J Algebr Comb (2009) 30: 447–475
DOI 10.1007/s10801-009-0172-5

Hamiltonian cycles in cubic Cayley graphs: the $\langle 2, 4k, 3\rangle$ case

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Received: 22 May 2008 / Accepted: 16 February 2009 / Published online: 10 March 2009

Abstract It was proved by Glover and Marušič (J. Eur. Math. Soc. 9:775–787, 2007), that cubic Cayley graphs arising from groups $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots\rangle$ having a $(2, s, 3)$-presentation, that is, from groups generated by an involution a and an element x of order s such that their product ax has order 3, have a Hamiltonian cycle when $|G|$ (and thus also s) is congruent to 2 modulo 4, and have a Hamiltonian path when $|G|$ is congruent to 0 modulo 4.

In this article the existence of a Hamiltonian cycle is proved when apart from $|G|$ also s is congruent to 0 modulo 4, thus leaving $|G|$ congruent to 0 modulo 4 with s either odd or congruent to 2 modulo 4 as the only remaining cases to be dealt with in order to establish existence of Hamiltonian cycles for this particular class of cubic Cayley graphs.

Keywords Hamiltonian cycle · Cayley graph · Cubic arc-transitive graph · Consistent cycle · Cyclic edge connectivity

1 Introductory remarks

In 1969, Lovász [27] asked if every finite, connected vertex-transitive graph has a Hamiltonian path, that is, a simple path going through all vertices of the graph. With

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the exception of K_2, only four connected vertex-transitive graphs that do not have a Hamiltonian cycle are known to exist. These four graphs are the Petersen graph, the Coxeter graph and the two graphs obtained from them by replacing each vertex by a triangle. The fact that none of these four graphs is a Cayley graph has led to a folklore conjecture that every Cayley graph with order greater than 2 has a Hamiltonian cycle.

Despite the fact that these questions have been challenging mathematicians for almost forty years, only partial results have been obtained thus far. For instance, it is known that connected vertex-transitive graphs of orders kp, where $k \le 5$, p^j, where $j \le 4$, and $2p^2$ contain a Hamiltonian path. Furthermore, for all of these families, except for the graphs of order $5p$, it is also known that they contain a Hamiltonian cycle (except for the above mentioned Petersen and Coxeter graph), see [1, 6, 25, 29–33, 38]. For the subclass of Cayley graphs a number of partial results are known (see for example [2, 12, 18, 19, 24, 28, 39, 40]). Also, it is known that every connected vertex-transitive graph, other than the Petersen graph, whose automorphism group contains a transitive subgroup with a cyclic commutator subgroup of prime-power order, has a Hamiltonian cycle. The result was proved in [14] and it uses results from a series of papers dealing with the same group-theoretic restrictions in the context of Cayley graphs [13, 28, 39].

It was proved in [18] that cubic Cayley graphs arising from groups having a $(2, s, 3)$-presentation, that is, for groups $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \dots \rangle$ generated by an involution a and an element x of order s such that their product ax has order 3, have a Hamiltonian cycle when $|G|$ (and thus also s) is congruent to 2 modulo 4 and have a Hamiltonian path when $|G|$ is congruent to 0 modulo 4. In this paper we proved the existence of a Hamiltonian cycle in a class of cubic Cayley graphs arising from groups having a $(2, s, 3)$-presentation when apart from $|G|$ also s is congruent to 0 modulo 4, thus leaving $|G|$ congruent to 0 modulo 4 with s either odd or congruent to 2 modulo 4 as the only remaining cases to establish existence of Hamiltonian cycles for this particular class of cubic Cayley graphs. The following is therefore the main result of this paper.

Theorem 1.1 *Let* $s \equiv 0 \pmod 4$ *be a positive integer and let* $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \dots \rangle$ *be a group with a* $(2, s, 3)$*-presentation. Then the Cayley graph* $Cay(G, \{a, x, x^{-1}\})$ *has a Hamiltonian cycle.*

For the motivation for studying this family of graphs see [18].

This article is organized as follows. In Section 2 notation and terminology is introduced and certain results, essential to the strategy of the proof of Theorem 1.1, are gathered. In Section 3 examples illustrating the method of the proof of Theorem 1.1 are given. The proof of Theorem 1.1 is carried out in Section 6. Our strategy is based on an embedding of $X = \mathrm{Cay}(G, S)$ in the closed orientable surface with s-gonal and hexagonal faces, in which we then look for a *Hamiltonian tree of faces*, that is, a tree of faces whose boundary is a Hamiltonian cycle in X. This Hamiltonian tree of faces is obtained in the following way. First, we associate to X a so called *hexagon graph* $\mathrm{Hex}(X)$, a cubic graph whose vertex set consists of all hexagons arising from the relation $(ax)^3$ with two such hexagons being adjacent if they share an edge in X (see Section 3 for more details). Second, we modify $\mathrm{Hex}(X)$ somewhat by deleting some of its vertices and edges to end up with a smaller cubic graph $\mathrm{Mod}(X)$ whose order is congruent to 2 modulo 4. Third, we find a subset of the vertex set of $\mathrm{Mod}(X)$ inducing a tree, whose complement is an independent set of vertices. This is possible by showing that $\mathrm{Mod}(X)$ is cyclically 4-edge connected (carried out in Sections 4 and 5 following a tedious analysis of various possibilities that can occur for $\mathrm{Hex}(X)$) and then using the result of Payan and Sakarovitch given in Proposition 2.1. Finally, the above tree in $\mathrm{Mod}(X)$ is then used to produce a Hamiltonian tree of faces in the embedding of X, and hence a Hamiltonian cycle in X (two examples illustrating this method are given in Section 3).

2 Preliminaries

2.1 Notation

Given a graph X we let $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}\,X$ be the vertex set, the edge set, the arc set and the automorphism group of X, respectively. For adjacent vertices u and v in X, we write $u \sim v$ and denote the corresponding edge by uv. If $u \in V(X)$ then $N(u)$ denotes the set of neighbors of u and $N_i(u)$ denotes the set of vertices at distance $i > 1$ from u. A graph X is said to be *cubic* if $|N(u)| = 3$ for any vertex $u \in V(X)$. A sequence $(u_0, u_1, u_2, \dots, u_k)$ of distinct vertices in a graph is called a *k-arc* if u_i is adjacent to u_{i+1} for every $i \in \{0, 1, \dots, k-1\}$. For $S \subseteq V(X)$ we let $X[S]$ denote the induced subgraph of X on S. By an n-cycle we shall always mean a cycle with n vertices. We will use the symbol $\mathbb{Z}_n$, both for the cyclic group of order n and the ring of integers modulo n. In the latter case, $\mathbb{Z}_n^*$ will denote the multiplicative group of units of $\mathbb{Z}_n$.

A subgroup $G \leq \mathrm{Aut}\,X$ is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* provided it acts transitively on the sets of vertices, edges and arcs of X, respectively. A graph is said to be *vertex-transitive*, *edge-transitive*, and *arc-transitive* if its automorphism group is vertex-transitive, edge-transitive and arc-transitive, respectively. An arc-transitive graph is also called *symmetric*. A subgroup $G \leq \mathrm{Aut}\,X$ is said to be *k-regular* if it acts transitively on the set of k-arcs and the stabilizer of a k-arc in G is trivial.

Given a group G and a generating set S of G such that $S = S^{-1}$ and $1 \notin S$, the *Cayley graph* $\mathrm{Cay}(G, S)$ of G relative to S has vertex set G and edge set $\{g \sim gs \mid g \in G, s \in S\}$. It is easy to see that the group G acts regularly on the vertex set G by left multiplication, and hence G may be viewed as a regular subgroup of the automorphism group of the Cayley graph $X = \mathrm{Cay}(G, S)$.

2.2 Cyclic stability and cyclic connectivity

Following [36] we say that, given a graph (or more generally a loopless multigraph) X, a subset S of $V(X)$ is *cyclically stable* if the induced subgraph $X[S]$ is acyclic, that is, a forest. The size $|S|$ of a maximum cyclically stable subset S of $V(X)$ is called the *cyclic stability number* of X.

Given a connected graph X, a subset $F \subseteq E(X)$ of edges of X is said to be *cycle-separating* if $X - F$ is disconnected and at least two of its components contain cycles.

We say that X is *cyclically k-edge-connected* if no set of fewer than k edges is cycle-separating in X. Furthermore, the *edge-cyclic connectivity* $\zeta(X)$ of X is the largest integer k not exceeding the Betti number $|E(X)| - |V(X)| + 1$ of X for which X is cyclically k-edge connected. (This distinction is indeed necessary as, for example, the theta graph Θ_2, K_4 and $K_{3,3}$ possess no cycle-separating sets of edges and are thus cyclically k-edge connected for all k, however their edge-cyclic connectivities are 2, 3 and 4, respectively.)

Bearing in mind the expression for the cyclic stability number given above the following result may be deduced from [36, Théorème 5].

Proposition 2.1 [36] *Let X be a cyclically 4-edge connected cubic graph of order n, and let S be a maximum cyclically stable subset of $V(X)$. Then $|S| = \lfloor (3n-2)/4 \rfloor$ and more precisely, the following hold.*

(i) *If $n \equiv 2 \pmod 4$ then $|S| = (3n-2)/4$, and $X[S]$ is a tree and $V(X) \setminus S$ is an independent set of vertices.*
(ii) *If $n \equiv 0 \pmod 4$ then $|S| = (3n-4)/4$, and either $X[S]$ is a tree and $V(X) \setminus S$ induces a graph with a single edge, or $X[S]$ has two components and $V(X) \setminus S$ is an independent set of vertices.*

The following result concerning cyclic edge connectivity of cubic vertex-transitive graphs is proved in [35, Theorem 17].

Proposition 2.2 [35] *The cyclic edge connectivity $\zeta(X)$ of a cubic connected vertex-transitive graph X equals its girth $g(X)$.*

2.3 Semiregular automorphisms and $I_k^s(t)$-paths

Let X be a graph and $\mathcal{W}$ a partition of the vertex set $V(X)$. Then the *quotient graph* $X_{\mathcal{W}}$ relative to the partition $\mathcal{W}$ is the graph with vertex set $\mathcal{W}$ and edge set induced naturally by the edge set $E(X)$. When $\mathcal{W}$ is the set of orbits of a subgroup H in $\mathrm{Aut}\, X$ the quotient graph $X_{\mathcal{W}}$ we will denoted by X_H. In particular, if $H = \langle h \rangle$ is generated by a single element h then we let $X_h = X_H$.

An automorphism of a graph is called (k,n)-*semiregular*, where $k \geq 1$ and $n \geq 2$ are integers, if it has k orbits of length n and no other orbit. Let $\mathcal{W}$ be the set of orbits of a semiregular automorphism $\rho \in \mathrm{Aut}\, X$. Then the subgraph of X induced by $W \in \mathcal{W}$ will be denoted by $X[W]$. Similarly, we let $X[W, W']$, $W, W' \in \mathcal{W}$, denote the bipartite subgraph of X induced by the edges having one endvertex in W and the other endvertex in W'. Moreover, for $W, W' \in \mathcal{W}$ we let $d(W)$ and $d(W, W')$, respectively, denote the valency of $X[W]$ and $X[W, W']$.

Let X be a connected graph admitting a (k,n)-semiregular automorphism ρ, where $k \geq 1$, $n \geq 2$ are integers, let $\mathcal{W} = \{W_i \mid i \in \mathbb{Z}_k\}$ be the set of orbits of ρ and let the vertices of X be labeled in such a way that $u_i^s \in W_i$ for $i \in \mathbb{Z}_k$ and $s \in \mathbb{Z}_n$. Then X may be represented by the notation of Frucht [16] emphasizing the k orbits of ρ in the following way. The k orbits of ρ are represented by k circles. The symbol n/T, where $T \subseteq \mathbb{Z}_n \setminus \{0\}$, inside a circle corresponding to the orbit W_i indicates that, for each $s \in \mathbb{Z}_n$, the vertex u_i^s is adjacent to all vertices u_i^{s+t}, $t \in T$. When $|T| \leq 2$ we use

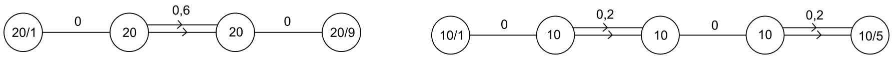

Fig. 1 The graph F080A on the left-hand side and the graph F050A on the right-hand side given in Frucht's notation relative to a (4, 20)-semiregular automorphism and a (5, 10)-semiregular automorphism, respectively

a simplified notation n/t, $n/(n/2)$ and n, respectively, when $T = \{t, -t\}$, $T = \{n/2\}$ and $T = \emptyset$. Finally, an arrow pointing from the circle representing the orbit W_i to the circle representing the orbit W_j, $j \neq i$, labeled by $y \in \mathbb{Z}_n$ means that vertices u_i^s and u_j^{s+y} are adjacent for all $s \in \mathbb{Z}_n$. When the label is 0, the arrow on the line may be omitted. Two examples, the graphs F080A and F050A, illustrating this notation are given in Figure 1. (Hereafter the notation FnA, FnB, etc. will refer to the corresponding graphs in the Foster census of all cubic arc-transitive graphs [5, 7]. This notation will be consistently used for those graphs for which no alternative standard notation is available.)

