_k\}. \end{cases}$$

The homology of the complex is denoted by

$$H_k = \ker \partial_k / \text{im } \partial_{k+1}.$$

Remark 4.2 Since ∂_d is injective, we have that $H_d = 0$.

Proposition 4.3 [33] *If $V_X := \bigcap_{x \in X} V_x = (0)$, then the homogeneous maximal ideal \mathfrak{m} kills all homology, i.e., $\mathfrak{m}H_i = 0$ for all i .*

The following result is Corollary 20.19 in [13].

Lemma 4.4 *If A, B, C are finitely generated graded modules, and*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is exact, then

1. $\text{reg}(A) \leq \max\{\text{reg}(B), \text{reg}(C) + 1\}$;
2. $\text{reg}(B) \leq \max\{\text{reg}(A), \text{reg}(C)\}$;
3. $\text{reg}(C) \leq \max\{\text{reg}(A) - 1, \text{reg}(B)\}$.

Proposition 4.5 *Suppose that $V_X = \bigcap_{x \in X} V_x = (0)$. Then H_k is concentrated in degree k (and in particular, it is finite dimensional).*

Proof We have $\text{reg}(C_i) \leq i$ by Theorem 4.1. Let Z_i and B_i be the kernel, respectively, the cokernel of ∂_i .

First, we prove that

$$\text{reg}(H_i) \leq \text{reg}(B_i) - 1 \tag{21}$$

for $i = 0, 1, \dots, d - 1$. Since $\mathfrak{m}H_i = 0$, H_i is just equal to a number of copies of K in various degrees. From the Koszul resolution it follows that

$$\text{deg}(\text{Tor}_j(H_i, K)) = \text{deg}(H_i) + j$$

for $j = 0, 1, 2, \dots, n$, hence $\text{reg}(H_i) = \text{deg}(H_i)$. The exact sequence

$$0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i \rightarrow 0 \tag{22}$$

gives rise to a long exact Tor sequence

$$0 \rightarrow \text{Tor}_n(B_i, K) \rightarrow \text{Tor}_n(Z_i, K) \rightarrow \text{Tor}_n(H_i, K) \rightarrow \text{Tor}_{n-1}(B_i, K) \rightarrow \dots$$

Since Z_i is a submodule of a free module, its projective dimension is $\leq n - 1$ and $\text{Tor}_n(Z_i, K) = 0$. Therefore

$$\text{deg}(\text{Tor}_{n-1}(B_i, K)) \geq \text{deg}(\text{Tor}_n(H_i, K)) = \text{reg}(H_i) + n.$$

It follows that

$$\text{reg}(B_i) + n - 1 \geq \text{deg}(\text{Tor}_{n-1}(B_i, K)) \geq \text{reg}(H_i) + n.$$

This proves (21).

From (22) and Lemma 4.4 it follows that

$$\text{reg}(Z_i) \leq \max\{\text{reg}(B_i), \text{reg}(H_i)\} = \text{reg}(B_i). \tag{23}$$

By induction on i we will show that $\text{reg}(B_{d-i}) \leq d - i + 1$, $\text{reg}(Z_{d-i}) \leq d - i + 1$ and $\text{reg}(H_{d-i}) \leq d - i$. For $i = 1$ we have $\text{reg}(B_{d-1}) = \text{reg}(C_d) = d$, $\text{reg}(Z_{d-1}) \leq d$ by (23) and $\text{reg}(H_{d-1}) \leq d - 1$ by (21).

Suppose that $i > 1$. We may assume by induction that Z_{d-i+1} is $(d - i + 2)$ -regular. From the exact sequence

$$0 \rightarrow Z_{d-i+1} \rightarrow C_{d-i+1} \rightarrow B_{d-i} \rightarrow 0$$

it follows that

$$\text{reg}(B_{d-i}) \leq \max\{\text{reg}(Z_{d-i+1}) - 1, \text{reg}(C_{d-i+1})\} \leq d - i + 1$$

by Lemma 4.4. Now we have $\text{reg}(Z_{d-i}) \leq d - i + 1$ by (23) and $\text{reg}(H_{d-i}) \leq d - i$ by (21). □

Suppose that G is a linearly reductive group and let \widehat{G} denote the set of isomorphism classes of irreducible representations of G . Let $\mathbb{Z}^{\widehat{G}}$ be the set of maps $\widehat{G} \rightarrow \mathbb{Z}$. Elements of $\mathbb{Z}^{\widehat{G}}$ may be thought of as G -Hilbert series. If M is a G -module such that every irreducible representation appears only finitely many times, then we define

$$\langle M \rangle = \langle M \rangle_G \in \mathbb{Z}^{\widehat{G}}.$$

For every irreducible representation U of G , $\langle M \rangle(U)$ is the multiplicity of U in M .

Lemma 4.6 *Suppose that G acts on Z such that every irreducible representation of G appears only finitely many times in $S(Z)$. Then we have*

$$\sum_{A \subset X} (-1)^{|A|} \langle J_A \rangle = \sum_{i=0}^d (-1)^i \langle C_i \rangle = \sum_{i=0}^{d-1} (-1)^i \langle H_i \rangle. \tag{24}$$

Proof The first equality follows from the definition (19). For every i we have exact sequences

$$0 \rightarrow Z_i \rightarrow C_i \rightarrow \text{im } B_{i-1} \rightarrow 0$$

and

$$0 \rightarrow B_i \rightarrow Z_i \rightarrow H_i \rightarrow 0.$$

So we have

$$\begin{aligned} \sum_i (-1)^i \langle C_i \rangle &= \sum_i (-1)^i \langle Z_i \rangle + \sum_i (-1)^i \langle B_{i-1} \rangle \\ &= \sum_i (-1)^i \langle Z_i \rangle - \sum_i (-1)^i \langle B_i \rangle = \sum_i (-1)^i \langle H_i \rangle. \end{aligned} \tag{25} \quad \square$$

5 Realizable polymatroids

5.1 The tensor trick

Let us fix a field K .

Definition 5.1 An *arrangement realization* of a polymatroid $\mathbf{X} = (X, \text{rk})$ over K is a finite dimensional K -vector space V together with a collection of subspaces V_x , $x \in X$ such that

$$\text{rk}(A) = \dim V - \dim V_A$$

for every $A \subseteq X$, where

$$V_A = \bigcap_{x \in X} V_x.$$

Let $\mathbf{X} = (X, \text{rk})$ be a polymatroid and set $d = |X|$. From now on, assume that K is a field of characteristic 0. Suppose that V is an n -dimensional K -vector space and V_x , $x \in X$ is a collection of subspaces that form a realization of \mathbf{X} .

Let W be another K -vector space and let $R(W) := K[V \otimes W^*]$ be the ring of polynomial functions on $V \otimes W^* = \text{Hom}(W, V)$. Note that $\text{GL}(W)$ acts regularly on $K[V \otimes W^*]$. Let $J_x(W) \subseteq R(W)$ be the vanishing ideal of $V_x \otimes W^* \subseteq V \otimes W^*$. For a subset $A \subseteq X$ we define

$$J_A(W) = \prod_{x \in A} J_x(W)$$

and we set $J(W) := J_X(W)$. Define

$$C_i(W) := \bigoplus_{\substack{A \subseteq X \\ |A|=i}} J_A(W).$$

As in (20), we have a complex

$$0 \rightarrow C_d(W) \rightarrow C_{d-1}(W) \rightarrow \dots \rightarrow C_1(W) \rightarrow C_0(W) \rightarrow 0. \tag{26}$$

Let $H_i(W)$ be the i -th homology group. By Lemma 4.6, we have

$$\sum_{i=0}^{d-1} (-1)^i \langle H_i(W) \rangle = \sum_{i=0}^d (-1)^i \langle C_i(W) \rangle = \sum_{A \subseteq X} (-1)^{|A|} \langle J_A(W) \rangle. \tag{27}$$

If $f = \sum_{\lambda} a_{\lambda} s_{\lambda} \in \mathbb{Z}[[e_1, e_2, \dots]]$, then we define

$$f \star W = \sum a_{\lambda} \langle S_{\lambda}(W) \rangle.$$

For example, we have

$$\sigma \star W = (s_0 + s_1 + s_2 + s_3 + \dots) \star W = \sum_{i=0}^{\infty} \langle S_i(W) \rangle = \langle S(W) \rangle.$$

If $f, g \in \mathbb{Z}[[e_1, e_2, \dots]]$, then

$$(f \cdot g) \star W = (f \star W) \otimes (g \star W).$$

5.2 Product ideals and the invariants $\mathcal{P}[\mathbf{X}], \mathcal{H}[\mathbf{X}](q, t)$

Theorem 5.2 *We have*

$$(\sigma^{n-\text{rk}(X)} \mathcal{P}[\mathbf{X}]) \star W = \sum_{A \subseteq X} (-1)^{|A|} \langle J_A(W) \rangle \tag{28}$$

and

$$(\sigma^n \mathcal{H}[X](\sigma^{-1}, -1)) \star W = \langle J(W) \rangle. \tag{29}$$

Proof We prove the statement by induction on $d = |X|$. If $X = \emptyset$, then $\mathcal{P}[\mathbf{X}] = 1$ and

$$\sigma^n \star W = \langle S(W)^{\otimes n} \rangle = \langle S(W \otimes V^*) \rangle = \langle K[V \otimes W^*] \rangle = \langle R(W) \rangle = \langle J_{\emptyset}(W) \rangle,$$

so (28) holds.

For every $A \subseteq X$, define

$$Z_A := \sum_{B \subseteq A} (-1)^{|B|} \langle J_B(W) \rangle.$$

By Möbius inversion, we get

$$\langle J_B(W) \rangle = \sum_{A \subseteq B} (-1)^{|A|} Z_A.$$

By induction we may assume that

$$(\sigma^{n-\text{rk}(A)} \mathcal{P}[\mathbf{X} | _A]) \star W = Z_A$$

for all proper subsets $A \subset X$.

