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On the uniqueness of promotion operators on tensor products of type A crystals

Jason Bandlow · Anne Schilling · Nicolas M. Thiéry

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Abstract The affine Dynkin diagram of type $A_n^{(1)}$ has a cyclic symmetry. The analogue of this Dynkin diagram automorphism on the level of crystals is called a promotion operator. In this paper we show that the only irreducible type A_n crystals which admit a promotion operator are the highest weight crystals indexed by rectangles. In addition we prove that on the tensor product of two type A_n crystals labeled by rectangles, there is a single connected promotion operator. We conjecture this to be true for an arbitrary number of tensor factors. Our results are in agreement with Kashiwara's conjecture that all 'good' affine crystals are tensor products of Kirillov-Reshetikhin crystals.

J. Bandlow
Department of Mathematics, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104-6395, USA
e-mail: jbandlow@math.upenn.edu
url: http://www.math.upenn.edu/~jbandlow

A. Schilling (✉)
Department of Mathematics, University of California, One Shields Avenue, Davis, CA 95616-8633, USA
e-mail: anne@math.ucdavis.edu
url: http://www.math.ucdavis.edu/~anne

N.M. Thiéry
Univ Paris-Sud, Laboratoire de Mathématiques d'Orsay, Orsay, 91405, France

N.M. Thiéry
CNRS, Orsay, 91405, France

Present address:
N.M. Thiéry
Department of Mathematics, University of California, One Shields Avenue, Davis, CA 95616, USA
e-mail: Nicolas.Thiery@u-psud.fr
url: http://Nicolas.Thiery.name

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1 Introduction

The Dynkin diagram of affine type $A_n^{(1)}$ has a cyclic symmetry generated by the map $i \mapsto i+1 \pmod{n+1}$. The promotion operator is the analogue of this Dynkin diagram automorphism on the level of crystals. Crystals were introduced by Kashiwara [7] to give a combinatorial description of the structure of modules over the universal enveloping algebra $U_q(\mathfrak{g})$ when q tends to zero. In short, a crystal is a non-empty set B endowed with raising and lowering crystal operators e_i and f_i indexed by the nodes of the Dynkin diagram $i \in I$, as well as a weight function wt. It can be depicted as an edge-colored directed graph with elements of B as vertices and i-arrows given by f_i. In type A_n, the highest weight crystal $B(\lambda)$ of highest weight λ is the set of all semi-standard Young tableaux of shape λ (see for example [15, 17]) with weight function given by the content of tableaux.

Definition 1.1 A *promotion operator* pr on a crystal B of type A_n is an operator $\mathrm{pr}: B \to B$ such that:

(1) pr shifts the content: If $\mathrm{wt}(b) = (w_1, \dots, w_{n+1})$ is the content of the crystal element $b \in B$, then $\mathrm{wt}(\mathrm{pr}(b)) = (w_{n+1}, w_1, \dots, w_n)$;
(2) Promotion has order $n+1$: $\mathrm{pr}^{n+1} = \mathrm{id}$;
(3) $\mathrm{pr} \circ e_i = e_{i+1} \circ \mathrm{pr}$ and $\mathrm{pr} \circ f_i = f_{i+1} \circ \mathrm{pr}$ for $i \in \{1, 2, \dots, n-1\}$.

If condition (2) is not satisfied, but pr is still bijective, then pr is a *weak* promotion operator.

Given a (weak) promotion operator on a crystal B of type A_n, one can define an associated (weak) *affine crystal* by setting

$$e_0 := \mathrm{pr}^{-1} \circ e_1 \circ \mathrm{pr} \quad \text{and} \quad f_0 := \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr}. \tag{1.1}$$

A promotion operator pr is called *connected* if the resulting affine crystal B is connected (as a graph). Two promotion operators are called *isomorphic* if the resulting affine crystals are isomorphic.

Our aim is the classification of all affine crystals that are associated to a promotion operator on a tensor product of highest weight crystals $B(\lambda)$ of type A_n.

Schützenberger [25] introduced a weak promotion operator $\mathfrak{pr}$ on tableaux using jeu-de-taquin (see Section 3.1). It turns out that $\mathfrak{pr}$ is the unique weak promotion operator on $B(\lambda)$; furthermore, $\mathfrak{pr}$ is a promotion operator if and only if λ is a rectangle (cf. Proposition 3.2 which is based on results by Haiman [4] and Shimozono [28]).

Let us denote by $\omega_1, \dots, \omega_n$ the fundamental weights of type A_n. One can identify the rectangle partition $\lambda := (s^r)$ of height r and width s with the weight $s\omega_r$. We henceforth call $\mathfrak{pr}$ on $B(s\omega_r)$ the *canonical promotion operator*. It can be extended to tensor products $B(s_1\omega_{r_1}) \otimes \cdots \otimes B(s_\ell\omega_{r_\ell})$ indexed by rectangles by setting $\mathfrak{pr}(b_1 \otimes \cdots \otimes b_\ell) := \mathfrak{pr}(b_1) \otimes \cdots \otimes \mathfrak{pr}(b_\ell)$. Let B be a crystal with an isomorphism Ψ to

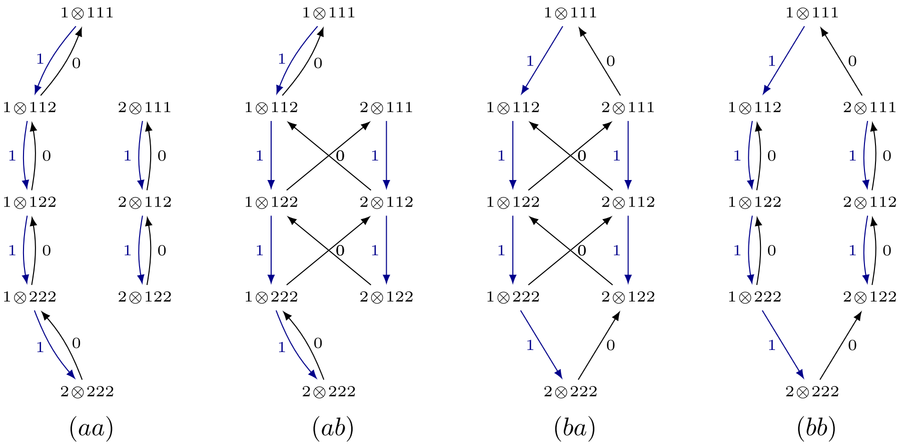

Fig. 1 The four affine crystals associated to the classical crystal $B(\omega_1) \otimes B(3\omega_1)$ for type A_1. The affine crystal $B^{1,1} \otimes B^{3,1}$ corresponds to (bb). The others are not 'good' crystals (see Definition 2.12): (aa) is not connected, (ab) is not simple, and (ba) does not satisfy the convexity condition on string lengths.

a direct sum of tensor products of highest weight crystals indexed by rectangles. A promotion operator is *induced by* Ψ if it is of the form $\Psi^{-1} \circ \mathfrak{pr} \circ \Psi$, where $\mathfrak{pr}$ is the canonical promotion on each summand. Note that throughout the paper, all tensor factors are written in reverse direction compared to Kashiwara's conventions, which is more compatible with operations on tableaux.

The main result of this paper is the following theorem.

Theorem 1.2 *Let $B = B(s'\omega_{r'}) \otimes B(s\omega_r)$ be the tensor product of two classical highest weight crystals of type A_n with $n \geq 2$, labeled by rectangles. If $(s, r) \neq (s', r')$, there is a unique promotion operator* $\mathrm{pr} = \mathfrak{pr}$. *If $(s, r) = (s', r')$, there are two promotion operators*: *The canonical one* $\mathrm{pr} = \mathfrak{pr}$ *which is connected and the one induced by* Ψ (*with* Ψ *as defined in* (2.3)) *which is disconnected.*

Remark 1.3 As illustrated in Figure 1, Theorem 1.2 does not hold for $n = 1$. Only (bb) yields a 'good' crystal according to the combinatorial Definition 2.12. It would be interesting to determine whether (ab) and (ba) correspond to crystals for $U_q'(\widehat{\mathfrak{sl}}_2)$-modules.

As suggested by further evidence discussed in Section 5, we expect this result to carry over to any number of tensor factors.

Conjecture 1.4 *Let $B := B(\lambda^1) \otimes \cdots \otimes B(\lambda^\ell)$ be a tensor product of classical highest weight crystals of type A_n with $n \geq 2$. Then, any promotion operator is induced by an isomorphism Ψ from B to some direct sum of tensor products of classical highest weight crystals of rectangular shape.*

Furthermore, there exists a connected promotion operator if and only if $\lambda^1, \ldots, \lambda^\ell$ are rectangles, and this operator is $\mathfrak{pr}$ *up to isomorphism.*

As shown by Shimozono [28], the affine crystal constructed from $B(s\omega_r)$ using the promotion operator $\mathfrak{pr}$ is isomorphic to the Kirillov-Reshetikhin crystal $B^{r,s}$ of type $A_n^{(1)}$. Kirillov-Reshetikhin crystals $B^{r,s}$ form a special class of finite dimensional affine crystals, indexed by a node r of the classical Dynkin diagram and a positive integer s. Finite-dimensional affine $U_q'(\mathfrak{g})$-crystals have been used extensively in the study of exactly solvable lattice models in statistical mechanics. It has recently been proven [12, 19] that (for nonexceptional types) the Kirillov-Reshetikhin module $W(s\omega_r)$, labeled by a positive multiple of the fundamental weight ω_r, has a crystal basis called the Kirillov-Reshetikhin crystal $B^{r,s}$. Kashiwara conjectured (see Conjecture 2.13) that any 'good' affine finite crystal is the tensor product of Kirillov-Reshetikhin crystals.

Note that Theorem 1.2 and Conjecture 1.4 are in agreement with Kashiwara's Conjecture 2.13. Namely, if one can assume that every 'good' affine crystal for type $A_n^{(1)}$ comes from a promotion operator, then Theorem 1.2 and Conjecture 1.4 imply that any crystal with underlying classical crystal being a tensor product is a tensor product of Kirillov-Reshetikhin crystals.

Promotion operators have appeared in other contexts as well. Promotion has been studied by Rhoades et al. [20, 23] in relation with Kazhdan-Lusztig theory and the cyclic sieving phenomenon. Hernandez [5] proved q-character formulas for cyclic Dynkin diagrams in the context of toroidal algebras. He studies a ring morphism R which is related to the promotion of the Dynkin diagram. Since q-characters are expected to be related to crystal theory, this is another occurrence of the promotion operator. Theorem 1.2 is also a first step in defining an affine crystal on rigged configurations. There exists a bijection between tuples of rectangular tableaux and rigged configurations [2, 13, 16]. A classical crystal on rigged configurations was defined and a weak promotion operator was conjectured in [27]. It remains to prove that this weak promotion operator has the correct order.

This paper is organized as follows. In Section 2 crystal theory for type A_n is reviewed, some basic properties of promotion operators are stated which are used later, and Kashiwara's conjecture is stated. In Section 3, the Schützenberger map $\mathfrak{pr}$ is defined on $B(\lambda)$ using jeu-de-taquin. It is shown that it is the only possible weak promotion operator on $B(\lambda)$, and that it is a promotion operator on $B(\lambda)$ if and only if λ is of rectangular shape. Section 4 is devoted to the proof of Theorem 1.2 and in Section 5 we provide evidence for Conjecture 1.4; in particular, we discuss unique factorization into a product of Schur polynomials indexed by rectangles.

2 Review of type A crystals

In this section, we recall some definitions and properties of type A crystals, state some lemmas which will be used extensively in the proof of Theorem 1.2, and state Kashiwara's conjecture.

2.1 Type A crystal operations

Crystal graphs of integrable $U_q(\mathfrak{sl}_{n+1})$-modules can be defined by operations on tableaux (see for example [15, 17]). Consider the type A_n Dynkin diagram with nodes indexed by $I := \{1, \ldots, n\}$. There is a natural correspondence between dominant weights in the weight lattice $P := \bigoplus_{i \in I} \mathbb{Z}\omega_i$, where ω_i is the i-th fundamental weight, and partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ with at most n parts. Suppose $\lambda = \omega_{r_1} + \cdots + \omega_{r_k}$ is a dominant weight. Then we can associate to λ the partition with columns of height $r_1, \ldots, r_k$. In particular, the fundamental weight $s\omega_r$ is associated to the partition of rectangular shape of width s and height r.

The highest weight crystal $B(\lambda)$ of type A_n is given by the set of all semi-standard tableaux of shape λ over the alphabet $\{1, 2, \ldots, n+1\}$ endowed with maps

$$\begin{aligned} e_i, f_i : & B(\lambda) \to B(\lambda) \cup \{\emptyset\} \quad \text{for } i \in I = \{1, 2, \ldots, n\}, \\ \mathrm{wt} : & B(\lambda) \to P. \end{aligned}$$

Throughout this paper, we use French notation for tableaux (that is, they are weakly increasing along rows from left to right and strictly increasing along columns from bottom to top). The *weight* of a tableau t is its content

$$\mathrm{wt}(t) := (m_1(t), m_2(t), \ldots, m_{n+1}(t)),$$

where $m_i(t)$ is the number of letters i appearing in t. The *lowering and raising operators* f_i and e_i can be defined as follows. Consider the *row reading word* $w(t)$ of t; it is obtained by reading the entries of t from left to right, top to bottom. Consider the subword of $w(t)$ consisting only of the letters i and $i+1$ and associate an open parenthesis ')' with each letter i and a closed parenthesis '(' with each letter $i+1$. Successively match all parentheses. Then f_i transforms the letter i that corresponds to the rightmost unmatched parenthesis ')' into an $i+1$. If no such parenthesis ')' exists, $f_i(t) = \emptyset$. Similarly, e_i transforms the letter $i+1$ that corresponds to the leftmost unmatched parenthesis '(' into an i. If no such parenthesis exists, $e_i(t) = \emptyset$.

For a tableau t, define $\varphi_i(t) = \max\{k \mid f_i^k(t) = \emptyset\}$ (resp. $\varepsilon_i(t) = \max\{k \mid e_i^k(t) = \emptyset\}$) to be the maximal number of times f_i (resp. e_i) can be applied to t. The quantity $\varphi_i(t) + \varepsilon_i(t)$ is the length of the *i-string* of t. Similarly, let $b_i(t)$ be the number of paired '()' parentheses in the algorithm for computing f_i and e_i. We call this the number of *i-brackets* in t.

