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On the representation theory of an algebra of braids and ties

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Abstract We consider the algebra $\mathcal{E}_n(u)$ introduced by Aicardi and Juyumaya as an abstraction of the Yokonuma–Hecke algebra. We construct a tensor space representation for $\mathcal{E}_n(u)$ and show that this is faithful. We use it to give a basis of $\mathcal{E}_n(u)$ and to classify its irreducible representations.

Keywords Diagram algebras · Symmetric group · Specht modules

1 Introduction

We initiate in this paper a systematic study of the representation theory of an algebra $\mathcal{E}_n(u)$ defined by Aicardi and Juyumaya. Let G be a Chevalley group over $\mathbb{F}_q$ with Borel group B and maximal unipotent subgroup U. The origin of $\mathcal{E}_n(u)$ is in the Yokonuma–Hecke algebra $\mathcal{Y}_n(u)$, which is defined similarly as the Iwahori–Hecke algebra but with B replaced by U. That is, $\mathcal{Y}_n(u)$ is the endomorphism algebra of the induced G-module $\mathrm{ind}_U^G 1$. It is a unipotent Hecke algebra in the sense of [20]. Yokonuma gave in [21] a presentation of $\mathcal{Y}_n(u)$ along the lines of the standard T_i-presentation of the Iwahori–Hecke algebra, but the introduction of $\mathcal{E}_n(u)$ is more naturally motivated by the new presentation of $\mathcal{Y}_n(u)$ found by Juyumaya in [15], see also [14, 16]. For type A_n, this new presentation has generators $T_i, i = 1, \dots, n-1$ and $f_i, i = 1, \dots, n$ where the f_i generate a product of cyclic groups and the T_i satisfy the usual braid relation of type A, but do *not* coincide with Yokonuma's T_i-generators. The quadratic relation takes the form

$$T_i^2 = 1 + (u-1)e_i(1+T_i),$$

for e_i a complicated expression involving f_i and f_{i+1}.

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The algebra $\mathcal{E}_n(u)$ is obtained by leaving out the f_i, but declaring the e_i new generators, denoted E_i. It was introduced by Aicardi and Juyumaya in [1]. They showed that $\mathcal{E}_n(u)$ is finite dimensional and that it has connections to knot theory via the Vasiliev algebra. They also constructed a diagram calculus for $\mathcal{E}_n(u)$ where the T_i are represented by braids in the usual sense and the E_i by ties. Using results from [3], they moreover showed that $\mathcal{E}_n(u)$ can be Yang–Baxterized in the sense of Jones [11].

In this paper we initiate a systematic study of the representation theory of $\mathcal{E}_n(u)$, obtaining a complete classification of its simple modules for generic choices of the parameter u. In [1], this was achieved only for $n = 2, 3$. An interesting feature of this classification is the construction of a tensor space module $V^{\otimes n}$ for $\mathcal{E}_n(u)$. It was in part inspired by the tensor module for the Ariki–Koike algebra in [2]—see also [19]. A main property of $V^{\otimes n}$ is its faithfulness that we obtain as a corollary to our Theorem 3 giving a basis G for $\mathcal{E}_n(u)$. The dimension of $\mathcal{E}_n(u)$ turns out to be $B_n n!$ where B_n is the Bell number, i.e. the number of set partitions of $\{1, 2, \ldots, n\}$.

The appearance of the Bell number is somewhat intriguing and may indicate a connection to the partition algebra defined independently by P. Martin in [17] and V. Jones in [12], but as we indicate in the remarks following Corollary 4, we do not think at present that the connection can be very direct.

Given the tensor module, the classification of the irreducible modules follows the principles laid out in James's famous monograph on the representation theory of the symmetric group [10].

Let us briefly explain the organization of the paper. Section 2 contains the definition of the algebra $\mathcal{E}_n(u)$. In Sect. 3 we start out by giving the construction of the tensor space $V^{\otimes n}$. We then construct the subset $G \subset \mathcal{E}_n(u)$ and show that it generates $\mathcal{E}_n(u)$. Finally we show that it maps to a linearly independent set in $\mathrm{End}(V^{\otimes n})$, thereby obtaining the faithfulness of $V^{\otimes n}$ and the dimension of $\mathcal{E}_n(u)$.

In Sect. 4 we recall the basic representation theory of the symmetric group and the Iwahori–Hecke algebra, and use the previous sections to construct certain simple modules for $\mathcal{E}_n(u)$ as pullbacks of the simple modules of these. In Sect. 5 we show that $\mathcal{E}_n(u)$ is selfdual by constructing a nondegenerate invariant form on it. This involves the Moebius function for the usual partial order on set partitions. In Sect. 6 we give the classification of the simple modules of $\mathcal{E}_n(u)$, to a large extent following the approach of James' book [10]. Thus, we especially introduce a parametrizing set $\mathcal{L}_n$ for the irreducible modules, analogs of the permutations modules and prove James' submodule theorem in the setup. The simple modules, the Specht modules, turn out to be a combination of the Specht modules for the Hecke algebra and for the symmetric group and hence $\mathcal{E}_n(u)$ can be seen as a combination of these two. Finally, in the last section we raise some questions connected to the results of the paper.