Let us assume that $V_X = (0)$. From (27) and Proposition 4.5 it follows that Z_X is a combination of $\langle S_{\lambda}(W) \rangle$ with $|\lambda| < d$. Consider

$$(\sigma^n \mathcal{H}[X](\sigma^{-1}, -1)) \star W - \langle J(W) \rangle$$

$$\begin{aligned}
 &= \sum_{A \subseteq X} (-1)^{|A|} (\sigma^{n-\text{rk}(A)} \mathcal{P}[\mathbf{X} \mid_A] \star W - \langle J(W) \rangle) \\
 &= (-1)^{|X|} (\sigma^{n-\text{rk}(X)} \mathcal{P}[\mathbf{X}] \star W - Z_X) + \sum_{A \subseteq X} (-1)^{|A|} Z_A - \langle J(W) \rangle \\
 &= (-1)^{|X|} (\sigma^{n-\text{rk}(X)} \mathcal{P}[\mathbf{X}] \star W - Z_X). \tag{30}
 \end{aligned}$$

In $(\sigma^n \mathcal{H}[X](\sigma^{-1}, -1)) \star W$ and $\langle J(W) \rangle$ only terms $\langle S_\lambda(W) \rangle$ appear with $|\lambda| \geq d$. On the other hand, in $\sigma^{n-\text{rk}(X)} \mathcal{P}[\mathbf{X}] \star W$ and Z_X only terms $\langle S_\lambda(W) \rangle$ appear with $|\lambda| < d$. It follows that the left-hand side and the right-hand side of (30) are equal to 0.

Suppose that $V_X \neq (0)$. Let V' be a complement of V_X in V of dimension $n - r(X)$. Define $V'_x = V' \cap V_x$ for all $x \in X$ and $V'_A = V' \cap V_A = \bigcap_{x \in A} V'_x$ for all $A \subseteq X$. We have that $V'_X = V' \cap V_X = (0)$ and $V'_A = V'_A \oplus V_X$ for all $A \subseteq X$. It follows that

$$\begin{aligned}
 \text{rk}(A) &= \dim V - \dim V_A = (\dim V' + \dim V_X) - (\dim V'_A + \dim V_X) \\
 &= \dim V' - \dim V'_A.
 \end{aligned}$$

Let $J'_X(W) \subseteq K[V' \otimes W^*]$ be the vanishing ideal of $V'_x \otimes W^*$ inside $V' \otimes W^*$. Define $J'_A(W) = \prod_{x \in A} J'_x(W)$ and set $J'(W) = J'_X(W)$. By the previous case,

$$(\sigma^{\text{rk}(X)} \mathcal{H}[X](\sigma^{-1}, -1)) \star W = \langle J'(W) \rangle$$

and

$$\mathcal{P}[\mathbf{X}] \star W = \sum_{A \subseteq X} (-1)^{|A|} \langle J'_A(W) \rangle.$$

It follows that

$$J(W) = J'(W) \otimes S(V_X^* \otimes W) = J'(W) \otimes S(W)^{\otimes(n-\text{rk}(X))}$$

and

$$\begin{aligned}
 (\sigma^n \mathcal{H}[X](\sigma^{-1}, -1)) \star W &= (\sigma^{\text{rk}(X)} \mathcal{H}[X](\sigma^{-1}, -1)) \star W \otimes \langle S(W)^{\otimes(n-\text{rk}(X))} \rangle \\
 &= \langle J'(W) \otimes S(W)^{\otimes(n-\text{rk}(X))} \rangle = \langle J(W) \rangle. \tag{31}
 \end{aligned}$$

Similarly, from

$$\mathcal{P}[\mathbf{X}] \star W = \sum_{A \subseteq X} (-1)^{|A|} \langle J'_A(W) \rangle$$

it follows that

$$\begin{aligned}
 (\sigma^{n-\text{rk}(X)} \mathcal{P}[\mathbf{X}]) \star W &= \sum_{A \subseteq X} (-1)^{|A|} \langle J'_A(W) \otimes S(W)^{\otimes(n-\text{rk}(X))} \rangle \\
 &= \sum_{A \subseteq X} (-1)^{|A|} \langle J_A(W) \rangle.
 \end{aligned}$$

□

Corollary 5.3 *Suppose that $V_X = (0)$. If we write*

$$\mathcal{P}[\mathbf{X}] = u_0 - u_1 + u_2 - \cdots + (-1)^{d-1}u_{d-1}$$

where u_i is a homogeneous symmetric polynomial of degree i for all i , then

$$u_i \star W = \langle H_i(W) \rangle.$$

Proposition 5.4 *We can write*

$$\mathcal{H}[\mathbf{X}](\sigma^{-1}, -1) = w_d - w_{d+1} + w_{d+2} - w_{d+3} + \cdots ,$$

where $d = |X|$ and w_i is a homogeneous symmetric polynomial of degree i . We have

$$w_{d+i} \star W = \langle \text{Tor}_i(J(W), K) \rangle.$$

Proof Since $J(W)$ is d -regular and generated in degree d , it has a linear minimal free resolution. We can choose this resolution to be $\text{GL}(W)$ -equivariant. Define

$$E_i(W) := \text{Tor}_i(J(W), K).$$

The minimal resolution has the form

$$0 \rightarrow E_\ell(W) \otimes R(W) \rightarrow \cdots \rightarrow E_1(W) \otimes R(W) \rightarrow E_0(W) \otimes R(W) \rightarrow J(W) \rightarrow 0,$$

where $\ell = \text{pd}(J(W))$ is the projective dimension of $J(W)$. We have

$$(\sigma^n \mathcal{H}[\mathbf{X}](\sigma^{-1}, -1)) \star W = \langle J(W) \rangle = \sum_{i=0}^{\ell} (-1)^i \langle E_i(W) \otimes R(W) \rangle,$$

so

$$\mathcal{H}[\mathbf{X}](\sigma^{-1}, -1) \star W = \left(\sum_{i=0}^{\infty} (-1)^i w_{d+i} \right) \star W = \sum_{i=0}^{\ell} (-1)^i \langle E_i(W) \rangle. \quad \square$$

Example 5.5 Let $V = \mathbb{C}$ and let $V_1 = V_2 = \cdots = V_d = \{0\}$. The rank function is the same as in Example 3.3.

$$\mathcal{H}[\mathbf{X}](q, t) = 1 + q \sum_{i=1}^d \binom{d}{i} t^i \left(\sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} s_j \right).$$

The ideal $J(W) = \mathfrak{m}(W)^d$ where $\mathfrak{m}(W)$ is the maximal homogeneous ideal in $K[V \otimes W^*] \cong K[W^*] \cong S(W)$.

For $d = 1$ we have

$$\mathcal{H}[\mathbf{X}](q, t) = 1 + qt.$$

It follows that

$$\mathcal{H}[\mathbf{X}](\sigma^{-1}, -1) = 1 - \sigma^{-1} = s_1 - s_{1,1} + s_{1,1,1} - \cdots .$$

This shows that the i -th free module in the free resolution is $S(W) \otimes S_{1,1,\dots,1} W \cong S(W) \otimes \bigwedge^i(W)$. So the minimal resolution is

$$\dots \rightarrow S(W) \otimes S_{1,1}(W) \rightarrow S(W) \otimes W \rightarrow \mathfrak{m}(W) \rightarrow 0,$$

which is of course the Koszul resolution of the maximal ideal $\mathfrak{m}(W)$. For $d = 2$, we get

$$\mathcal{H}[\mathbf{X}](q, t) = 1 + 2qt + qt^2(1 - s_1)$$

and

$$\mathcal{H}[\mathbf{X}](\sigma^{-1}, -1) = 1 - \sigma^{-1}(1 + s_1) = s_2 - s_{2,1} + s_{2,1,1} - \dots$$

So this means the equivariant minimal free resolution of $\mathfrak{m}(W)^2$ looks like

$$\dots \rightarrow S(W) \otimes S_{2,1,1}(W) \rightarrow S(W) \otimes S_{2,1}(W) \rightarrow S(W) \otimes S_2(W) \rightarrow \mathfrak{m}(W)^2 \rightarrow 0.$$

5.3 Nonnegativity results for the coefficients of $\mathcal{P}[\mathbf{X}]$ and $\mathcal{H}[\mathbf{X}](q, t)$

Corollary 5.6 *Suppose that $\mathbf{X} = (X, \text{rk})$ is realizable over a field K of characteristic 0.*

1.

$$\sigma^{\text{rk}(X)} \mathcal{H}[\mathbf{X}](\sigma^{-1}, -1) = \sum_{\lambda} a_{\lambda} s_{\lambda}, \tag{32}$$

where λ runs over all partitions with $|\lambda| \geq d$ and $a_{\lambda} \geq 0$ for all λ ;

2.

$$\mathcal{P}[\mathbf{X}] = \sum_{\lambda} (-1)^{|\lambda|} b_{\lambda} s_{\lambda},$$

where λ runs over all partitions with $|\lambda| < d$ and $b_{\lambda} \geq 0$ for all λ ;

3.

$$\mathcal{H}[\mathbf{X}](\sigma^{-1}, -1) = \sum_{\lambda} (-1)^{|\lambda|} c_{\lambda} s_{\lambda},$$

where λ runs over all partitions λ with $|\lambda| \geq d$ with more than $|\lambda|/\text{rk}(X)$ parts, and $c_{\lambda} \geq 0$ for all λ .

Proof Assume, as before, that V together with $V_x, x \in X$ form a realization of \mathbf{X} . We may also assume that $V_X = (0)$.

(1) From Remark 2.5 it follows that no s_{λ} with $|\lambda| < d$ appears in the left-hand side of (32). If we choose $\dim W \geq |\lambda|$ then $S_{\lambda}(W) \neq 0$ and $\langle S_{\lambda}(W) \rangle$ appears with a nonnegative coefficient on the right-hand side of (31). Therefore, the coefficient of s_{λ} in $\sigma^{\text{rk}(X)} \mathcal{H}[\mathbf{X}](\sigma^{-1}, -1)$ is nonnegative.