Example 2.1 Let

$$t = \begin{array}{|c|c|c|c} \hline 2 & 3 & 3 & \\ \hline 1 & 2 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\,.$$

Then $w(t) = 2331223$ and

$$f_2(t) = \begin{array}{|c|c|c|c} \hline 3 & 3 & 3 & \\ \hline 1 & 2 & 2 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \text{and} \quad e_2(t) = \begin{array}{|c|c|c|c} \hline 2 & 3 & 3 & \\ \hline 1 & 2 & 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\,.$$

Definition 2.2 For $J \subset I = \{1, 2, \ldots, n\}$, the element $b \in B$ is *J-highest weight* if $e_i(b) = \emptyset$ for all $i \in J$. It is *highest weight* if it is I-highest weight. Similarly, $b \in B$ is *J-lowest weight* if $f_i(b) = \emptyset$ for all $\in J$.

2.2 Crystal isomorphisms

Let B and B' be two crystals over the same Dynkin diagram. Then a bijective map $\Phi : B \to B'$ is a *crystal isomorphism* if for all $b \in B$ and $i \in I$,

$$f_i \Phi(b) = \Phi(f_i b) \quad \text{and} \quad e_i \Phi(b) = \Phi(e_i b)\,,$$

where by convention $\Phi(\emptyset) = \emptyset$. More generally, let B and B' be crystals over two isomorphic Dynkin diagrams D and D' with nodes respectively indexed by I and I', and let $\tau : I \to I'$ be an isomorphism from D to D'. Then Φ is a *τ-twisted-isomorphism* if for all $b \in B$ and $i \in I$,

$$f_{\tau(i)} \Phi(b) = \Phi(f_i b) \quad \text{and} \quad e_{\tau(i)} \Phi(b) = \Phi(e_i b)\,.$$

It was proven by Stembridge [29] that in the expansion of the product of two Schur functions indexed by rectangles, each summand s_λ occurs with multiplicity zero or one. This implies in particular that in the decomposition of type A_n crystals

$$B(s'\omega_{r'}) \otimes B(s\omega_r) \cong \bigoplus_\lambda B(\lambda) \tag{2.1}$$

each irreducible component $B(\lambda)$ occurs with multiplicity at most one. Hence there is a *unique* crystal isomorphism

$$B(s'\omega_{r'}) \otimes B(s\omega_r) \cong B(s\omega_r) \otimes B(s'\omega_{r'}). \tag{2.2}$$

Recall that all tensor factors are written in reverse direction compared to Kashiwara's conventions.

For two equal rectangular tensor factors, there is a unique additional crystal isomorphism

$$\Psi : B(s\omega_r)^{\otimes 2} \cong B((s-1)\omega_r) \otimes B((s+1)\omega_r) \oplus B(s\omega_{r-1}) \otimes B(s\omega_{r+1}). \tag{2.3}$$

Its existence follows from the well-known Schur function equality [11, 14]:

$$s^2_{(s^r)} = s_{((s-1)^r)} s_{((s+1)^r)} + s_{(s^{r-1})} s_{(s^{r+1})}.$$

This isomorphism can be described explicitly as follows. For $b' \otimes b \in B(s\omega_r)^{\otimes 2}$ consider the tableau $b'.b$ given by the Schensted row insertion of b into b'. By [29], there is a unique pair of tableaux $\tilde{b}' \otimes \tilde{b}$ either in $B((s-1)\omega_r) \otimes B((s+1)\omega_r)$ or in $B(s\omega_{r-1}) \otimes B(s\omega_{r+1})$ such that $\tilde{b}'.\tilde{b} = b'.b$. Define $\Psi(b' \otimes b) = \tilde{b}' \otimes \tilde{b}$.

Example 2.3 Let

$$b' \otimes b = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array} \quad \text{so that} \quad b'.b = \begin{array}{|c|c|c|c}\cline{1-1} 3 & \multicolumn{3}{c}{} \\ \cline{1-3} 2 & 2 & \multicolumn{1}{c|}{2} & \\ \hline 1 & 1 & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\,.$$

Then

$$\Psi(b' \otimes b) = \tilde{b}' \otimes \tilde{b} = \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \quad \text{since} \quad \tilde{b}'.\tilde{b} = b'.b.$$

If on the other hand

$$b' \otimes b = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array} \quad \text{then} \quad b'.b = \begin{array}{|c|c|cc}\cline{1-2} 3 & 3 & & \\ \cline{1-2} 2 & 2 & & \\ \hline 1 & 1 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array}.$$

Hence

$$\Psi(b' \otimes b) = \tilde{b}' \otimes \tilde{b} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array}.$$

2.3 Duality

For each A_n crystal $B(\lambda)$ of highest weight λ, there exists a dual crystal $B(\lambda^{\complement})$, where $\lambda^{\complement}$ is the *complement partition* of λ in a rectangle of height $n+1$ and width λ_1. The crystal $B(\lambda)$ and its dual $B(\lambda^{\complement})$ are twisted-isomorphic, with $\tau(i) = n+1-i$.

Proposition 2.4 *The A_n crystal $B = B(s_1\omega_{r_1}) \otimes \cdots \otimes B(s_\ell \omega_{r_\ell})$ is twisted-isomorphic to the A_n crystal $B(s_1\omega_{n+1-r_1}) \otimes \cdots \otimes B(s_\ell \omega_{n+1-r_\ell})$.*

Proof This follows from the fact that the tensor product of twisted-isomorphic crystals must be twisted-isomorphic. □

Lemma 2.5 (Duality Lemma) *All promotion operators on $B = B(s'\omega_{r'}) \otimes B(s\omega_r)$ of type A_n are in one-to-one connectedness-preserving correspondence with the promotion operators on $B(s'\omega_{n+1-r'}) \otimes B(s\omega_{n+1-r})$. As a consequence, to classify all promotion operators on B, it suffices to classify them for $n \le r + r' - 1$.*

Proof By Proposition 2.4, $B(s'\omega_{r'}) \otimes B(s\omega_r)$ is twisted-isomorphic to $B(s'\omega_{n+1-r'}) \otimes B(s\omega_{n+1-r})$. Notice that under this twisted-isomorphism Φ, a promotion pr on B becomes $\Phi \circ \mathrm{pr}$ and satisfies the conditions of Definition 1.1 of the inverse of a promotion. Hence each pr induces a promotion on the dual of B. It is clear that connectedness is preserved.

Now suppose $n > r + r' - 1$. Summing the heights of the dual tensor product and subtracting one, we obtain

$$(n+1-r') + (n+1-r) - 1 = 2n - (r+r') + 1 > n,$$

which satisfies the condition of the lemma. Hence it suffices to classify promotion operators for $n \le r + r' - 1$. □

2.4 Properties of promotion operators

In this section we discuss some further properties of promotion operators. We begin with two remarks about consequences of the axioms for a promotion operator as defined in Definition 1.1 which will be used later. In particular, in Remark 2.7 a reformulation of the three conditions in Definition 1.1 is provided which in practice might be easier to verify. Then we prove two Lemmas: the Highest Weight Lemma 2.8 and the Two Path Lemma 2.10.

Remark 2.6 Let pr be a promotion operator. Then, $\mathrm{pr}^k \circ e_i = e_{i+k} \circ \mathrm{pr}^k$ whenever $i, i+k \neq 0 \pmod{n+1}$, and similarly for f_i.

Proof Iterate condition (3) of Definition 1.1, using condition (2) to go around $i = 0$. □

Remark 2.7 Let $B := B_1 \otimes \cdots \otimes B_\ell$ be a tensor product of type A_n highest weight crystals (or more generally a crystal of type A_n with some weight space of dimension 1; this includes the simple crystals of Definition 2.11), and pr a weak promotion operator on B which satisfies:

(2') $\mathrm{pr}^2 \circ e_n = e_1 \circ \mathrm{pr}^2$, and $\mathrm{pr}^2 \circ f_n = f_1 \circ \mathrm{pr}^2$.

Assume that the associated weak affine crystal graph is connected. Then, pr is a promotion operator.

Proof We need to prove condition (2): $\mathrm{pr}^{n+1} = \mathrm{id}$. First note that condition (2') together with the definition of e_0 in (1.1) implies that condition (3): $\mathrm{pr} \circ e_i = e_{i+1} \circ \mathrm{pr}$ (with $i+1$ taken $\pmod{n+1}$) holds even for $i = n$. By repeated application, one obtains $\mathrm{pr}^{n+1} \circ e_i = e_i \circ \mathrm{pr}^{n+1}$ for all i (and similarly for f_i). In other words, pr^{n+1} is an automorphism of the weak affine crystal graph.

We now check that such an automorphism has to be trivial. First note that it preserves classical weights. For all $1 \leq j \leq \ell$, let u_j be the highest vector of B_j. Then, $u := u_1 \otimes \cdots \otimes u_\ell$ is the unique element of B of weight $\mathrm{wt}(u_1) + \cdots + \mathrm{wt}(u_\ell)$, and therefore is fixed by pr^{n+1}. Take finally any $v \in B$. By the connectivity assumption $v = F(u)$, where F is some concatenation of crystal operators. Therefore, $\mathrm{pr}^{n+1}(v) = \mathrm{pr}^{n+1} \circ F(u) = F(\mathrm{pr}^{n+1}(u)) = F(u) = v$. □

For the remainder of this section B is a crystal of type A_n on which a promotion operator pr is defined. Recall that for $J \subset \{1, 2, \ldots, n\}$, the element $b \in B$ is J-*highest weight* if $e_i(b) = \emptyset$ for all $i \in J$.

Lemma 2.8 (Highest Weight Lemma) *If* $\mathrm{pr}(b)$ *is known for all* $\{1, 2, \ldots, n-1\}$-*highest weight elements* $b \in B$, *then* pr *is determined on all of* B.

Proof Any element $b' \in B$ is connected to a $\{1, 2, \ldots, n-1\}$ highest weight element b using a sequence $e_{i_1} \cdots e_{i_k}$ with $i_j \in \{1, 2, \ldots, n-1\}$. Hence $\mathrm{pr}(b') = e_{i_1+1} \cdots e_{i_k+1}(\mathrm{pr}(b))$, which is determined if $\mathrm{pr}(b)$ is known, since $i_j + 1 \in \{2, \ldots, n\}$. □

Definition 2.9 The *orbit* of $b \in B$ under the promotion operator pr is the family

$$b \xrightarrow{\mathrm{pr}} \mathrm{pr}(b) \xrightarrow{\mathrm{pr}} \mathrm{pr}^2(b) \xrightarrow{\mathrm{pr}} \cdots \xrightarrow{\mathrm{pr}} \mathrm{pr}^n(b) \xrightarrow{\mathrm{pr}} b\,,$$

(or any cyclic shift thereof).

Lemma 2.10 (Two Path Lemma) *Suppose $x, y, b \in B$ such that the following conditions hold*:

(1) *The entire orbits of x and y are known*;
(2) *b is connected to x by a chain of crystal edges*, *with all edge colors from some set I_x*;
(3) *b is connected to y by a chain of crystal edges*, *with all edge colors from some set I_y*;
(4) $I_x \cap I_y = \emptyset$.

Then the entire orbit of b under promotion is determined.

Proof By Remark 2.6, we have $\mathrm{pr}^k \circ e_i = e_{i+k} \circ \mathrm{pr}^k$ (and similarly for f_i) whenever $i, i+k \neq 0 \pmod{n+1}$. Since by assumption the entire orbit of x is known and b is connected to x by a chain consisting of edges from the set I_x, all powers $\mathrm{pr}^k(b)$ are determined except for $k \in \{n+1-i\}_{i \in I_x}$. Similarly, the entire orbit of y is known and b is connected to y by a chain consisting of edges from the set I_y, all powers $\mathrm{pr}^k(b)$ are determined except for $k \in \{n+1-i\}_{i \in I_y}$. Since $I_x \cap I_y = \emptyset$, the entire orbit of b is determined. □

2.5 Kashiwara's conjecture

Let B be a $U_q(\mathfrak{g})$-crystal with index set I (for the purpose of this paper it suffices to assume that $\mathfrak{g}$ is of type $A_n^{(1)}$, but the statements in this subsection hold more generally). We denote by $\mathfrak{g}_J$ the subalgebra of $\mathfrak{g}$ restricted to the index set $J \subset I$. The crystal B is said to be *regular* if, for any $J \subset I$ of finite-dimensional type, B as a $U_q(\mathfrak{g}_J)$-crystal is isomorphic to a crystal associated with an integrable $U_q(\mathfrak{g}_J)$-module. Stembridge [30] provides a local characterization of when a $\mathfrak{g}$-crystal is a crystal corresponding to a $U_q(\mathfrak{g})$-module.

In [1, 7], Kashiwara defined the notion of extremal weight modules. Here we briefly review the definition of an *extremal weight crystal* $\tilde{B}(\lambda)$ for $\lambda \in P$. Let W be the Weyl group associated to $\mathfrak{g}$ and s_i the simple reflection associated to α_i. Let B be a crystal corresponding to an integrable $U_q(\mathfrak{g})$-module. A vector $u_\lambda \in B$ of weight $\lambda \in P$ is called an *extremal vector* if there exists a family of vectors $\{u_{w\lambda}\}_{w\in W}$ satisfying

$$u_{w\lambda} = u_\lambda \text{ for } w = e, \tag{2.4}$$

$$\text{if } \langle \alpha_i^\vee, w\lambda \rangle \geq 0, \text{ then } e_i u_{w\lambda} = \emptyset \text{ and } f_i^{\langle \alpha_i^\vee, w\lambda \rangle} u_{w\lambda} = u_{s_i w\lambda}, \tag{2.5}$$

$$\text{if } \langle \alpha_i^\vee, w\lambda \rangle \leq 0, \text{ then } f_i u_{w\lambda} = \emptyset \text{ and } e_i^{-\langle \alpha_i^\vee, w\lambda \rangle} u_{w\lambda} = u_{s_i w\lambda}, \tag{2.6}$$

where $\alpha_i^\vee$ are the simple coroots. Then $\tilde{B}(\lambda)$ is an extremal weight crystal if it is generated by an extremal weight vector u_λ.