The so called $I_k^s(t)$-paths will play an important part in the subsequent sections. We define these graphs as follows. Let X be a cubic graph admitting a (k, s)-semiregular automorphism ρ, where $k \geq 2$, $s \geq 3$ are integers. Let W_i, $i \in \mathbb{Z}_k$, be the orbits of ρ and let the vertices of X be labeled in such a way that $u_i^s \in W_i$ for $i \in \mathbb{Z}_k$ and $s \in \mathbb{Z}_n$. Then X is said to be an *$I_k^s(t)$-path* if the corresponding quotient graph X_ρ is a path $W_0 W_1 \ldots W_{k-1}$, and, in particular, $N(u_0^s) = \{u_0^{s\pm 1}, u_1^s\}$, $N(u_{2i-1}^s) = \{u_{2i-2}^s, u_{2i}^s, u_{2i}^{s+2}\}$, $N(u_{2i}^s) = \{u_{2i-1}^s, u_{2i-1}^{s-2}, u_{2i+1}^s\}$, for $i \in \{1, \ldots, \lfloor (k-2)/2 \rfloor\}$, and $N(u_{k-1}^s) = \{u_{k-2}^s, u_{k-2}^{s-2}, u_{k-1}^{s+t}\}$, where $t = s/2$, if k is odd and $N(u_{k-1}^s) = \{u_{k-2}^s, u_{k-1}^{s\pm t}\}$, where $t = s/2 + 1$, if k is even. For example, the Pappus graph F018A is the $I_3^6(3)$-path, the graph F050A is the $I_5^{10}(5)$-path, but F080A is not an $I_k^s(t)$-path (see Figure 1). As for the generalized Petersen graphs (see the next subsection) note that GP(4, 1) is the $I_2^4(3)$-path, GP(8, 3) is the $I_2^8(5)$-path, and GP(12, 5) is the $I_2^{12}(7)$-path.

2.4 On cubic arc-transitive graphs

Let $n \geq 3$ be a positive integer, and let $k \in \{1, \ldots, n-1\} \setminus \{n/2\}$. The *generalized Petersen graph* $\mathrm{GP}(n, k)$ is defined to have the vertex set $V(\mathrm{GP}(n, k)) = \{u_j \mid j \in \mathbb{Z}_n\} \cup \{v_j \mid j \in \mathbb{Z}_n\}$ and the edge set $E(\mathrm{GP}(n, k)) = \{u_j u_{j+1} \mid j \in \mathbb{Z}_n\} \cup \{v_j v_{j+k} \mid j \in \mathbb{Z}_n\} \cup \{u_j v_j \mid j \in \mathbb{Z}_n\}$. Note that $\mathrm{GP}(n, k)$ is cubic, and it is easy to see that $\mathrm{GP}(n, k) \cong \mathrm{GP}(n, n-k)$. The following proposition due to Frucht, Graver and Watkins is well known.

Proposition 2.3 [17] *The only arc-transitive generalized Petersen graphs are the cube* GP(4, 1), *the Petersen graph* GP(5, 2), *the Moebius-Kantor graph* GP(8, 3), *the dodecahedron* GP(10, 2), *the Desargues graph* GP(10, 3), GP(12, 5) *and* GP(24, 5).

Note that the respective Foster census notations for the above generalized Petersen graphs are: F008A, F010A, F016A, F020A, F020B, F024A and F048A.

The following result on cubic arc-transitive graphs of girth 6 and 7 due to Feng and Nedela [15, Lemmas 4.2 and 4.3] will be used in subsequent sections.

Proposition 2.4 [15] *Let X be a connected cubic arc-transitive graph of girth $g \in \{6, 7\}$, and let c be the number of g-cycles passing through an edge of X. If $c > 2$ then X is isomorphic to one of the following five graphs: the Heawood graph* F014A, *the Moebius-Kantor graph* GP(8, 3), *the Pappus graph* F018A, *the Desargues graph* GP(10, 3), *and the Coxeter graph* F028A.

Given a graph X and $G \leq \mathrm{Aut}\, X$, a walk $\vec{D} = (u_0, \ldots, u_r)$ in X is called G-*consistent* (or just *consistent* if the subgroup G is clear from the context) if there exists $g \in G$ such that $u_i^g = u_{i+1}$ for $i \in \{0, 1, \ldots, r-1\}$. The automorphism g is called a *shunt automorphism* of $\vec{D}$. In 1971 Conway [11] proved that an arc-transitive group of automorphisms G of a cubic graph has exactly 2 orbits in its action on the set of all G-consistent cycles. (Complete proofs of this result have been provided by Biggs [4] and recently by Miklavič, Potočnik and Willson [34].)

Since Conder and Nedela [9] proved that with the exception of the Heawood graph, the Pappus graph, and the Desargues graph, a cubic arc-transitive graph of girth 6 is either 1-regular or 2-regular the following result, proved in [26], completely determines the lengths of consistent cycles in cubic arc-transitive graphs of girth 6.

Proposition 2.5 [26] *Let X be a cubic arc-transitive graph of girth* 6. *Then the following statements hold.*

(i) *X is the Heawood graph* F014A, *the Moebius-Kantor graph* GP(8, 3), *the Pappus graph* F018A *or the Desargues graph* GP(10, 3), *then the corresponding* Aut X*-consistent cycles are, respectively, of length* 6 *and* 8, *of length* 8 *and* 12, *of length* 6 *and* 12, *or of length* 6 *and* 10.
(ii) *$X \not\cong$* GP(8, 3) *is* 2*-regular and* Aut X*-consistent cycles are of even length equal to* 6 *and $s \geq 8$; moreover if $s \equiv 2 \pmod 4$ then X is either the $I^s_{s/2}(s/2)$-path or the $I^s_{s/6}(s/2)$-path, and if $s \equiv 0 \pmod 4$ then X is either the $I^s_{s/2}(s/2+1)$-path or the $I^s_{s/6}(s/2+1)$-path.*
(iii) *X is* 1*-regular and all* Aut X*-consistent cycles are of length* 6.

When dealing with hamiltonian properties of cubic Cayley graphs arising from groups having $(2, s, 3)$-presentations it is important to note that such groups act 1-regularly on the associated hexagon graphs. It is therefore useful to know whether a given cubic arc-transitive graph contains a 1-regular subgroup or not. Bearing this in mind we end this subsection with two propositions that will be needed in the analysis of possible structures of hexagon graphs given in Section 4. The first one was proved in [18, Proposition 3.4], whereas the second one may be extracted from [23].

Proposition 2.6 [18] *The only cubic arc-transitive graphs of girth less than* 6 *whose automorphism group admits a* 1*-regular subgroup are the theta graph Θ_2, the complete graph K_4, the complete bipartite graph $K_{3,3}$, the cube* GP(4, 1) *and the dodecahedron* GP(10, 2).

Proposition 2.7 [23] *Let $\tilde{X}$ be a connected regular $\mathbb{Z}_r$-cover of the dodecahedron* F020A. *If $\tilde{X}$ is arc-transitive*, *then $r = 1, 2, 3, 4, 6$ or* 12. *Moreover, $\tilde{X}$ is isomorphic to* GP(10, 2), F040A, F060A, F080A, F120B, *or* F240C, *respectively.*

3 Examples illustrating the method

The methods used in this paper began for the authors with [19]. Let G be a group having a $(2, s, 3)$-presentation where $s \geq 3$, that is, $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \dots\rangle$ generated by an involution a and an element x of order s such that their product ax has order 3, and let X be the Cayley graph $\mathrm{Cay}(G, S)$ of G relative to the set of generators $S = \{a, x, x^{-1}\}$. As observed in [18], X has a canonical Cayley map given by an embedding in the closed orientable surface of genus

$$g = 1 + (s - 6)|G|/12s \qquad (1)$$

whose faces are $|G|/s$ disjoint s-gons and $|G|/3$ hexagons. This map is given by using the same rotation of the x, a, x^{-1} edges at every vertex and results in one s-gon and two hexagons adjacent at each vertex. (A Cayley map is an embedding of a Cayley graph onto an oriented surface having the same cyclic rotation of generators around each vertex. Cayley maps have been studied extensively, see [3, 8, 21, 22, 37].) We associate with $X = Cay(G, S)$ the so called *hexagon graph* $\mathrm{Hex}(X)$ whose vertex set consists of all the hexagons in X arising from the relation $(ax)^3$, with two hexagons adjacent in $\mathrm{Hex}(X)$ if they share an edge in X. It is easily seen that $\mathrm{Hex}(X)$ is isomorphic to the orbital graph of the left action of G on the set of left cosets $\mathcal{H}$ of the subgroup $H = \langle ax\rangle$, arising from the suborbit $\{aH, x^{-1}H, ax^2H\}$ of length 3. More precisely, the graph has vertex set $\mathcal{H}$, with adjacency defined as follows: a coset yH, $y \in G$, is adjacent to the three cosets $yaH(= yxH)$, $yx^{-1}H$ and $yax^2H(= yaxaH)$. Clearly, G acts 1-regularly on $\mathrm{Hex}(X)$, and so $\mathrm{Hex}(X)$ is a cubic arc-transitive graph. Observe also that the s-cycle $(H, xH, x^2H, \dots, x^{s-1}H, H)$ in $\mathrm{Hex}(X)$ is a G-consistent cycle with the shunt automorphism $x \in G$.

Now let $s \equiv 0 \pmod 4$ and let $\mathcal{S}$ be a fixed s-gonal face in X embedded in an orientable surface of genus g (see (1)). Denote the vertices in the hexagon graph $\mathrm{Hex}(X)$ which correspond to the s hexagons around $\mathcal{S}$ (in the associated Cayley map of X) in such a way that $(H, xH, x^2H, \dots, x^{s-1}H, H)$ is the corresponding s-cycle in $\mathrm{Hex}(X)$. Then the *modified hexagon graph* $\mathrm{Mod_H}(X)$ is a graph with the vertex set

$$V(\mathrm{Mod_H}(X)) = V(\mathrm{Hex}(X)) \setminus \{x^{2i+1}H, x^{2i}ax^2H \mid i \in \mathbb{Z}_s\} \\ \cup \{ax^2H, x^{s/2}ax^2H\} \cup \{\mathcal{S}\}$$

and the edge set

$$E(\mathrm{Mod_H}(X)) = E(X[V(\mathrm{Hex}(X)) \setminus \{x^{2i+1}H, x^{2i}ax^2H \mid i \in \mathbb{Z}_s\} \\ \cup \{ax^2H, x^{s/2}ax^2H\}]) \cup \{x^{2i}H\mathcal{S} \mid i \in \mathbb{Z}_s\}.$$

The construction of the modified hexagon graph $\mathrm{Mod_H}(X)$ can be viewed in the context of the original cubic Cayley graph X as follows. Color all hexagonal faces in X, choose an s-gonal face $\mathcal{S}$ in X, and then uncolor every other hexagonal face surrounding $\mathcal{S}$. Then uncolor the hexagonal faces that share an edge with a colored hexagonal face surrounding $\mathcal{S}$, except for a pair of antipodal hexagonal faces. Finally, color the s-gonal face $\mathcal{S}$. The modified hexagon graph $\mathrm{Mod_H}(X)$ is then the graph whose vertex set consists of all the colored faces, with two faces being adjacent in $\mathrm{Mod_H}(X)$ if they share an edge in X.

The method for constructing Hamiltonian cycles in the cubic Cayley graph X consists in identifying in a modified hexagon graph $\mathrm{Mod_H}(X)$, a subset S of vertices $V = V(\mathrm{Mod_H}(X))$ inducing a tree, the complement $V \setminus S$ of which is an independent set of vertices. This tree gives rise to a tree of faces in the Cayley map X (in particular, as we shall see, one face in this tree of faces is an s-gonal face and all the others are hexagonal). This tree of faces is a topological disk and its boundary is the desired Hamiltonian cycle. This method will work for all but two cases when the hexagon graph is isomorphic to GP(12, 5) or GP(24, 5). The method for constructing Hamiltonian cycles in the corresponding cubic Cayley graph can be found with a somewhat different method (see Figures 4 and 5 in Section 5).

We give here examples of two cubic Cayley graph, arising from groups having a $(2, s, 3)$-presentation where $s \equiv 0 \pmod 4$.

Example 3.1 In the right-hand side picture in Figure 2 we show a tree of faces, whose boundary is a Hamiltonian cycle in the spherical Cayley map of the Cayley graph X of the group $G = S_4$ with a $(2, 4, 3)$-presentation $\langle a, x \mid a^2 = x^4 = (ax)^3 = 1\rangle$, where $a = (12)$ and $x = (1234)$. The two pictures next to it show this same tree in the corresponding modified hexagon graph $\mathrm{Mod_H}(X)$, and the corresponding modified hexagon graph $\mathrm{Mod_H}(X)$ as a graph of faces in the spherical Cayley map of X, respectively. The left-hand side picture in Figure 2 shows the associated hexagon graph, which is the cube GP(4, 1), and the picture next to it the theta graph Θ_2, the graph obtained from the modified hexagon graph $\mathrm{Mod_H}(X)$ by suppressing the vertices of valency 2.