(2) This follows from Corollary 5.3.

(3) The nonnegativity of c_{λ} follows from Proposition 5.4. If $\ell = \text{pd}(J(W))$ is the projective dimension of $J(W)$, then we have

$$\ell = \text{pd}(J(W)) = \text{pd}(R(W)/J(W)) - 1 < \dim V \dim W = \text{rk}(X) \dim W.$$

Suppose $\lambda = (\lambda_1, \dots, \lambda_k)$ and the coefficient of s_λ in $\mathcal{H}[\mathbf{X}](\sigma^{-1}, -1)$ is nonzero. If W is k -dimensional, then $S_\lambda(W) \neq 0$, so $E^{|\lambda|}(W) \neq 0$ and $|\lambda| \leq \ell < \text{rk}(X)k$. \square

Conjecture 5.7 *Corollary 5.6 is true, even if $\mathbf{X} = (X, \text{rk})$ is a polymatroid that is not realizable.*

5.4 The Rees ring and the invariant $\tilde{H}[\mathbf{X}](q, t, y)$

Instead of looking at the $GL(W)$ -Hilbert series of $J(W)$, one could also consider the $GL(W)$ -Hilbert series of the Rees ring

$$R(W)[yJ(W)] = R(W) \oplus yJ(W) \oplus y^2J(W)^2 \oplus \dots,$$

where y is an indeterminate. This Hilbert series is

$$\sigma^n \sum_{i=0}^{\infty} \mathcal{H}[\mathbf{X}^i](\sigma^{-1}, -1)y^i,$$

where

$$\mathbf{X}^i = \underbrace{\mathbf{X} \oplus \mathbf{X} \oplus \dots \oplus \mathbf{X}}_i.$$

It is therefore natural to define the invariant

$$\tilde{\mathcal{H}}[\mathbf{X}](q, t, y) := \sum_{i=0}^{\infty} \mathcal{H}[\mathbf{X}^i](q, t)y^i.$$

Another interesting ring is the subalgebra $T(W)$ of $R(W)$ generated by

$$(W \otimes Z_1)(W \otimes Z_2) \dots (W \otimes Z_d).$$

The degree kd part in $T(W)$ (or degree k after rescaling) is equal to the degree (kd, d) part in $R(W)$. If we take

$$\sigma^n \tilde{\mathcal{H}}[\mathbf{X}](\sigma^{-1}, -1, z^{-1}),$$

replace s_λ by $z^{|\lambda|d}s_\lambda$ for all λ and then set $z = 0$, then we obtain the Hilbert series of $T(W)$.

It was proven in [6] that the algebra $T(W)$ is Koszul when Z_1, Z_2, \dots, Z_d are transversal. If Conjecture 4.2 in that paper is true, then $T(W)$ is Koszul for arbitrary subspaces Z_1, \dots, Z_d . Such a Koszul duality would lead to new interesting interpretations of the coefficients of $\tilde{\mathcal{H}}$.

6 The polarized Schur functor

6.1 The space $S_\lambda(Z_1, \dots, Z_d)$

Assume again that $\mathbf{X} = (X, \text{rk})$ is a polymatroid, K is a field of characteristic 0, and that we have a realization given by a vector space V and subspaces $V_x, x \in X$. Define

$Z = V^*$, and for every $x \in X$, let $Z_x = V_x^\perp$ be the set of all linear functionals on V vanishing on V_x . Also, for any $A \subseteq X$, let

$$Z_A = V_A^\perp = \sum_{x \in A} Z_x.$$

We have

$$\text{rk}(A) = \dim V - \dim V_A = \dim Z_A$$

for all $A \subseteq X$.

Let Σ_d be the symmetric group on d letters. Its irreducible representations are T_λ where λ runs over all partitions of d .

Schur-Weyl duality gives a decomposition

$$Z^{\otimes d} := \underbrace{Z \otimes Z \otimes \cdots \otimes Z}_d \cong \bigoplus_{\lambda} S_{\lambda} Z \otimes T_{\lambda}$$

as a representation of $GL(Z) \times \Sigma_d$. Let

$$\pi_{\lambda} : Z^{\otimes d} \rightarrow S_{\lambda} Z \otimes T_{\lambda}$$

be the $GL(Z) \times \Sigma_d$ -equivariant projection. There is a unique $GL(Z) \times \Sigma_d$ -equivariant linear map

$$\theta_{\lambda} : Z^{\otimes d} \otimes T_{\lambda}^* \rightarrow S_{\lambda}(Z)$$

such that

$$\theta_{\lambda}(z \otimes \varphi) = (\text{id} \otimes \varphi)\pi_{\lambda}(z)$$

for every $z \in Z^{\otimes n}$ and $\varphi \in T_{\lambda}^*$. Note that $T_{\lambda}^* \cong T_{\lambda}$ as representations of Σ_d .

Definition 6.1 We define

$$S_{\lambda}(Z_1, Z_2, \dots, Z_d) = \theta_{\lambda}(Z_1 \otimes Z_2 \otimes \cdots \otimes Z_d \otimes T_{\lambda}).$$

Remark 6.2 For a permutation $\tau \in \Sigma_d$ we have

$$\begin{aligned} S_{\lambda}(Z_1, \dots, Z_d) &= \theta_{\lambda}(Z_1 \otimes \cdots \otimes Z_d) \otimes T_{\lambda} \\ &= \theta_{\lambda}(\tau^{-1}(Z_1 \otimes \cdots \otimes Z_d \otimes T_{\lambda})) \\ &= \theta_{\lambda}(\tau^{-1}(Z_1 \otimes \cdots \otimes Z_d) \otimes T_{\lambda}) \\ &= \theta_{\lambda}(Z_{\tau(1)} \otimes \cdots \otimes Z_{\tau(d)}) \otimes T_{\lambda} \\ &= S_{\lambda}(Z_{\sigma(1)}, \dots, Z_{\sigma(d)}). \end{aligned} \tag{33}$$

In other words, $S_{\lambda}(Z_1, \dots, Z_d)$ does not depend on the order of Z_1, \dots, Z_d .

Note that

$$S_\lambda(\underbrace{Z, Z, \dots, Z}_d) = S_\lambda(Z).$$

6.2 The connection between $S_\lambda(Z_1, \dots, Z_d)$ and $\mathcal{H}[\mathbf{X}](q, t)$

Proposition 6.3 *Let us write*

$$\sigma^n \mathcal{H}[\mathbf{X}](\sigma^{-1}, -1) = \sum_\lambda a_\lambda s_\lambda,$$

where λ runs over all partitions with $|\lambda| \geq d$. Then we have

$$a_\lambda = \dim S_\lambda(Z_1, Z_2, \dots, Z_d, \underbrace{Z, \dots, Z}_{|\lambda|-d}).$$

Proof Let $r = |\lambda|$ and $\mathfrak{m}(W)$ be the maximal homogeneous ideal of $R(W)$. The degree r part of $J(W)$ is

$$J_1(W)J_2(W) \cdots J_d(W)\mathfrak{m}(W)^{r-d}.$$

Set $U = W \otimes V^* = W \otimes Z$ and $U_i = W \otimes Z_i$. Then Cauchy’s formula tells us that

$$R(W) = S(W \otimes Z) = \bigoplus_\lambda S_\lambda W \otimes S_\lambda Z.$$

The degree r part of $J(W)$ is

$$U_1 \cdot U_2 \cdots U_d \cdot U^{r-d} \subset S_r(U) = \bigoplus_{|\lambda|=r} S_\lambda W \otimes S_\lambda Z.$$

So if

$$\pi_r^U : \underbrace{U \otimes U \otimes \cdots \otimes U}_r \rightarrow S_r(U)$$

is the canonical projection, then the degree r part of $J(W)$ is

$$\pi_r(U_1, U_2, \dots, U_d, \underbrace{U, \dots, U}_{r-d}).$$

Let $\gamma_\lambda : S_r(U) \rightarrow S_\lambda W \otimes S_\lambda Z$ be the projection. The isotypic component of $J(W)$ for the representation $S_\lambda(W)$ is

$$\gamma_\lambda(\pi_r(U_1 \otimes \cdots \otimes U_d \otimes U^{r-d})).$$

We have

$$U^{\otimes r} = (Z \otimes W)^{\otimes r} = Z^{\otimes r} \otimes W^{\otimes r} = \bigoplus_\lambda S_\lambda(W) \otimes T_\lambda \otimes Z^{\otimes r}$$

$$\cong \bigoplus_{\lambda} S_{\lambda}(W) \otimes Z^{\otimes r} \otimes T_{\lambda}.$$

If we first project $U^{\otimes r}$ onto $S_{\lambda}(W) \otimes Z^{\otimes r} \otimes T_{\lambda}$ and then we apply

$$\text{id} \otimes \pi_{\lambda}^Z : S_{\lambda}(W) \otimes Z^{\otimes r} \otimes T_{\lambda} \rightarrow S_{\lambda} W \otimes S_{\lambda} Z$$

then we get a nonzero $\text{GL}(V) \times \text{GL}(Z) \times \Sigma_r$ equivariant linear map

$$U^{\otimes r} \rightarrow S_{\lambda} W \otimes S_{\lambda} Z.$$

This map must be, up to a non-zero scalar, equal to the composition $\gamma_{\lambda} \circ \pi_r$. It follows that

$$\begin{aligned} \gamma_{\lambda}(\pi_r(U_1 \otimes \dots \otimes U_d \otimes U^{r-d})) &= \text{id} \otimes \pi_{\lambda}(S_{\lambda}(W) \otimes Z_1 \otimes \dots \otimes Z_d \otimes Z^{r-d} \otimes T_{\lambda}) \\ &= S_{\lambda} W \otimes S_{\lambda}(Z_1, \dots, Z_d, \underbrace{Z, \dots, Z}_{r-d}). \end{aligned}$$