For an affine Kac-Moody algebra $\mathfrak{g}$, let δ denote the null root in the weight lattice P and c the canoncial central element. Then define $P_{\mathrm{cl}} = P/\mathbb{Z}\delta$ and $P^0 = \{\lambda \in P \mid \langle c, \lambda\rangle = 0\}$.

Definition 2.11 [1] A finite regular crystal B with weights in P_{cl}^0 is a *simple crystal* if B satisfies

(1) There exists $\lambda \in P_{\mathrm{cl}}^0$ such that the weight of any extremal vector of B is contained in $W_{\mathrm{cl}}\lambda$;
(2) The weight space of B of weight λ has dimension one.

Definition 2.12 (Kashiwara [9, Section 8]) A *'good' crystal* B has the properties that

(1) B is the crystal base of a $U_q'(\mathfrak{g})$-module;
(2) B is simple;
(3) Convexity condition: For any $i, j \in I$ and $b \in B$, the function $\varepsilon_i(f_j^k b)$ in k is convex.

Note that the third condition of Definition 2.12 is only necessary for rank 2 crystals. For higher rank crystals this follows from regularity and Stembridge's local characterization of crystals [30].

Conjecture 2.13 (Kashiwara [10, Introduction]) *Any 'good' finite affine crystal is the tensor product of Kirillov-Reshetikhin crystals.*

3 Promotion

In this section we introduce the Schützenberger operator $\mathfrak{pr}$ involving jeu-de-taquin on highest weight crystals $B(\lambda)$. This is used to show that promotion operators exist on $B(\lambda)$ if and only if λ is a rectangle. We then extend the definition of $\mathfrak{pr}$ to tensor products and discuss its relation to connectedness.

3.1 Existence and uniqueness on $B(\lambda)$

Schützenberger [25] defined a weak promotion operator $\mathfrak{pr}$ on standard tableaux. Here we define the obvious extension [28] on semi-standard tableaux on the alphabet $\{1, 2, \dots, n+1\}$ using jeu-de-taquin [26] (see for example also [3]):

(1) Remove all letters $n+1$ from tableau t (this removes a horizontal strip from t);
(2) Using jeu-de-taquin, slide the remaining letters into the empty cells (starting from left to right);
(3) Fill the vacated cells with zeroes;
(4) Increase each entry by one.

The result is denoted by $\mathfrak{pr}(t)$.

Example 3.1 Take $n = 3$. Then

$$t = \begin{array}{|c|c|c|}\hline 3 & 4 & 4 \\ \hline 2 & 3 & 3 \\ \hline 1 & 1 & 2 \\ \hline \end{array} \xrightarrow{(1)+(2)} \begin{array}{|c|c|c|}\hline 3 & 3 & 3 \\ \hline 1 & 2 & 2 \\ \hline \bullet & \bullet & 1 \\ \hline \end{array} \xrightarrow{(3)+(4)} \begin{array}{|c|c|c|}\hline 4 & 4 & 4 \\ \hline 2 & 3 & 3 \\ \hline 1 & 1 & 2 \\ \hline \end{array} = \mathfrak{pr}(t).$$

One can consider the reverse operation (which is also sometimes called *demotion*):

(1) Remove all letters 1 from tableau t (this removes the first part of the first row);
(2) Using jeu-de-taquin, slide the remaining letters into the empty cells;
(3) Fill the vacated cells with $n+2$s;
(4) Decrease each entry by one.

The result is denoted by $\mathfrak{pr}^{-1}(t)$. We will argue in the proof of the following proposition why these operations are actually well-defined and inverses of each other.

Proposition 3.2 *Let λ be a partition with at most n parts and let $B(\lambda)$ be a type A_n highest weight crystal. Then*, $\mathfrak{pr}$ *is the unique weak promotion operator on $B(\lambda)$. Furthermore*, $\mathfrak{pr}$ *is a promotion operator if and only if λ is a rectangle.*

Using standardization, the second part of the proposition follows from results of Haiman [4] who shows that, for standard tableaux on $n+1$ letters, $\mathfrak{pr}$ has order $n+1$ if and only if λ is a rectangle (and provides a generalization of this statement for shifted shapes). Shimozono [28] proves that $\mathfrak{pr}$ is the unique promotion operator on $B(s\omega_r)$ of type A_n. The resulting affine crystal is the Kirillov-Reshetikhin crystal $B^{r,s}$ of type $A_n^{(1)}$ [12, 28]. We could not find the statement of the uniqueness of the weak promotion operator in the literature.

For the sake of completeness, we include a complete and elementary proof of Proposition 3.2; the underlying arguments are similar in spirit to those in [4], except that we are using crystal operations on semi-standard tableaux instead of dual equivalence on standard tableaux. We first recall the following properties of jeu-de-taquin (see for example [3, 8, 17, 18]).

Remarks 3.3 Fix the ordered alphabet $\{1, 2, \dots, n+1\}$.

(a) Jeu-de-taquin is an operation on skew tableaux which commutes with crystal operations.

(b) Let λ/μ be a skew partition, and T the set of semi-standard skew-tableaux of shape λ/μ, endowed with its usual type A_n crystal structure. Let f be a function which maps each skew tableau in T to a semi-standard tableau of partition shape, and which commutes with crystal operations. For example, one can take for f the straightening function which applies jeu-de-taquin to $t \in T$ until it has partition shape. Let C be a connected crystal component of T. Then, by commutativity with crystal operations, there exists a unique partition ν such that $f(C)$ is the full type A_n crystal $B(\nu)$ of tableaux over this alphabet. Since $B(\nu)$ has no automorphism, this isomorphism is unique, and f has to be straightening using jeu-de-taquin.

(c) Let λ be a rectangle, and $\mu \subset \lambda$. Consider the complement partition $\mu^{\mathtt{C}}$ of μ in the rectangle λ. Then, the type A_n crystal of skew tableaux of shape λ/μ is isomorphic to the crystal of tableaux of shape $\mu^{\complement}$; this can be easily seen by rotating each tableau t of shape $\mu^{\complement}$ by 180° and mapping each letter i to $n+2-i$. By uniqueness of the isomorphism, the isomorphism and its inverse are both given by applying jeu-de-taquin, either sliding up or down. In particular, jeu-de-taquin down takes any tableau of shape λ/μ to a tableau of shape $\mu^{\complement}$, and vice-versa.

Example 3.4 Let $\lambda := (6^4)$ and $\mu := (5,2)$. The complement partition of μ in λ is $\mu^{\complement} = (6,6,4,1)$. We now apply jeu-de-taquin up from a tableau of shape $\mu^{\complement}$, and obtain a skew tableau of shape λ/μ. Applying jeu-de-taquin down yields back the original tableau. As is well-known for jeu-de-taquin, the end result does not depend on the order in which the inner corners are filled; here we show one intermediate step, after filling successively the three inner corners (2,4), (5,3), and (6,3). The color of the dots at the bottom (resp. at the top) indicates at which step each empty cell has been created by jeu-de-taquin up (resp. down).

$$\begin{array}{|c|c|c|c|c|c|}\hline 6 & \circ & \circ & \bullet & \bullet & \bullet \\ \hline 4 & 4 & 5 & 5 & \circ & \circ \\ \hline 3 & 3 & 3 & 3 & 4 & 6 \\ \hline 2 & 2 & 2 & 2 & 3 & 4 \\ \hline \end{array} \overset{J-D-T}{\longleftrightarrow} \begin{array}{|c|c|c|c|c|c|}\hline 4 & 6 & \circ & \circ & \circ & \circ \\ \hline 3 & 3 & 4 & 5 & 5 & 6 \\ \hline 2 & 2 & 3 & 3 & 4 & 4 \\ \hline \bullet & \bullet & \bullet & 2 & 2 & 3 \\ \hline \end{array} \overset{J-D-T}{\longleftrightarrow} \begin{array}{|c|c|c|c|c|c|}\hline 3 & 4 & 4 & 5 & 6 & 6 \\ \hline 2 & 3 & 3 & 3 & 4 & 5 \\ \hline \bullet & \bullet & 2 & 2 & 2 & 4 \\ \hline \circ & \circ & \circ & \circ & \bullet & 3 \\ \hline \end{array}.$$

Proof of Proposition 3.2 We first check that $\mathfrak{pr}$ is well-defined; the only non-trivial part is at step 3 where we must ensure that the previously vacated cells form the beginning of the first row. Fix a partition λ, and consider the set T of all tableaux whose $n+1$s are in a given horizontal border strip of length k. Step (1) puts them in bijection with the tableaux of the type A_{n-1} crystal $B(\lambda')$ where λ' is λ with the border strip removed. Let f be the function on $B(\lambda')$ which implements the jeu-de-taquin step (2) of the definition of $\mathfrak{pr}$. Since jeu-de-taquin commutes with crystal operations, $B(\lambda')$ is an irreducible crystal, and since crystal operations preserve shape, all tableaux in $f(B(\lambda'))$ have the same skew-shape λ/μ. Considering $f(t)$ where t is the anti-Yamanouchi tableau of shape λ' shows that $\mu = (1^k)$ as desired because the jeu-de-taquin slides follow successive hooks (the *anti-Yamanouchi* tableau of shape $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ is the unique tableau of shape λ' which contains λ'_i entries $m+1-i$). For example:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 4 & & \multicolumn{5}{c}{} \\ \cline{1-5} 3 & 4 & 4 & & & \multicolumn{2}{c}{} \\ \cline{1-6} 2 & 3 & 3 & 4 & 4 & & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ \hline \end{array} \to \begin{array}{|c|c|c|c|c|c|c|}\hline 4 & 4 & \multicolumn{5}{c}{} \\ \cline{1-5} 3 & 3 & 4 & & & \multicolumn{2}{c}{} \\ \cline{1-6} 2 & 2 & 3 & 4 & 4 & & \multicolumn{1}{c}{} \\ \hline \bullet & 1 & 2 & 3 & 3 & 4 & 4 \\ \hline \end{array} \to \begin{array}{|c|c|c|c|c|c|c|}\hline 4 & 4 & \multicolumn{5}{c}{} \\ \cline{1-5} 3 & 3 & 4 & 4 & & \multicolumn{2}{c}{} \\ \cline{1-6} 2 & 2 & 3 & 3 & 4 & & \multicolumn{1}{c}{} \\ \hline \bullet & \bullet & 1 & 2 & 3 & 4 & 4 \\ \hline \end{array} \to \begin{array}{|c|c|c|c|c|c|c|}\hline 4 & 4 & \multicolumn{5}{c}{} \\ \cline{1-5} 3 & 3 & 4 & 4 & 4 & \multicolumn{2}{c}{} \\ \cline{1-6} 2 & 2 & 3 & 3 & 3 & & \multicolumn{1}{c}{} \\ \hline \bullet & \bullet & \bullet & 1 & 2 & 4 & 4 \\ \hline \end{array} \to \begin{array}{|c|c|c|c|c|c|c|}\hline 4 & 4 & \multicolumn{5}{c}{} \\ \cline{1-5} 3 & 3 & 4 & 4 & 4 & \multicolumn{2}{c}{} \\ \cline{1-6} 2 & 2 & 3 & 3 & 3 & 4 & \multicolumn{1}{c}{} \\ \hline \bullet & \bullet & \bullet & \bullet & 1 & 2 & 4 \\ \hline \end{array}.$$

Note further that applying down jeu-de-taquin to $f(t)$ reverses the process, and yields back t. It follows that $\mathfrak{pr}^{-1}$ as described above is indeed a left inverse and therefore an inverse for $\mathfrak{pr}$. Finally, $\mathfrak{pr}$ satisfies conditions (1) and (3) of Definition 1.1 by construction, so it is a weak promotion operator.

We now prove that a weak promotion operator pr on $B(\lambda)$ is necessarily $\mathfrak{pr}$. Consider the action of pr^{-1} on a tableau t. By condition (1) of Definition 1.1, it has to strip away the 1s, subtract one from each remaining letter, transform the result into a semi-standard tableau of some shape $\mu'(t) \subset \lambda$, and complete with $n+1$s. Let B' be the set of all skew-tableaux in $B(\lambda)$ after striping and subtraction, endowed with the A_{n-1} crystal structure induced by the $\{2,\dots,n\}$ crystal structure of $B(\lambda)$. Write f^{-1} for the function which reorganizes the letters. By condition (3) of Definition 1.1, f^{-1} is an A_{n-1}-crystal morphism, so by Remark 3.3 (b) it has to be jeu-de-taquin. Therefore $\mathrm{pr}^{-1} = \mathfrak{pr}^{-1}$, or equivalently $\mathrm{pr} = \mathfrak{pr}$.

It remains to prove that $\mathfrak{pr}$ is a promotion operator if and only if λ is a rectangle.

Assume first that λ is a rectangle. By Remark 3.3 (c), for each k, jeu-de-taquin down provides a suitable bijection f^{-1} from skew tableaux of shape $\lambda/(k)$ and tableaux of shape $(k)^{\complement}$. The inverse bijection f is jeu-de-taquin up. We show $\mathfrak{pr}^2 \circ e_n = e_1 \circ \mathfrak{pr}^2$, which by Remark 2.7 finishes the proof that $\mathfrak{pr}$ is a promotion operator. Let t be a semi-standard tableau, l_1, l_2, and l_3 be respectively the number of bracketed pairs $(n+1, n)$, of unbracketed $n+1$s, and unbracketed ns. Then, from Remark 3.3 (c) one can further deduce that in $\mathfrak{pr}^2(t)$ there are l_1 bracketed pairs (2, 1), l_2 unbracketed 2s, and l_3 unbracketed 1s. We revisit Example 3.4 in this context. We have $l_1 = 2$, $l_2 = 1$, and $l_3 = 2$; due to label shifts, we have on the left $\bullet = n+1$ and $\circ = n$, in the middle $\bullet = 1$ and $\circ = n+1$, and on the right $\bullet = 2$ and $\circ = 1$:

$$\begin{array}{|c|c|c|c|c|c|}\hline 5&\circ&\circ&\bullet&\bullet&\bullet\\\hline 3&3&4&4&\circ&\circ\\\hline 2&2&2&2&3&5\\\hline 1&1&1&1&2&3\\\hline\end{array} \xrightarrow{\mathfrak{pr}} \begin{array}{|c|c|c|c|c|c|}\hline 4&6&\circ&\circ&\circ&\circ\\\hline 3&3&4&5&5&6\\\hline 2&2&3&3&4&4\\\hline \bullet&\bullet&\bullet&2&2&3\\\hline\end{array} \xrightarrow{\mathfrak{pr}} \begin{array}{|c|c|c|c|c|c|}\hline 4&5&5&6&7&7\\\hline 3&4&4&4&5&6\\\hline \bullet&\bullet&3&3&3&5\\\hline \circ&\circ&\circ&\circ&\bullet&3\\\hline\end{array}.$$

It follows in particular that e_1 applies to $\mathfrak{pr}^2(t)$ if and only if $l_2 > 0$ if and only if e_n applies to t; furthermore both the action of e_1 and e_n decrease l_2 by one and increase l_3 by one. This does not change $\mu = (l_1, l_1 + l_2 + l_3)$, and therefore the jeu-de-taquin action on the rest of the tableaux. Therefore $\mathfrak{pr}^2(e_n(t)) = e_1(\mathfrak{pr}^2(t))$, as desired.