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2 Definition of $\mathcal{E}_n(u)$

In this section we introduce the algebra $\mathcal{E}_n(u)$, the main object of our work. Let $\mathcal{A}$ be the principal ideal domain $\mathbb{C}[u,u^{-1}]$ where u is an unspecified variable. We first define the algebra $\mathcal{E}_n^{\mathcal{A}}(u)$ as the associative unital $\mathcal{A}$-algebra on the generators $T_1,\dots,T_{n-1}$ and $E_1,\dots,E_{n-1}$ and relations

$$\begin{array}{ll}
(E1) & T_iT_j = T_jT_i \quad \text{if } |i-j|>1,\\
(E2) & E_iE_j = E_jE_i \quad \forall i,j,\\
(E3) & E_iT_j = T_jE_i \quad \text{if } |i-j|>1,\\
(E4) & E_i^2 = E_i,\\
(E5) & E_iT_i = T_iE_i,\\
(E6) & T_iT_jT_i = T_jT_iT_j \quad \text{if } |i-j|=1,\\
(E7) & E_jT_iT_j = T_iT_jE_i \quad \text{if } |i-j|=1,\\
(E8) & E_iE_jT_j = E_iT_jE_i = T_jE_iE_j \quad \text{if } |i-j|=1,\\
(E9) & T_i^2 = 1+(u-1)E_i(1+T_i).
\end{array}$$

It follows from $(E9)$ that T_i is invertible with inverse

$$T_i^{-1} = T_i + \big(u^{-1}-1\big)E_i(1+T_i)$$

so the presentation of $\mathcal{E}_n(u)$ is not efficient, since the generators E_i for $i\geq 2$ can be expressed in terms of E_1. However, for the sake of readability, we prefer the presentation as it stands.

We then define $\mathcal{E}_n(u)$ as

$$\mathcal{E}_n(u) := \mathcal{E}_n^{\mathcal{A}}(u)\otimes_{\mathcal{A}}\mathbb{C}(u),$$

where $\mathbb{C}(u)$ is considered as an $\mathcal{A}$-module through inclusion.

This algebra is our main object of study. It was introduced by Aicardi and Juyumaya, in [1], although the relation $(E9)$ varies slightly from theirs since we have changed T_i to $-T_i$. They show, among other things, that it is finite dimensional.

From $\mathcal{E}_n^{\mathcal{A}}(u)$ we can consider the specialization to a fixed value u_0 of u which we denote $\mathcal{E}_n(u_0)$. However, we shall in this paper only need the case $u_0=1$, corresponding to

$$\mathcal{E}_n(1) = \mathcal{E}_n^{\mathcal{A}}(u)\otimes_{\mathcal{A}}\mathbb{C},$$

where $\mathbb{C}$ is made into an $\mathcal{A}$-module by taking u to 1. Letting S_n denote the symmetric group on n letters, there is a natural algebra homomorphism $\iota:\mathbb{C}S_n\to\mathcal{E}_n(1)$, $(i,i+1)\mapsto T_i$. It can be shown to be injective, using the results of the paper.

3 The tensor space

For the rest of the paper we shall write $K=\mathbb{C}(u)$. Let V be the K-vector space

$$V = \operatorname{span}_K\big\{v_i^j \,|\, i,j=1,2,\dots,n\big\}.$$

We consider the tensor product $V^{\otimes 2}$ and define $E \in \mathrm{End}_K(V^{\otimes 2})$ by the rules

$$E\bigl(v_{i_1}^{j_1} \otimes v_{i_2}^{j_2}\bigr) = \begin{cases} v_{i_1}^{j_1} \otimes v_{i_2}^{j_2} & \text{if } j_1 = j_2, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore we define $T \in \mathrm{End}_K(V^{\otimes 2})$ by the rules

$$T\bigl(v_{i_1}^{j_1} \otimes v_{i_2}^{j_2}\bigr) = \begin{cases} v_{i_2}^{j_2} \otimes v_{i_1}^{j_1} & \text{if } j_1 \neq j_2, \\ u v_{i_2}^{j_1} \otimes v_{i_1}^{j_2} & \text{if } j_1 = j_2, i_1 = i_2, \\ v_{i_2}^{j_2} \otimes v_{i_i}^{j_1} & \text{if } j_1 = j_2, i_1 < i_2, \\ u v_{i_2}^{j_2} \otimes v_{i_1}^{j_1} + (u-1) v_{i_1}^{j_1} \otimes v_{i_2}^{j_2} & \text{if } j_1 = j_2, i_1 > i_2. \end{cases}$$

We extend these operators to operators E_i, T_i acting in the tensor space $V^{\otimes n}$ by letting E, T act in the factors $(i, i+1)$. In other words, E_i acts as a projection in the factors at the positions $(i, i+1)$ with equal upper index, whereas T_i acts as a transposition if the upper indices are different and as a Jimbo matrix for the action of the Iwahori–Hecke algebra in the usual tensor space if the upper indices are equal, see [13].

Theorem 1 *With the above definitions $V^{\otimes n}$ becomes a module for the algebra $\mathcal{E}_n(u)$.*

Proof We must show that the operators satisfy the defining relations $(E1), \dots, (E9)$. Here the relations $(E1), \dots, (E5)$ are almost trivially satisfied, since E_i acts as a projection.

To prove the braid relation $(E6)$ we may assume that $n = 3$ and must evaluate both sides of $(E6)$ on the basis vectors $v_{i_1}^{j_1} \otimes v_{i_2}^{j_2} \otimes v_{i_3}^{j_3}$ of $V^{\otimes 3}$. The case where j_1, j_2, j_3 are distinct corresponds to the symmetric group case and $(E6)$ certainly holds. Another easy case is $j_1 = j_2 = j_3$, where $(E6)$ holds by Jimbo's classical result [13].

We are then left with the case $j_1 = j_2 \neq j_3$ and its permutations. In order to simplify notation, we omit the upper indices of the factors of the equal j's and replace the third j by a prime, e.g. $v_{i_1}^{j_1} \otimes v_{i_2}^{j_2} \otimes v_{i_3}^{j_3}$ is written $v_{i_1} \otimes v_{i_2} \otimes v'_{i_3}$ and so on.