So, as $\text{GL}(W)$ -modules, we have an isomorphism

$$J(W) \cong \bigoplus_{\lambda} S_{\lambda}(Z_1, Z_2, \dots, Z_d, \underbrace{Z, \dots, Z}_{|\lambda|-d}) \otimes S_{\lambda}(W).$$

Since a_{λ} is the multiplicity of $S_{\lambda} W$ in $J(W)$, we get

$$a_{\lambda} = \dim S_{\lambda}(Z_1, \dots, Z_d, \underbrace{Z, \dots, Z}_{r-d}).$$

□

For $A \subseteq X$, let us define

$$S_{\lambda,A} := S_{\lambda}(V_{x_1}, \dots, V_{x_k}, \underbrace{V, \dots, V}_{|\lambda|-k}),$$

where $k = |A|$ and $A = \{x_1, \dots, x_k\}$. If $|\lambda| < k$, then we define $S_{\lambda,A} = 0$. Define

$$C_{\lambda,k} = \bigoplus_{|A|=k} S_{\lambda,A}.$$

Then we get

$$C_k = \bigoplus_{\lambda} C_{\lambda,k} \otimes S_{\lambda}(W).$$

The maps in the complex (26) are $\text{GL}(W)$ -equivariant, and by taking the isotypic component for $S_{\lambda}(W)$ we get a complex

$$0 \rightarrow C_{\lambda,\ell} \otimes S_{\lambda}(W) \rightarrow \dots \rightarrow C_{\lambda,1} \otimes S_{\lambda}(W) \rightarrow C_{\lambda,0} \otimes S_{\lambda}(W) \rightarrow 0,$$

where $\ell = \min\{d, |\lambda|\}$. Since all maps in this complex are $GL(W)$ -equivariant, the complex is obtained from a complex

$$0 \rightarrow C_{\lambda,\ell} \rightarrow \dots \rightarrow C_{\lambda,1} \rightarrow C_{\lambda,0} \rightarrow 0 \tag{34}$$

by tensoring it by $S_\lambda(W)$. The map $\partial_k : C_{\lambda,k} \rightarrow C_{\lambda,k-1}$ can be written as $\partial_k = \sum_{A,B} \partial_k^{A,B}$, where

$$\partial_k^{A,B} : S_{\lambda,A} \rightarrow S_{\lambda,B}.$$

Suppose that $A = \{i_1, i_2, \dots, i_k\}$ with $i_1 < i_2 < \dots < i_k$, then we have

$$\partial_k^{A,B} := \begin{cases} 0 & \text{if } B \not\subseteq A; \\ (-1)^r \text{id} & \text{if } B = \{i_1, \dots, i_{r-1}, i_{r+1}, \dots, i_k\}. \end{cases}$$

Let $H_{\lambda,i}$ be the i -th homology group of (34). From

$$H_i(W) = \bigoplus_{\lambda} H_{\lambda,i} \otimes S_\lambda(W)$$

and Corollary 5.3 now follows the following statement.

Corollary 6.4 *Suppose that $V_X = 0$, which means that $Z_X = Z$. Write*

$$\mathcal{P}[X] = \sum_{\lambda} (-1)^{|\lambda|} b_{\lambda} S_{\lambda}.$$

Then we have

$$\dim H_{\lambda,i} = \begin{cases} 0 & \text{if } |\lambda| \neq i; \\ b_{\lambda} & \text{if } |\lambda| = i. \end{cases}$$

The dimension of

$$S_{\lambda}Z = S_{\lambda}(\underbrace{Z, Z, \dots, Z}_d)$$

(where $d = |\lambda|$) is exactly the number of Young Tableau of shape λ and entries in the set $\{1, 2, \dots, n\}$. In fact, given a basis of Z , an explicit basis of $S_{\lambda}Z$ can be given in terms of these Young tableaux (see [14, §8.1, Theorem 1]).

Problem 6.5 *Give a combinatorial interpretation of*

$$\dim S_{\lambda}(Z_1, Z_2, \dots, Z_d),$$

perhaps in terms of certain fillings of Young diagrams. Moreover, can one give an explicit basis of $S_{\lambda}(Z_1, \dots, Z_d)$?

Such a combinatorial setup might still have a meaning for non-realizable polymatroids. An explicit bases of $S_d(Z_1, \dots, Z_d)$ was given in [6, Corollary 5.10] in case the subspaces Z_1, \dots, Z_d of Z are generic.

Also, one can ask the same questions for $H_{\lambda} := H_{\lambda,|\lambda|}$. Such results might prove Conjecture 5.7.

7 Quasi-symmetric functions associated to polymatroids

7.1 The Hopf algebras *Mat* and *PolyMat*

Although most of the Hopf algebras in this section can be defined over the integers \mathbb{Z} , we will choose to define them over \mathbb{Q} for simplicity. In [32] the matroid Hopf algebra *Mat* was introduced (see also [9–11]). This construction easily generalizes to polymatroids.

Let us first introduce the Hopf algebra of polymatroids, *PolyMat*. For a polymatroid $\mathbf{X} = (X, \text{rk})$, we denote its isomorphism class by $[\mathbf{X}]$. As a \mathbb{Q} -vector space, *PolyMat* has a basis consisting of all isomorphism classes of polymatroids. We define a product by

$$[\mathbf{X}] \cdot [\mathbf{Y}] := [\mathbf{X} \oplus \mathbf{Y}].$$

Also, a coproduct $\Delta : \text{PolyMat} \rightarrow \text{PolyMat} \otimes_{\mathbb{Q}} \text{PolyMat}$ is defined by

$$\Delta[\mathbf{X}] = \sum_{A \subseteq X} [\mathbf{X}|_A] \otimes [\mathbf{X}/A].$$

This coproduct is coassociative, but in general not cocommutative. The unit is $[\emptyset]$ where \emptyset denotes the empty polymatroid. A counit $\epsilon : \text{PolyMat} \rightarrow \mathbb{Q}$ is given by

$$\epsilon([\mathbf{X}]) = \begin{cases} 1 & \text{if } \mathbf{X} = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

The bialgebra *PolyMat* has a grading such that $[\mathbf{X}]$ has degree $|X|$ for every polymatroid $\mathbf{X} = (X, \text{rk})$. This makes *PolyMat* into a connected graded bialgebra. It was shown in [27] that one can define an antipode such that *PolyMat* becomes a Hopf algebra.

Let *Mat* be the subspace spanned by all $[\mathbf{X}]$ where \mathbf{X} is a matroid. Then *Mat* is sub-Hopf algebra of *PolyMat*.

7.2 The Hopf algebra *NSym*

Let $NSym_{\mathbb{Q}}\langle p_1, p_2, p_3, \dots \rangle$ be the ring of noncommutative polynomials in the indeterminates p_1, p_2, p_3, \dots . We define a Hopf algebra structure on *NSym* as follows. The comultiplication $\Delta : NSym \rightarrow NSym \otimes NSym$ by

$$\Delta(p_i) = p_i \otimes 1 + 1 \otimes p_i$$

for all i . The counit $\epsilon : NSym \rightarrow \mathbb{Q}$ is defined by

$$\epsilon(p_i) = 0$$

for all i . The antipode is defined by

$$p_i \mapsto -p_i$$

for all i . A basis of $NSym$ is given by all noncommutative monomials in p_1, p_2, \dots . It is also convenient to have a different basis. We define h_1, h_2, \dots by the following equality of generating functions in $NSym[[t]]$. Define

$$H(t) = h_1t + h_2t^2 + h_3t^3 + \dots$$

and

$$P(t) = p_1t + p_2t^2 + p_3t^3 + \dots.$$

Then h_1, h_2, h_3, \dots are defined by

$$1 + H(t) = \exp(P(t)).$$

Here $\exp(t)$ denotes the power series of the exponential function

$$\exp(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots.$$

So we have

$$h_k = \sum_{r=1}^k \frac{1}{r!} \left(\sum_{\substack{i_1, \dots, i_r \\ i_1 + \dots + i_r = k}} p_{i_1} p_{i_2} \dots p_{i_r} \right). \quad (35)$$

If $\alpha = (i_1, \dots, i_r)$ is a sequence of positive integers, then we will write p_α instead of $p_{i_1} p_{i_2} \dots p_{i_r}$ and h_α instead of $h_{i_1} h_{i_2} \dots h_{i_r}$. The length of α is $\ell(\alpha) := r$, and we define $|\alpha| = i_1 + i_2 + \dots + i_r$. We can rewrite (35) as

$$h_k = \sum_{\substack{\alpha \\ |\alpha|=k}} \frac{p_\alpha}{\ell(\alpha)!}. \quad (36)$$

Inverting gives

$$P(t) = \log(1 + H(t)),$$

where

$$\log(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \dots,$$

so

$$p_k = \sum_{r=1}^k \frac{(-1)^{r-1}}{r} \sum_{\substack{i_1, \dots, i_r \\ i_1 + \dots + i_r = k}} h_{i_1} h_{i_2} \dots h_{i_r}.$$

Again, we can rewrite this as

$$p_k = \sum_{\alpha} \frac{(-1)^{\ell(\alpha)-1} h_\alpha}{\ell(\alpha)}. \quad (37)$$

From

$$\Delta(P(t)) = P(t) \otimes 1 + 1 \otimes P(t)$$

it follows that

$$\begin{aligned} \Delta(1 + H(t)) &= \Delta(\exp(P(t))) = \Delta(\exp(P(t)) \otimes 1 + 1 \otimes P(t)) \\ &= \exp(P(t) \otimes 1) \exp(1 \otimes P(t)) = ((1 + H(t)) \otimes 1) \cdot (1 \otimes (1 + H(t))) \\ &= (1 + H(t)) \otimes (1 + H(t)) \end{aligned}$$

inside the ring

$$NSym \otimes NSym[[t]] = NSym[[t]] \otimes_{\mathbb{Q}[[t]]} NSym[[t]].$$

If we use the convention $h_0 = 1$, then we have

$$\Delta(h_k) = \sum_{i=0}^k h_i \otimes h_{k-i}.$$

The Hopf algebra $NSym$ is not commutative, but it is cocommutative.