To conclude, let us assume that λ is not a rectangle. We show that $\mathfrak{pr}^2 \circ e_n \neq e_1 \circ \mathfrak{pr}^2$, which by Remark 2.7 implies that $\mathfrak{pr}$ cannot be a promotion operator. The prototypical example is $n = 2$ and $\lambda = (2, 1)$, where the following diagram does not commute:

$$\begin{tikzcd}
\begin{array}{|c|c}\hline 3&\\\hline \underline{1}&\multicolumn{1}{c|}{3}\\\hline\end{array} \arrow[r, "\mathfrak{pr}"] \arrow[d, "e_n"] & \begin{array}{|c|c}\hline \underline{2}&\\\hline 1&\multicolumn{1}{c|}{1}\\\hline\end{array} \arrow[r, "\mathfrak{pr}"] & & \begin{array}{|c|c}\hline \underline{3}&\\\hline 2&\multicolumn{1}{c|}{2}\\\hline\end{array} \arrow[d, "e_1"] \\
\begin{array}{|c|c}\hline 2&\\\hline \underline{1}&\multicolumn{1}{c|}{3}\\\hline\end{array} \arrow[r, "\mathfrak{pr}"] & \begin{array}{|c|c}\hline 3&\\\hline 1&\multicolumn{1}{c|}{\underline{2}}\\\hline\end{array} \arrow[r, "\mathfrak{pr}"] & \begin{array}{|c|c}\hline 2&\\\hline 1&\multicolumn{1}{c|}{\underline{3}}\\\hline\end{array} \neq & \begin{array}{|c|c}\hline \underline{3}&\\\hline 1&\multicolumn{1}{c|}{2}\\\hline\end{array}
\end{tikzcd}$$

Interpretation: the underlined cell is the unique cell containing a 1 (resp. a 2, resp. a 3) on the left hand side (resp. middle, resp. right hand side), and we can track how it moves under promotion. Note that the value in the cell is such that promotion will always move the cell weakly up or to the right, and neither e_1 nor e_n affects it. At the first promotion step, depending on whether we apply e_n or not, the cell moves to the right, or up. But then, due to the inner corner of the partition it cannot switch to the other side, and therefore the diagram cannot close.

The same phenomenon occurs for any shape having (at least one) inner corner. Consider the uppermost inner corner, and construct the tableau:

$$\begin{array}{|c|c|c|c|ccc}
\hline
n-1 & \cdots & n-1 & n+1 \\
\hline
n-2 & \cdots & n-2 & n-1 \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
n-k & \cdots & n-k & n-k+1 \\
\hline
 & & & \underline{n-k} & \multicolumn{1}{|c|}{n+1} & \cdots \\
\cline{4-6}
 & & & & & \cdots \\
 & & < n-k & & & \cdots \\
 & & & & & \cdots \\
\hline
\end{array} \tag{3.1}$$

We assume that this tableau does not contain any letter n so that e_n applies to it and transforms the $n+1$ in the top row into an n. Let j be the width of the upper rectangle and assume that the tableau does not contain any further letters $n-k$ (only the j copies in the first j columns). Applying $\mathfrak{pr}$ without application of e_n, promotion slides the underlined $n-k$ up, and even after an additional application of $\mathfrak{pr}$ all the j cells containing $n-k$ in the original tableau, are in the upper rectangle. First applying e_n and then $\mathfrak{pr}$ has the effect of sliding the cell containing the underlined $n-k$ to the right; this cell cannot come back in the upper rectangle with another application of $\mathfrak{pr}$. Hence $\mathfrak{pr}^2 \circ e_n \neq e_1 \circ \mathfrak{pr}^2$. □

3.2 Promotion on tensor products

Now take $B := B(s_1\omega_{r_1}) \otimes \cdots \otimes B(s_\ell\omega_{r_\ell})$ a tensor product of ℓ classical highest weight crystals labeled by rectangles. For $b_1 \otimes \cdots \otimes b_\ell \in B$, define $\mathfrak{pr} : B \to B$ by

$$\mathfrak{pr}(b_1 \otimes \cdots \otimes b_\ell) = \mathfrak{pr}(b_1) \otimes \cdots \otimes \mathfrak{pr}(b_\ell). \tag{3.2}$$

Lemma 3.5 *$\mathfrak{pr}$ on $B = B(s_1\omega_{r_1}) \otimes \cdots \otimes B(s_\ell\omega_{r_\ell})$ is a connected promotion operator.*

Proof Since $\mathfrak{pr}$ on each tensor factor $B(s_i\omega_i)$ satisfies conditions (1) and (2) of Definition 1.1, $\mathfrak{pr}$ on B also satisfies conditions (1) and (2). Since $\mathfrak{pr}$ on each tensor factor $B(s_i\omega_i)$ satisfies condition (3) and the bracketing is well-behaved with respect to acting on each tensor factor, we also have condition (3) for $\mathfrak{pr}$ on B. The affine crystal resulting from $\mathfrak{pr}$ on $B(s\omega_r)$ is the Kirillov-Reshetikhin crystal $B^{r,s}$ of type $A_n^{(1)}$ [12, 28]. Since $B^{r,s}$ is simple, the affine crystal resulting from $\mathfrak{pr}$ on B is connected by [9, Lemmas 4.9 and 4.10]. □

Lemma 3.5 shows that a promotion operator with the properties of Definition 1.1 exists on $B = B(s_1\omega_{r_1}) \otimes \cdots \otimes B(s_\ell\omega_{r_\ell})$. Theorem 1.2 states that for $\ell = 2$ this is the only connected promotion operator.

4 Inductive proof of Theorem 1.2

In this section we provide the proof of Theorem 1.2. Throughout this section $B := B(s'\omega_{r'}) \otimes B(s\omega_r)$. For $n < \max(r, r')$ this crystal is either nonexistent or trivial.

4.1 Outline of the proof

Aside from distinguishing the cases where $(s', r') = (s, r)$, our proof does not depend in a material way on the values of s and s'. The basic tool in our proof is an induction which allows us to relate the cases described by the triple (r', r, n) to those described by $(r' - 1, r - 1, n - 1)$, provided that

(1) $n \leq r' + r - 1$ and
(2) we do not have $r' = r = 1$.

As follows from Lemma 2.5, any crystal which does not satisfy these hypotheses is isomorphic to one which does, with the exception of the case where $r' = r = n = 1$. (This case does not satisfy the result of the Theorem as was discussed in Remark 1.3). The general idea for the proof is to use repeated applications of induction and duality to successively reduce the rank of the crystal. Note that both techniques preserve the fact that the rank is greater or equal to the maximum of the heights of the two rectangles r and r'. We take as base cases those crystals where either r or r' is equal to zero. In these cases, we have only a single tensor factor and the statement of Theorem 1.2 was shown by Shimozono [28].

This approach, however, does not cover those cases which inductively reduce to the case $(1, 1, 1)$. The only case which *directly* reduces to $(1, 1, 1)$ is $(2, 2, 2)$. By duality, the case $(2, 2, 2)$ is equivalent to the case $(1, 1, 2)$. We prove this case directly, as a separate base case, and thus complete the proof.

The proof is laid out as follows. In Section 4.2, we discuss the base case of the A_2 crystals with $r = r' = 1$. In Section 4.3, we present the basic lemma (Lemma 4.8) for our inductive arguments. In Section 4.4, we show how to apply the induction in the case where $r' \geq r$ and $r' > 1$ for different tensor factors. Note that by (2.2) we can always assume that $r' \geq r$. In Section 4.5 we treat the case of equal tensor factors.

4.2 Row tensor row case, $n = 2$

In this subsection we prove Theorem 1.2 for the row tensor row case with $n = 2$. In this case, the isomorphism of Equation (2.3) becomes:

$$\Psi : B(s\omega_1) \otimes B(s\omega_1) \hookrightarrow\!\!\!\!\!\rightarrow B((s-1)\omega_1) \otimes B((s+1)\omega_1) \oplus B(s\omega_2)\,. \tag{4.1}$$

Proposition 4.1 *Let $B := B(s'\omega_1) \otimes B(s\omega_1)$ be the tensor product of two single row classical highest weight crystals of type A_2 with $s, s' \geq 1$. If $s \neq s'$, there is a unique promotion operator* $\mathrm{pr} = \mathfrak{pr}$ *which is connected. If $s = s'$, there are two promotion operators*: $\mathfrak{pr}$ *which is connected, and* $\mathfrak{pr}' := \Psi^{-1} \circ \mathfrak{pr} \circ \Psi$, *induced by the canonical promotions on the classical crystals $B((s-1)\omega_1) \otimes B((s+1)\omega_1)$ and $B(s\omega_2)$, which is disconnected.*

We may assume without loss of generality that $s' \leq s$. After a preliminary Lemma 4.2, we show that if the pr-orbits coincide with the $\mathfrak{pr}$-orbits on the inversionless component, then $\mathrm{pr} = \mathfrak{pr}$ (Proposition 4.3). Here the inversionless component is the component $B((s + s')\omega_1)$ in the decomposition (2.1) of B. Then, we proceed with the analysis of pr-orbits on the inversionless component (Lemma 4.6). When $s' < s$, there is a single possibility which implies $\mathrm{pr} = \mathfrak{pr}$. When $s' = s$, there are two possibilities, and we argue that one implies $\mathrm{pr} = \mathfrak{pr}$, while the other implies $\mathrm{pr} = \mathfrak{pr}'$ via the isomorphism Ψ.

Lemma 4.2 (Content Lemma) *If $v' \otimes v \in B$ does not contain any 3s, then* $\mathrm{pr}(v' \otimes v) = \mathfrak{pr}(v') \otimes \mathfrak{pr}(v)$.

Proof By assumption $w = v' \otimes v$ contains only the letters 1 and 2. The only 1-bracketing that can be achieved is by 2s in the left tensor factor that bracket with 1s in the right tensor factor. Hence knowing $\varphi_1(w)$ and $\varepsilon_1(w)$ determines w completely. Since pr rotates content, $\mathrm{pr}(w)$ contains only 2s and 3s. Since furthermore $\varphi_2(\mathrm{pr}(w)) = \varphi_1(w)$ and $\varepsilon_2(\mathrm{pr}(w)) = \varepsilon_1(w)$, this completely determines $\mathrm{pr}(w)$. Since $\mathfrak{pr}$ is a valid promotion operator by Lemma 3.5, pr must agree with $\mathfrak{pr}$ on these elements. □

Proposition 4.3 *Let $B := B(s'\omega_1) \otimes B(s\omega_1)$, and* pr *be a promotion on a classical type A_n crystal $C := B \oplus B'$ of which B is a direct summand (typically $C := B$). Assume that the orbits under promotion on the inversionless component of B coincide with those for the canonical promotion $\mathfrak{pr}$ of B. Then* pr *coincides with $\mathfrak{pr}$ on B.*

We start with the elements with only one letter in some tensor factor.

Lemma 4.4 *Under the hypothesis of Proposition 4.3, the* pr*-orbit of an element $v' \otimes v$ is its $\mathfrak{pr}$-orbit whenever either v' or v contains a single letter.*

Proof Assume that $v' = k^{s'}$ (resp. $v = k^s$). Then, $v' \otimes v$ is in the $\mathfrak{pr}$-orbit of the inversionless element $1^{s'} \otimes \mathfrak{pr}^{1-k}(v)$ (resp. of $\mathfrak{pr}^{3-k}(v') \otimes 3^s$) which by hypothesis is also its pr-orbit. □

Next come elements with exactly two letters in each tensor factor.

Lemma 4.5 *Under the hypothesis of Proposition 4.3, the* pr*-orbit of an element $w := v' \otimes v$ is its $\mathfrak{pr}$-orbit whenever v' and v each contain precisely two distinct letters.*

Proof By Lemma 4.4, it remains to consider the cases when both v' and v contain two letters.

(1) If $w = 1^a 2^b \otimes 2^c 3^d$ it is inversionless and we are done by hypothesis. The $\mathfrak{pr}$-orbit includes the elements $2^a 3^b \otimes 1^d 3^c$ and $1^b 3^a \otimes 1^c 2^d$.
(2) Assume $w = 2^a 3^b \otimes 1^d 2^c$. Applying f_2 a sufficient number of times gives $3^{a+b} \otimes 1^d 2^{c_1} 3^{c_2}$. If we instead apply e_1 a sufficient number of times to w, we get the elements $1^{a_1} 2^{a_2} 3^b \otimes 1^{d+c}$. In both cases Lemma 4.4 applies, and by the Two Path Lemma 2.10, the pr-orbit of w is its $\mathfrak{pr}$-orbit.
(3) The orbits of the elements considered previously include the elements $1^b 3^a \otimes 2^d 3^c$ and $1^a 2^b \otimes 1^c 3^d$.
(4) Assume $w = 1^b 3^a \otimes 1^d 3^c$. Applying e_2^a yields $1^b 2^a \otimes 1^d 3^c$, and applying f_1^d yields $1^b 3^a \otimes 2^d 3^c$. Both elements have already been treated, and by the Two Path Lemma 2.10, the pr-orbit of w is its $\mathfrak{pr}$-orbit.

(5) The orbits of the elements considered previously include $w = 1^a 2^b \otimes 1^c 2^d$ and $w = 2^a 3^b \otimes 2^c 3^d$. Hence all cases are covered.

□

We are now in the position to prove Proposition 4.3.