We may assume that the lower indices of the unprimed factors are 1 or 2 since the action of T just depends on the order. Furthermore we may assume that the lower index of the primed factor is always 1 since T always acts as a transposition between a primed and an unprimed factor. This gives 12 cases. On the other hand, the cases where the two unprimed factors have equal lower indices are easy, since both sides of $(E6)$ act through $u\sigma_{13}$, where σ_{13} is the permutation of the first and third factor of the tensor product. So we are left with the following 6 cases

$$\begin{array}{lll} v_1 \otimes v_2 \otimes v'_1, & v_1 \otimes v'_1 \otimes v_2, & v'_1 \otimes v_1 \otimes v_2, \\ v_2 \otimes v_1 \otimes v'_1, & v_2 \otimes v'_1 \otimes v_1, & v'_1 \otimes v_2 \otimes v_1. \end{array}$$

Both sides of $(E6)$ act through σ_{13} on the first three of these subcases whereas the last three subcases involve each one Hecke–Jimbo action. For instance

$$T_1T_2T_1(v_2 \otimes v_1 \otimes v_1') = uv_1' \otimes v_1 \otimes v_2 + (u-1)v_1' \otimes v_2 \otimes v_1,$$

which is the same as acting with $T_2T_1T_2$. The other subcases are similar.

Let us now verify that $(E7)$ holds for our operators. We may once again assume that $n=3$ and must check $(E7)$ on all basis elements $v_{i_1}^{j_1} \otimes v_{i_2}^{j_2} \otimes v_{i_3}^{j_3}$. Once again, the cases of j_1, j_2, j_3 all distinct or all equal are easy. We then need only consider $j_1 = j_2 \neq j_3$ and its permutations and can once again use the prime/unprime notation as in the verification of $(E6)$.

Let us first verify that $E_1T_2T_1 = T_2T_1E_2$. We first observe that E_2 acts as the identity on exactly those basis vectors that are of the form $v_{i_1}' \otimes v_{i_2} \otimes v_{i_3}$. Hence

$$T_2T_1E_2(v_{i_1}' \otimes v_{i_2} \otimes v_{i_3}) = v_{i_2} \otimes v_{i_3} \otimes v_{i_1}' = E_1T_2T_1(v_1' \otimes v_{i_2} \otimes v_{i_3}).$$

The missing basis vectors are of the form $v_{i_1} \otimes v_{i_2}' \otimes v_{i_3}$ or $v_{i_1} \otimes v_{i_2} \otimes v_{i_3}'$ and are hence killed by E_2 and therefore $T_2T_1E_2$. But one easily checks that they are also killed by $E_1T_2T_1$.

The relation $E_2T_1T_2 = T_1T_2E_1$ is verified similarly.

Let us then check the relation $(E8)$. Once again we take $n=3$ and consider the action of $E_1E_2T_2$, $E_1T_2E_1$ and $T_2E_1E_2$ in the basis vector $v_{i_1}^{j_1} \otimes v_{i_2}^{j_2} \otimes v_{i_3}^{j_3}$. If the j_1, j_2, j_3 are distinct, the action of the three operators is zero, and if $j_1 = j_2 = j_3$ they all act as T_2. Hence we may once again assume that exactly two of the j's are equal.

But it is easy to check that each of the three operators acts as zero on all vectors of the form $v_{i_1}' \otimes v_{i_2} \otimes v_{i_3}$, $v_{i_1} \otimes v_{i_2}' \otimes v_{i_3}$ and $v_{i_1} \otimes v_{i_2} \otimes v_{i_3}'$. and so we have proved that $E_1E_2T_2 = E_1T_2E_1 = T_2E_1E_2$.

Similarly one proves that $E_2E_1T_1 = E_2T_1E_2 = T_1E_2E_1$.

Finally we check the relation $(E9)$, which by $(E5)$ can be transferred into

$$T_i^2 = 1 + (u-1)(1+T_i)E_i.$$

It can be checked taking $n=2$. We consider vectors of the form $v_{i_1}^{j_1} \otimes v_{i_2}^{j_2}$. If $j_1 \neq j_2$ then E_i acts as zero and we are done. And if $j_1 = j_2$, the relation reduces to the usual Hecke algebra square. The theorem is proved. □

Since the above proof is only a matter of checking relations, it also works over $\mathcal{E}_n^{\mathcal{A}}(u)$ and hence we get

Remark 1 There is a module structure of $\mathcal{E}_n^{\mathcal{A}}(u)$ on $V^{\otimes n}$.

Our next goal is to prove that $V^{\otimes n}$ is a faithful representation of $\mathcal{E}_n(u)$. Our strategy for this will be to construct a subset G of $\mathcal{E}_n^{\mathcal{A}}(u)$ that generates $\mathcal{E}_n^{\mathcal{A}}(u)$ as an $\mathcal{A}$-module and maps to a linearly independent subset of $\mathrm{End}_{\mathcal{A}}(V^{\otimes n})$ under the representation. We will then also have determined the dimension of $\mathcal{E}_n(u)$.

Let us start out by stating the following useful lemma.

Lemma 1 *The following formulas hold in $\mathcal{E}_n(u)$ and $\mathcal{E}_n^{\mathcal{A}}(u)$.*

(a) $T_j E_i T_j^{-1} = T_i^{-1} E_j T_i$ *if* $|i - j] = 1$,
(b) $T_i^{-1} T_j E_i = E_j T_i^{-1} T_j$ *if* $|i - j] = 1$,
(c) $T_j E_i T_j^{-1} = T_i E_j T_i^{-1}$ *if* $|i - j] = 1$.

Proof The formula (a) is just a reformulation of $(E7)$ whereas the formula (b) follows from

$$T_i^{-1} = T_i + \big(u^{-1} - 1\big) E_i (1 + T_i)$$

combined with $(E7)$ and $(E8)$. Formula (c) is a variation of (b). □

For $1 \le i < j \le n$ we define E_{ij} by E_i if $j = i + 1$, and otherwise

$$E_{ij} := T_i T_{i+1} \cdots T_{j-2} E_{j-1} T_{j-2}^{-1} \cdots T_{i+1}^{-1} T_i^{-1}.$$

We shall from now on use the notation $\mathbf{n} := \{1, 2, \ldots, n\}$. For any nonempty subset $I \subset \mathbf{n}$ we extend the definition of E_{ij} to

$$E_I := \prod_{(i,j)\in I\times I, i<j} E_{ij},$$

where by convention $E_I := 1$ if $|I| = 1$. We now aim at showing that this product is independent of the order in which it is taken.