7.3 The Hopf algebra $QSym$

Let $QSym$ be the Hopf algebra of quasi-symmetric functions (see [15, 16, 18, 19]). For a sequence $\alpha = (\alpha_1, \dots, \alpha_r)$ of positive integers we define an element $M_\alpha \in \mathbb{Q}[x_1, x_2, \dots]$ by

$$M_\alpha := \sum_{0 < i_1 < i_2 < \dots < i_r} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_r^{\alpha_r}.$$

The ring $QSym$ is the subring of $\mathbb{Q}[x_1, x_2, x_3, \dots]$ spanned by all M_α . The \mathbb{Q} -vector space $QSym$ is closed under multiplication. We will view $QSym$ as the graded dual vector space of $NSym$ where the $\{M_\alpha\}$ form a dual basis of the $\{h_\alpha\}$. As such, $QSym$ is a Hopf algebra in a natural way. Also, let $\{P_\alpha\}$ be a dual basis of $\{p_\alpha\}$. We have that

$$P_\alpha P_\beta = \sum_{\gamma} P_\gamma,$$

where γ runs over all

$$\binom{\ell(\alpha) + \ell(\beta)}{\ell(\alpha)}$$

shuffles of α and β . If $\alpha = (\alpha_1, \dots, \alpha_r)$, then

$$\Delta(P_\alpha) = \sum_{\beta, \gamma: \beta\gamma=\alpha} P_\beta \otimes P_\gamma.$$

The antipode on $QSym$ is given by

$$P_\alpha \mapsto (-1)^{\ell(\alpha)} P_\alpha.$$

From (36) it follows that

$$h_\alpha = h_{i_1} \cdots h_{i_r} = \sum_{\substack{\beta_1, \dots, \beta_r \\ |\beta_1|=i_1, \dots, |\beta_r|=i_r}} \frac{P_{\beta_1 \beta_2 \cdots \beta_r}}{\ell(\beta_1)! \cdots \ell(\beta_r)!}, \tag{38}$$

where $\alpha = (i_1, \dots, i_r)$. Dualizing (38) gives

$$P_\beta = \sum_r \sum_{\substack{\beta_1, \dots, \beta_r \\ \beta = \beta_1 \cdots \beta_r}} \frac{M_{|\beta_1|, \dots, |\beta_r|}}{\ell(\beta_1)! \ell(\beta_2)! \cdots \ell(\beta_r)!}.$$

From (37) it follows that

$$p_\alpha = p_{i_1} \cdots p_{i_r} = \sum_{\substack{\beta_1, \dots, \beta_r \\ |\beta_1|=i_1, \dots, |\beta_r|=i_r}} \frac{(-1)^{\ell(\beta_1) + \cdots + \ell(\beta_r) - r} h_{\beta_1 \beta_2 \cdots \beta_r}}{\ell(\beta_1) \cdots \ell(\beta_r)}. \tag{39}$$

Dualizing (39) yields

$$M_\beta = \sum_r \sum_{\substack{\beta_1, \dots, \beta_r \\ \beta = \beta_1 \cdots \beta_r}} (-1)^{\ell(\beta) - r} \frac{P_{|\beta_1|, \dots, |\beta_r|}}{\ell(\beta_1) \cdots \ell(\beta_r)}. \tag{40}$$

7.4 Combinatorial Hopf algebras and the invariant $\mathcal{F}[\mathbf{X}]$

Billera, Jia and Reiner defined a homomorphism of Hopf algebras

$$\mathcal{F} : Mat \rightarrow QSym$$

(see [3, 26]). One way to define this map is using a universal property of $QSym$.

A combinatorial Hopf algebra (over \mathbb{Q}) is a pair (\mathcal{H}, ζ) where $\mathcal{H} = \bigoplus_{d \geq 0} \mathcal{H}_d$ is a graded Hopf algebra with $\mathcal{H}_0 = \mathbb{Q}$ and \mathcal{H}_d is finite dimensional for all d , and $\zeta : \mathcal{H} \rightarrow \mathbb{Q}$ is a character (i.e., an algebra homomorphism). A morphism $\varphi : (\mathcal{H}', \zeta') \rightarrow (\mathcal{H}, \zeta)$ is a Hopf-algebra morphism $\varphi : \mathcal{H}' \rightarrow \mathcal{H}$ such that $\zeta \circ \varphi = \zeta'$.

Aguiar, Bergeron and Sottile proved that there exists a terminal object in the category of combinatorial Hopf algebras over \mathbb{Q} , namely $(QSym, \zeta)$ where $\zeta = \zeta_{QSym}$ is defined by

$$\zeta(M_\alpha) = \begin{cases} 1 & \text{if } \ell(\alpha) \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

We can define a character $\zeta = \zeta_{Mat}$ on Mat by

$$\zeta([\mathbf{X}]) = \begin{cases} 1 & \text{if } \mathbf{X} \text{ completely splits into loop and coloop matroids;} \\ 0 & \text{otherwise.} \end{cases}$$

Since $(QSym, \zeta_{QSym})$ is terminal, there is a unique homomorphism

$$\mathcal{F} : (Mat, \zeta_M) \rightarrow (QSym, \zeta_{QSym})$$

of combinatorial Hopf algebras.

Although \mathcal{F} is a powerful invariant for matroids, it cannot distinguish between a loop and an isthmus.

7.5 The new quasi-symmetric function invariant $\mathcal{G}[\mathbf{X}]$

It sometimes is convention to shift the indices by 1, so for a vector $a = (a_1, a_2, \dots, a_d)$ of nonnegative integers, we define

$$U_{(a_1, a_2, \dots, a_d)} := P_{a_1+1, a_2+1, \dots, a_d+1}.$$

Definition 7.1 We define a \mathbb{Q} -linear map

$$\mathcal{G} : \text{PolyMat} \rightarrow \text{QSym}$$

defined by

$$\mathcal{G}[\mathbf{X}] = \sum_{\underline{X}} U_{r(\underline{X})},$$

where \underline{X} runs over all maximal chains

$$\underline{X} : \emptyset = X_0 \subset X_1 \subset \dots \subset X_d = X$$

and

$$r(\underline{X}) := (\text{rk}(X_1) - \text{rk}(X_0), \text{rk}(X_2) - \text{rk}(X_1), \dots, \text{rk}(X_d) - \text{rk}(X_{d-1})).$$

We call $r(\underline{X})$ the rank sequence for \underline{X} . The multiset of all $r(\underline{X})$ where \underline{X} runs over all maximal chains in X , we will call *the rank sequences for \mathbf{X}* . If $\mathbf{X} = (X, \text{rk})$ then there are exactly $|X|!$ rank sequences.

Lemma 7.2 *The linear map \mathcal{G} is a homomorphism of Hopf algebras.*

Proof If \mathbf{X} has a rank sequence $\gamma = r(\underline{X})$ and $\gamma = \alpha\beta$, then α is a rank sequence for $\mathbf{X} \upharpoonright_A$ and β is a rank sequence for \mathbf{X}/A , where $A = X_i$ and $i = \ell(\alpha)$ is the length of α . So we have

$$\begin{aligned} \mathcal{G} \otimes \mathcal{G} \circ \Delta([\mathbf{X}]) &= \sum_{A \subseteq X} \mathcal{G}[\mathbf{X} \upharpoonright_A] \otimes \mathcal{G}[\mathbf{X}/A] \\ &= \sum_{A \subseteq X} \sum_{\alpha} \sum_{\beta} U_{\alpha} \otimes U_{\beta} = \Delta\left(\sum_{\gamma} U_{\gamma}\right) = \Delta(\mathcal{G}[\mathbf{X}]), \end{aligned} \tag{41}$$

where α runs over all rank sequences for $\mathbf{X} \upharpoonright_A$, β runs over all rank sequences of \mathbf{X}/A and γ runs over all rank sequences for \mathbf{X} .

To see that \mathcal{G} commutes with the product, note that the rank sequences for $\mathbf{X} \oplus \mathbf{Y}$ are exactly all shuffles of rank sequences for \mathbf{X} and \mathbf{Y} .

It is easy to verify that \mathcal{G} is compatible with the unit and counit. □

For a vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, define

$$\alpha^\vee = (1 - \alpha_d, 1 - \alpha_{d-1}, \dots, 1 - \alpha_1).$$

Lemma 7.3 For a matroid $\mathbf{X} = (X, \text{rk})$ we have

$$\mathcal{G}[\mathbf{X}^\vee] = \sum_{\underline{X}} U_{r(\underline{X})}^\vee.$$

Proof For a maximal chain \underline{X} , define a chain \underline{X}^\vee by $X_i^\vee := X \setminus X_{d-i}$. Note that

$$\text{rk}^\vee(X_i) = |X| - \text{rk}(X) + \text{rk}(X_{d-i})$$

and

$$\text{rk}^\vee(X_i^\vee) - \text{rk}^\vee(X_{i-1}^\vee) = 1 - (\text{rk}(X_{d-i+1}) - \text{rk}(X_{d-i})).$$

If α runs over all rank sequence for \mathbf{X} , then α^\vee runs over all rank sequences for \mathbf{X}^\vee . \square

7.6 \mathcal{G} specializes to \mathcal{F}

Let us define another character $\gamma : QSym \rightarrow \mathbb{Q}$ by

$$\gamma(P_\alpha) = 0$$

if α is not weakly increasing. Otherwise, write $\alpha = (\alpha_1^{k_1}, \alpha_2^{k_2}, \dots, \alpha_s^{k_s})$ with

$$\alpha_1 < \alpha_2 < \dots < \alpha_s,$$

and define

$$\gamma(P_\alpha) = \frac{1}{k_1!k_2! \dots k_s!}.$$

Suppose that $\alpha' = (\alpha_1^{l_1}, \dots, \alpha_s^{l_s})$. Then

$$P_\alpha P_{\alpha'} = \binom{l_1 + k_1}{k_1} \binom{l_2 + k_2}{k_2} \dots \binom{l_s + k_s}{k_s} P_\delta + P',$$

where $\delta = (\alpha_1^{k_1+l_1}, \dots, \alpha_s^{k_s+l_s})$ and P' is a linear combination of P_δ 's where δ is not weakly increasing. The binomials appear from the fact there are $\binom{l_i+k_i}{k_i}$ ways to shuffle $\alpha_i^{k_i}$ and $\alpha_i^{l_i}$. If we apply γ we get

$$\gamma(P_\alpha P_{\alpha'}) = \gamma(P_\delta) = \frac{\binom{l_1+k_1}{k_1} \dots \binom{l_s+k_s}{k_s}}{(l_1 + k_1)! \dots (l_s + k_s)!} = \frac{1}{k_1! \dots k_s!} \cdot \frac{1}{l_1! \dots l_s!} = \gamma(P_\alpha) \gamma(P_{\alpha'}).$$

This shows that γ is multiplicative. Since $(QSym, \zeta)$ is the terminal object for the combinatorial Hopf algebras, there is a unique morphism of combinatorial Hopf algebras

$$\theta : (QSym, \gamma) \rightarrow (QSym, \zeta).$$

Theorem 7.4 *We have*

$$\theta \circ \mathcal{G} \upharpoonright_{Mat} = \mathcal{F},$$

where $\mathcal{G} \upharpoonright_{Mat}$ is the restriction of \mathcal{G} to Mat .