Proof of Proposition 4.3 By the Highest Weight Lemma 2.8, we only need to determine promotion of each $\{1\}$-highest weight element. They are of the form $1^a 2^b 3^c \otimes 1^d 3^e$ with $b \le d$. We claim that its promotion orbit is given as follows:

$$w_0 = 1^a 2^b 3^c \otimes 1^d 3^e \xrightarrow{(1)} w_1 = 1^c 2^a 3^b \otimes 1^e 2^d \xrightarrow{(2)} w_2 = 1^b 2^c 3^a \otimes 2^e 3^d \xrightarrow{(3)} w_0 \,.$$

Applying e_1 a sufficient number of times to w_1 yields a word whose second factor contains a single letter. Using Lemma 4.4, we deduce that $\mathrm{pr}(w_1) = w_2 = \mathfrak{pr}(w_1)$ as claimed (arrow (2)). Applying f_1^b to w_2 yields $2^{b+c} 3^a \otimes 2^c 3^d$ whose pr-orbit is its $\mathfrak{pr}$-orbit by Lemma 4.5. Therefore $\mathrm{pr}(w_2) = w_0 = \mathfrak{pr}(w_2)$ as claimed (arrow (3)). Arrow (1) follows from $\mathrm{pr}^3 = \mathrm{id}$. □

We now turn to the analysis of promotion orbits on the inversionless component (see Figure 2).

Lemma 4.6 *When $s' \neq s$, the* pr*-orbit of every element in the inversionless component agrees with* $\mathfrak{pr}$. *When $s' = s$, there are precisely two cases; either* pr *agrees with* $\mathfrak{pr}$ *on the orbit of every element in this component, or* pr *agrees with* $\mathfrak{pr}'$ *on the orbit of every element in this component.*

Proof Draw the crystal graph for the inversionless component with $1^{s'} \otimes 1^s$ at the top, 1-arrows going down and 2-arrows going right (see Figure 2). When there is no ambiguity, we drop the $\otimes$ sign and consider elements in B as words. The orbits of the elements w in the inversionless component are considered in the following order:

(1) Corners: $w \in \{1^{s'} \otimes 1^s, 2^{s'} \otimes 2^s, 3^{s'} \otimes 3^s\}$.
(2) Diagonal: $w := 1^a 2^{s'-a} \otimes 2^{s-a} 3^a$ with $1 \le a < s'$.
(3) Middle row: $w := 1^{s'} \otimes 2^{s-a} 3^a$ with $s' \le a < s$.
(4) Lower leftmost column: $w := 1^a 2^{s'-a} \otimes 2^s$ with $1 \le a \le s'$ and $a < s$.
(5) Left of lower row: $w := 2^{s'} \otimes 2^a 3^{s-a}$ with $1 \le a < s$.
(6) Rest of leftmost column and lower row: $w := 1^a 2^b$ or $w := 2^a 3^b$, except when $a = b = s = s'$.
(7) General elements: $w := 1^a 2^b 3^c$ not in any of the other cases.
(8) Row and column of $1^s \otimes 3^s$ when $s = s'$.

(1): By content, $1^{s'} \otimes 1^s \xrightarrow{\mathrm{pr}} 2^{s'} \otimes 2^s \xrightarrow{\mathrm{pr}} 3^{s'} \otimes 3^s \xrightarrow{\mathrm{pr}} 1^{s'} \otimes 1^s$, which agrees with the $\mathfrak{pr}$-orbit.

(2): The orbit of $w := 1^a 2^{s'-a} \otimes 2^{s-a} 3^a$ for $1 \le a < s'$ is forced by bracketing arguments. Recall that $b_i(w)$ denotes the number of () brackets in the construction of f_i and e_i on w. Start with the element $w_1 := 2^a 3^{s'-a} \otimes 1^a 3^{s-a}$. Note that $b_1(w_1) = a$. Thus $b_2(\mathrm{pr}(w_1)) = a$. This implies that in $w_2 := \mathrm{pr}(w_1)$ all 3s must be in the left

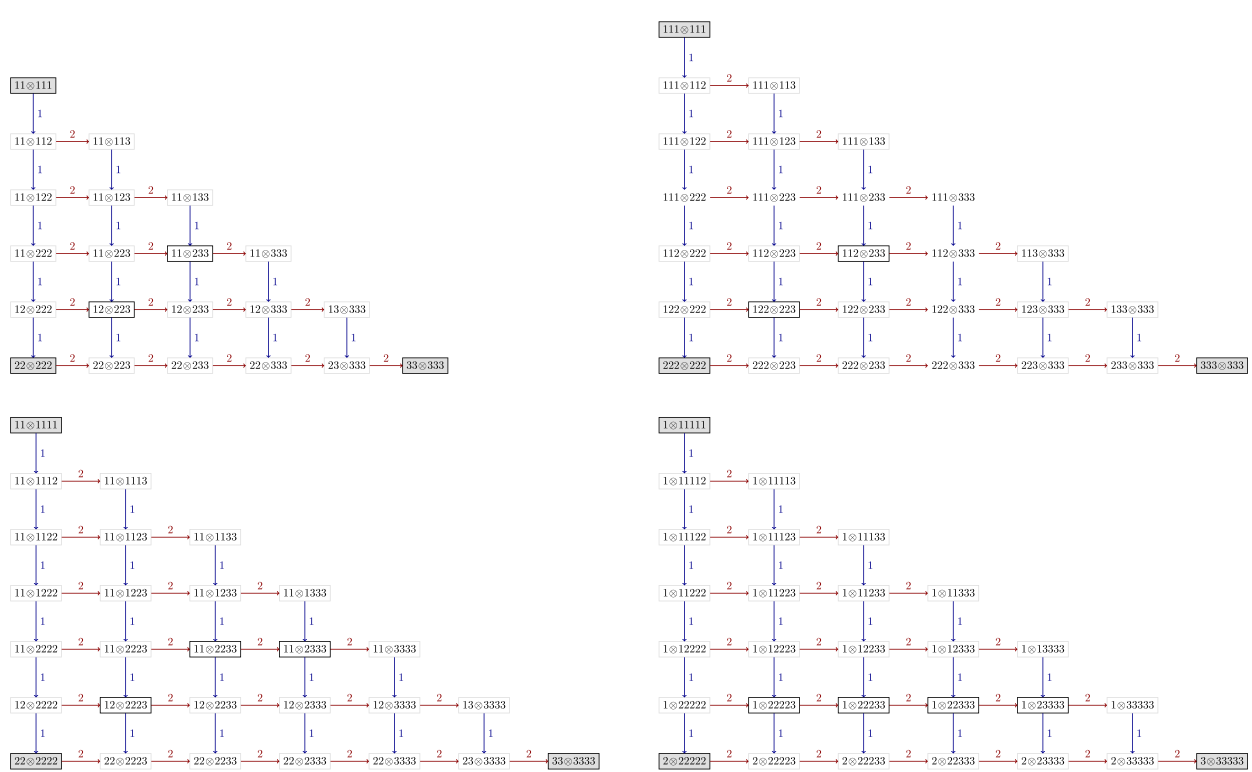

Fig. 2 The inversionless classical component of the $B(s'\omega_1) \otimes B(s\omega_1)$ A_2-crystal. The pr-orbit is uniquely determined by content for the grayed elements, by bracketing for the framed elements, and then by the Two Path Lemma for the lightly framed elements. This covers all elements except when $s = s'$ (upper right), where there are potentially two choices for $1^s \otimes 3^s$, and consequently for any element in its row or column.

tensor factor and all 2s must be in the right tensor factor. This forces $w_2 = 1^{s'-a}3^a \otimes 1^{s-a}2^a$. Now we have $b_1(w_2) = 0$. Thus if we define $w_0 := \mathrm{pr}(w_2) = \mathrm{pr}^{-1}(w_1)$, we must have $b_2(w_0) = 0$. However, we also have that $b_2(w_1) = 0$. Hence we have $b_1(w_0) = 0$. These facts imply that in w_0 all 1s precede all 2s and all 2s precede all 3s. Therefore $w_0 = 1^a2^{s'-a} \otimes 2^{s-a}3^a = w$, and the pr-orbit of w is its $\mathfrak{pr}$-orbit $w \xrightarrow{\mathrm{pr}} w_1 \xrightarrow{\mathrm{pr}} w_2 \xrightarrow{\mathrm{pr}} w$.

(3): The argument for $w := 1^{s'} \otimes 2^{s-a}3^a$ with $s' \le a < s$ is very similar to (2). Start with $w_1 := 2^{s'} \otimes 1^a3^{s-a}$. Since $b_1(w_1) = s'$, we must have $b_2(w_2) = s'$ for $w_2 := \mathrm{pr}(w_1)$. This forces the first tensor factor of w_2 to be $3^{s'}$, which completely fixes $w_2 = 3^{s'} \otimes 1^{s-a}2^a$. We have $b_1(w_2) = 0$, and so $b_2(w_0) = 0$ where $w_0 := \mathrm{pr}(w_2) = \mathrm{pr}^{-1}(w_1)$. However, we also have $b_2(w_1) = 0$, and so $b_1(w_0) = 0$. Thus, as above, w_0 must have no inversions, and we have $w_0 = 1^{s'} \otimes 2^{s-a}3^a$ and the pr-orbit of w is its $\mathfrak{pr}$-orbit.

(4): Applying f_1^a to $w := 1^a2^{s'-a} \otimes 2^s$ gives the corner element $2^{s'} \otimes 2^s$. Applying f_2^a to w yields the diagonal element $1^a2^{s'-a} \otimes 2^{s-a}3^a$ (or middle row element for $a = s' < s$). Hence by the Two Path Lemma 2.10 the pr-orbit of w is its $\mathfrak{pr}$-orbit.

(5): Let $w := 2^{s'} \otimes 2^{s-a}3^a$ for $1 \le a < s$, Then $e_2^a(w) = 2^{s'} \otimes 2^s$ whose orbit is known by the corner case, and $e_1^{s'}$ or e_1^a applied to w yields a diagonal or row element. Hence again by the Two Path Lemma 2.10 the pr-orbit of w is its $\mathfrak{pr}$-orbit.

(6): By Lemma 4.2, the element $w := 1^a2^b$ is mapped to 2^a3^b under promotion. From Step (4) and (5) either the pr-orbit of 1^a2^b or of 2^a3^b is its $\mathfrak{pr}$-orbit, except when $a = b = s = s'$.

(7): For a general element $w := 1^a2^b3^c$, $f_1^a(w)$ yields the element $2^{a+b}3^c$ of the lowest row, and $e_2^c(w)$ yields the element 1^a2^{b+c} of the leftmost column edge. By (4), (5), (6) and the Two Path Lemma 2.10 the pr-orbit of w is its $\mathfrak{pr}$-orbit, except when $s = s'$ and w is in the row or column of $1^s \otimes 3^s$.

(8): Assume $s = s'$, and consider $\mathrm{pr}(1^{s-1}2 \otimes 3^s)$. Using (7) for the pr-orbit of $1^{s-1}3 \otimes 3^s$ and $\mathrm{pr}^3 = \mathrm{id}$, we have:

$$\begin{aligned}
\mathrm{pr}(1^{s-1}2 \otimes 3^s) &= \mathrm{pr}(e_2(1^{s-1}3 \otimes 3^s)) = \mathrm{pr}(e_2(\mathrm{pr}(23^{s-1} \otimes 2^s))) \\
&= \mathrm{pr}^2(e_1(23^{s-1} \otimes 2^s)) = \mathrm{pr}^{-1}(e_1(23^{s-1} \otimes 2^s)) \\
&= \mathrm{pr}^{-1}(13^{s-1} \otimes 2^s)\,.
\end{aligned} \tag{4.2}$$

By the bracketing structure between the 2s and 3s in $13^{s-1} \otimes 2^s$, there are only two possible choices for $\mathrm{pr}^{-1}(13^{s-1} \otimes 2^s)$, as desired:

$$2^{s-1}3 \otimes 1^s \text{ and } 12^{s-1} \otimes 1^{s-1}3\,. \tag{4.3}$$

By the Two Path Lemma 2.10, each of those choices completely determines the pr-orbits on the row and column of $1^s \otimes 3^s$. The first choice is clearly compatible with $\mathfrak{pr}$. The second choice is compatible with $\mathfrak{pr}'$:

$$12^{s-1} \otimes 1^{s-1}3 \quad \xrightarrow{\Psi} \quad 2^{s-1} \otimes 1^s3 \quad \xrightarrow{\mathfrak{pr}} \quad 3^{s-1} \otimes 12^s \quad \xrightarrow{\Psi^{-1}} \quad 13^{s-1} \otimes 2^s.$$

$\square$

Proof of Proposition 4.1 If $s \neq s'$, the statement of the Proposition follows from Lemma 4.6 and Proposition 4.3. It remains to settle the case $s = s'$.

Case 1: pr coincides with $\mathfrak{pr}$ on the inversionless component. Then, by Proposition 4.3: $\mathrm{pr} = \mathfrak{pr}$.

Case 2: pr coincides with $\mathfrak{pr}'$ on the inversionless component. Note that Ψ maps this component to the inversionless component of $B((s-1)\omega_1) \otimes B((s+1)\omega_1)$. Therefore, applying Proposition 4.3 to $B((s-1)\omega_1) \otimes B((s+1)\omega_1)$ yields that pr coincides with $\mathfrak{pr}'$ on $\Psi^{-1}(B((s-1)\omega_1) \otimes B((s+1)\omega_1))$. Then, pr stabilizes both $\Psi^{-1}(B((s-1)\omega_1) \otimes B((s+1)\omega_1))$ and $\Psi^{-1}(B(s\omega_2))$; since the latter piece admits a unique promotion, $\mathrm{pr} = \mathfrak{pr}'$.

Since $\mathfrak{pr} \neq \mathfrak{pr}'$, the two cases above are necessarily exclusive, and by Lemma 4.6 they cover all the choices for pr on the inversionless component. □

4.3 Induction

The remainder of our proof uses induction to relate any promotion operator on $B := B(s'\omega_{r'}) \otimes B(s\omega_r)$ of type A_n to a promotion operator on $D := B(s'\omega_{r'-1}) \otimes B(s\omega_{r-1})$ of type A_{n-1}. This is done in Proposition 4.8. Before we can state and prove this proposition we first need some more notation and a preliminary lemma.