Let us denote by s_i the transposition $(i, i + 1)$. Write $E_{\{j,k\}}$ for $E_{\min\{j,k\},\max\{j,k\}}$. Then we have

Lemma 2 *We have for all i, j, k that*

(a) $T_i E_{jk} T_i^{-1} = E_{\{s_i j, s_i k\}}$,
(b) $T_i^{-1} E_{jk} T_i = E_{\{s_i j, s_i k\}}$.

Proof Let us prove (a). We first consider the case where i is not any of the numbers $j-1, j, k-1$ or k. In that case we must show that T_i and $E_{j,k}$ commute. For $i < j-1$ and $i > k$ this is clear since T_i then commutes with all of the factors of $E_{j,k}$. And for $j < i < k - 1$ one can commute T_i through $E_{j,k}$ using $(E6)$ and $(E3)$.

For $i = j - 1$ the formula follows directly from the definition of $E_{j,k}$. For $i = k$ we get that T_i commutes with all the T_l factors of $E_{j,k}$ and hence it reduces to showing that

$$T_k E_{k-1} T_k^{-1} = T_{k-1} E_k T_{k-1}^{-1},$$

which is true by formula (c) of Lemma 1. For $i = k - 1$ the formula follows from the definitions and $(E7)$.

Finally, we consider the case $i = j$. To deal with this case, we first rewrite E_{jk}, using (c) of Lemma 1 repeatedly starting with the innermost term, in the form

$$E_{jk} = T_{k-1}T_{k-2}\cdots T_{j+1}E_j T_{j+1}^{-1}\cdots T_{k-2}^{-1}T_{k-1}^{-1}. \tag{1}$$

The formula of the lemma now follows from relation $(E7)$.

Formula (b) is proved the same way. □

With this preparation we obtain the commutativity of the factors involved in E_I. We have that

Lemma 3 *The E_{ij} are commuting idempotents of $\mathcal{E}_n(u)$ and $\mathcal{E}_n^{\mathcal{A}}(u)$.*

Proof The E_{ij} are obviously idempotents in $\mathcal{E}_n(u)$ and $\mathcal{E}_n^{\mathcal{A}}(u)$ so we just have to prove that they commute.

Thus, given E_{ij} and E_{kl} we show by induction on $(j-i)+(l-k)$ that they commute with each other. The induction starts for $(j-i)+(l-k)=2$, in which case $E_{ij}=E_i$ and $E_{kl}=E_k$, that commute by $(E2)$.

Suppose now $(j-i)+(l-k)>2$ and that E_{ij}, E_{kl} is not a pair of the form $E_{s-1,s+2}, E_{s,s+1}$ for any s. One checks now there is an r such that $E_{s_r\{i,j\}}, E_{s_r\{k,l\}}$ is covered by the induction hypothesis. But then, using (a) from the previous lemma together with the induction hypothesis, we find that

$$\begin{aligned} E_{ij}E_{kl} &= T_r^{-1}E_{s_r\{i,j\}}T_rT_r^{-1}E_{s_r\{k,l\}}T_r = T_r^{-1}E_{s_r\{i,j\}}E_{s_r\{k,l\}}T_r \\ &= T_r^{-1}E_{s_r\{k,l\}}E_{s_r\{i,j\}}T_r = T_r^{-1}E_{s_r\{k,l\}}T_rT_r^{-1}E_{s_r\{i,j\}}T_r = E_{kl}E_{ij} \end{aligned}$$

as needed. Finally, if our pair is of the form $E_{s-1,s+2}, E_{s,s+1}$ we use $(E8)$ to finish the proof the lemma as follows

$$\begin{aligned} E_{s-1,s+2}E_{s,s+1} &= T_{s-1}T_sE_{s+1}T_s^{-1}T_{s-1}^{-1}E_s = E_sT_{s-1}T_sE_{s+1}T_s^{-1}T_{s-1}^{-1} \\ &= E_{s,s+1}E_{s-1,s+2}. \end{aligned}$$
□

We have now proved that the product involved in E_I is independent of the order taken. We then go on to show that many of the factors of this product can be left out.

Lemma 4 *Let $I \subset \mathbf{n}$ with $|I| \geq 2$ and set $i_0 := \min I$. Then*

$$E_I = \prod_{i: i\in I\setminus\{i_0\}} E_{i_0 i}.$$

Proof It is enough to show the lemma for I of cardinality three. By a direct calculation using the definition of E_{kl} one sees that this case reduces to $I=\{1,2,i\}$. Set now

$$\begin{aligned} E^1 &:= E_1T_1T_2\cdots T_{i-1}E_iT_{i-1}^{-1}\cdots T_2^{-1}T_1^{-1}, \\ E^2 &:= T_2T_3\cdots T_{i-1}E_iT_{i-1}^{-1}\cdots T_3^{-1}T_2^{-1}. \end{aligned}$$