Proof We claim that

$$\zeta = \gamma \circ \mathcal{G} \upharpoonright_{Mat}.$$

Suppose that $\mathbf{X} = (X, \text{rk})$ is a matroid with $d := |X|$ and $n := \text{rk}(X) \leq d$. Then $\gamma(\mathcal{G}[\mathbf{X}])$ is equal to $\frac{N}{n!(d-n)!}$, where N counts the number of maximal chains

$$X_0 = \emptyset \subset X_1 \subset \dots \subset X_d = X$$

with

$$0 = \text{rk}(X_0) = \dots = \text{rk}(X_{d-n}) = 0 \tag{42}$$

and

$$\text{rk}(X_{d-n+i}) = i \tag{43}$$

for $i = 1, 2, \dots, n$. Let $Y = X_{d-n}$ and $Z = X \setminus Y$. For a subset $A \subseteq X$, we have

$$\text{rk}(A) \geq \text{rk}(X) - \text{rk}(X \setminus A)$$

and

$$\text{rk}(X \setminus A) \leq \text{rk}(Y \setminus A) + \text{rk}(Z \setminus A) = \text{rk}(Z \setminus A) \leq |Z| - |Z \cap A| = n - |Z \cap A|.$$

It follows that

$$\text{rk}(A) \geq \text{rk}(X) - \text{rk}(X \setminus A) = n - (n - |Z \cap A|) = |Z \cap A|.$$

We also have

$$\text{rk}(A) \leq \text{rk}(A \cup Y) \leq \text{rk}(Y) + |A \cup Y| - |Y| = |A \cap Z|.$$

We conclude that

$$\text{rk}(A) = |A \cap Z|$$

for all $A \subseteq X$. This implies that

$$(X, \text{rk}) = \underbrace{\mathbf{0} \cdot \mathbf{0} \cdot \dots \cdot \mathbf{0}}_{d-n} \cdot \underbrace{\mathbf{1} \cdot \mathbf{1} \cdot \dots \cdot \mathbf{1}}_n, \tag{44}$$

where $\mathbf{0}$ is the loop matroid, and $\mathbf{1}$ is the isthmus matroid. In particular, if (X, rk) does not split completely, then $\gamma(\mathcal{G}[\mathbf{X}]) = 0$.

Suppose that $\mathbf{X} = (X, \text{rk})$ splits completely as in (44). Without loss of generality, we may assume that $X = \{1, 2, \dots, d\}$, and $\text{rk}(A) = |A \cap Z|$ where $Y = \{1, 2, \dots, d - n\}$ and $Z = X \setminus Y$.

A flag

$$X_0 = \emptyset \subset X_1 \subset \dots \subset X_d = X$$

satisfies (42) and (43) if and only if $X_{d-n} = Y$. There are $(d - n)!$ flags

$$\emptyset = X_0 \subset \dots \subset X_{d-n} = Y$$

and $n!$ flags

$$Y = X_{d-n} \subset X_{d-n+1} \subset \dots \subset X_d = X.$$

It follows that $N = n!(d - n)!$, and

$$\gamma(\mathcal{G}[\mathbf{X}]) = \frac{N}{n!(d - n)!} = 1.$$

It follows that $\gamma \circ \mathcal{G} |_{Mat} = \zeta = \zeta([\mathbf{X}])$. By the uniqueness, we get $\theta \circ \mathcal{G} |_{Mat} = \mathcal{F}$. \square

Note that

$$\mathcal{G}(Mat) \subseteq QSym_2,$$

where $QSym_2$ is the sub-Hopf algebra of $QSym$ spanned by all Q_α 's where α is a sequence of 0's and 1's. The algebra $QSym_2$ is the graded dual of the Hopf algebra $\mathbb{Q}\langle p_1, p_2 \rangle$. Now θ restricts to a homomorphism

$$\theta_2 : QSym_2 \rightarrow QSym.$$

Proposition 7.5 *The homomorphism θ_2 is surjective, and the kernel of θ_2 is the principal ideal generated by $P_{(2)} - P_{(1)} = U_{(1)} - U_{(0)}$.*

Proof The surjectivity follows from the fact that \mathcal{F} is surjective. We choose the grading on $QSym_2$ where P_α has degree $\ell(\alpha)$. There are 2^d basis elements P_α of degree d . So the Hilbert series of the $QSym_2$ is

$$1 + 2t + 2^2t^2 + \dots = \frac{1}{1 - 2t}.$$

Note that $QSym_2$ is not finitely generated as a commutative algebra.

On $QSym$, we choose the grading where P_α has degree $|\alpha|$. There is one basis element of degree 0, namely $P_{()}$ and for $d > 0$ there are 2^{d-1} basis elements of degree d , because there are 2^{d-1} decompositions of d . So the Hilbert series of $QSym$ with this grading is

$$1 + t + 2t^2 + 2^2t^3 + \dots = 1 + \frac{t}{1 - 2t} = \frac{1 - t}{1 - 2t}.$$

Therefore, the Hilbert series of the kernel of θ_2 is

$$\frac{1}{1 - 2t} - \frac{1 - t}{1 - 2t} = \frac{t}{1 - 2t}.$$

The kernel contains the principal ideal $(P_{(2)} - P_{(1)})$. It is not hard to see that $P_{(2)} - P_{(1)}$ is not a zero divisor, so the Hilbert series of the principal ideal is $\frac{t}{1-2t}$. Since this is equal to the Hilbert series of the kernel of θ_2 we must have

$$\ker \theta_2 = (P_{(2)} - P_{(1)}). \quad \square$$

7.7 \mathcal{G} specializes to \mathcal{H}

Theorem 7.6 *There exists a homomorphism $\tau : QSym \rightarrow Sym[q, t]$ of commutative algebras such that $\tau(\mathcal{G}[\mathbf{X}]) = \mathcal{H}[\mathbf{X}]$ for every polymatroid \mathbf{X} .*

Proof We will inductively define a symmetric function $\mathcal{P}(\alpha)$ for any vector $\alpha = (\alpha_1, \dots, \alpha_d)$ of nonnegative integers as follows. We define $\mathcal{P}() = 1$. Then $\mathcal{P}(\alpha_1, \dots, \alpha_d)$ is the unique symmetric function of degree $< d$ such that

$$\sum_{i=0}^d \binom{d}{i} \mathcal{P}(\alpha_1, \dots, \alpha_i) (-1)^i \sigma^{-\alpha_1 - \dots - \alpha_i} \tag{45}$$

vanishes in degree $< d$. For a vector $\alpha = (\alpha_1, \dots, \alpha_d)$ and $i < d$, let $\alpha^{[i]} = (\alpha_1, \dots, \alpha_i)$ be the truncated vector. So (45) becomes

$$\sum_{i=0}^d \binom{d}{i} \mathcal{P}(\alpha^{[i]}) (-1)^i \sigma^{-|\alpha^{[i]}|}. \tag{46}$$

Define

$$\tilde{\mathcal{P}}[\mathbf{X}] = \frac{1}{d!} \sum_{\underline{X}} \mathcal{P}(r(\underline{X})) \tag{47}$$

for every polymatroid $\mathbf{X} = (X, \text{rk})$ such that $d = |X|$. Here \underline{X} runs over all maximal chains in X .

We claim that $\mathcal{P}[\mathbf{X}] = \tilde{\mathcal{P}}[\mathbf{X}]$. The claim is clearly true when $|X| = 0$ or $|X| = 1$. Note that $\tilde{\mathcal{P}}[\mathbf{X}]$ is a symmetric polynomial of degree $< d = |X|$. To prove the claim it suffices to show that

$$\sum_{A \subseteq X} \tilde{\mathcal{P}}[\mathbf{X}|_A] (-1)^{|A|} \sigma^{-\text{rk}(A)} \tag{48}$$

vanishes in degree $< d$. The symmetric polynomial (47) is equal to

$$\sum_{i=0}^d \sum_{\substack{A \subseteq X \\ |A|=i}} \frac{1}{i!} \sum_{\underline{A}} \mathcal{P}(r(\underline{A})) (-1)^i \sigma^{-\text{rk}(A)}, \tag{49}$$

where \underline{A} runs over all maximal chains in A . Every such chain \underline{A} can be extended to $(d - i)!$ maximal chains in X . Therefore, (49) is equal to

$$\sum_{i=0}^d \frac{1}{i!(d-i)!} \sum_{\underline{X}} \mathcal{P}(r(\underline{X})^{[i]}) (-1)^i \sigma^{-|r(\underline{X})^{[i]}|}$$

$$= \frac{1}{d!} \sum_{\underline{X}} \sum_{i=0}^d \binom{d}{i} \mathcal{P}(r(\underline{X})^{[i]}) (-1)^i \sigma^{-|r(\underline{X})^{[i]}|} \quad (50)$$

which vanishes in degree $< d$.