Example 3.2 In the right-hand side picture in Figure 3 we give the genus 2 Cayley map of a Cayley graph X of the group $G = Q_8 \rtimes S_3$ with a $(2, 8, 3)$-presentation $\langle a, x \mid a^2 = x^8 = (ax)^3 = 1, \ldots\rangle$. (For the description of the elements a and x we refer the reader to [18, Example 2.4].) Note that this map is given by identifying antipodal octagons (as numbered). Note also that the sixth octagon is omitted

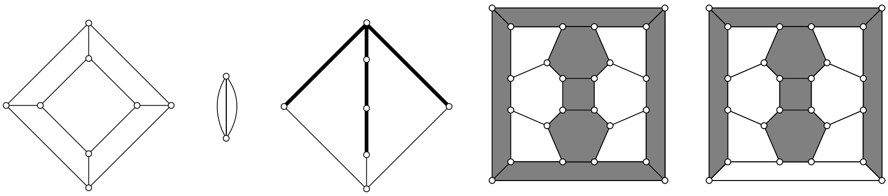

Fig. 2 A Hamiltonian tree of faces in the spherical Cayley map of the Cayley graph of S_4 giving rise to a Hamiltonian cycle, the associated modified hexagon graph $\mathrm{Mod_H}(X)$ shown in two ways, as a graph of faces and as a graph, the graph obtained from $\mathrm{Mod_H}(X)$ by suppressing the vertices of valency 2, and the hexagon graph $\mathrm{Hex}(X)$

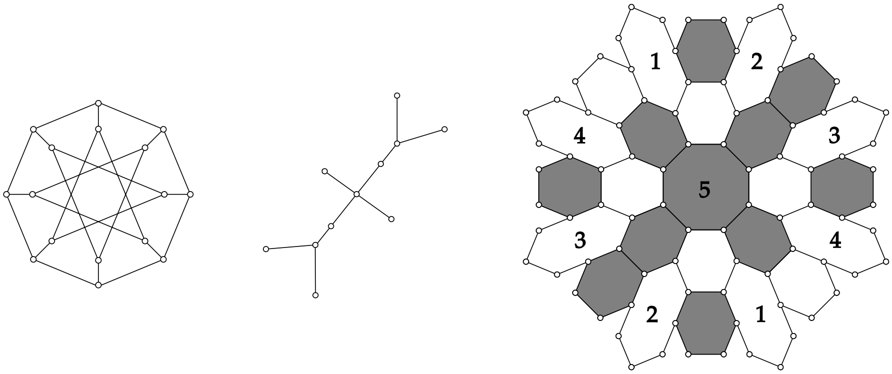

Fig. 3 A Hamiltonian tree of faces in the genus 2 Cayley map of a Cayley graph of $Q_8 \rtimes S_3$, the associated hexagon graph and the corresponding modified hexagon graph

from this picture, but occurs at the outer edges of the outer hexagons. We show a tree of faces in this map whose boundary is a Hamiltonian cycle. The middle picture shows the corresponding modified hexagon graph $\mathrm{Mod_H}(X)$ and the left-hand side picture shows the corresponding hexagon graph, the Moebius-Kantor graph $\mathrm{GP}(8,3)$.

4 On cubic arc-transitive graphs admitting a 1-regular subgroup

Throughout this section let $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \dots\rangle$ be a group having a $(2, s, 3)$-presentation and let Y be the orbital graph of the left action of G on the set of left cosets $\mathcal{H}$ of the subgroup $H = \langle ax\rangle$ arising from the self-paired suborbit $\{aH, x^{-1}H, ax^2H\}$ of length 3. More precisely, the graph has vertex set $\mathcal{H}$, with adjacency defined as follows: a coset yH, $y \in G$, is adjacent to the three cosets $yxH(= yaH)$, $yx^{-1}H$ and $yax^2H(= yaxaH)$. Clearly, G acts 1-regularly on Y. (Hereafter, the elements of G will thus also be thought of as automorphisms of Y.) Conversely, let Y be a cubic arc-transitive graph admitting a 1-regular action of a subgroup G of $\mathrm{Aut}\,Y$. Let $v \in V(Y)$ and let h be a generator of $H = G_v \cong \mathbb{Z}_3$. Then there must exist an element $a \in G$ such that $G = \langle a, h\rangle$ and such that Y is isomorphic to the orbital graph of G relative to the suborbit $\{aH, haH, h^2aH\}$. Moreover, a short computation shows that a may be chosen to be an involution, and letting $x = ah$ we get the desired presentation for G. Therefore each cubic arc-transitive graph admitting a 1-regular subgroup can be obtained as the orbital graph of a group having a $(2, s, 3)$-presentation.

Observe that the s-cycle $\mathcal{C} = (H, xH, x^2H, \dots, x^{s-1}H, H)$ in Y is a G-consistent cycle with the shunt automorphism $x \in G$. A thorough analysis of all possible local structures of the part of Y in the immediate vicinity of $\mathcal{C}$ with respect to the orbits of the shunt automorphism $x \in G$ is given below.

We define the *first ring* R_1 as the set of vertices of the s-cycle $\mathcal{C}$ in Y, that is, $R_1 = \{x^i H \mid i \in \mathbb{Z}_s\}$. Then we let the *second ring* $R_2 = \{x^i ax^2 H \mid i \in \mathbb{Z}_s\}$ be the set of vertices adjacent to the vertices in R_1. Next, the *third ring* R_3 is the set of vertices adjacent to the vertices in R_2 which do not belong to $R_1 \cup R_2$. Analogously, we can define the t-th ring R_t for any positive integer t. Let $\mathcal{R}$ denote the set of all these rings in Y.

Let as remark that when Y is the hexagon graph $\mathrm{Hex}(X)$ of a cubic Cayley graph X arising from a group having a $(2, s, 3)$-presentation, the vertices in the first ring R_1 correspond to the s hexagons surrounding a fixed s-gonal face in the Cayley map of X, and the hexagons adjacent to these hexagons make up R_2. In general, however, these two rings are not necessarily different.

In Lemmas 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6 we deal with the orbital graph, denoted by Y through this section, associated with a 1-regular action of the group $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots\rangle$ on the set of left cosets of the subgroup $H = \langle ax\rangle$ relative to the suborbit $\{aH, x^{-1}H, ax^2H\}$ of length 3. The structure of this graph will be described in terms of the quotient Y_x and the different possibilities that may occur for the first three rings $R_1, R_2, R_3 \in \mathcal{R}$. Throughout the proofs we will frequently use the fact that the product of the two generators a and x of G is of order 3, that is,

$$(ax)^3 = 1. \tag{2}$$

Lemma 4.1 *One of the following occurs*:

(i) *either* $R_1 \cap R_2 = \emptyset$; *or*
(ii) $R_2 = R_1$ *and* $Y \cong \theta_2$ *in which case* $G = \langle a, x | a^2 = x^6 = (ax)^3 = 1, a = x^3\rangle$; *or*
(iii) $R_2 = R_1$ *and* $Y \cong K_{3,3}$ *in which case* $G = S_3 \times \mathbb{Z}_3 = \langle a, x | a^2 = x^6 = (ax)^3 = 1, ax^2 = x^2a\rangle$.

Proof Assume first that $a \in \langle x\rangle$. Then s is even and $a = x^{s/2}$. From (2) it follows that $s = 6$, and therefore $Y \cong \theta_2$. We may therefore assume that $a \notin \langle x\rangle$. Suppose that the intersection $R_1 \cap R_2 \neq \emptyset$. Then we must have $x^j ax^2 H = H$ for some $j \in \mathbb{Z}_s$. In other words, $x^j ax^2 \in H = \{1, ax, (ax)^2\}$. But then Y is a circulant and we must have $j = s/2$, and so Y is the wheel $W_s = \mathrm{Cay}(\mathbb{Z}_s, \{\pm 1, s/2\})$. However, the only arc-transitive wheels W_s are the complete graph K_4 and the complete bipartite graph $K_{3,3}$. In the first case we have that $s = 4$ and $j = 2$, which implies that $x^2ax^2 \in H$. But by short computation one can easily see that this is impossible. Hence Y is the complete bipartite graph $K_{3,3}$, and thus $s = 6$ and $j = 3$. Now the fact that $x^3ax^2 \in H$ brings about three possibilities. First, if $x^3ax^2 = 1$ then $a \in \langle x\rangle$, a contradiction. Second, if $x^3ax^2 = ax$ then $x^3axa = 1$, and so $x^3x^{-1}ax^{-1} = 1$. Hence, $xa = 1$, again a contradiction. Finally, if $x^3ax^2 = (ax)^2 = x^{-1}a$ then $ax^2a = x^{-4} = x^2$ and so a centralizes x^2. Therefore $G = \langle a, x \mid a^2 = x^6 = (ax)^3 = 1, ax^2 = x^2a\rangle$ and a short computation shows that $G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_6 \cong S_3 \times \mathbb{Z}_3$. □

In view of Lemma 4.1 the first two rings R_1 and R_2 either coincide in which case the graph Y is either θ_2 or $K_{3,3}$, or they are disjoint. In the rest of this section we will deal with this second possibility. The following two easy consequences of this assumption will be used throughout the rest of this section. First,

$$a \notin \langle x \rangle. \tag{3}$$

And second, since, when $R_1 \cap R_2 = \emptyset$, for all i the coset $x^i aH \in R_1$ is different from $ax^2H \in R_2$, we have that

$$\langle x^2 \rangle \text{ is not normal in } G. \tag{4}$$

Lemma 4.2 *Let $R_1 \cap R_2 = \emptyset$. If W is an orbit of $x \in G$ of length s then every orbit adjacent to W is of length either s or $s/3$. Moreover, if $|R_2| \neq s$ then $s = 3$ and $Y \cong K_4$, or $s = 6$ and $Y \cong$* GP(4, 1).

Proof Let W be an orbit of $x \in G$ of length s. Suppose that there exists an orbit W' adjacent to W of length different from s. Let yH be a vertex in W'. Using the facts that Y is cubic and that x is of order s one can easily see that this orbit is either of length $s/2$ or of length $s/3$. Moreover, the former cannot occur since then $yH = x^{s/2}yH$ forces $x^{s/2}$ to be of order 3, which is clearly impossible. Thus W' is of length $s/3$.

Now let us assume that $|R_2| \neq s$. Then, by the above result, 3 divides s and $|R_2| = s/3$. If $s = 3$ it follows that $|R_2| = 1$ giving us the complete graph K_4. Next, if $s = 6$ then $|R_2| = 2$ giving us the cube GP(4, 1). Finally, if $s > 6$ then the fact that $(H, ax^2H, x^{s/3}H, x^{s/3+1}H, xax^2H, xH, H)$ is a 6-cycle in Y and Proposition 2.5 combined together imply that Y is the generalized Petersen graph GP(8, 3) and $s = 12$. But then $ax^2H = x^4ax^2H$ which implies that $x^{-2}ax^4ax^2 \in H$. If $x^{-2}ax^4ax^2 = 1$ then $x^4 = 1$, a contradiction. If $x^{-2}ax^4ax^2 = ax$ then $x^{-1}ax^4axax = 1$, and so using (2) we get $axa = x^3$, contradicting (4). Finally, if $x^{-2}ax^4ax^2 = x^{-1}a$ then $ax^{-1}ax^4ax^2 = 1$, and so $xax^5ax^2 = 1$, and so $ax^5a = x^{-3}$, a contradiction since the element on the left-hand side has order 12 and the element on the right-hand side has order 4. □

Let $N(R_2) = \{v \in N(u) \mid u \in R_2\}$ be the set of neighbors of the vertices in R_2. The next lemma describes the graph Y under the additional assumption that $R_2 \cap N(R_2) \neq \emptyset$.

Lemma 4.3 *Let $R_1 \cap R_2 = \emptyset$ and $R_2 \cap N(R_2) \neq \emptyset$. Then either*

(i) *$\mathcal{R} = \{R_1, R_2\}$, and Y is isomorphic to one of the seven arc-transitive generalized Petersen graphs*: GP(4, 1), GP(5, 2), GP(8, 3), GP(10, 2), GP(10, 3), GP(12, 5) *and* GP(24, 5); *or*
(ii) *$\mathcal{R} = \{R_1, R_2, R_3\}$, $s = 6$ and Y is isomorphic to the Heawood graph* F014A.

Proof Since for the complete graph K_4 and $s = 3$, and for the cube and $s = 6$, we have that $R_2 \cap N(R_2) = \emptyset$, Lemma 4.2 implies that $|R_2| = s$.

Recall that $R_2 = \{x^i ax^2H \mid i \in \mathbb{Z}_s\}$. Consider $N(ax^2H) = \{H, ax^3H, (ax^2)^2H\}$ and assume that $R_2 \cap N(R_2) \neq \emptyset$. If $\mathcal{R} = \{R_1, R_2\}$ then clearly $Y[R_2]$ is of valency 2. Therefore Y is an arc-transitive generalized Petersen graph, and by Proposition 2.3, one of the graphs in part (i).

We may now assume that $Y[R_2]$ is of valency 1 and so isomorphic to $s/2K_2$. In particular, s is even in this case. Observe also that $s \neq 4$. Namely, for $s = 4$ the graph Y is of girth less than or equal to 4, and so Proposition 2.6 and Lemma 4.1 combined together imply that Y is isomorphic to the cube GP(4, 1) for which $Y[R_2] \neq s/2K_2$. Assuming now that $s = 6$ we have that the 2-arc $HxHx^2H$ lies on at least two different 6-cycles: $(H, xH, x^2H, x^3H, x^4H, x^5H, H)$ and $(H, xH, x^2H, x^3H, x^3ax^2H, ax^2H, H)$. By Proposition 2.4 it follows that $Y \cong$ F014A. We may therefore assume that $s \geq 8$. We will now show that this case cannot occur.

Since $Y[R_2] \cong s/2K_2$ it follows that the vertex $ax^2H \in R_2$ is adjacent to $x^{s/2}ax^2H \in R_2$. Therefore, either $x^{s/2}ax^2H = ax^3H$ or $x^{s/2}ax^2H = (ax^2)^2H$. We analyze each of these two cases separately. For the sake of convenience we will use a shorthand notation: $J = x^{s/2}H \in R_1$ and $K = x^{s/2+1}H \in R_1$.

Case 1 $x^{s/2}ax^2H = ax^3H$.