Then the left hand side of the lemma is E^1E^2 while the right hand side is E^1, so we must show that $E^1E^2=E^1$. But using formula (a) of Lemma 1 repeatedly this identity reduces to

$$E_1 T_1 E_2 T_1^{-1} E_2 = E_1 T_1 E_2 T_1^{-1},$$

which is true by relations $(E5)$ and $(E8)$. □

In order to generalize the previous results appropriately we need to recall some notation. A *set partition* $A = \{I_1, I_2, \dots, I_k\}$ of $\mathbf{n}$ is by definition an equivalence relation on $\mathbf{n}$ with classes I_j. This means that the I_j are disjoint, nonempty subsets of $\mathbf{n}$ with union $\mathbf{n}$. We also refer to the I_j as the *blocks* of A. The number of distinct set partitions of $\mathbf{n}$ is called the nth Bell number and is written B_n. For example $B_1 = 1$, $B_2 = 2$ and $B_3 = 5$. The five set partitions of $\mathbf{3} = \{1, 2, 3\}$ are

$$\big\{\{1\}, \{2\}, \{3\}\big\}, \qquad \big\{\{1\}, \{2, 3\}\big\}, \qquad \big\{\{2\}, \{1, 3\}\big\},$$
$$\big\{\{3\}, \{1, 2\}\big\}, \qquad \big\{\{1, 2, 3\}\big\}.$$

Let us denote by $\mathcal{P}_n$ the set of all set partitions of $\mathbf{n}$. There is natural partial order on $\mathcal{P}_n$, denoted $\subset$. It is defined by $A = \{I_1, I_2, \dots, I_k\} \subset B = \{J_1, J_2, \dots, J_l\}$ if and only if each J_i is a union of certain I_i.

Let R be a subset of $\mathbf{n} \times \mathbf{n}$. Write $i \smile_R j$ if $(i, j) \in R$ and write $\sim_R$ for the equivalence relation induced by $i \smile_R j$. Then $i \sim_R j$ iff $i = j$ or there is a chain $i = i_1, i_2, \dots, i_k = j$ such that $i_s \smile_R i_{s+1}$ or $i_{s+1} \smile_R i_s$ for all s. Let $\langle R\rangle$ denote the set partition corresponding to $\sim_R$. For example, if $R = \emptyset$ we get that $\langle R\rangle$ is the trivial set partition whose blocks are all of cardinality one.

For $A = \{I_1, \dots, I_k\} \in \mathcal{P}_n$ we define

$$E_A := \prod_i E_{I_i}.$$

It follows from Lemma 3 that the product is independent of the order in which it is taken.

For $w \in S_n$ we define $wA := \{wI_1, wI_2, \dots, wI_k\} \in \mathcal{P}_n$. If $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced form of w, we write as usual $T_w = T_{i_1} T_{i_2} \cdots T_{i_n}$. Then we have

Corollary 1 *With $A \in \mathcal{P}_n$ and w as above the following formula holds*:

$$T_w E_A T_w^{-1} = E_{wA}.$$

Proof This is a consequence of Lemma 2(a) and the definitions. □

The next lemma is an important ingredient in the construction of the basis for $\mathcal{E}_n(u)$.

Lemma 5 *Suppose $R \subset \mathbf{n} \times \mathbf{n}$. Then the following formula is valid*:

$$\prod_{i,j:(i,j)\in R} E_{\{i,j\}} = E_{\langle R\rangle}.$$

Proof Writing $E_R := \prod_{i,j:(i,j)\in R} E_{\{i,j\}}$ we must prove that $E_R := E_{\langle R\rangle}$. Clearly, all the factors of E_R are also factors of $E_{\langle R\rangle}$. We show that the extra factors of $E_{\langle R\rangle}$ do not change the product of E_R. For this, suppose first that the following equations hold for $i<j<k$

$$E_{ij}E_{ik} = E_{ij}E_{jk} = E_{ik}E_{jk} = E_{ij}E_{jk}E_{ik}. \tag{2}$$

Assume now that $i, j \in \mathbf{n}$ satisfy $i \sim_R j$. Then, by definition, there is a chain $i = i_1, i_2, \dots, i_k = j$ with $(i_s, i_{s+1}) \in R$ or $(i_{s+1}, i_s) \in R$ for all s. Let $1 \le l < k$ and assume recursively that we have $E_R = E_R E_{\{i,i_l\}}$. Then using (2) we get that also $E_R := E_R E_{\{i,i_{l+1}\}}$. Continuing, we find that $E_R := E_R E_{\{ij\}}$, and so indeed the extra factors of $E_{\langle R\rangle}$ do not change the product E_R. Thus we are reduced to proving (2).

The equation $E_{ij}E_{ik} = E_{ij}E_{jk}E_{ik}$ was shown in the previous lemma so we only need show that $E_{ik}E_{jk} = E_{ij}E_{jk}E_{ik}$ and $E_{ij}E_{jk} = E_{ij}E_{jk}E_{ik}$.

We consider the involution inv of $\mathcal{E}_n^{\mathcal{A}}(u)$ given by the formulas

$$\mathrm{inv}(T_i) = T_{n-i}, \qquad \mathrm{inv}(E_i) = E_{n-i}.$$

Using (1) we find that

$$\mathrm{inv}(E_{ij}) = E_{n-j,n-i}.$$

But then $E_{ik}E_{jk} = E_{ij}E_{jk}E_{ik}$ follows from $E_{ij}E_{ik} = E_{ij}E_{jk}E_{ik}$.

We then show that $E_{ij}E_{jk} = E_{ij}E_{jk}E_{ik}$. By the above, it can be reduced to showing the identity

$$E_{ij}E_{jk} = E_{ij}E_{ik}.$$

Using the definition of the E_{ij} it can be reduced to the case $i=1$, $j=2$, i.e. $E_1E_{2k} = E_1E_{1k}$. Using formula (a) of Lemma 1 it becomes the valid identity $E_1E_2 = E_1T_1E_2T_1^{-1}$. □

From the lemma we get the following compatibility between the partial order on $\mathcal{P}_n$ and the E_A.

Corollary 2 *Assume $A, B \in \mathcal{P}_n$ and let $C \in \mathcal{P}_n$ be minimal with respect to $A \subseteq C$ and $B \subseteq C$. Then $E_AE_B = E_C$.*

We are now in position to construct the subset G of $\mathcal{E}_n^{\mathcal{A}}(u)$. We define

$$G := \{E_AT_w \mid A \in \mathcal{P}_n, w \in S_n\}. \tag{3}$$

With the theory developed so far we can state the following theorem.