Let $C_i(B)$ be the subgraph of B consisting of the vertices with $s+s'$ copies of the letter i, along with the arrows e_j, f_j with $j \in \{1, 2, \dots, n\} \setminus \{i-1, i\}$. Recall the promotion operator $\mathfrak{pr}$ of Section 3.1. In addition, if $(s, r) = (s', r')$, we can define the promotion operator $\mathfrak{pr}' = \Psi^{-1} \circ \mathfrak{pr} \circ \Psi$. If we want to emphasize that these promotion operators act on the crystal B (respectively D), we write $\mathfrak{pr}_B$ and $\mathfrak{pr}'_B$ (respectively $\mathfrak{pr}_D$ and $\mathfrak{pr}'_D$).

Lemma 4.7 *If $(s, r) = (s', r')$, there exist at least two promotion operators $\mathfrak{pr}_B$ and $\mathfrak{pr}'_B$ on B. Furthermore, they are distinct when restricted to maps on $C_1(B) \to C_2(B)$ or $C_n(B) \to C_{n+1}(B)$.*

Proof Set $C_i := C_i(B)$. We note first that for content reasons, any promotion operator must bijectively map $C_i \to C_{i+1}$ (and $C_{n+1} \to C_1$). By Section 3.1 we know that there are at least two promotion operators $\mathfrak{pr}_B$ and $\mathfrak{pr}'_B$ defined on B. It remains to show they differ when restricted to C_1 and C_n.

Consider the element (where we write the columns of tableaux using exponential notation to indicate the multiplicity of each column)

$$w_1 := \begin{pmatrix} r \\ \vdots \\ 2 \\ 1 \end{pmatrix}^s \otimes \begin{pmatrix} n+1 \\ \vdots \\ n+3-r \\ 1 \end{pmatrix}^s \in C_1.$$

Under $\mathfrak{pr}$, w_1 maps to

$$w_2 := \begin{pmatrix} r+1 \\ r \\ \vdots \\ 3 \\ 2 \end{pmatrix}^s \otimes \begin{pmatrix} n+1 \\ \vdots \\ n+4-r \\ 2 \\ 1 \end{pmatrix}^s .$$

However it is not hard to see that

$$\Psi(w_1) = \begin{pmatrix} r \\ r-1 \\ \vdots \\ 1 \end{pmatrix}^{s-1} \otimes \begin{pmatrix} r \\ r-1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} n+1 \\ \vdots \\ n+3-r \\ 1 \end{pmatrix}^s \in B((s-1)\omega_r) \otimes B((s+1)\omega_r).$$

Similarly,

$$\Psi(w_2) = \begin{pmatrix} r+1 \\ r \\ \vdots \\ 5 \\ 4 \\ 2 \end{pmatrix}^s \otimes \begin{pmatrix} n+1 \\ \vdots \\ n+4-r \\ 3 \\ 2 \\ 1 \end{pmatrix}^s \in B(s\omega_{r-1}) \otimes B(s\omega_{r+1}).$$

Since $\mathfrak{pr}'$ preserves the components under Ψ by definition, it cannot agree with $\mathfrak{pr}$ on w_1. Hence $\mathfrak{pr} \neq \mathfrak{pr}'$ on C_1.

For the restriction on C_n, consider the element

$$w_n := \begin{pmatrix} n \\ r-1 \\ \vdots \\ 1 \end{pmatrix}^s \otimes \begin{pmatrix} n+1 \\ n \\ \vdots \\ n+2-r \end{pmatrix}^s \in C_n.$$

Note that $\Psi(w_n)$ is in the $B((s-1)\omega_r) \otimes B((s+1)\omega_r)$ component. On the other hand, the image of w_n under $\mathfrak{pr}$ is

$$w_{n+1} := \begin{pmatrix} n+1 \\ r \\ \vdots \\ 2 \end{pmatrix}^s \otimes \begin{pmatrix} n+1 \\ \vdots \\ n+3-r \\ 1 \end{pmatrix}^s ,$$

which is in the $B(s\omega_{r-1}) \otimes B(s\omega_{r+1})$ component under Ψ. So we conclude that $\mathfrak{pr} \neq \mathfrak{pr}'$ on C_n. □

For the next proposition, $B := B(s'\omega_{r'}) \otimes B(s\omega_r)$ is an A_n crystal, with $n \geq r' \geq r \geq 1$, and $D := B(s'\omega_{r'-1}) \otimes B(s\omega_{r-1})$ is an A_{n-1} crystal. Here we interpret ω_0 as the zero weight.

Proposition 4.8 *If $\mathfrak{pr}_D$ is the only promotion operator defined on D, then any promotion operator* pr *on B agrees with $\mathfrak{pr}_B$ on both $C_1(B)$ and $C_n(B)$. If $\mathfrak{pr}_D$ and $\mathfrak{pr}'_D$ are the only promotion operators defined on D, then any promotion operator* pr *on B either agrees with $\mathfrak{pr}_B$ on both $C_1(B)$ and $C_n(B)$ or it agrees with $\mathfrak{pr}'_B$ on both $C_1(B)$ and $C_n(B)$.*

Proof For the purpose of this proof we set $C_i := C_i(B)$. Note that while the graphs C_1 and C_{n+1} are twisted-isomorphic to D, the other graphs C_i for $i \neq 1, n+1$ do not have enough arrows.

We claim there is a unique isomorphism from the A_{n-1} crystal C_{n+1} to D. We define one such isomorphism ϕ_{n+1}, by simply removing the top row from each factor. (It is easy to see that this is removing all letters $n+1$). By the decomposition (2.1), this isomorphism is unique.

Similary, there is a unique twisted isomorphism from the $\{2, \dots, n\}$ crystal C_1 to the $\{1, \dots, n-1\}$ crystal D with twist given by $\tau : i \mapsto i-1$. We define one such isomorphism, ϕ_1, by removing the bottom row from each factor and subtracting one from each letter. Now given any τ-twisted isomorphism $\phi : C_1 \to D$, we get $\phi^{-1} \circ \phi_1 : D \to D$ is a crystal isomorphism. By (2.1) this must be the identity, which implies that $\phi = \phi_1$.

Note that this implies that there is a unique twisted isomorphism from $C_{n+1} \to C_1$ with twist given by $\tau : i \mapsto i+1$. This is given by $\phi_1^{-1} \circ \phi_{n+1}$, and any other τ-twisted isomorphism would give a nontrivial automorphism of D. Since any promotion operator pr on B must give such a τ-twisted isomorphism when restricted to C_{n+1}, we have shown that the promotion operator pr restricted to $\mathrm{pr} : C_{n+1} \to C_1$ is determined.

Now let pr be any promotion operator on B, restricted to the union of the sets $C_1, \dots, C_{n+1}$. Let $\mathfrak{pr}_B$ be the standard promotion operator on B, and $\mathfrak{pr}_D$ be the standard promotion operator on D. Define a map $\phi_2 : C_2 \to D$ by

$$\phi_2 := \mathfrak{pr}_D \circ \phi_1 \circ \mathfrak{pr}_B^{-1}.$$

(All functions written here will be acting on the left.)

Define a map from D to itself by

$$\rho := \phi_2 \circ \mathrm{pr} \circ \phi_1^{-1}.$$

Note that ρ is a map which takes D to itself and affects content and arrows according to axioms 1 and 3 of promotion operators as defined in Definition 1.1. We will show that ρ also satisfies the axiom that $\rho^n = \mathrm{id}$, hence proving that ρ is a promotion operator on D.

We determine the order of ρ by constructing a commutative diagram:

$$\begin{tikzcd}
C_1 \arrow[r, "\mathrm{pr}"] \arrow[d, "\phi_1"'] & C_2 \arrow[r, "\mathrm{pr}"] \arrow[d, "\phi_2"'] & C_3 \arrow[r, "\mathrm{pr}"] \arrow[d, "\phi_3"'] & \cdots \arrow[r, "\mathrm{pr}"] & C_{n+1} \arrow[r, "\mathrm{pr}"] \arrow[d, "\phi_{n+1}"'] & C_1 \arrow[dl, "\phi_1"'] \\
D \arrow[r, "\rho"] & D \arrow[r, "\rho"] & D \arrow[r, "\rho"] & \cdots \arrow[r, "\rho"] & D &
\end{tikzcd}$$

The maps ϕ_i in this diagram (for $3 \le i \le n$) are defined by requiring this diagram to commute. Specifically, we have

$$\phi_i := \rho \circ \phi_{i-1} \circ \mathrm{pr}^{-1}.$$

Notice that, by the uniqueness of ϕ_{n+1}, we must have $\rho \circ \phi_n \circ \mathrm{pr}^{-1} = \phi_{n+1}$ on C_{n+1}. Since pr is a promotion operator, the composition pr^{n+1} along the top row of this diagram must be equal to the identity. Thus we can collapse the diagram to

$$\begin{tikzcd} C_1 \arrow[d, "\phi_1"'] \arrow[dr, "\phi_1"] & \\ D \arrow[r, "\rho^n"] & D \end{tikzcd}$$

which implies $\rho^n = \mathrm{id}$. This completes the proof that ρ is a promotion operator.

Now assume that the only choice for a promotion operator on D is $\mathfrak{pr}_D$. Recall

$$\begin{aligned} \rho &= \phi_2 \circ \mathrm{pr} \circ \phi_1^{-1} \\ &= \left(\mathfrak{pr}_D \circ \phi_1 \circ \mathfrak{pr}_B^{-1}\right) \circ \mathrm{pr} \circ \phi_1^{-1}. \end{aligned}$$

Since $\rho = \mathfrak{pr}_D$ we multiply both sides by $\mathfrak{pr}_D^{-1}$ on the left to obtain

$$\mathrm{id}_D = \phi_1 \circ \mathfrak{pr}_B^{-1} \circ \mathrm{pr} \circ \phi_1^{-1}.$$

Conjugating by ϕ_1 gives

$$\mathrm{id}_{C_1} = \mathfrak{pr}_B^{-1} \circ \mathrm{pr}$$

which implies

$$\mathrm{pr} = \mathfrak{pr}_B \quad \text{on } C_1.$$

Next assume that there are two choices for the promotion operator on D, namely $\mathfrak{pr}_D$ and $\mathfrak{pr}'_D$. If $\rho = \mathfrak{pr}_D$, then by the same arguments as above, we conclude that $\mathrm{pr} = \mathfrak{pr}_B$ on C_1. If $\rho = \mathfrak{pr}'_D$, then $\mathrm{pr} = \phi_2^{-1} \circ \rho \circ \phi_1$ must be different from $\mathfrak{pr}_B$ on C_1. Furthermore, there are no more than these two possibilities for pr on C_1. By Lemma 4.7, $\mathfrak{pr}_B$ and $\mathfrak{pr}'_B$ are different on C_1. Hence if $\rho = \mathfrak{pr}'_D$, then $\mathrm{pr} = \mathfrak{pr}'_B$ on C_1.

We now wish to show that $\mathrm{pr}^{-1} = \mathfrak{pr}_B^{-1}$ on C_{n+1} if the only choice for a promotion operator on D is $\mathfrak{pr}_D$. We keep our definitions of ϕ_1 and ϕ_{n+1} and define $\phi_n : C_n \to D$ by

$$\phi_n := \mathfrak{pr}_D^{-1} \circ \phi_{n+1} \circ \mathfrak{pr}_B.$$

We now redefine $\rho : D \to D$ (though it will in fact coincide with the old definition) by

$$\rho := \phi_{n+1} \circ \mathrm{pr} \circ \phi_n^{-1}.$$

We again conclude that ρ satisfies the content and arrow properties of a promotion operator, and we determine its order with the following diagram:

$$\begin{tikzcd}
C_{n+1} \arrow[d, "\phi_{n+1}"'] & C_n \arrow[l, "\mathrm{pr}"'] \arrow[d, "\phi_n"'] & C_{n-1} \arrow[l, "\mathrm{pr}"'] \arrow[d, "\phi_{n-1}"'] & \cdots \arrow[l, "\mathrm{pr}"'] & C_1 \arrow[l, "\mathrm{pr}"'] \arrow[d, "\phi_1"'] & C_{n+1} \arrow[l, "\mathrm{pr}"'] \arrow[dl, "\phi_{n+1}"] \\
D & D \arrow[l, "\rho"] & D \arrow[l, "\rho"] & \cdots \arrow[l, "\rho"] & D \arrow[l, "\rho"] &
\end{tikzcd}$$

(Again the undefined vertical arrows are defined soley to make the diagram commute.) As before, we conclude that $\rho^n = \mathrm{id}$ and hence $\mathrm{pr} = \mathfrak{pr}_B$ on C_n. By very similar arguments as before, pr on C_n is either $\mathfrak{pr}_B$ and $\mathfrak{pr}'_B$ if there are the two choices $\mathfrak{pr}_D$ and $\mathfrak{pr}'_D$ for promotion operators on D.

If $\mathfrak{pr}_D$ is the only promotion operator on D, then we are done. If $\mathfrak{pr}_D$ and $\mathfrak{pr}'_D$ are the two choices for promotion operators on D, then it remains to show that pr is $\mathfrak{pr}_B$ on both C_1 and C_n or that pr is $\mathfrak{pr}'_B$ on both C_1 and C_n. Recall w_1 and w_n as defined in the proof of Lemma 4.7. Note that w_1 is related to w_n by a series of f_j operators (not including f_n). Thus $\mathrm{pr}(w_1)$ determines $\mathrm{pr}(w_n)$ and conversely. □

4.4 Rectangle tensor rectangle case

In this section we prove Theorem 1.2 for $(s, r) \neq (s', r')$, $r' \geq r \geq 1$, $r' > 1$, and $n \leq r + r' - 1$. The proof is by induction on n, showing that there is a unique promotion on the A_n crystal $B = B(s'\omega_{r'}) \otimes B(s\omega_r)$, assuming that the statement is true by induction for smaller n.