Then $x^{-3}ax^{s/2}ax^2 \in H$. If $x^{-3}ax^{s/2}ax^2 = 1$, then using (2) we get $x^{s/2} = axa = x^{-1}ax^{-1}$, and so $a = x^{s/2+2}$, contradicting (3). If $x^{-3}ax^{s/2}ax^2 = ax$, then we have $x^{s/2} = ax^3ax^{-1}a$, and so $(ax^3ax^{-1}a)^2 = 1$ which implies that $ax^2a = x^{-2}$, contradicting (4). Therefore we are left with the possibility $x^{-3}ax^{s/2}ax^2 = x^{-1}a$, which gives us

$$x^{s/2} = ax^2ax^{-2}a. \tag{5}$$

Now by assumption, J is a neighbor of ax^3H, and so either $J = ax^4H$ or $J = ax^3ax^2H$. But by (5) the first possibility implies that $ax^2H = x^{-2}aH \in R_1$, a contradiction since $ax^2 \in R_2$. Hence $J = ax^3ax^2H$, and so $K = xJ$, a neighbor of J, is either ax^3ax^3H or $ax^3(ax^2)^2H$.

Subcase 1.1 $K = ax^3ax^3H$.

Then $ax^3ax^3H = K = xJ = xax^3ax^2H$ which implies that $(ax^3ax^2)^{-1}x^{-1} \times (ax^3ax^2)x \in H$. Using (2) we get that $x^{-2}ax^{-2}ax^4ax^3 \in H$. If $x^{-2}ax^{-2}ax^4ax^3 = 1$ then, by computation, $ax^3 = x^3a$, and so $ax^3H = x^3aH = x^4H$. But $ax^3H \in R_2$ and $x^4H \in R_1$, contradicting $R_1 \cap R_2 = \emptyset$. Suppose next that $x^{-2}ax^{-2}ax^4ax^3 = ax$. Then $x^{-2}ax^{-2}ax^4ax^2 = a$, and so $x^{-2}ax^{-2}(ax^2)x^2ax^2 = a$. Consequently, ax^2 is an involution, and thus $ax^2a = x^{-2}$, contradicting (4). Finally, if $x^{-2}ax^{-2}ax^4ax^3 = x^{-1}a$ then by computation, using (2), we get $ax^5a = x^{-5}$. Thus $ax^5H = x^{-5}aH \in R_1$. But ax^5H is a neighbor of $ax^4H \in R_3$ not belonging to $\mathcal{R} \setminus \{R_1, R_2\}$, a contradiction.

Subcase 1.2 $K = ax^3ax^2ax^2H$.

Then $ax^3ax^2ax^2H = K = xJ = xax^3ax^2H$, and so $x^{-2}ax^{-3}ax^{-1}ax^3ax^2ax^2 \in H$ which implies that $z = x^{-2}ax^{-2}ax^4ax^2ax^2 = (x^4)^{ax^2ax^2} \in H$. Therefore, since $s \notin \{4, 6\}$ this implies that $z \in \{ax, x^{-1}a\}$, and moreover x is of order 12. Now, by (5), $x^6 = ax^2ax^{-2}a$, and so

$$ax^{-2}a = x^{-2}ax^6, \tag{6}$$

as well as $x^4 = ax^2ax^{-2}ax^{-2}$. Substituting (6) and the latter into the above expression for z, we get that $z = x^6ax^{-4}ax^2$. If $z = ax$ then, by computation, we get that $ax^5a = x^5$, and so $ax^5H = x^5aH \in R_1$ which leads to the same contradiction as in Subcase 1.1. Finally, if $z = x^{-1}a$ then we have that $x^6ax^{-4}ax^2a = x^{-1}$ and using (6) we get $ax^2a = x^3$, which is clearly impossible.

Case 2 $x^{s/2}ax^2H = (ax^2)^2H$.

Then $x^{-2}ax^{-2}ax^{s/2}ax^2 \in H$. If $x^{-2}ax^{-2}ax^{s/2}ax^2 = 1$, then we get $x^{-2}ax^{s/2} = 1$, and so $a = x^{s/2-2}$, contradicting (3). If $x^{-2}ax^{-2}ax^{s/2}ax^2 = x^{-1}a$, then using (2) we get $x^{s/2} = ax^2axax^{-2}a = x^{-1}ax^{-4}a$, and so $(x^{-1}ax^{-4}a)^2 = 1$. Using (2) we get $ax^2a = x^{-2}$, contradicting (4). Therefore we are left with the possibility $x^{-2}ax^{-2}ax^{s/2}ax^2 = ax$, which implies

$$x^{s/2} = ax^2ax^3ax. \tag{7}$$

Hence $J = x^{s/2}H = ax^2ax^3axH = ax^2ax^3H$, and so $K = xJ$, a neighbor of J, is either ax^2ax^4H or $ax^2ax^3ax^2H$.

Subcase 2.1 $K = ax^2ax^4H$.

Then $ax^2ax^4H = K = xJ = xax^2ax^3H$, and so $x^{-4}ax^{-2}axax^2ax^3 \in H$. Since $(axax^2)^2 = 1$ we get that $x^{-4}ax^{-2}axax^2ax^3 = x^{-4}ax^{-4}ax^2 \in H$. Therefore $z = x^{-2}ax^{-4}ax^2 \in x^2H$, and so either $z = x^2$, or $z = x^2ax$ or $z = xa$. In the first case, by computation, $ax^2a = x^{-4}$, contradicting (4). In the second case using (2) we get that $x^{-2}ax^{-5}ax^{-3} = 1$, implying $ax^5a = x^{-5}$, which leads to the same contradiction as in Subcase 1.1. In the last case $(x^{-4})^{ax^2} = z = xa$ is of order 3, and so since $s \notin \{4, 6\}$ we have that $s = 12$. Now using (7) we get that $ax^2a = x^5ax^{-3}$. Plugging this into $(x^{-4})^{ax^2} = z = xa$ we get that $x^{-2}axax^{-3} = x$, and consequently $a = x^8 \in \langle x\rangle$, contradicting (3).

Subcase 2.2 $K = ax^2ax^3ax^2H$.

Then $ax^2ax^3ax^2H = K = xJ = xax^2ax^3H$, and so $x^{-3}ax^{-2}ax^{-1}ax^2ax^3ax^2 \in H$. Using (2) we get that $z = x^{-2}ax^4ax^3ax^2 \in H$. If $z = 1$ then $a = x^{-7}$, contradicting (3). If $z = ax$ then again using (2) we get that

$$x^{-2}ax^4ax^2ax^{-1} = 1, \tag{8}$$

and so $x^{-2}ax^4ax^2 = xa$ which implies that x^4 is of order 3. As in Subcase 2.1 we have that $s = 12$, and thus by (7) we have $x^6 = ax^2ax^3ax$. But then $ax^2a = x^5ax^{-3}$ and plugging this into (8) we get that $ax^9a = x^6$, a contradiction since x^9 and x^6 are clearly not of the same order. Finally, if $z = x^{-1}a$ then $ax^{-1}ax^4ax^3ax^2 = 1$ and using (2) we get $xax^5ax^3ax^2 = 1$ and consequently $xax^3(x^2ax^3ax)x = 1$. Applying (7) we get that $xax^3ax^{s/2+1} = 1$, and so $ax^3 = x^{-s/2-2}a$. But then $ax^3H = x^{-s/2-2}aH = x^{-s/2-1}H \in R_1$, a contradiction since $ax^3H \in R_3$. This completes the proof of Lemma 4.3. □

We next consider the situation where $R_2 \cap N(R_2) = \emptyset$, that is, R_2 is an independent set of vertices, and the third ring R_3 is a single orbit of $x \in G$, which implies that there exists $j \in \mathbb{Z}_s$ such that $x^j ax^3 H = (ax^2)^2 H$.

Lemma 4.4 *Let $R_1 \cap R_2 = \emptyset$ and $R_2 \cap N(R_2) = \emptyset$. If R_3 is a single orbit of $x \in G$, then one of the following occurs*:

(i) *either a centralizes x^5, $s \in \{5, 10, 15, 20, 30, 60\}$, and Y has four orbits of $x \in G$ and is a regular $\mathbb{Z}_{s/5}$-cover of the dodecahedron* GP(10, 2), *and thus one of the following graphs*: GP(10, 2), F040A, F060A, F080A, F120B, F240C; *or*
(ii) $(ax^2)^2(ax^{-2})^2 = 1$, *and either $s = 6$ or $s \geq 8$ is even*; *in the latter case if $s \equiv 2 \pmod 4$ then Y is either the $I^s_{s/2}(s/2)$-path or the $I^s_{s/6}(s/2)$-path, and if $s \equiv 0 \pmod 4$ then Y is either the $I^s_{s/2}(s/2+1)$-path or the $I^s_{s/6}(s/2+1)$-path*; *or*
(iii) $(ax^2)^2(ax^{-2})^2 = x^{s/2}$.

Proof Since R_3 is a single orbit of $x \in G$ there exists $j \in \mathbb{Z}_s$ such that $x^j ax^3 H = (ax^2)^2 H$. This fact gives us three possibilities to consider. As we shall see, the first one will give us a direct contradiction, the second one the six graphs in (i), and the third one will give us either (ii) or (iii).

By assumption $x^j ax^3 \in (ax^2)^2 H$. Suppose first that $x^j ax^3 = (ax^2)^2$. Then $x^j ax = ax^2 a$, and so by (2) $x^j = ax^2ax^{-1}a = ax^3ax$, and finally $x^{j-1} = ax^3a$. But then $ax^3H = x^{j-1}aH = x^jH \in R_1$, a contradiction since $ax^3H \in R_3$. Suppose next that $x^j ax^3 = (ax^2)^2(ax)^2$. Then using (2) we see that $x^jax^3 = (ax^2)^2x^{-1}a = ax^2axa = ax^2x^{-1}ax^{-1} = axax^{-1}$ and so $x^j = axax^{-4}a = x^{-1}ax^{-5}a$ and so $x^{j+1} = ax^{-5}a$. But then $x^{j+1} = ax^{-4}x^{-1}a = ax^{-4}axax$ which implies that $axax^{-4}a = ax^{-4}axa$ and thus $ax^{-5} = x^{-5}a$. In other words, $a \in C_G(\langle x^5 \rangle)$, and hence the subgroup $K = \langle x^5 \rangle$ is normal in G. By (4) it follows that 5 is a divisor of s. Thus we may assume that $s = 5r$ for some positive integer r. Then $G = \langle a, x \mid a^2 = x^{5r} = (ax)^3 = 1 \rangle$ implies that $G/K = \langle aK, xK \mid (aK)^2 = (xK)^5 = (axK)^3 = 1 \rangle$ has a (2, 5, 3)-presentation, forcing $G/K \cong A_5$, for the latter is the only group having a (2, 5, 3)-presentation. Using the fact that G/K must act 1-regularly on the quotient graph Y_K we get that Y_K is a cubic arc-transitive graph of order 20. There are two such graphs, the dodecahedron GP(10, 2) and the Desargues graph GP(10, 3) (see [5]). But since the latter does not admit a 1-regular subgroup it follows that Y_K is isomorphic to GP(10, 2), and so Y is a regular $\mathbb{Z}_r$-cover of GP(10, 2). Applying Proposition 2.7 we get that $s = 5r \in \{5, 10, 15, 20, 30, 60\}$, as required. The regular $\mathbb{Z}_4$-cover of GP(10, 2) is the graph F080A shown in Figure 1.

We now turn to the last remaining case. Suppose that $x^jax^3 = (ax^2)^2ax$. It follows that $x^{j-2} = (ax^2)^2(ax^{-2})^2$. Also, using (2) we get $x^{j+1} = xax^2ax^2ax^{-2}a = ax^{-3}ax^{-3}a$, and so $x^{j-2} = (ax^{-3})^3$. We now show that $x^{j-2} = (ax^2)^2(ax^{-2})^2$ is either 1 or $x^{s/2}$ (and hence $j \in \{2, s/2+2\}$).

First, using the fact that $x^{j-2} = (ax^2)^2(ax^{-2})^2$, we get that $ax^{j-2} = x^{2-j}a$. Next using the fact that $x^{j-2} = (ax^{-3})^3$, we get $a(ax^{-3})^3 = (x^3a)^3a$, which gives $x^{-3}ax^{-3}ax^{-3} = x^3ax^3ax^3$, and so $(x^{-3}ax^{-3}ax^{-3})^2 = 1$. Therefore $x^{-3}ax^{-3} \times ax^{-6}ax^{-3}ax^{-3} = 1$ which implies that $x^{j+1}x^{-6}x^{j+1} = 1$, and thus $x^{2j-4} = 1$, forcing $x^{j-2} \in \{1, x^{s/2}\}$.