Theorem 2 *The set G generates $\mathcal{E}_n^{\mathcal{A}}(u)$ over $\mathcal{A}$.*

Proof Consider a word $w = X_{i_1}X_{i_2}\cdots X_{i_k}$ in the generators T_i and E_i, i.e. $X_{i_j} = T_{i_j}$ or $X_{i_j} = E_{i_j}$ for all j. Using Lemma 2 we can move all the E_i to the front position, at each step changing the index set by its image under some reflection, and are finally left with a word in the T_i, which is possibly not reduced. If it is not so, it is equivalent under the braid relations ($E6$) to a word with two consecutive T_i, see [7] Chap. 8. Expanding the T_i^2 gives rise to a linear combination of 1, E_i and $T_i E_i$, where the E_i can be commuted to the front position the same way as before. Continuing this way we eventually reach a word in reduced form, that is a linear combination of elements of the form $\prod_{(i,j)\in R, w\in S_n} E_{ij} T_w$ for some subset R of $\mathbf{n}\times\mathbf{n}$, satisfying $(i,j)\in R$ only if $i<j$. Using Lemma 5 we may rewrite it as a linear combination of $E_{\langle R\rangle} T_w$ and the proof is finished. □

With these results at hand we can prove the following main theorem.

Theorem 3 *The set G is a basis of $\mathcal{E}_n^{\mathcal{A}}(u)$ and induces bases of $\mathcal{E}_n(u)$ and $\mathcal{E}_n(1)$.*

Proof By the previous theorem it is enough to show that G is an $\mathcal{A}$-linearly independent subset of $\mathcal{E}_n^{\mathcal{A}}(u)$ and induces K and $\mathbb{C}$-linearly independent subsets of $\mathcal{E}_n(u)$ and $\mathcal{E}_n(1)$.

Assume that there exists a nontrivial linear dependence $\sum_{g\in G}\lambda_g g=0$ where $\lambda_g\in\mathcal{A}$ for all g. Let $\lambda\in\mathcal{A}$ be the greatest common divisor of the λ_g and write $\lambda=(v-1)^M\lambda_1$ with $\lambda_1\in\mathcal{A}$ and $\lambda_1(1)\neq 0$. Setting $\mu_g:=\lambda_g/(v-1)^M\in\mathcal{A}$ we obtain an $\mathcal{A}$-linear dependence $\sum_{g\in G}\mu_g g=0$ satisfying $\mu_g(1)\neq 0$ for at least one g. By specializing, we obtain from this a nontrivial $\mathbb{C}$-linear dependence $\sum_{g\in G}\mu_g(1)g=0$ in $\mathcal{E}_n(1)$.

Denoting by $\psi:\mathcal{E}_n^{\mathcal{A}}(u)\to\mathrm{End}_{\mathcal{A}}(V^{\otimes n})$ the representation homomorphism we get by specializing a homomorphism $\psi_1:\mathcal{E}_n(1)\to\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$. We use it to obtain the nontrivial linear dependence $\sum_{g\in G}\mu_g(1)\psi_1(g)=0$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$. It is now enough to show that $\{\psi_1(g)|g\in G\}$ is a $\mathbb{C}$-linearly independent set of $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$.

But for $u=1$, the action of T_i in $V^{\otimes n}$ is just permutation of the factors $(i,i+1)$. Hence, in this case, E_{kl} acts as a projection in the space of equal upper indices in the kl'th factors of $V^{\otimes n}$. In formulas

$$\begin{aligned} &E_{kl}\big(v_{i_1}^{j_1}\otimes\cdots\otimes v_{i_k}^{j_k}\otimes\cdots\otimes v_{i_l}^{j_l}\otimes\cdots\otimes v_{i_n}^{j_n}\big)\\ &\quad=\begin{cases} v_{i_1}^{j_1}\otimes\cdots\otimes v_{i_k}^{j_k}\otimes\cdots\otimes v_{i_l}^{j_l}\otimes\cdots\otimes v_{i_n}^{j_n} & \text{if } j_k=j_l,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus, for a set partition $A=\{I_1,I_2,\dots,I_s\}\in\mathcal{P}_n$ we get that E_A acts as the projection π_A on the space of equal upper indices in factors corresponding to each of the I_k. In formulas

$$\begin{aligned} &E_A\big(v_{i_1}^{j_1}\otimes\cdots\otimes v_{i_r}^{j_r}\otimes\cdots\otimes v_{i_s}^{j_s}\otimes\cdots\otimes v_{i_n}^{j_n}\big)\\ &\quad=\begin{cases} 0 \quad \text{if there exist } r,s,k \text{ such that } r,s\in I_k \text{ and } j_r\neq j_s,\\ v_{i_1}^{j_1}\otimes\cdots\otimes v_{i_r}^{j_r}\otimes\cdots\otimes v_{i_s}^{j_s}\otimes\cdots\otimes v_{i_n}^{j_n} \quad \text{otherwise.}\end{cases}\end{aligned}$$

Let us now consider a linear dependence:

$$\sum_{w\in S_n, A\in\mathcal{P}_n}\lambda_{w,A}T_w\pi_A=0, \tag{4}$$

with $\lambda_{w,A} \in \mathbb{C}$. Take $A_0 \in \mathcal{P}_n$ such that $\lambda_{w,A_0} \neq 0$ for some $w \in S_n$ and A_0 is minimal with respect to this condition, where minimality refers to the partial order on $\mathcal{P}_n$ introduced above. Suppose that $A_0 = \{I_1, I_2, \dots, I_s\}$. If we take a basis vector of $V^{\otimes n}$

$$v^{A_0} = v_{i_1}^{j_1} \otimes \cdots \otimes v_{i_k}^{j_k} \otimes \cdots \otimes v_{i_l}^{j_l} \otimes \cdots \otimes v_{i_n}^{j_n}$$

such that $j_k = j_l$ if and only if k, l belong to the same I_i, then we get on evaluation in (4), using the minimality of A_0, that

$$\sum_{w \in S_n} \lambda_{w,A_0} T_w v^{A_0} = 0.$$

We now furthermore take v^{A_0} such that its lower i-indices are all distinct. But then $\{T_w v^{A_0}, w \in S_n\}$ is a linearly independent set and we conclude that $\lambda_{w,A_0} = 0$ for all w, which contradicts the choice of A_0.