Lemma 4.9 *For $n < r + r' - 2$, the promotion operator on the A_n crystal $B := B(s'\omega_{r'}) \otimes B(s\omega_r)$ is determined.*

Proof It suffices to show that the promotion operator on $\{1, \ldots, n-1\}$ highest weight elements is determined. The right tensor factor of such an element w must be of the form

$$\begin{pmatrix} r \\ r-1 \\ \vdots \\ 1 \end{pmatrix}^a \begin{pmatrix} n+1 \\ r-1 \\ \vdots \\ 1 \end{pmatrix}^b. \tag{4.4}$$

Hence the bottom row of the left tensor factor can only contain the letters $1, r$ and $r+1$ (since $r' > 1$ the letter $n+1$ is not possible in the first row). But if r or $r+1$ appears in the bottom of a column in the left tensor factor, then $r + r' - 1$ is the smallest possible number which could appear at the top of that column by columnstrictness. But since $n < r + r' - 2$, this letter is not in our crystal. Hence the bottom row of the left tensor factor consists only of 1s, so w is in $C_1(B)$ and by Proposition 4.8 promotion is given by either $\mathfrak{pr}$ or $\mathfrak{pr}'$. □

For the rest of the proof, we assume that $n = r' + r - 2$ or $n = r' + r - 1$. The unique element of weight $s\omega_r$ in $B(s\omega_r)$ is called *Yamanouchi*; it is the tableau with row i filled with letter i for $1 \le i \le r$. After a preliminary lemma, we first show that promotion on all $\{1, 2, \dots, n-1\}$ highest weight vectors, where the right factor is Yamanouchi, is determined. Then we prove the claim for general $\{1, 2, \dots, n-1\}$ highest weight elements.

Lemma 4.10 *Suppose w is a $\{1, \dots, n-1\}$ highest weight element whose right factor is Yamanouchi or whose right factor has height one. Let* pr, pr$'$ *be any two* (a priori *different*) *promotion operators. Let w_1, w_n be defined by*

$$w_n \overset{\mathrm{pr}'}{\to} w \overset{\mathrm{pr}'}{\to} w_1.$$

$$\textit{If} \quad w_n \overset{\mathrm{pr}}{\longrightarrow} w_0 \overset{\mathrm{pr}}{\longrightarrow} w_1 \quad \textit{then} \quad w_0 = w.$$

Proof We first assume that the right factor has height one. We claim that w is completely specified by:

(1) The fact that w is $\{1, \dots, n-1\}$ highest weight;
(2) The content of w;
(3) The content of the right factor of w.

Suppose w is $\{1, \dots, n-1\}$ highest weight. Then the right factor of w must be of the form $1^a(n+1)^b$. Suppose also that $m_{n+1}(w) = b + c$. The A_{n-1} crystal consisting of those elements of B with c copies of $n+1$ in the left factor and b copies of $n+1$ in the right factor is isomorphic to $B((s'-c)\omega_{r'} + c\omega_{r'-1}) \otimes B((s-b)\omega_1)$. This has a decomposition into classical components according to the multiplication of Schur functions $s_{(s'^{r-1}, s'-c)} s_{(s-b)}$. Since this product is indexed by a "near rectangle" and a rectangle, Theorem 2.1 of [29] gives that this product, when expanded into Schur functions, is multiplicity free. Hence, there is at most one highest weight vector of a given content. This proves the claim.

It is clear that (1) and (2) can be reconstructed from w_1. From (1), we know the right factor of w is of the form $1^a(n+1)^b$, and the right factor of w_0 is of the form $1^{a'}(n+1)^{b'}$. So it suffices show that $b' = b$. For this, we note that the $\{2, \dots, n\}$ lowest weight element associated to w has precisely $s' + b$ copies of $n+1$. Thus the $\{1, \dots, n-1\}$ lowest weight element associated to w_n has $s' + b$ copies of n, and the $\{2, \dots, n\}$ lowest weight element corresponding to w_0 must have $s' + b$ of copies of $n+1$. But this is only the case if $b' = b$.

Now we consider the case that $r > 1$ and the right factor of w is Yamanouchi. We claim that w is completely specified by:

(1) The fact that w is $\{1, \dots, n-1\}$ highest-weight;
(2) The content of w;

(3) The fact that the right factor of w is Yamanouchi.

In this case, we know that all $c := m_{n+1}(w)$ copies of $n+1$ occur in the top row of the left factor. Thus the A_{n-1} crystal consisting of all elements with c copies of $n+1$ in the left factor and none in the right is isomorphic to $B((s'-c)\omega_{r'} + c\omega_{r'-1}) \otimes B(s\omega_r)$. Again, the corresponding product of Schur functions is indexed by a near rectangle and a rectangle, and so the product is multiplicity free. Hence there is at most one highest weight vector of a given content.

It is clear that (1) and (2) can be reconstructed from w_1. From (1), we know the right factor of w_0 must be of the form (4.4). We must show that $b = 0$ in (4.4).

Assume first that $r' > r$. Let m_r be the number of letters r in w. We note that the number of letters r in the $\{2, \ldots, n\}$ highest weight associated to w must also be precisely m_r; the only elements that can change at all are the letters $n+1$, and because they are in a row of height $> r$, they cannot become letters r. Thus the number of letters $r-1$ in the $\{1, \ldots, n-1\}$ highest weight associated to w_n is also m_r. From this we conclude that in any w_0, the number of rs in the associated $\{2, \ldots, n\}$ highest weight is m_r. If there were an $n+1$ in the right factor of w, this would not be true, so we can conclude that $b = 0$.

Finally suppose that $r' = r$. Let $k \geq 0$ be the number of columns of the left factor of w whose top two entries are of the form $\substack{n+1 \\ r-1}$. If $k > 0$, the top two rows of the left factor of w is of the general form:

$$\begin{pmatrix} \leq n & \ldots & \leq n & n+1 & \ldots & n+1 & n+1 & \ldots & n+1 \\ r-1 & \ldots & r-1 & r-1 & \ldots & r-1 & \geq r & \ldots & \geq r \\ & & & & \ldots & & & & \end{pmatrix}.$$

If $k \leq s$, we note that the number of rs in the $\{2, \ldots, n\}$ highest weight associated to w is again m_r. Hence we can repeat the $r' > r$ argument above to conlude that $b = 0$. If $k > s$, we set w' to be the $\{r, \ldots, n\}$ lowest weight associated to w. We have $m_{n+1}(w') = s'$, since every element in the top row of the left factor can be raised to an $n+1$, and every r in the right factor will be 'blocked' by at least one of the k letters $(n+1)$ above an $r-1$ on the left on its way up. Translating this property to w_n and back to w_0, we see that the $\{r, \ldots, n\}$ lowest weight associated to any w_0 must contain the letter $n+1$ precisely s' times. But the number of $(n+1)$s in this lowest weight must be at least $s' + b$; hence $b = 0$. $\square$

Lemma 4.11 *The promotion operator is determined on the set of* $\{1, \ldots, n-1\}$ *highest weight elements in* B *for which the right factor is Yamanouchi.*

Proof For $n = r' + r - 2$, the bottom row of the left factor can only contain the letter 1 by the same arguments as in the proof of Lemma 4.9. Hence all $\{1, \ldots, n-1\}$ highest weight elements are in $C_1(B)$ and by Proposition 4.8 the statement follows by induction.

For $n = r' + r - 1$, let w be a $\{1, \ldots, n-1\}$ highest weight element with a Yamanouchi right factor. The general strategy is to consider the $\mathfrak{pr}$-orbit of w:

$$w \to w_1 \to \cdots \to w_n \to w.$$

We show that $w_1, \ldots, w_{r'}$ are associated to $\{1, \ldots, n-1\}$ highest weight elements in $C_1(B)$ whose promotion we know by induction. We then show that promotion inverse of $w_{r'+1}, \ldots, w_n$ is determined by showing that the associated $\{2, \ldots, n+1\}$ lowest weight elements are in $C_{n+1}(B)$. By Lemma 4.10 we also know that w is the unique element which is simultaneously $\mathrm{pr}(w_n)$ and $\mathrm{pr}^{-1}(w_1)$.

The right factor of w is Yamanouchi and hence of the form

$$\begin{pmatrix} r \\ \vdots \\ 1 \end{pmatrix}^s .$$

Note also that the bottom row of the left factor of any element of this crystal must consist of letters $\leq r+1$; if there is a larger letter in the first row, this would force a letter larger than $n+1 = r+r'$ in the top row. In particular, if $r+1$ appears in the bottom row of the left factor, then the column has an $n+1$ on the top. Now consider the element $w_i := \mathfrak{pr}^i(w)$, for $1 \leq i \leq r'$. The right factor of w_i is

$$\begin{pmatrix} r+i \\ \vdots \\ 1+i \end{pmatrix}^s .$$

The bottom row of the left factor contains only letters $< r+i$, which are not bracketed with the right factor. In particular, we see that the bottom row can always be lowered via e_j operators (without e_n) to a row of 1s. Since the bottom row of the right factor can also always be lowered to 1s (without using e_n), we see that w_i can be lowered to $C_1(B)$ and so the promotion $w_i \xrightarrow{\mathrm{pr}} w_{i+1}$ for $1 \leq i \leq r'$ is determined by Proposition 4.8.

Now notice that for $r' \leq i \leq n$, the top row of the right factor of w_i consists of only $(n+1)$s. Thus the associated $\{2, \ldots, n\}$ lowest weight element is in $C_{n+1}(B)$, and so by induction Proposition 4.8 we have determined promotion inverse. Hence we have computed $w_1 \xrightarrow{\mathrm{pr}} w_2 \xrightarrow{\mathrm{pr}} \cdots \xrightarrow{\mathrm{pr}} w_n$ and so by Lemma 4.10 we know the orbit of w. $\square$

Lemma 4.12 *The promotion operator is determined on the set of* $\{1, \ldots, n-1\}$ *highest weight elements in* B *with a right factor of*

$$\begin{pmatrix} n+1 \\ r-1 \\ r-2 \\ \vdots \\ 1 \end{pmatrix}^s . \tag{4.5}$$

Proof As before we set $w_i = \mathfrak{pr}^i(w)$. Every element in $\{w_1, w_2, \ldots, w_{n+1-r}\}$ is associated via a sequence of e_j (not including e_n) to a $\{1, \ldots, n-1\}$ highest weight element with a Yamanouchi right factor. So promotion of these elements is determined by Lemma 4.11. The remaining elements $\{w_{n+2-r}, \ldots, w_n, w\}$ are associated with a $\{2, \ldots, n\}$ lowest weight element in $C_{n+1}(B)$, so promotion inverse is determined by Proposition 4.8. Thus the orbit of w is determined. □

Lemma 4.13 *The promotion operator is determined on the set of* $\{1, \ldots, n-1\}$ *highest weight elements in* B.

Proof If w is a $\{1, \ldots, n-1\}$ highest weight element and has a right factor which is Yamanouchi or of the form (4.5), the result follows from Lemmas 4.11 and 4.12. Hence we may assume that the top row of the right factor of w contains both the letters r and $n+1$. Then the letters in the top row of the right factor of $w_i := \mathfrak{pr}^i(w)$ are given by:

$$\begin{array}{llll} 0:(r,n+1) & 1:(r,r+1) & \ldots & n-r:(n-1,n) \\ n-r+1:(n,n+1) & n-r+2:(n+1,n+1) & \ldots & n:(n+1,n+1). \end{array}$$

Notice that the right factor of w_i for $1 \le i < n-r+1$ can be transformed to the Yamanouchi element using a sequence of e_j (not including e_n). Hence by Lemma 4.11 promotion on this element is known. In the case that $r=1$, we have determined $w_1 \xrightarrow{\mathrm{pr}} \ldots \xrightarrow{\mathrm{pr}} w_n$, and hence by Lemma 4.10 we have determined the orbit of w. If $r > 1$, then the top row of the right factor of w_i for $n-r+1 < i \le n$ consists only of $n+1$, and hence a sequence of e_j (not including e_n) can transform these w_i into a $\{1, 2, \ldots, n-1\}$ highest weight element with right factor of the form (4.5), whose promotion orbit is already determined. In w_{n-r+1}, the right factor has the form

$$\begin{pmatrix} n \\ \vdots \\ n-r+1 \end{pmatrix}^b \begin{pmatrix} n+1 \\ \vdots \\ n-r+2 \end{pmatrix}^a .$$

Notice that every letter in the right factor is fully bracketed except for the letters n. Thus every letter $1, 2, \ldots, n$ in the left factor of w_{n-r+1} is unbracketed (with respect to the right factor). In particular, every letter in the first row of the left factor is unbracketed, so we can reduce them to 1s. This gives an element of $C_1(B)$ as the associated $\{1, \ldots, n-1\}$ highest weight element, and hence promotion is known by induction by Proposition 4.8. Thus the orbit of w is determined, and this completes the proof. □

By Lemma 4.13 promotion on all $\{1, 2, \ldots, n-1\}$ highest weight elements is determined. Hence by the Highest Weight Lemma 2.8 promotion is determined on all of B. This concludes the induction step in the proof of Theorem 1.2 when $(s, r) \neq (s', r')$.

4.5 Equal Tensor Factors

Let $B := B(s\omega_r) \otimes B(s\omega_r)$ be the tensor product of two identical classical highest weight crystals of type A_n with $n \ge 2$ and $r > 1$. We show in this section that there are two promotion operators on B, given by the connected operator $\mathfrak{pr}$ and the disconnected operator $\mathfrak{pr}' = \Psi^{-1} \circ \mathfrak{pr} \circ \Psi$.

By Proposition 4.8, there are at most two possibilities for the action of promotion on the subsets of B given by $C_1 := C_1(B)$ and $C_n := C_n(B)$. If promotion restricted to these subsets is given by $\mathfrak{pr}$, then all the arguments from Section 4.4 apply as before and we are done. So for the rest of this section we consider the case where promotion on C_1 and C_n is given by $\mathfrak{pr}'$. As before by the Highest Weight Lemma 2.8, it suffices to determine promotion on all $\{1, \dots, n-1\}$ highest weight elements.

Lemma 4.14 *Suppose* pr *on* B *coincides with* $\mathfrak{pr}'$ *on* C_1 *and* C_n. *If* $w \in B$ *is a* $\{1, \dots, n-1\}$ *highest weight element, with* $\Psi(w) \in B_1 := B((s-1)\omega_r) \otimes B((s+1)\omega_r)$, *and the right factor of* $\Psi(w)$ *is Yamanouchi, then the orbit of* w *is given by* $\mathfrak{pr}'$.