If $x^{j-2} = x^{s/2}$ then part (iii) is satisfied. Thus we may assume that $x^{j-2} = 1$. Then $(ax^2)^2(ax^{-2})^2 = 1$ and one can easily see that $(H, xH, x^2H, x^2ax^2H, x^2ax^3H, ax^2H, H)$ is a 6-cycle in Y. If $s = 6$ part (ii) is satisfied. Assume that $s > 6$. Since none of the five cubic arc-transitive graphs of girth less than 6 has the above presentation, it follows that Y has girth 6. Since the s-cycle in R_1 is G-consistent and $|\mathcal{R}| > 2$. Proposition 2.5 and Lemma 4.2 combined together imply that $s \geq 8$ is even and one of the following two possibilities occur: if $s \equiv 2 \pmod 4$ then Y is either the $I^s_{s/2}(s/2)$-path or the $I^s_{s/6}(s/2)$-path, and if $s \equiv 0 \pmod 4$ then Y is either the $I^s_{s/2}(s/2+1)$-path or the $I^s_{s/6}(s/2+1)$-path. This completes the proof of Lemma 4.4. □

Lemma 4.5 *Let $R_1 \cap R_2 = \emptyset$, $R_2 \cap N(R_2) = \emptyset$, and let R_3 be a single orbit of $x \in G$. If $R_3 \cap N(R_3) \neq \emptyset$ then either $s = 6$ and Y is the Pappus graph* F018A, *or $s = 18$ and Y is the $I^{18}_3(9)$-path.*

Proof All conclusions of Lemma 4.4 hold. We only need to show that because of the additional assumption that $R_3 \cap N(R_3) \neq \emptyset$ parts (i) and (iii) of Lemma 4.4 cannot occur and that part (ii) leads to the Pappus graph F018A and the $I^{18}_3(9)$-path.

By regularity Y has only three orbits of $x \in G$, all of length s, and is thus of order $3s$. Moreover, since Y is cubic we have that $(ax^2)^2H \in R_3$ is adjacent to $x^{s/2}(ax^2)^2H \in R_3$. Now part (i) of Lemma 4.4 cannot occur since the regular $\mathbb{Z}_r$-covers of $\mathrm{GP}(10,2)$ have four orbits of $x \in G$. Next, assume that part (ii) of Lemma 4.4 holds. Using the fact that $k = 3$ we see that Y is either the $I^6_3(3)$-path (the Pappus graph F018A) or the $I^{18}_3(9)$-path. Finally assume that part (iii) of Lemma 4.4 holds. Then $x^{s/2+2}ax^3H = (ax^2)^2H$ and $x^{s/2+2}ax^2H = ax^2ax^3H$. The latter implies that the two neighbors ax^2H and ax^2ax^3H of $(ax^2)^2H$ are both in R_2, forcing $x^{s/2}(ax^2)^2H = (ax^2)^3H$. Hence $x^{s/2}(ax^2)^2H = (ax^2)^2(ax^{-2})^2(ax^2)^2H = (ax^2)^3H$, and so $(ax^{-2})^2(ax^2)^2 \in ax^2H$. We have three possibilities. If $(ax^{-2})^2(ax^2)^2 = ax^2$ then $(ax^{-2})^2ax^2 = 1$ which implies that $a = x^{s/2-2}$, contradicting (3). Next, if $(ax^{-2})^2(ax^2)^2 = ax^2x^{-1}a$ then $x^{-2}ax^{-2}ax^2ax^2 = xa$, a contradiction since a and xa are clearly not of the same order. Finally, suppose that $(ax^{-2})^2(ax^2)^2 = ax^2ax$. Then by computation, using (2) we get that $ax^3a = x^{-6}$, and hence $\langle x^3\rangle$ is normal in G, which in view of (4) implies that 3 divides s. Thus the quotient graph $Y_{\langle x^3\rangle}$ is a cubic arc-transitive graph of order $3s/(s/3) = 9$, a contradiction. □

Using the notation from [20], the *Y-graph* $Y(s,1,l,k)$, where $l,k \in \mathbb{Z}_s$ are coprime with s, is the graph admitting a $(4,s)$-semiregular element with orbits $W_i = \{u_i^j \mid j \in \mathbb{Z}_s\}$, $i \in \mathbb{Z}_4$, and edge set $E = \{u_0^ju_0^{j+1} \mid j \in \mathbb{Z}_s\} \cup \{u_1^ju_1^{j+l} \mid j \in \mathbb{Z}_s\} \cup \{u_2^ju_2^{j+k} \mid j \in \mathbb{Z}_s\} \cup \{u_0^ju_3^j \mid j \in \mathbb{Z}_s\} \cup \{u_1^ju_3^j \mid j \in \mathbb{Z}_s\} \cup \{u_2^ju_3^j \mid j \in \mathbb{Z}_s\}$. For example, $Y(7,1,2,4)$ is the well known Coxeter graph F028A. If in the above definition the condition $l,k \in \mathbb{Z}_s$ be coprime with s is omitted, then the graph $Y(s,1,l,k)$ is called the *generalized Y-graph*. (In spite of notational similarity, the Y-graphs and the generalized Y-graphs are not to be confused with the more general orbital graph Y.)

The next lemma gives a complete classification of those graphs Y for which $|\mathcal{R}| = 3$.

Lemma 4.6 *Let $R_1 \cap R_2 = \emptyset$. If $|\mathcal{R}| = 3$ then one of the following occurs*:

(i) $s = 6$, *and either* $Y \cong$ F014A *or* $Y \cong$ F018A *or* $Y \cong$ GP(12, 5);
(ii) $s = 12$ *and* $Y \cong$ GP(24, 5);
(iii) $s = 18$ *and* Y *is the* $I_3^{18}(9)$-*path*.

Proof Clearly, $|R_1| = s = |R_2|$. If $R_2 \cap N(R_2) \neq \emptyset$ then Lemma 4.3 implies that $s = 6$ and $Y \cong$ F014A. We may therefore assume that $N(R_2) \cap R_2 = \emptyset$. If R_3 is a single orbit of $x \in G$ then Lemma 4.5 implies that either $s = 6$ and Y is the Pappus graph, or $s = 18$ and Y is the $I_3^{18}(9)$-path. Hence, we may assume that R_3 is a union of two orbits, say W and W', of $x \in G$, and that $ax^3H \in W$ and $(ax^2)^2H \in W'$.

We first show that W and W' are both of length s and thus that Y is of order $4s$. Suppose on the contrary that at least one of W and W' is not of length s. Then by Lemma 4.2 this orbit is of length $s/3$ and so s is divisible by 3. If both are of length $s/3$ then $(ax^2H, ax^3H, x^{s/3}ax^2H, (ax^2)^2H, ax^2H)$ is a 4-cycle in Y, a contradiction. Hence we may assume that just one of W and W' is not of length s, and so Y is of order $3s + s/3 = 10s/3$. Furthermore $s \geq 9$. Namely, if $s \in \{3, 6\}$ then the Petersen graph GP(5, 2), the dodecahedron GP(10, 2) and the Desargues graph GP(10, 3) are the only possible candidates for the graph Y. But GP(5, 2) and GP(10, 3) do not admit a 1-regular subgroup whereas GP(10, 2) has consistent cycles of length 5 and 10, contradicting the fact that 3 divides s. We now distinguish two cases.

Case 1 $|W| = s/3$.

Then $ax^3H = x^{s/3}ax^3H = x^{2s/3}ax^3H$ and $N(ax^3H) = \{ax^2H, ax^4H, ax^3ax^2H\} = \{ax^2H, x^{s/3}ax^2H, x^{2s/3}ax^2H\}$. Therefore, $ax^4H = x^{rs/3}ax^2H$ where $r \in \{1, 2\}$, and thus $x^{-4}ax^{rs/3}ax^2 \in H$. If $x^{-4}ax^{rs/3}ax^2 = 1$ then by computations $ax^2a = x^{rs/3}$, contradicting (4). Further, if $x^{-4}ax^{rs/3}ax^2 = ax$ then $ax^5a = x^{rs/3-1}$, and so $\langle x^5 \rangle$ is normal in G. In view of (4) 5 divides s and Y is a regular $\mathbb{Z}_{s/5}$-cover of a graph of order $10s/(3s/5) = 50/3 \notin \mathbb{Z}$, a contradiction. Finally, if $x^{-4}ax^{rs/3}ax^2 = x^{-1}a$ then we have that $x^{-3}ax^{rs/3}ax^2 = a$ is an involution, and so $(x^{-3}ax^{rs/3}ax^2)^2 = 1$. This implies that $ax^{rs/3+2}a = x^{-rs/3-2}$, and so $\langle x^{rs/3+2} \rangle$ is normal in G. Since 3 divides s the greatest common divisor $d = (rs/3 + 2, s)$ of $rs/3 + 2$ and s belongs to the set $\{1, 2, 3, 6\}$. Therefore Y is a regular $\mathbb{Z}_{s/d}$-cover of a cubic arc-transitive graph of order $10s/(3s/d) = 10d/3$, where $d \in \{3, 6\}$, and so isomorphic to one of the graphs: GP(5, 2), GP(10, 2) and GP(10, 3). But GP(5, 2) and GP(10, 3) are not possible for the same reason given in the paragraph preceding Case 1, whereas GP(10, 2) is not possible since it does not have consistent cycles of length $d = 6$. (Consistent cycles in GP(10, 2) are of length 5 and 10.)

Case 2 $|W'| = s/3$.

Then $(ax^2)^2H = x^{s/3}(ax^2)^2H = x^{2s/3}(ax^2)^2H$ and so

$$N((ax^2)^2H) = \{ax^2H, (ax^2)^2xH, (ax^2)^3H\} = \{ax^2H, x^{s/3}ax^2H, x^{2s/3}ax^2H\}.$$

It follows that $(ax^2)^2xH = x^{rs/3}ax^2H$, where $r \in \{1, 2\}$, and hence $x^{-2}ax^{rs/3}ax^2 \in ax^3H$. We have three possibilities. If $x^{-2}ax^{rs/3}ax^2 = ax^3$ then by computation, using (2), we get $axa = x^{rs/3+1}$, and so $\langle x\rangle$ is normal in G, contradicting (4). If $x^{-2}ax^{rs/3}ax^2 = ax^3x^{-1}a = ax^2a$ then x^2 is of order 3, and therefore $s = 6$, a contradiction. If $x^{-2}ax^{rs/3}ax^2 = ax^3ax$ then

$$x^{rs/3} = ax^2ax^3ax^{-1}a. \tag{9}$$

Repeated use of (2) gives us $x^{rs/3} = ax^2ax^4ax$ and then

$$x^{rs/3} = ax^{-2}ax^3a. \tag{10}$$

Combining together (9) and (10) we get $ax^3a = x^{-4}ax^4$, and thus x^3 is an involution. Therefore, $s = 6$, a contradiction.

We have therefore shown that W and W' are both of length s as claimed, and consequently Y is of order $4s$. Thus we have $W = \{x^iax^3H \mid i \in \mathbb{Z}_s\}$ and $W' = \{x^i(ax^2)^2H \mid i \in \mathbb{Z}_s\}$. Since by assumption $|\mathcal{R}| = 3$ the following three possibilities can occur: $d(W, W') = 0$ and $d(W) = d(W') = 2$; or $d(W, W') = 1$ and $d(W) = d(W') = 1$; or $d(W, W') = 2$ and $d(W) = d(W') = 0$.

Suppose first that $d(W, W') = 1$ and $d(W) = d(W') = 1$. Then x^iax^3H and $x^i(ax^2)^2H$, $i \in \mathbb{Z}_s$, are adjacent to $x^{i+s/2}ax^3H$ and $x^{i+s/2}(ax^2)^2H$, respectively. Moreover, since $d(W, W') = 1$ there exists $j \in \mathbb{Z}_s$ such that for each $i \in \mathbb{Z}_s$ the vertex x^iax^3H is adjacent to $x^{i+j}(ax^2)^2H$. But then one can easily see that

$$(ax^3H, x^j(ax^2)^2H, x^{j+s/2}(ax^2)^2H, x^{s/2}ax^3H, ax^3H)$$

is a 4-cycle in Y, a contradiction.

Suppose now that $d(W, W') = 0$ and $d(W) = d(W') = 2$. Then Y is one of the generalized Y-graphs $Y(s, 1, l, k)$ defined in the paragraph preceding the statement of this lemma. We will show that none of these graphs can occur. Since $d(W) = d(W') = 2$ there exists $j \in \mathbb{Z}_s$ such that $ax^4H = x^jax^3H$ and so $x^{-4}ax^jax^3 \in H$. First, if $x^{-4}ax^jax^3 = 1$ then $x^a = x^j$, contradicting (4). Second, if $x^{-4}ax^jax^3 = x^{-1}a$ then $x^j = ax^3ax^{-3}a$, which implies that $j = s/2$, contradicting the fact that $d(W) = d(W') = 2$. Third, if $x^{-4}ax^jax^3 = ax$ then $x^jax^2 = ax^4a$, and thus

$$x^j = ax^4ax^{-2}a, \tag{11}$$

and moreover, $x^jax^2H = ax^5H$. It follows that either $x^jH = ax^6H$ or $x^jH = ax^5ax^2H$.

Suppose first that $x^jH = ax^6H$. Then $x^{-j}ax^6 \in H$ and hence three possibilities can occur. First, if $x^{-j}ax^6 = 1$ then $a = x^{j-6}$, contradicting (3). Second, if $x^{-j}ax^6 = ax$ then $ax^5a = x^j$ and thus $ax^5H = x^jaH = x^{j+1}H \in R_1$, a contradiction since $ax^5H \in R_2$. Finally, if $x^{-j}ax^6 = x^{-1}a$ then $ax^6a = x^{j-1}$ and so $\langle x^6\rangle$ is normal in G. In view of (4), $\langle x^6\rangle \neq \langle x\rangle$ and $\langle x^6\rangle \neq \langle x^2\rangle$. Therefore, Y is a $\mathbb{Z}_{s/d}$-cover of a cubic arc-transitive graph of order $4s/(s/d) = 4d$ where $d \in \{3, 6\}$. But $d = 3$ is not possible since there is no cubic arc-transitive graph Z of order $4d = 12$ (see [5]). If $d = 6$ then Y is a $\mathbb{Z}_{s/6}$-cover of an arc-transitive generalized Y-graph Z of order $4d = 24$. But since GP(12, 5), the only cubic arc-transitive graph of order 24, is not a generalized Y-graph, this case cannot occur.