This shows that the set $\{T_w \pi_A | w \in S_n, A \in \mathcal{P}_n\}$ is linearly independent. To get the linear independence of $\{\pi_A T_w | w \in S_n, A \in \mathcal{P}_n\}$ we apply Corollary 1.

We have shown that G induces a $\mathbb{C}$-independent subset of $\mathcal{E}_n(1)$ and we then conclude, as described above, that it is an $\mathcal{A}$-independent subset of $\mathcal{E}_n^{\mathcal{A}}(u)$. Since K is the quotient field of $\mathcal{A}$ it also induces a K-independent subset of $\mathcal{E}_n(u)$ and the theorem is proved. □

Corollary 3 *We have* $\dim \mathcal{E}_n(u) = n! B_n$, *where* B_n *is the Bell number, i.e. the number of set partitions of* $\mathbf{n}$. *For example* $\dim \mathcal{E}_2(u) = 4$, $\dim \mathcal{E}_3(u) = 30$, *etc.*

The appearance of set partitions in the above, notably Corollary 2, might indicate a connection between $\mathcal{E}_n(u)$ and the partition algebra $A_n(K)$ introduced independently by P. Martin in [17] and V. Jones in [12], see also [6] for an account of the representation theory of $A_n(K)$. On the other hand, the special relation $(E9)$ of $\mathcal{E}_n(u)$ does complicate the direct comparison $\mathcal{E}_n(u)$ with known variations of the partition algebra and at present we do not believe that there can be any straightforward connection. The relation $(E9)$ reveals the origin of $\mathcal{E}_n(u)$ in the Yokonuma–Hecke algebra. Since $u \neq 1$, it behaves like a kind of skein relation in the diagram calculus for $\mathcal{E}_n(u)$, which seems awkward to interpret in a partition algebra context. Note that $\mathcal{E}_n(u)$ becomes infinite dimensional if $(E9)$ is left out.

Corollary 4 *The tensor space* $V^{\otimes n}$ *is a faithful* $\mathcal{E}_n(u)$*-module.*

Proof We proved that G is a basis of $\mathcal{E}_n(u)$ that maps to a linearly independent set in $\mathrm{End}_K(V^{\otimes n})$. □

4 Representation theory, first steps

We initiate in this section the representation theory of $\mathcal{E}_n(u)$. We construct two families of irreducible representations of $\mathcal{E}_n(u)$ as pullbacks of irreducible representations of the symmetric group and of the Hecke algebra.

Let $I \subset \mathcal{E}_n(u)$ be the two-sided ideal generated by E_i for all i; actually E_1 is enough to generate I. Let furthermore $J \subset \mathcal{E}_n(u)$ be the two-sided ideal generated by $E_i - 1$ for all i; once again $E_1 - 1$ is enough to generate J. Recall that S_n denotes the symmetric group on n letters. Let $H_n(u)$ be the Hecke algebra over K of type A_{n-1}. It is the K-algebra generated by $T_1, \ldots, T_{n-1}$ with relations $T_iT_j = T_jT_i$ if $|i-j| > 1$ and

$$T_iT_{i\pm1}T_i = T_{i\pm1}T_iT_{i\pm1} \quad (T_i - u)(T_i + 1) = 0,$$

where i is any index such that the expressions make sense.

Lemma 6 (a) *There is an isomorphism* $\varphi : KS_n \to \mathcal{E}_n(u)/I$, $s_i \mapsto T_i$.
(b) *There is an isomorphism* $\psi : H_n(u) \to \mathcal{E}_n(u)/J$, $T_i \mapsto T_i$.

Proof We first prove a). In $\mathcal{E}_n(u)/I$ we have $T_i^2 = 1$ and hence we obtain a surjection $\varphi : KS_n \to \mathcal{E}_n(u)/I$ by mapping s_i to T_i. Consider once again the vector space $V = \mathrm{span}_K\{v_i^j \,|\, i, j = 1, \ldots, n\}$ and its tensor space $V^{\otimes n}$ as a representation of $\mathcal{E}_n(u)$. We consider the following subspace $M \subset V^{\otimes n}$.

$$M = \mathrm{span}_K\left\{v_{i_1}^{j_1} \otimes \cdots \otimes v_{i_n}^{j_n} \,|\, \text{the upper indices are all distinct}\right\}.$$

It is easy to check from the rules of the action of $\mathcal{E}_n(u)$ that M is a submodule of $V^{\otimes n}$. Since the E_i act as zero in M we get an induced homomorphism $\rho : \mathcal{E}_n(u)/I \to \mathrm{End}_K(M)$, where $\rho(T_i)$ is the switching of the i'th and $i+1$'th factors of the tensor product. But then the image of $\rho \circ \varphi$ has dimension $n!$ and we conclude that φ indeed is an isomorphism.