Proof We first note that the conditions of the lemma imply that the right factor of w is Yamanouchi: Suppose $\Psi(w) := v_1 \otimes v_2 \in B_1$ is $\{1, \dots, n-1\}$ highest weight with v_2 being Yamanouchi. Then every letter $n+1$ in $v_1.v_2$ is in a row higher than row r. Furthermore, since Ψ is a crystal isomorphism, $v_1 \otimes v_2$ is also $\{1, \dots, n-1\}$ highest weight. Now let $v_1' \otimes v_2' := \Psi^{-1}(v_1 \otimes v_2)$ (so $v_1'.v_2' = v_1.v_2$). This must still be $\{1, \dots, n-1\}$ highest weight, and thus the right tensor factor must be of the form (4.4). However, any $n+1$ in v_2' would certainly give an $n+1$ at height r in $v_1'.v_2'$. Thus the only possibility for $v_1'.v_2'$ to agree with $v_1.v_2$ is if v_2' is Yamanouchi.

Now, we label the elements of the orbit of w under $\mathfrak{pr}'$ by

$$w \to w_1' \to \cdots \to w_n' \to w.$$

Recall that Ψ is a crystal isomorphism and hence commutes with the crystal operators and preserves content. In particular, $\Psi^{-1}(C_1(B_1)) \subset C_1(B)$. By the proof of Lemma 4.11 we know that, for $1 \le i \le r$, $\Psi(w_i')$ is connected to $C_1(B_1)$ by a series of classical crystal operators (not involving e_n). Hence w_i' is connected to $C_1(B)$. From the same lemma, it also follows that for $r \le i \le n$, $\Psi(w_i')$ is connected to $C_n(B_1)$ by a series of classical crystal operators (not involving f_n); hence w_i' is connected to $C_n(B)$. Thus the partial orbit

$$w_1' \to w_2' \to \cdots \to w_n'$$

is determined. By Lemma 4.10, the entire orbit is now determined. □

Lemma 4.15 *Suppose* pr *is a promotion operator on* B *which coincides with* $\mathfrak{pr}'$ *on* C_1 *and* C_n. *If* $w \in B$ *is such that* $\Psi(w) \in B_1$, *then* $\mathrm{pr}(w) = \mathfrak{pr}'(w)$.

Proof It remains to show this for those $\{1, \dots, n-1\}$ highest weight elements whose image under Ψ is in B_1 and does not have a Yamanouchi right factor. First consider those elements w where $\Psi(w)$ has only a single repeated column on the right. Again, we label the orbit under $\mathfrak{pr}'$ of w by

$$w_0 := w \to w_1' \to \cdots \to w_n' \to w_0.$$

By the proof of Lemma 4.12, $\Psi(w_i')$ for $0 \le i \le n$ is connected by special sequences of crystal operators to elements whose promotion is already determined. Thus this is also true for w_i'. In particular, promotion of w is determined. This logic can also be applied to the remaining $\{1, \dots, n-1\}$ highest weight elements under consideration following the proof of Lemma 4.13. □

The fact that pr agrees with $\mathfrak{pr}'$ on B_1 implies that $\mathrm{pr}(B_2) = B_2$, where $B_2 := B(s\omega_{r-1}) \otimes B(s\omega_{r+1})$. By Section 4.4 we already know that promotion on a tensor product of two distinct rectangles is given by $\mathfrak{pr}$; thus we have in this case that $\mathrm{pr} = \mathfrak{pr}$ on B_2 and thus $\mathrm{pr} = \mathfrak{pr}'$ on B.

5 Evidence for Conjecture 1.4

In this section, we present evidence for Conjecture 1.4. In Section 5.1 we present theoretical results that support the claims of the conjecture and in Section 5.2 we discuss computer evidence.

5.1 Unique factorization into rectangular Schur functions

We have seen in Lemma 3.5 that $\mathfrak{pr}$ is a valid promotion operator on a tensor product of classical highest weight crystals of type A_n indexed by rectangles; furthermore $\mathfrak{pr}$ yields a connected affine crystal.

In the remainder of this section, we further argue that two distinct tensor products of classical highest weight crystals of type A_n indexed by rectangles have non-isomorphic classical structures, as desired for Conjecture 1.4 (otherwise, the two associated promotion operators could induce two non-isomorphic connected affine crystals). This statement translates as follows at the level of symmetric polynomials.

Proposition 5.1 *Let $n \ge 1$. If a symmetric polynomial $P := P(x_1, \dots, x_{n+1})$ can be factored as a product $P = s_{(c_1^{r_1})} \cdots s_{(c_k^{r_k})}$ of nontrivial rectangular Schur polynomials with $1 \le r_i \le n$, then this is the unique factorization of P as a product of rectangular Schur polynomials.*

This turns out to be a special case of the following theorem.

Theorem 5.2 (Rajan [22]) *Let $\mathfrak{g}$ be any simple Lie algebra, and $V_1, \dots, V_n$ and $W_1, \dots, W_m$ be nontrivial, finite-dimensional, irreducible $\mathfrak{g}$-modules. If $V_1 \otimes \cdots \otimes V_n \cong W_1 \otimes \cdots \otimes W_m$, then $n = m$ and $V_i \cong W_{\tau(i)}$ for some permutation τ.*

In type A, Purbhoo and van Willigenburg [21] give a combinatorial proof for products of two arbitrary Schur functions. The following combinatorial proof of Proposition 5.1 handles products of an arbitrary number of rectangular Schur functions.

Proof of Proposition 5.1 We may impose a total order on rectangular partitions by defining $(c^r) \ge (c'^{r'})$ if $r > r'$ or $r = r'$ and $c \ge c'$. We show that the factor in P

indexed by the largest rectangle in this order is uniquely determined. Hence induction on the largest factor proves the proposition.

Without loss of generality we may assume that $(c_1^{r_1}), \ldots, (c_k^{r_k})$ are ordered in weakly decreasing order. We use two facts, easily derived from the Littlewood–Richardson rule. Let $Q = \prod_{i=1}^k s_{\lambda^{(i)}}$ be any product of Schur functions. Let $(\nu^{(j)})_{j=1}^m$ be the list of partitions which index the expansion of Q into the sum of Schur functions (the order of this list does not matter). Then

(1) For all pairs (i, j) with $1 \leq i \leq k$ and $1 \leq j \leq m$ the diagram of $\nu^{(j)}$ contains the diagram of $\lambda^{(i)}$.
(2) If μ is a diagram consisting of the $\lambda^{(i)}$ concatenated to form a partition shape, then μ is one of the $\nu^{(j)}$.

Using these properties, we shall see that $(c_1^{r_1})$, defined to be the index of rectangle $\lambda^{(1)}$, can be determined from the collection of $\nu^{(j)}$. We first find r_1. Note that property (1) implies that the height of every diagram $\nu^{(j)}$ is at least r_1. But by property (2), there is some shape $\nu^{(j)}$ consisting of the shapes $\lambda^{(i)}$ concatenated from left to right. In particular, this shape has height exactly equal to r_1. So r_1 is the minimum of the heights of the $\nu^{(i)}$.

Since all other rectangles $(c_i^{r_i})$ for $1 \leq i \leq k$ have height $r_i \leq r_1$, we may assume without loss of generality that $n = r_1$. Each term s_ν in the Schur expansion of P can be associated with a highest weight crystal element in $B := B(c_1^{r_1}) \otimes \cdots \otimes B(c_k^{r_k})$ of weight ν. Take the collection of all terms s_ν in the Schur expansion of P such that the first $n-1$ parts of ν agree with the first $n-1$ parts of the partition obtained by concatenating all rectangles $(c_1^{r_1}), \ldots, (c_k^{r_k})$. This implies in particular that the corresponding highest weight crystal elements in B are all Yamanouchi in the first $n-1$ rows. If $c_1, \ldots, c_m$ are the widths of the rectangles of height $n = r_1$, then the terms s_ν, with ν given as above, are in one-to-one correspondence with the Schur expansion of the following product of complete symmetric functions $h_{c_1} \cdots h_{c_m}$ in two variables.

However, note that for $n = 1$ we have $h_j(x, 1) = \frac{1-x^{j+1}}{1-x}$, so its roots are the nontrivial $(j+1)$-th roots of unity. Let us consider two factorizations $h_{c_1} \cdots h_{c_m} = h_{c'_1} \cdots h_{c'_{m'}}$, and show that they must coincide. Consider the largest h_j occurring either on the left or the right hand side, and consider a primitive $(j+1)$-th root of unity ξ. Then, ξ is a root of $h_{c_i}(1, x)$ for some i, and by maximality of j, $c_i = j$. We can therefore factor out h_j from the left hand side – and similarly from the right hand side – and apply induction. $\square$

Example 5.3 Let us illustrate the proof of Proposition 5.1 in terms of an example. Take $P = s_{22} s_{11}^2 s_3$. In the Schur expansion of P there is a term s_{74}, which is obtained by concatenating the four rectangles. All terms are labeled by partitions with at least two parts. This tells us that the height of the largest rectangle is $r_1 = 2$.

To determine the width of the largest rectangle, we consider the highest weight crystal elements that are Yamanouchi in the first $r_1 - 1 = 1$ rows:

$$\begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array},$$

$$\begin{array}{|c|c|}\hline 2&3\\\hline 1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 3&3\\\hline 1&1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}.$$

The second row of these elements gives precisely the expansion of the $\mathfrak{sl}_2$ or two variable expansion of $h_2 h_1^2$, which is unique. Hence $c_1 = 2$, and the largest rectangle is (2^2).

5.2 Computer exploration

The research was partially driven by computer exploration. In particular, we implemented a branch-and-bound algorithm to search for all (connected) (weak) promotion operators on a given classical crystal. The algorithm goes down a search tree, deciding progressively to which $\{2, \ldots, n\}$-component each connected $\{1, \ldots, n-1\}$-component is mapped by promotion. Branches are cut as soon as it can be decided that the yet partially defined promotion cannot satisfy condition (2') of Remark 2.7, or cannot be connected. The algorithm can also take advantage of the symmetries of the classical crystal (not fully though, by lack of appropriate group theoretical tools in MuPAD), and uses some heuristics for the decision order. The branch cutting works reasonably well; for the difficult case of $B(1)^{\otimes 4}$ in type A_2, where the total search space is a priori of size 144473849856000, with $2!3!3! = 72$ symmetries, the algorithm actually explores 115193 branches in 5 hours and 26 minutes (on a 2 GHz Linux PC), using 16M of memory. The result is 8 isomorphic connected promotion operators: 9 symmetries out of the 72 could be exploited to cut the search space.

Example 5.4 We start by loading the MuPAD-Combinat package, and setting the notation for tensor products:.

```
» package("MuPAD-Combinat"):
» operators::setTensorSymbol("#"):
```

Consider the A_2 classical crystal $C := B(1) \otimes B(1) \otimes B(1)$:

```
» B1 := crystals::tableaux(["A",2], Shape = [1]):
» C := B1 # B1 # B1:
```

The decomposition into classical components is given by $s_1^3 = s_3 + s_{111} + 2s_{21}$ (note the multiplicity of s_{21}). There are four promotion operators:

```
» promotions := C::promotions():
» nops(promotions)
                                4
```

Let us construct the corresponding crystal graphs:

```
»   affineCrystals :=
»   [crystals::affineFromClassicalAndPromotion(C, promotion)
»    $ promotion in promotions]:
```

Among them, two are connected:

```
»   [ A::isConnected() $ A in affineCrystals ]
                                   [TRUE, FALSE, FALSE, TRUE]
```

But they are in fact isomorphic via the exchange of the two $(2,1)$-classical components:

```
» nops((affineCrystals[1])::isomorphisms(affineCrystals[4]))
                                        1
```

The other two affine crystals are disconnected, and induced by the decomposition $s_1^3 = s_1 s_{11} + s_1 s_2$ of s_1^3 into a sum of products of rectangles. The use of the options `Connected` and `Symmetries` cuts down the search tree. It turns out that for our current crystal, the symmetries are fully exploited, and we only get one isomorphic copy of the connected promotion operator:

```
» nops(C::promotions(Connected, Symmetries))
                                        1
```

Now consider the A_2 classical crystal $C := B(2,1) \otimes B(2,1)$.

```
» B21 := crystals::tableaux(["A",2], Shape = [2,1]):
» C := B21 # B21:
```

The highest weights of the classical crystal are given by the following Schur polynomial expansion:

$$s_{21}^2 = s_{42} + s_{411} + s_{33} + 2s_{321} + s_{222}. \tag{5.1}$$

Beware that, since $n = 2$, the term s_{2211} is zero. Also, the crystal for s_{411} is isomorphic to that for s_3, and similarly s_{222} is trivial. Finally, note the multiplicity of s_{321}.

There are no connected promotion operators:

```
» nops(C::promotions(Connected))
                                        0
```

Indeed, f has no factorization into products of rectangle Schur polynomials. On the other hand, there are eight disconnected promotion operators:

```
» nops(C::promotions())
                                        8
```

They are induced by the following four decompositions:

$$\begin{aligned} s_{21}^2 &= s_{22} s_1 s_1 + s_3 s_{111} \\ &= s_{22} s_{11} + s_{22} s_2 + s_3 s_{111} \\ &= s_{22} s_2 + s_2 s_1 s_{111} + s_{33} \\ &= s_{11} s_{11} s_2 + s_{33} \end{aligned} \tag{5.2}$$

combined with the automorphism which exchanges the two $(3,2,1)$-classical components.

The examples of Figure 1 for $n = 1$ were found with this algorithm. On the other hand, we ran systematic tests on the following crystals with $n \geq 2$:

- All tensor products of rows with up to 3 factors and up to 6 cells (except $B(2)^{\otimes 3}$) in type A_2 and up to 7 cells (except $B(3) \otimes B(2)^{\otimes 2}$) in type A_3 and A_4;
- $B(3,2,1) \otimes B(1)$, $B(2,1) \otimes B(2,1)$, $B(2,1) \otimes B(1) \otimes B(1)$, $B(2,2) \otimes B(1,1,1)$, $B(2,2) \otimes B(1,1,1) \otimes B(1)$, $B(1)^{\otimes 4}$, in type A_2 and A_3.

They all agree with Conjectures 1.4 and 2.13. Namely, for tensor products of rectangles, there is a unique connected promotion operator, up to isomorphism; for other tensor products, there is none.

In the smaller examples, we further checked that the total number of promotions was exactly the number of automorphisms of the underlying classical crystal (that is $\prod m_\lambda!$ where m_λ is the number of classical components of highest weight λ) times the number of decompositions of the symmetric function $\sum m_\lambda s_\lambda$ into sums of products of rectangular Schur functions.

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