Now suppose that $x^jH = ax^5ax^2H$. Then $x^{-j}ax^5ax^2 \in H$. First, if $x^{-j}ax^5ax^2 = 1$ then $ax^5a = x^{j-2}$ and $ax^5H = x^{j-2}aH = x^{j-1}H \in R_1$, a contradiction since $ax^5H \in R_2$. Second, if $x^{-j}ax^5ax^2 = ax$ then $x^{-j}ax^5axa = 1$. Using (2) we get that $ax^4a = x^{j+1}$, and so $ax^4aH = x^{j+1}H \in R_1$, contradicting the fact that $ax^4aH \in R_2$. And finally, if $x^{-j}ax^5ax^2 = x^{-1}a$ then $ax^5ax^2 = x^{j-1}a$. Applying (11) we get $ax^5ax^2 = x^{-1}ax^4ax^{-2}aa = x^{-1}ax^4ax^{-2}$, and so $xax^5 = ax^4ax^{-4}a$ which implies that $xax^5xax^5 = 1$, and consequently $\langle x^6\rangle$ is normal in G. As in the preceding paragraph this gives a contradiction. This completes the case in which $d(W, W') = 0$ and $d(W) = d(W') = 2$.

We may therefore assume that $d(W, W') = 2$ and $d(W) = d(W') = 0$. Then there exists $j \in \mathbb{Z}_s$ such that $ax^4H = x^j(ax^2)^2H$ and so $x^{-2}ax^{-2}ax^{-j}ax^4 \in H$. First, if $x^{-2}ax^{-2}ax^{-j}ax^4 = 1$ then $ax^{-2}ax^{-j}ax^2 = 1$, and so $x^{-j}ax^2 = ax^2a$, forcing $x^{-j}ax^2H = ax^2aH$. But this is impossible since $x^{-j}ax^2H \in R_2$ and $ax^2aH = ax^3H \in R_3$.

Second, if

$$x^{-2}ax^{-2}ax^{-j}ax^4 = ax \tag{12}$$

then by (2) we have $x^{-1}ax^{-2}ax^{-j}ax^4 = xax = ax^{-1}a$, and so $ax^{-1}ax^{-2}ax^{-j}ax^4 = x^{-1}a$. Again using (2) twice, we get $xax^{-1}ax^{-j}ax^4 = x^{-1}a$ and $x^2ax^{-j+1}ax^4 = x^{-1}a$. Rewriting we get $ax^{-j+1}a = x^{-3}ax^{-4}$, forcing $ax^{-j} = x^{-3}ax^{-4}ax^{-1} = x^{-3}ax^{-3}axa$, and so $ax^{-j}a = x^{-3}ax^{-3}ax$. From (12), $x^{-2}ax^{-5}ax^{-3}ax^4 = a$, and thus $x^{-4}ax^{-3}ax^4 = xax^2a = ax^{-1}axa$, forcing x^{-3} to be an involution which implies that $s = 6$. It follows that Y is of order 24 and thus $Y \cong \mathrm{GP}(12, 5)$.

Finally, if $x^{-2}ax^{-2}ax^{-j}ax^4 = x^{-1}a$ and so $x^{-1}ax^{-2}ax^{-j}ax^4 = a$, we get, by repeated use of (2), that $ax^6a = x^{j-1}$ and thus that $\langle x^6\rangle$ is normal in G. In view of (4), $\langle x^6\rangle \neq \langle x\rangle$ and $\langle x^6\rangle \neq \langle x^2\rangle$. Therefore, Y is a $\mathbb{Z}_{s/d}$-cover of a cubic arc-transitive graph of order $4s/(s/d) = 4d$ where $d \in \{3, 6\}$. But $d = 3$ is not possible since there is no cubic arc-transitive graph Z of order $4d = 12$ (see [5]). If $d = 6$ then Y is a $\mathbb{Z}_{s/6}$-cover of a cubic arc-transitive graph Z of order $4d = 24$, and so $Z \cong \mathrm{GP}(12, 5)$. Since $d(W, W') = 2$ there also exists some $i \in \mathbb{Z}_s$ such that $ax^3ax^2H = x^i(ax^2)^2H$. Then $x^i(ax^2)^2 \in ax^3ax^2H$ and three possibilities need to be considered. First, if $x^i(ax^2)^2 = ax^3ax^2$ then $axa = x^i$, contradicting (4). Second, if $x^i(ax^2)^2 = ax^3ax^2ax$ then $x^iax^2axa = ax^3ax^2$, and by a repeated use of (2) we get $x^{i-1}a = ax^3ax^4$, and so $x^{i-1}ax^{-6} = ax^3ax^{-2}$. Since $x^{j-1} = ax^6a$, this gives us $x^{i-j}ax^2 = ax^3a$ which implies that $x^{i-j}ax^2H = ax^3aH = ax^4H$, a contradiction since $x^{i-j}ax^2H \in R_2$ and $ax^4 \in R_3$. We may therefore assume that the third possibility occurs and hence $x^i(ax^2)^2 = ax^3ax^2x^{-1}a$. Then

$$x^i = ax^3axax^{-2}ax^{-2}a \tag{13}$$

and using (2) we get

$$x^i = ax^2ax^{-3}ax^{-2}a. \tag{14}$$

Squaring this equation we get that $x^{2i} = ax^2ax^{-6}ax^{-2}a$, and using the fact that $ax^6a = x^{j-1}$ we obtain $x^{2i} = x^{-6}$ which implies that $i = -3$ or $i = s/2 - 3$. If $i = -3$ then $x^{-3} = x^i = ax^3axax^{-2}ax^{-2}a$. By computation, $x^{-2}ax^{-3}ax^{-3}ax^2 =$

$axax^{-1}a$ which implies that $x^{-3}ax^{-3}$ is an involution. It follows that

$$ax^6a = x^{-6}. \tag{15}$$

Since, by (14), $ax^2 = x^{-3}ax^2ax^3a$ we get that $x^2ax^4ax^3ax^2 = x^{-1}a$ which implies that

$$x^4 = ax^{-3}ax^{-2}ax^{-3}a. \tag{16}$$

Since, by (2), $axa = x^{-1}ax^{-1}$ using (15) we get $x^{-6} = (x^{-1}ax^{-1})^6$ and so $x^{-4} = (ax^{-2})^5a$. Using (16) we see that $ax^3ax^2ax^3a = x^{-4} = (ax^{-2})^5a$, and so $x^5ax^2ax^5 = (ax^{-2})^3a$, and consequently $x^6ax^2ax^6 = x(ax^{-2})^3ax$. Plugging (15) into this equation we get $ax^{-10}a = x(ax^{-2})^3ax$, and so using (2) we see that $x^{-8} = ax^{-3}ax^{-2}ax^{-3}a$. Now (16) implies that $x^{-8} = x^4$ and therefore $s = 12$ and Y is of order $4s = 48$, which implies that $Y \cong \mathrm{GP}(24, 5)$. Suppose finally that $i = s/2 - 3$. Then combining (2) and (13) we have that $x^{-3}ax^{s/2-3}$ is an involution which implies that $ax^{s/2-6}a = x^{6-s/2}$. Squaring this equation we get that

$$ax^{-12}a = x^{12}. \tag{17}$$

Using (2) we now get that $x^{-12} = ax^{12}a = (x^{-1}ax^{-1})^{12}$, and so $x^{-10} = (ax^{-2})^{11}a$. On the other hand, by (17), we have $x^{-10} = x^2ax^{12}a$. Hence $x^2ax^{12}a = x^{-10} = (ax^{-2})^{11}a$ which implies that $ax^2ax^{14} = x^{-2}(ax^{-2})^9a$. Since $ax^2a = x^{-1}ax^{-2}ax^{-1}$ we get that $x^{-1}ax^{-2}ax^{13} = x^{-2}(ax^{-2})^9a$, and so $(x^2a)^9xax^{-2}ax^{13} = a$. By repeated use of (2) we have $x^{-1}ax^{-12}ax^{13} = 1$, and thus $ax^{-12}a = x^{-12}$. The latter combined together with (17) implies that $s = 24$ and Y is of order $4s = 96$. There exist two cubic arc-transitive graphs of order 96, F096A and F096B (see [5, 10]). However, none of these two graphs can occur. Namely, the latter does not have consistent 24-cycles, whereas the former is 2-regular of girth 6, and thus Proposition 2.5 implies that the quotient Y_x is a path. This completes the proof of Lemma 4.6. □

Lemmas 4.1, 4.3, 4.4, 4.5 and 4.6 combined together imply the following corollary.

Corollary 4.7 *Let $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots\rangle$ be a group with a $(2, s, 3)$-presentation, let $H = \langle ax\rangle$, let Y be the orbital graph of G associated with its 1-regular action on the set of left cosets of H relative to the suborbit $\{aH, x^{-1}H, ax^2H\}$ of length 3, and let $\mathcal{R}$ be the set of rings in Y. If $s \equiv 0 \pmod 4$ then the following hold.*

(i) *If $|\mathcal{R}| = 2$ then either $s = 4$ and $Y \cong \mathrm{GP}(4, 1)$, or $s = 8$ and $Y \cong \mathrm{GP}(8, 3)$, or $s = 12$ and $Y \cong \mathrm{GP}(12, 5)$, or $s = 24$ and $Y \cong \mathrm{GP}(24, 5)$.*
(ii) *If $|\mathcal{R}| = 3$ then $s = 12$ and $Y \cong \mathrm{GP}(24, 5)$.*
(iii) *If $|\mathcal{R}| \geq 3$ and R_3 is a single orbit of $x \in G$ then $|\mathcal{R}| > 3$, and one of the following holds:*
  (a) *$|\mathcal{R}| = 4$ and Y is a regular $\mathbb{Z}_{s/5}$-cover of the dodecahedron $GP(10, 2)$, where either $s = 20$ and $Y \cong$ F080A or $s = 60$ and $Y \cong$ F240C; or*
  (b) *Y is either the $I^s_{s/2}(s/2 + 1)$-path or the $I^s_{s/6}(s/2 + 1)$-path; or*

(c) $x^{s/2} = (ax^2)^2(ax^{-2})^2$.

(iv) *In all other cases* $|\mathcal{R}| > 3$ *and* R_3 *is a union of two orbits of* $x \in G$.

The following result on cyclic edge-connectivity of cubic arc-transitive graphs will be of crucial importance in the subsequent sections.

Lemma 4.8 *Let* $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots\rangle$ *be a group with a* $(2, s, 3)$*-presentation, let* $H = \langle ax\rangle$*, and let* Y *be the orbital graph of* G *associated with its* 1*-regular action on the set of left cosets of* H *relative to the suborbit* $\{aH, x^{-1}H, ax^2H\}$ *of length* 3*. Then the following statements hold.*

(i) *If the cyclic connectivity of* Y *is* $k \leq 5$*, then it is the theta graph* Θ_2*, the complete graph* K_4*, the complete bipartite graph* $K_{3,3}$*, the cube* GP(4, 1) *or the dodecahedron* GP(10, 2).

(ii) *If the cyclic connectivity of* Y *is* 6*, then one of the following holds*:

(a) $s = 6$*, or*

(b) $Y \cong$ GP(8, 3)*, or*

(c) $s \geq 8$ *is even, and moreover, if* $s \equiv 2 \pmod 4$ *then* Y *is either the* $I^s_{s/2}(s/2)$*-path or the* $I^s_{s/6}(s/2)$*-path, and if* $s \equiv 0 \pmod 4$ *then* Y *is either the* $I^s_{s/2}(s/2+1)$*-path or the* $I^s_{s/6}(s/2+1)$*-path.*

(iii) *If the cyclic connectivity of* Y *is* 7*, then* $s = 7$.

(iv) *In all other cases* Y *is cyclically* 8*-edge-connected.*

Proof Since $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots\rangle$ acts 1-regularly on the graph Y, Proposition 2.2 and Proposition 2.6 combined together imply (i).

Now suppose that Y is cyclically 6-edge-connected and that $s \neq 6$. Then, by Proposition 2.2, Y is of girth 6. Furthermore, since the s-cycle in R_1 is G-consistent and $s \neq 6$, Proposition 2.5 and Lemma 4.2 combined together imply that Y is either GP(8, 3) or GP(10, 3) or an $I^s_k(t)$-path where either $k = s/2$ or $k = s/6$, $t = s/2$ if k is odd, and $t = s/2 + 1$ if k is even. But GP(10, 3) is excluded since it does not admit a 1-regular subgroup. This proves (ii).

Next suppose that Y is cyclically 7-edge-connected. By Proposition 2.2, Y is of girth 7. Moreover, since the automorphism group of the Coxeter graph F028A does not contain a 1-regular subgroup, Proposition 2.4 implies that each 2-arc in Y lies on the unique 7-cycle. Let $m = |V(Y)|$. Then Y has $3m/2$ edges and $3m$ 2-arcs. It follows that Y has $3m/7$ different 7-cycles. Clearly, G acts transitively on the set of 7-cycles in Y. Since G is 1-regular it is of order $|G| = 3m$, and a standard counting argument gives us that the stabilizer of a 7-cycle in G is isomorphic to $\mathbb{Z}_7$. Choose an arbitrary 7-cycle C. Since 7 is a prime there exists a one-step rotation of C, and consequently C is G-consistent. Hence Y has G-consistent cycles of length 7 and s, and because of the 1-regularity of G we have that $s = 7$, completing the proof of Lemma 4.8. □

Since the s-cycle in the first ring R_1 of Y is G-consistent, and since consistent cycles in GP(10, 2) are of length 5 and 10, Lemma 4.8 gives us the following corollary.