For a vector $\alpha = (\alpha_1, \dots, \alpha_d)$, define

$$\tau(U_\alpha) = \sum_{i=0}^d \frac{1}{i!(d-i)!} \mathcal{P}(\alpha^{[i]}) q^{|\alpha^{[i]}|} t^i.$$

If $\mathbf{X} = (X, \text{rk})$ is a polymatroid with $|X| = d$, then we have

$$\tau(\mathcal{G}[\mathbf{X}]) = \tau\left(\sum_{\underline{X}} U_{r(\underline{X})}\right) = \sum_{\underline{X}} \tau(U_{r(\underline{X})}) = \sum_{\underline{X}} \sum_{i=0}^d \frac{1}{i!(d-i)!} \mathcal{P}(r(\underline{X})^{[i]}) q^{|r(\underline{X})^{[i]}|} t^i. \quad (51)$$

For every subset $A \subseteq X$ with $|A| = i$, and every maximal chain \bar{A} in A there are exactly $(d-i)!$ maximal chains \bar{X} in X extending \bar{A} . Therefore, (51) is equal to

$$\sum_{i=0}^d \sum_{A \subseteq X; |A|=i} \frac{1}{i!} \sum_{\bar{A}} \mathcal{P}(r(\bar{A})) q^{\text{rk}(A)} t^{|A|} = \sum_{A \subseteq X} \mathcal{P}[\mathbf{X} | A] q^{\text{rk}(A)} t^{|A|} = \mathcal{H}[\mathbf{X}](q, t). \quad \square$$

Corollary 7.7 *The quasi-symmetric function $\mathcal{F}[\mathbf{X}]$ specializes to $\mathcal{P}[\mathbf{X}]$ for matroids \mathbf{X} .*

Proof We define $\xi : QSym_2 \rightarrow Sym$ by

$$\xi(Q_\alpha) = t^{\ell(\alpha)} \tau(Q_\alpha)(1, t^{-1})|_{t=0}.$$

One easily verifies that ξ is a homomorphism of algebras, and

$$\xi(\mathcal{G}[\mathbf{X}]) = \mathcal{H}[\mathbf{X}](1, t^{-1})t^{|\mathbf{X}|} |_{t=0} = \mathcal{P}[\mathbf{X}]$$

for every matroid $\mathbf{X} = (X, \text{rk})$. Since $Q_{(1)} - Q_{(0)}$ lies in the kernel of ξ , ξ factors through $\theta : QSym_2 \rightarrow QSym \cong QSym_2 / (Q_{(1)} - Q_{(0)})$, say $\xi = \eta \circ \theta$. Then we have

$$\mathcal{P}[\mathbf{X}] = \xi(\mathcal{G}[\mathbf{X}]) = \eta(\theta(\mathcal{G}[\mathbf{X}])) = \eta(\mathcal{F}[\mathbf{X}]). \quad \square$$

7.8 Speyer's invariant

For a matroid \mathbf{X} David Speyer defined an interesting polynomial $g_{\mathbf{X}}(t)$. It has the multiplicative property ($g_{\mathbf{X}_1 \oplus \mathbf{X}_2}(t) = g_{\mathbf{X}_1}(t)g_{\mathbf{X}_2}(t)$), it is invariant under matroid-duality and has various other nice properties.

Conjecture 7.8 *The invariant \mathcal{G} specializes to Speyer's invariant.*

8 Polymatroid base polytopes

8.1 The valuative property of \mathcal{G}

We will denote $\{1, 2, \dots, n\}$ by \underline{n} . For a polymatroid $\mathbf{X} = (\underline{n}, \text{rk})$ we define its base polytope $Q(\text{rk}) = Q_{\mathbf{X}}(\text{rk}) \subset \mathbb{R}^n$ by

$$Q(\text{rk}) = \{v \in \mathbb{R}^n \mid \sum_{i=1}^n v_i = \text{rk}(\underline{n}) \text{ and } \forall A \subseteq \underline{n}, \sum_{i \in A} v_i \leq \text{rk}(A)\}.$$

The i -th basis vector is denoted by e_i .

Theorem 8.1 ([20]) *A compact convex polytope in \mathbb{R}^n is the base polytope of a polymatroid if and only if every vertex of the polytope has nonnegative integer coordinates, and every edge is parallel to $e_j - e_k$ for some $j \neq k$.*

For a compact convex polytope $\Pi \subset \mathbb{R}^n$, its characteristic function $[\Pi] : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$[\Pi](x) = \begin{cases} 1 & \text{if } x \in \Pi; \\ 0 & \text{if } x \notin \Pi. \end{cases}$$

Let $\mathcal{K}(\mathbb{R}^n)$ be the \mathbb{R} -vector space spanned by all $[\Pi]$ where Π is a compact convex polytope. The *Euler characteristic* is a linear function $\chi : \mathcal{K}(\mathbb{R}^n) \rightarrow \mathbb{R}$ such that $\chi([\Pi]) = 1$ for every compact convex polytope Π (see [2, Theorem 7.4] where χ is defined for the slightly larger algebra of closed convex sets).

Definition 8.2 Suppose that V is a \mathbb{Q} -vector space. A \mathbb{Q} -linear map $f : \text{PolyMat} \rightarrow V$ is called *valuative* if it has the following property. For a finite set X and polymatroids $\mathbf{X} = (X, \text{rk}_i), i = 1, 2, \dots, r$ and rational numbers $a_1, \dots, a_r \in \mathbb{Q}$ such that

$$\sum_{i=1}^r a_i [Q(\text{rk}_i)] = 0$$

we have that

$$\sum_{i=1}^r a_i f[\mathbf{X}_i] = 0.$$

Moreover, let us call f *additive* if it is valuative and $f([\mathbf{X}]) = 0$ whenever the polymatroid base polytope $Q(\text{rk})$ of $\mathbf{X} = (X, \text{rk})$ has dimension $< n - 1$.

Theorem 8.3

$$\mathcal{G} : \text{PolyMat} \rightarrow \mathcal{QSym}$$

is *valuative*.

The proof of the theorem is in the next subsection.

Corollary 8.4 *Since \mathcal{G} specializes to \mathcal{H} and \mathcal{P} , these invariants are valuative as well.*

A *polymatroid base decomposition* is a decomposition

$$Q(\text{rk}) = \bigcup_{i=1}^r Q(\text{rk}_i) \tag{52}$$

such that

$$Q(\text{rk}_i) \cap Q(\text{rk}_j)$$

is a common face of $Q(\text{rk}_i)$ and $Q(\text{rk}_j)$ for $i \neq j$. Let us call such a decomposition *proper* if $r > 1$ and $Q(\text{rk}_i) \not\subseteq Q(\text{rk}_j)$ for all $i \neq j$. The polytope $Q(\text{rk})$ is called *indecomposable* if it does not have a proper decomposition. For a fixed base field K , a polymatroid is called *rigid* if it has only finitely many realizations over K as a subspace arrangement up to isomorphism. The work of Lafforgue implies that a realizable matroid is *rigid* if and only if its matroid base polytope is indecomposable (see [24, 25]). It is therefore of interest to know whether a given matroid polytope is indecomposable. Valutive and additive invariants can be useful to determine whether a matroid polytope is decomposable. For a valutive invariant f , we have, by the inclusion-exclusion principle

$$f(\text{rk}) = \sum_{k=1}^r (-1)^{k-1} \sum_{i_1 < i_2 < \dots < i_k} f(\text{rk}_{i_1, i_2, \dots, i_k}),$$

where $\text{rk}_{i_1, \dots, i_k}$ is the rank function whose polymatroid polytope is

$$Q(\text{rk}_{i_1}) \cap \dots \cap Q(\text{rk}_{i_k}).$$

If f is additive, then we have

$$f(\text{rk}) = \sum_{i=1}^r f(\text{rk}_i).$$

Additive invariants can also be constructed from the Billera-Jia-Reiner quasi-symmetric function (see [3]).

Conjecture 8.5 *Is \mathcal{G} universal with respect to the valutive property? I.e., is it true that for every \mathbb{Q} -linear valutive map $f : \text{PolyMat} \rightarrow V$ there exists a \mathbb{Q} -linear map $\psi : \text{QSym} \rightarrow V$ such that $\psi \circ \mathcal{G} = f$?*

8.2 The proof of Theorem 8.3

The basis vectors of \mathbb{R}^n are denoted by e_1, \dots, e_n . Let Δ be the $(n - 2)$ -dimensional simplex spanned by $e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n$.

Lemma 8.6 *Choose ε such that $0 < \varepsilon < 1$. For $v \in \mathbb{Z}^n$, and a rank function $\text{rk} : \text{Pow}(X) \rightarrow \mathbb{R}$, the following statements are equivalent.*

1. $\sum_{i=1}^s v_i = \text{rk}(\underline{s})$ for $s = 1, 2, \dots, n$;

- 2. $v \in Q(\text{rk})$, and $v + \varepsilon(e_j - e_k) \notin Q(\text{rk})$ for all $j < k$;
- 3. $(v + \varepsilon\Delta) \cap Q(\text{rk}) = \emptyset$ and $v \in Q(\text{rk})$.

Proof (1) \Rightarrow (2): Suppose that (1) holds. Suppose that $A = \{i_1, \dots, i_s\}$ with $i_1 < \dots < i_s$. Then we have

$$\text{rk}(\{i_1, \dots, i_t\}) - \text{rk}(\{i_1, i_2, \dots, i_{t-1}\}) \leq \text{rk}(\{1, 2, \dots, i_t\}) - \text{rk}(\{1, 2, \dots, i_t - 1\}) = v_{i_t} \tag{53}$$

by the submodular property of the rank function.