Corollary 4.9 *Let $s \equiv 0 \pmod 4$, let $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots\rangle$ be a group with a $(2, s, 3)$-presentation, let $H = \langle ax\rangle$, and let Y be the orbital graph of G associated with its 1-regular action on the set of left cosets of H relative to the suborbit $\{aH, x^{-1}H, ax^2H\}$ of length 3. Then the following hold.*

(i) *The cyclic connectivity of Y is 4 if and only if it is the cube* GP(4, 1).
(ii) *The cyclic connectivity of Y is 6 if and only if $s \geq 8$ and Y is either the Moebius-Kantor graph* GP(8, 3) *or the $I^s_{s/2}(s/2 + 1)$-path or the $I^s_{s/6}(s/2 + 1)$-path.*
(iii) *In all other cases Y is cyclically 8-edge-connected.*

5 The modified hexagon graph

Throughout this section let X be the Cayley graph $\mathrm{Cay}(G, S)$ of $G = \langle a, x \mid a^2 = x^s = (ax)^3 = 1, \ldots\rangle$ relative to the set of generators $S = \{a, x, x^{-1}\}$, where $s \equiv 0 \pmod 4) \geq 4$, and let $\mathrm{Hex}(X)$ and $\mathrm{Mod_H}(X)$ be the corresponding hexagon graph and the modified hexagon graph, respectively. Since G acts 1-regularly on $\mathrm{Hex}(X)$ and since the s-cycle in $\mathrm{Hex}(X)$ which corresponds to the s hexagons around the fixed s-gonal face $\mathcal{S}$ in X is G-consistent, the results from Section 4 about the so called rings may be applied to $\mathrm{Hex}(X)$.

Recall that the modified hexagon graph $\mathrm{Mod_H}(X)$ is obtained from $\mathrm{Hex}(X)$ by first deleting all "odd" vertices in the first ring R_1, all "even" vertices (with the exception of two antipodal vertices) in the second ring R_2, and then by adding an extra vertex representing the central s-gonal face $\mathcal{S}$ (inside R_1) and finally by connecting this vertex to all "even" vertices in R_1. The modified hexagon graph $\mathrm{Mod_H}(X)$ may be interpreted as the graph whose vertices are certain faces in the associated Cayley map of X: the s-gonal face $\mathcal{S}$ and all the hexagons of X except for the hexagons corresponding to vertices $x^{2i+1}H$, $i \in \mathbb{Z}_s$ and $x^{2i}ax^2H$, $i \in \mathbb{Z}_s \setminus \{0, s/4\}$, in $\mathrm{Hex}(X)$. Further, Corollary 4.7 implies that the hexagons deleted from $\mathrm{Hex}(X)$ in order to obtain $\mathrm{Mod_H}(X)$ form an independent set of vertices in $\mathrm{Hex}(X)$. Consequently, every vertex of X lies on at least one face (in the associated Cayley map) which corresponds to some vertex in $\mathrm{Mod_H}(X)$. In particular, when $\mathrm{Mod_H}(X)$ is a tree it is a Hamiltonian tree of faces in the associated Cayley map, that is, its boundary gives rise to a Hamiltonian cycle in X.

Observe that Corollary 4.7 implies that $\mathrm{Mod_H}(X)$ is connected except when $s = 12$ and $\mathrm{Hex}(X) \cong \mathrm{GP}(12, 5)$, or $s = 24$ and $\mathrm{Hex}(X) \cong \mathrm{GP}(24, 5)$. In the first case $\mathrm{Mod_H}(X)$ is a tree union two independent vertices, and in the second case $\mathrm{Mod_H}(X)$ is a tree union eight independent vertices. However, the reader can check that in both cases by adding a suitable set of vertices of $\mathrm{Hex}(X)$ to $\mathrm{Mod_H}(X)$ a new graph can be obtained from $\mathrm{Mod_H}(X)$ which has an induced tree whose complement is an independent set of vertices, and thus giving rise to a Hamiltonian cycle in X (see Figures 4 and 5).

If $s = 12$ and $\mathrm{Hex}(X) \cong \mathrm{GP}(24, 5)$ then $\mathrm{Mod_H}(X)$ is the graph shown on the right-hand side in Figure 6. In this graph there exists an induced tree whose complement is an independent set of vertices giving rise to a Hamiltonian cycle in X (see Figure 6).

All these combined together with Examples 3.1 and 3.2 give us the following proposition.

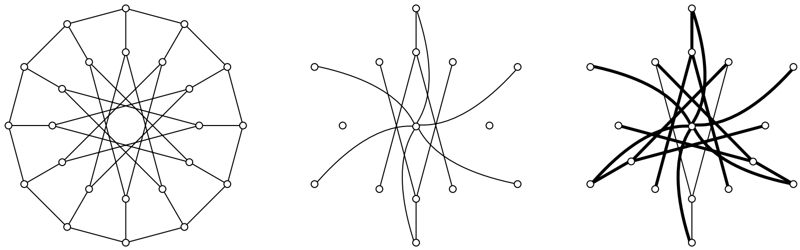

Fig. 4 The generalized Petersen graph GP(12, 5), the modified hexagon graph for $s = 12$ and a new modified hexagon graph with the induced tree whose complement is a vertex

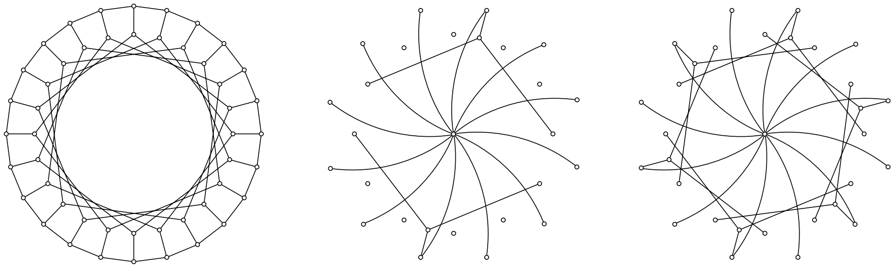

Fig. 5 The generalized Petersen graph GP(24, 5), the modified hexagon graph for $s = 24$ and a new modified hexagon graph

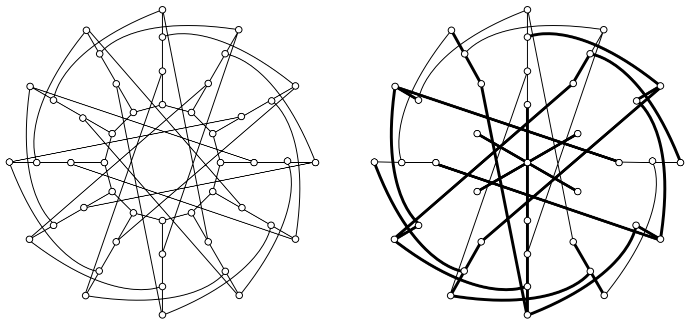

Fig. 6 The generalized Petersen graph GP(24, 5) and the modified hexagon graph $\mathrm{Mod_H}(X)$ for $s = 12$ with the induced tree whose complement is a set of five independent vertices

Proposition 5.1 *If* $\mathrm{Hex}(X)$ *is the generalized Petersen graph then* X *has a Hamiltonian cycle.*

Note that with the exception of the vertex $\mathcal{S}$, which is of valency $s/2$, all vertices in $\mathrm{Mod_H}(X)$ are of valency 1, 2 or 3, and that among the vertices adjacent to $\mathcal{S}$ all but two are of valency 1. Now let $\mathrm{Mod}(X)$ be the graph obtained from $\mathrm{Mod_H}(X)$ by deleting the vertices of valency 1 and by suppressing the vertices of valency 2. The obtained graph $\mathrm{Mod}(X)$ is called the *modified graph*. Observe, that whenever

$\mathrm{Mod}_{\mathrm{H}}(X)$ is connected and contains a cycle, $\mathrm{Mod}(X)$ is a connected cubic graph (see also Example 3.1). The following lemma tells us that either $\mathrm{Hex}(X)$ is the generalized Petersen graph, or the modified graph $\mathrm{Mod}(X)$ is cubic of order congruent to 2 modulo 4.

Lemma 5.2 *If* $\mathrm{Hex}(X)$ *does not belong to the family of the generalized Petersen graphs, then* $\mathrm{Mod}(X)$ *is a connected cubic graph of order congruent to* 2 *modulo* 4.

Proof Let $\mathcal{R}$ be the set of rings in $\mathrm{Hex}(X)$. By Corollary 4.7 we may assume that $|\mathcal{R}| \geq 3$, and furthermore it may be seen that $\mathrm{Mod}_{\mathrm{H}}(X)$ is connected and contains a cycle, and thus $\mathrm{Mod}(X)$ is a connected cubic graph. It remains to show that its order is congruent to 2 modulo 4. We will do this by analyzing each of the possibilities arising from parts (iii) and (iv) of Corollary 4.7.

If $\mathrm{Hex}(X)$ is such that R_3 is a single orbit of x then, by Corollary 4.7, $|\mathcal{R}| \geq 4$, and $\mathrm{Mod}_{\mathrm{H}}(X)$ contains the vertex representing the central s-gon $\mathcal{S}$, $s/2$ vertices from the first ring R_1 of which $s/2-2$ are of valency 1 and two are of valency 2, $s/2+2$ vertices from the second ring R_2 of which $s/2$ are of valency 2 and two are of valency 3, and all the vertices from the other rings. However, upon deletion of vertices of valency 1 and suppressing the vertices of valency 2 in $\mathrm{Mod}_{\mathrm{H}}(X)$ we get that the modified graph $\mathrm{Mod}(X)$ has no vertex from R_1, 2 vertices from R_2, $s/2$ vertices from R_3, $s/2+4$ vertices from R_4 and all the vertices from the other rings (if they exist). In other words, $|V(\mathrm{Mod}(X))| = |V(\mathrm{Hex}(X))| - s - (s-2) - s/2 - (s/2 - 4) = |V(\mathrm{Hex}(X))| - 3s + 6$ which is clearly congruent to 2 modulo 4, since $s \equiv 0 \pmod 4$.

Finally, if R_3 is a union of two orbits of $x \in G$ then by deleting the vertices of valency 1 and suppressing the vertices of valency 2 in $\mathrm{Mod}_{\mathrm{H}}(X)$, we have deleted the vertex $\mathcal{S}$, all the vertices from R_1, $s-2$ vertices from R_2 and $s/2-2$ vertices from each of the orbits in R_3. Hence, since $|V(\mathrm{Hex}(X))| \equiv 0 (\mathrm{mod}\ 4)$ and $s \equiv (0 \bmod 4)$ we get that $|V(\mathrm{Mod}(X))| = |V(\mathrm{Hex}(X))| - s - (s-2) - 2(s/2-2) = |V(\mathrm{Hex}(X))| - 3s + 6$ is congruent to 2 modulo 4, as claimed. □

The next two lemmas and Corollary 4.7 combined together will imply that the modified graph $\mathrm{Mod}(X)$ is cyclically 4-edge-connected when $\mathrm{Hex}(X)$ is not one of the arc-transitive generalized Petersen graphs $\mathrm{GP}(s,k)$. This will in turn essentially complete the proof of Theorem 1.1.

Lemma 5.3 *Let* $\mathrm{Hex}(X)$ *be a cubic* 2*-regular graph of girth* 6*. Then one of the following occurs*:

(i) $\mathrm{Hex}(X) \cong \mathrm{GP}(8,3)$; *or*
(ii) $\mathrm{Hex}(X) \cong \mathrm{GP}(12,5)$; *or*
(iii) $s \geq 8$ *and* $\mathrm{Hex}(X)$ *is either the* $I^s_{s/2}(s/2+1)$*-path or the* $I^s_{s/6}(s/2+1)$*-path*; *moreover* $\mathrm{Mod}(X)$ *is cyclically* 4*-edge-connected.*

Proof Proposition 2.5 implies that $s \geq 8$ and $\mathrm{Hex}(X)$ is either the Moebius-Kantor graph $\mathrm{GP}(8,3)$ or an $I^s_k(s/2+1)$-path where either $k = s/2$ or $k = s/6$. Moreover, if $s = 12$ and $k = s/6 = 2$ then the $I^s_k(s/2+1)$-path is the generalized Petersen graph

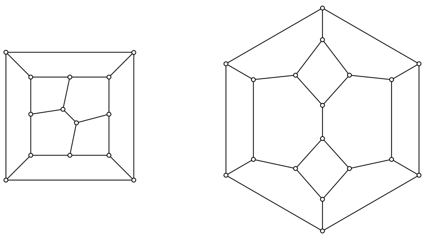

Fig. 7 The modified graph Mod(X) when the hexagon graph Hex(X) is the $I_4^8(5)$-path on the left-hand side picture, and when Hex(X) is the $I_4^{12}(7)$-path on the right-hand side picture

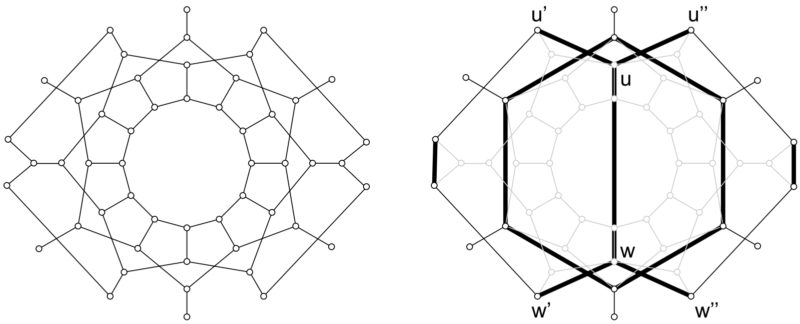

Fig. 8 The local structure of the $I_6^{12}(7)$-path on the left-hand side picture and the local structure of the corresponding modified graph Mod(X) on the right-hand side picture

GP(12, 5). In all other cases the fact that $s \equiv 0 \pmod 4$ implies $k \geq 4$. Hence, we may now assume that Hex(X) is the $I_k^s(s/2+1)$-path different from the $I_2^{12}(7)$-path. By Corollary 4.9 Hex(X) is cyclically 6-edge-connected. To complete the proof of Lemma 5.3 we now show that Mod(X) is cyclically 4-edge-connected.

For this purpose recall that the s-cycle in the first orbit in an $I_k^s(s/2+1)$-path is the first ring R_1 (see the proof of Lemma 4.4). Moreover, Hex(X) has exactly k rings and each ring R_i, $i \in \{1, 2, \dots, k\}$, is a single orbit of x.

It is a simple exercise to show that when Hex(X) is the $I_4^8(5)$-path or the $I_4^{12}(7)$-path, the modified graph Mod(X) is cyclically 4-edge-connected (see Figure 7). We may therefore assume that $s \geq 12$ and $k \geq 6$. In view of our assumptions, the local structure of the modified graph Mod(X) is as illustrated in Figure 8 for the $s = 12$ case. Let $M = E(\mathrm{Mod}(X)) \setminus E(\mathrm{Hex}(X))$ denote the set of new edges of Mod(X), that is, edges of Mod(X) which are not edges of Hex(X) (edges in bold on the right-hand side picture in Figure 8). Note that the edges in M form a disjoint union of a tree of order 6 (having two vertices of valency 3, and four vertices of valency 1), two $(s/4-2)$-paths, and an $s/2$-cycle Q. Label the six vertices of the above tree by u, u', u'', w, w' and w'', as shown in Figure 8. There exist two edges of Hex(X) connecting the 3-arc $u'uww'$ to one of the above two $(s/4-2)$-paths so as to get an $(s/4+3)$-cycle (see Figure 8). In a similar fashion an $(s/4+3)$-cycle is obtained starting with the 3-arc $u''uww''$ (see Figure 8). The central $s/2$-cycle Q in Mod(X) (formed by the edges in M) contains vertices of the form $x^{2i+1}ax^3H \in V(\mathrm{Hex}(X))$, $i \in \mathbb{Z}_s$ (belonging to the third ring R_3). Each edge $x^{2i+1}ax^3H, x^{2i+3}ax^3H$, $i \in \mathbb{Z}_s$, of this $s/2$-cycle Q also lies on a 5-cycle, denote it by f_i, arising from the unique

6-cycle passing through the 2-arc $(x^{2i+1}ax^3H, x^{2i+1}ax^2H, x^{2i+3}ax^3H)$ in $\mathrm{Hex}(X)$. (The uniqueness of this 6-cycle is guaranteed by Proposition 2.4.) Observe further that two 5-cycles f_i and f_j have a common edge if and only if $j \in \{i-1, i+1\}$.

Suppose that $\mathrm{Mod}(X)$ is not cyclically 4-edge-connected. Then there exists a cycle-separating subset $T \in E(\mathrm{Mod}(X))$ of size $t \leq 3$, and by deleting the edges in T the graph $\mathrm{Mod}(X)$ decomposes into two disjoint parts, say C and C', each containing a cycle. Clearly the edges in T are pairwise nonincident. In fact $t \in \{2, 3\}$ since $\mathrm{Mod}(X)$ is clearly 2-edge connected. Since every edge in T has one endvertex in C and the other endvertex in C', each cycle in $\mathrm{Mod}(X)$ must have an even number of edges from T, and thus no edge or two edges from T.

Case 1 $M \cap T = \emptyset$.

Then each edge in T is also an edge of $\mathrm{Hex}(X)$. Clearly, each edge of $\mathrm{Hex}(X)$ lies on two different 6-cycles in $\mathrm{Hex}(X)$, and thus each edge in T lies on two cycles of length less than or equal to 6 in $\mathrm{Mod}(X)$. Now $t = 2$ is not possible because otherwise $\mathrm{Hex}(X)$ would contain two 6-cycles having two edges in common, which is clearly not the case. But $t = 3$ is also not possible. Namely, in this case there would exist a ring of three 6-cycles glued together by the three edges from T but this would imply that $s = 6$, a contradiction.

Case 2 $M \cap T \neq \emptyset$.

The set M has four essentially different types of edges: first, the central edge uw of the tree of order 6; second, any of the remaining four edges of this tree; third, any edge from the $s/2$-cycle Q; and finally, any edge from the two $(s/4-2)$-paths. This gives us four essential possibilities for the intersection $M \cap T \neq \emptyset$. A careful analysis of each of these possibilities shows that a contradiction is obtained using an argument (based on the specific structure of 6-cycles in $\mathrm{Hex}(X)$) similar to the one used in Case 1. We give a detailed analysis for one of these possibilities and leave the essentially identical analysis of the other three to the reader.

Assume that an edge from the $s/2$-cycle Q is contained in T. In other words, there exists $i \in \mathbb{Z}_s$ such that the edge $e_i = (x^{2i+1}ax^3H, x^{2i+3}ax^3H) \in T$. For this to be an edge in a cycle-separating subset there must exist some $j \in \mathbb{Z}_s \setminus \{i\}$, such that also the edge $e_j = (x^{2j+1}ax^3H, x^{2j+3}ax^3H)$ of Q belongs to T. Recall that e_i and e_j lies, respectively, on the 5-cycle f_i and the 5-cycle f_j. Since $t \leq 3$ it follows that f_i and f_j have a common edge. But as remarked above this can only occur if e_i and e_j are neighboring edges on Q and thus incident, a contradiction. □

In a given graph X embedded in a closed orientable surface of genus g, two faces f and f' are said to be adjacent if they share a common edge. A *path of length r of faces* $f_0 f_1 \ldots f_r$ in X is a sequence of $r+1$ faces such that f_i is adjacent to f_{i+1} for every $i \in \mathbb{Z}_{r+1} \setminus \{r\}$, and $f_i \neq f_j$ for $i \neq j$. Analogously we define a *cycle of faces* $f_0 f_1 \ldots f_r f_0$. We say that two faces are at distance r if the shortest path of faces between them is of length r.

Lemma 5.4 *If* Hex(X) *does not belong to the family of the generalized Petersen graphs*, *then the modified graph* Mod(X) *is cyclically* 4-*edge-connected*.

Proof If Hex(X) is an $I_k^s(t)$-path then, by Lemma 5.3, Mod(X) is cyclically 4-edge-connected. By Corollary 4.9 we may therefore assume that Hex(X) is cyclically 8-edge-connected.

We consider an embedding of X in a closed orientable surface of genus $g = 1 + (s-6)|G|/12s$ with s-gonal and hexagonal faces. (More precisely, there are $|G|/s$ s-gons and $|G|/3$ hexagons.) Consequently Hex(X) and Mod(X) can also be embedded in the same surface of genus g. Moreover, the faces in Hex(X) are all s-gonal, whereas Mod(X) has also faces of other lengths. We say that a face in Mod(X) is an *old face* if it is also a face of Hex(X), and it is a *new face* otherwise, that is, if it is the end result of the corresponding modification process. Let the edge in Mod(X) arising from the 2-path $(H, \mathcal{S}, x^{s/2}H)$ in $\mathrm{Mod_H}(X)$ be denoted by e_v.

Now suppose on the contrary that Mod(X) is not cyclically 4-edge-connected. Then, as in the proof of Lemma 5.3, there exists a cycle-separating subset $T \in E(\mathrm{Mod}(X))$ of size $t \le 3$, and by deleting the edges in T the graph Mod(X) decomposes into two disjoint parts, say C and C', each containing a cycle. Clearly the edges in T are pairwise nonincident. In fact $t \in \{2, 3\}$ since Mod(X) is clearly 2-edge connected. Since every edge in T has one endvertex in C and the other endvertex in C', each face in Mod(X) must have an even number of edges from T, and thus no edge or two edges from T. Thus there exists a cycle consisting of t faces, call it Q, in Mod(X) where the adjacencies of these faces is given via the t edges in T.

Consider first the generic case, where $|\mathcal{R}| > 3$ and where the third ring R_3 is a union of two orbits of $x \in G$. Then the local structure of the modified graph Mod(X) (the structure of the area where Mod(X) differs from Hex(X)) embedded in the surface of genus g is as illustrated in Figure 9 for the case $s = 8$. Note that beside s-gonal faces, the modified graph Mod(X) contains also s faces of size $s-1$ and two new faces glued together with the edge e_v, the latter called *big faces* (see Figure 9).

If Q consists of t old faces, then T is also a cycle-separating subset of $E(\mathrm{Hex}(X))$, $t \le 3$, contradicting the fact that Hex(X) is cyclically 8-edge-connected. Similarly, one can easily see that also the case when Q consists of t faces of which some are

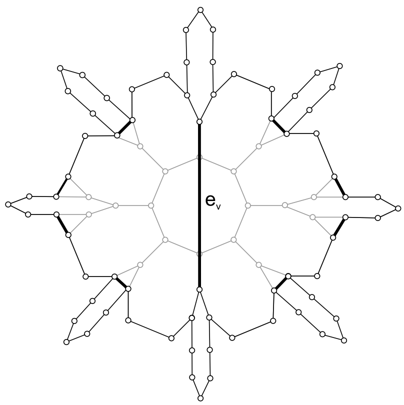

Fig. 9 The local structure of Mod(X) embedded in the closed orientable surface if $|\mathcal{R}| > 3$ and the third ring R_3 is a union of two orbits of $x \in G$ for $s = 8$

old faces and some are new faces of size $s-1$, but none of the two big faces, leads to a contradiction. We may therefore assume that Q contains at least one of the two big faces. If Q contains both big faces then clearly $e_v \in T$. Since Q consists of $t \leq 3$ faces it follows that either these two big faces have another common edge beside e_v, or there exists a face f, of size s or $s-1$, which together with the two big faces forms the cycle Q. But given any two edges e and e' in $\mathrm{Mod}(X)$, the first one lying on one and the other lying on the other big face in $\mathrm{Mod}(X)$, the corresponding two s-gonal faces in $\mathrm{Hex}(X)$ containing e and e', that were modified in the modification process, are at most distance 5 apart (see Figure 9). Therefore, there exists a cycle of $t-2+5=t+3\leq 6$ faces in $\mathrm{Hex}(X)$ which implies that $\mathrm{Hex}(X)$ contains a cycle-separating subset of size less than 7, a contradiction. We may therefore assume that Q contains just one of the two big faces in $\mathrm{Mod}(X)$. Then the two edges on this big face (which together with an appropriate face, if $t=2$, or an appropriate pair of faces, if $t=3$, of $\mathrm{Mod}(X)$ form a cycle of faces Q) are again at most distance 5 apart in $\mathrm{Hex}(X)$. Consequently there exists a cycle of $t-1+5\leq 7$ faces in $\mathrm{Hex}(X)$, again contradicting the fact that $\mathrm{Hex}(X)$ is cyclically 8-edge connected.

We are therefore left with the non-generic case for the graph $\mathrm{Hex}(X)$, that is, by Corollary 4.7, $|\mathcal{R}|>3$ and R_3 consists of a single orbit of $x\in G$. An argument similar to the one used in the generic case shows that no cycle-separating set consisting of at most 3 edges can exist. We omit the details. □

Combining together Lemma 5.2, Lemma 5.4 and Proposition 2.1 we obtain the following corollary.

Corollary 5.5 *If* $\mathrm{Hex}(X)$ *does not belong to the family of the generalized Petersen graphs, then there exists a maximum cyclically stable subset* S *in the modified graph* $\mathrm{Mod}(X)$ *such that the induced subgraph* $\mathrm{Mod}(X)[S]$ *is a tree and* $V(\mathrm{Mod}(X))\setminus S$ *is an independent set of vertices.*

6 Proving Theorem 1.1

We are now ready to prove the main theorem of this paper.

Proof of Theorem 1.1 Let $s\equiv 0 \pmod 4 \geq 4$ and let X be the Cayley graph of $G=\langle a,x \mid a^2=x^s=(ax)^3=1,\ldots\rangle$ with a $(2,s,3)$-presentation, relative to the set of generator $S=\{a,x,x^{-1}\}$ and embedded in a closed orientable surface of genus $g=1+(s-6)|G|/12s$ with s-gonal and hexagonal faces. If its hexagon graph $\mathrm{Hex}(X)$ is a generalized Petersen graph then X has a Hamiltonian cycle by Proposition 5.1. Assume that $\mathrm{Hex}(X)$ is not a generalized Petersen graph. By Corollary 5.5 there exists a maximum cyclically stable subset S which induces a tree $\mathrm{Mod}(X)[S]$ and whose complement $V(\mathrm{Mod}(X))\setminus S$ is an independent set of vertices. We add those vertices of $\mathrm{Mod_H}(X)$, that were removed from $\mathrm{Mod_H}(X)$ to get $\mathrm{Mod}(X)$, to the tree $\mathrm{Mod}(X)[S]$, and obtain an induced tree of the modified hexagon graph $\mathrm{Mod_H}(X)$ whose complement is an independent set of vertices. As illustrated in Section 3 this tree gives rise to a Hamiltonian tree of faces in the Cayley map of the graph X whose boundary is a Hamiltonian cycle in X. □

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