Summing (53) for $t = 1, 2, \dots, s$ gives

$$\text{rk}(\{i_1, \dots, i_s\}) \leq v_{i_1} + \dots + v_{i_s} = \sum_{i \in A} v_i.$$

This implies that $v \in Q(\text{rk})$. If $j < k$ and $w = v + \varepsilon(e_j - e_k)$, then we have

$$\sum_{i=1}^j w_i = \sum_{i=1}^j v_i + \varepsilon = \text{rk}(\underline{j}) + \varepsilon > \text{rk}(\underline{j}),$$

so $w \notin Q(\text{rk})$. This proves that (2) holds.

(2) \Rightarrow (1): Conversely, assume that (2) holds. A subset $S \subseteq \underline{n}$ is called *tight* if $\sum_{i \in S} v_i = \text{rk}(S)$. Clearly, \underline{n} and \emptyset are tight. If S, T are tight, then

$$\begin{aligned} \text{rk}(S \cup T) + \text{rk}(S \cap T) &\leq \text{rk}(S) + \text{rk}(T) \\ &= \sum_{i \in S} v_i + \sum_{i \in T} v_i \\ &= \sum_{i \in S \cap T} v_i + \sum_{i \in S \cup T} v_i \leq \text{rk}(S \cap T) + \text{rk}(S \cup T), \end{aligned} \tag{54}$$

so all inequalities are equalities, and $S \cup T$ and $S \cap T$ are tight as well.

Suppose that $j < k$ and set $w = v + \varepsilon(e_j - e_k)$. Because $g \notin Q(\text{rk})$, there exists a set $A_{j,k}$ such that

$$\sum_{i \in A_{j,k}} w_i > \text{rk}(A_{j,k}).$$

Since

$$\sum_{i \in A_{j,k}} v_i \leq \text{rk}(A_{j,k}),$$

we must have $j \in A_{j,k}$ and $k \notin A_{j,k}$. We obtain

$$\text{rk}(A_{j,k}) \geq \sum_{i \in A_{j,k}} v_i = \sum_{i \in A_{j,k}} w_i - \varepsilon > \text{rk}(A_{j,k}) - \varepsilon.$$

Because v is an integer vector, the first inequality is an equality and $A_{j,k}$ is tight. To prove (1) we need to show that \underline{i} is tight for $i = 0, 1, \dots, n$. We do this by induction on

i , the case $i = 0$ being trivial. Suppose that $i > 0$ and $\underline{i-1}$ is tight. Then $\underline{i-1} \cup A_{i,k}$ is tight for $k = i + 1, \dots, n$. We have

$$\underline{i} = \bigcap_{k=i+1}^n (\underline{i-1} \cup A_{i,k})$$

because $\underline{i} \subseteq \underline{i-1} \cup A_{i,k}$ for all i , and $k \notin \underline{i-1} \cup A_{i,k}$. Hence \underline{i} is tight.

(3) \Rightarrow (2): This implication is clear because $(e_j - e_k) \in \Delta$ for all $j < k$.

(2) \Rightarrow (3): Suppose $v \in Q(\text{rk})$ and $v + \varepsilon(e_j - e_k) \notin Q(\text{rk})$ for all $j < k$. Suppose that $v + \delta(e_j - e_k) \in Q(\text{rk})$ for some j, k with $j < k$ and $\delta > 0$. Set $z := e_j - e_k$. If the inequality

$$\sum_{i \in A} v_i \leq \text{rk}(A) \tag{55}$$

is an equality, then

$$\text{rk}(A) + \delta \sum_{i \in A} z_i = \sum_{i \in A} (v_i + \delta z_i) \leq \text{rk}(A)$$

because $v + \delta z \in Q(\text{rk})$. So we obtain

$$\sum_{i \in A} z_i \leq 0.$$

Therefore, we have

$$\sum_{i \in A} (v_i + \varepsilon z_i) \leq \text{rk}(A).$$

If (55) is not tight, then

$$\sum_{i \in A} v_i \leq \text{rk}(A) - 1$$

and

$$\sum_{i \in A} (v_i + \varepsilon z_i) \leq \text{rk}(A) - 1 + \varepsilon \sum_{i \in A} z_i \leq \text{rk}(A) - 1 + \varepsilon \leq \text{rk}(A).$$

So we conclude that

$$\sum_{i \in A} (v_i + \varepsilon z_i) \leq \text{rk}(A)$$

for all subsets $A \subseteq \underline{n}$. So $v + \varepsilon z \in Q(\text{rk})$, but this contradicts our assumptions. We conclude that $v + \delta(e_j - e_k) \notin Q(\text{rk})$ for every $j < k$ and every $\delta > 0$.

Suppose that v lies in the interior of a face of positive dimension of $Q(\text{rk})$. This face is parallel to $e_j - e_k$ for some $j < k$. This means that there exists a $\delta > 0$ such that $v + \delta(e_j - e_k), v - \delta(e_j - e_k) \in Q(\text{rk})$ for some $\delta > 0$. This gives a contradiction, therefore v must be a vertex of the polytope $Q(\text{rk})$. Let v_1, v_2, \dots, v_r be other vertices of $Q(\text{rk})$ such that the edges of $Q(\text{rk})$ meeting at v are vv_1, vv_2, \dots, vv_r . For every

$v_i, v - v_i$ is a positive multiple of $e_k - e_j$ for some $j < k$. This means that $Q(\text{rk})$ is contained in cone

$$C := v + \mathbb{R}_{\geq 0}(e_2 - e_1) + \mathbb{R}_{\geq 0}(e_3 - e_2) + \cdots + \mathbb{R}_{\geq 0}(e_n - e_{n-1}),$$

where $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers. We conclude that

$$(v + \varepsilon\Delta) \cap Q(\text{rk}) \subseteq (v + \varepsilon\Delta) \cap C = \emptyset.$$

So (3) follows. □

For $v \in \mathbb{Z}^n$, define a valuation $\mu_v : \mathcal{K}(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$\mu_v(h) = h(v) - \lim_{\varepsilon \downarrow 0} \chi([v + \varepsilon\Delta] \cdot h).$$

Let

$$r = (r_1, r_2, \dots, r_n),$$

where $r_i = \text{rk}(i) - \text{rk}(i - 1)$ for all i .

Corollary 8.7 *We have*

$$\mu_v([Poly(\text{rk})]) = \begin{cases} 1 & \text{if } v = r \\ 0 & \text{otherwise.} \end{cases}$$

Proof Suppose that $v = r$. By Lemma 8.6, we have $v \in Q(\text{rk})$ and $(v + \varepsilon\Delta) \cap Q(\text{rk}) = \emptyset$. Therefore, we get

$$\chi([v + \varepsilon\Delta] \cdot [Q(\text{rk})]) = \chi([(v + \varepsilon\Delta) \cap Q(\text{rk})]) = \chi([\emptyset]) = \chi(0) = 0$$

and $[Q(\text{rk})](v) = 1$, so $\mu_v([Q(\text{rk})]) = 1$.

Suppose that $v \neq r$. Assume that $v \notin Q(\text{rk})$. Since $Q(\text{rk})$ is closed, there exists a $\delta > 0$ such that

$$(v + (\varepsilon\Delta)) \cap Q(\text{rk})$$

for all ε with $0 < \varepsilon < \delta$. This implies that $\mu_v([Q(\text{rk})]) = 0$.

Suppose that $v \in Q(\text{rk})$. Then $(v + \varepsilon\Delta) \cap Q(\text{rk})$ is a closed nonempty convex polytope. Hence we have

$$\chi([v + \varepsilon\Delta] \cdot [Q(\text{rk})]) = 1.$$

Therefore, we conclude that $\mu_v([Q(\text{rk})]) = 1 - 1 = 0$. □

Proof of Theorem 8.3 The symmetric group Σ_n acts on \mathbb{R}^n by permuting the coordinates. Define

$$\mu_v^\sigma(h) = \mu_v(h \circ \sigma)$$

for every $\sigma \in \Sigma_n$ and every $h \in \mathcal{K}(\mathbb{R})$. We have that

$$\begin{aligned} & \mu_v^\sigma([Q(\text{rk})]) \\ &= \mu_v([Q(\text{rk} \circ \sigma)]) \\ &= \begin{cases} 1 & \text{if } v_i = \text{rk}(\{\sigma(1), \dots, \sigma(i)\}) - \text{rk}(\{\sigma(1), \dots, \sigma(i-1)\}) \text{ for all } i; \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (56)$$

Define

$$M_v = \sum_{\sigma \in \Sigma_n} \mu_v^\sigma.$$

From the definition of \mathcal{G} follows that

$$\mathcal{G}[\mathbf{X}] = \sum_v M_v([Q(\text{rk})])U_v.$$

From the linearity of M_v and \mathcal{G} it follows that

$$\sum_i a_i \mathcal{G}[\{1, \dots, n, \text{rk}_i\}] = 0$$

whenever

$$\sum_i a_i [Q(\text{rk}_i)] = 0.$$

This completes the proof of the theorem. □

9 Future directions

For a polymatroid \mathbf{X} we defined symmetric functions $\mathcal{P}[\mathbf{X}]$ and $\mathcal{H}[\mathbf{X}]$. In the case where the polymatroid comes from a subspace arrangement, we gave interpretations of the coefficients of these symmetric functions in terms of the Hilbert series and the minimal free resolution of the associated product ideal, and in terms of the polarized Schur functor. We hope for similar interpretations and nonnegativity results in the case where the polymatroid is not realizable (Conjecture 5.7). We also defined a quasi-symmetric function $\mathcal{G}[\mathbf{X}]$. This invariant has many interesting properties, and it specializes to $\mathcal{P}[\mathbf{X}]$, $\mathcal{H}[\mathbf{X}]$ and to the Billera-Jia-Reiner quasi-symmetric function $\mathcal{F}[\mathbf{X}]$. We would like to know whether $\mathcal{G}[\mathbf{X}]$ specializes to Speyer's invariant in [34] (Conjecture 7.8). The invariant \mathcal{G} behaves valuatively with respect to (poly-)matroid base polytope decompositions. We wonder whether \mathcal{G} is universal with this property (Conjecture 8.5).

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