In order to prove (b) we basically proceed in the same way. In the quotient $\mathcal{E}_n(u)/J$ we have $T_i^2 = 1 + (u-1)(1 + T_i)$ which implies the existence of a surjection $\psi : H_n(u) \to \mathcal{E}_n(u)/J$ mapping T_i to T_i. To show that ψ is injective we this time consider the submodule

$$N = \mathrm{span}_K\left\{v_{i_1}^{j_1} \otimes \cdots \otimes v_{i_n}^{j_n} \,|\, \text{the upper indices are all equal to } 1\right\}.$$

All E_i act as 1 in N and so we get a induced map $\rho' : \mathcal{E}_n(u)/J \to \mathrm{End}_K(N)$. The composition $\rho' \circ \psi$ is the regular representation of $H_n(u)$ and hence $\dim \mathrm{Im}(\rho' \circ \psi) = n!$ which proves that also ψ is an isomorphism. □

We now recall the well known basic representation theory of KS_n and of $H_n(u)$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be an integer partition of $|\lambda| := n$ and let $Y(\lambda)$ be its Young diagram. Let t^λ (resp. t_λ) be the λ-tableau in which the numbers $\{1, 2, \ldots, n\}$ are filled in by rows (resp. columns). Denote by $R(\lambda)$ (resp. $C(\lambda)$) the row (resp. column) stabilizer of t^λ. Define now

$$r_\lambda = \sum_{w \in R(\lambda)} w, \qquad c_\lambda = \sum_{w \in C(\lambda)} (-1)^{l(w)} w, \qquad s_\lambda = c_\lambda r_\lambda.$$

Then s_λ is the Young symmetrizer and $S(\lambda) = KS_ns_\lambda$ is the Specht module associated with λ. Since $\mathrm{Char}\, K = 0$, the Specht modules are simple and classify the simple modules of KS_n.

To give the Specht modules for $H_n(u)$, we use Gyoja's Hecke algebra analog of the Young symmetrizer [9, 18]. In our setup it looks as follows: For $X \subset S_n$, define

$$\iota(X) = \sum_{w \in X} T_w, \qquad \epsilon(X) = \sum_{w \in X} (-u)^{-l(w)} T_w.$$

If for example $X = S_n$, we have

$$T_w \iota(S_n) = u^{l(w)} \iota(S_n), \qquad T_w \epsilon(S_n) = (-1)^{l(w)} \epsilon(S_n),$$

for all T_w. We now define

$$x_\lambda = \iota\big(R(\lambda)\big), \qquad y_\lambda = \epsilon\big(R(\lambda)\big).$$

Let $w_\lambda \in S_n$ be the element such that $w_\lambda t^\lambda = t_\lambda$. Then the Hecke algebra analog of the Young symmetrizer is

$$e_\lambda = T_{w_\lambda^{-1}} y_{\lambda'} T_{w_\lambda} x_\lambda = c_\lambda(u) r_\lambda(u),$$

where $c_\lambda(u) := T_{w_\lambda^{-1}} y_{\lambda'} T_{w_\lambda}$ and $r_\lambda(u) := x_\lambda(u)$. The permutation module and the Specht module associated with λ are defined as $M_u(\lambda) := H_n(u) x_\lambda$ and $S_u(\lambda) = H_n(u) e_\lambda$. Since u is generic, $S_u(\lambda)$ is irreducible.

For future reference, we recall the following result, see e.g. [4, 18].

Lemma 7 *Suppose that $c_\lambda(u) M_u(\mu) \neq 0$. Then $\mu \trianglelefteq \lambda$.*

Here $\trianglelefteq$ refers to the dominance order on partitions of n, defined by $\lambda = (\lambda_1, \lambda_2, \dots) \trianglelefteq \mu = (\mu_1, \mu_2, \dots)$ iff $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$ for all i. The dominance order is only a partial order, but we shall embed it into the total order $<$ on partitions of n, defined by $\lambda = (\lambda_1, \lambda_2, \dots) < \mu = (\mu_1, \mu_2, \dots)$ iff $\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \nu_2 + \cdots + \mu_i$ for some i and $\lambda_1 + \lambda_2 + \cdots + \lambda_j = \mu_1 + \mu_2 + \cdots + \mu_j$ for $j < i$. We extend $<$ to a total order on all partitions by declaring $\lambda < \mu$ if $|\lambda| < |\mu|$.

It is known that $y_{\lambda'} T_w x_\lambda \neq 0$ only if $w = w_\lambda$ see [4, 18]. Using it we find that

$$c_\lambda(u) z r_\lambda(u) = C_z c_\lambda(u) r_\lambda(u) \quad \text{for all } z \in H_n(u), \tag{5}$$

for a constant $C_z \in K$. It follows that $s_\lambda(u)$ is a preidempotent, i.e. an idempotent up to a nonzero scalar. There is a similar formula

$$c_\lambda z r_\lambda = C_z c_\lambda r_\lambda \quad \text{for all } z \in K S_n \tag{6}$$

in the symmetric group case.

Using the Specht module $S(\lambda)$ for $K S_n$ or $S_u(\lambda)$ for $H_n(u)$ we use φ or ψ to obtain a simple module for $\mathcal{E}_n(u)$, by pulling back. On the other hand, these two series of simple modules do not exhaust all the simple modules for $\mathcal{E}_n(u)$ as we shall see in the next sections.

5 $\mathcal{E}_n(u)'$ as a $\mathcal{E}_n(u)$-module

In this section we return to $\mathcal{E}_n(u)$. We show that it is selfdual as a left module over $\mathcal{E}_n(u)$ itself. As a consequence of this we get that all simple modules occur as left ideals in $\mathcal{E}_n(u)$.

Denote by $*: \mathcal{E}_n(u) \to \mathcal{E}_n(u)$ the K-linear antiautomorphism given by $T_i^* = T_i$ and $E_i^* = E_i$. To check that $*$ exists we must verify that $*$ leaves the defining relations $(E1), \dots, (E9)$ invariant. This is obvious for all of them, except possibly for $(E7)$ where it follows by interchanging i and j. There is a similar antiautomorphism for $\mathcal{E}_n(1)$, also denoted $*$.

We now make the linear dual $\mathcal{E}_n(u)'$ of $\mathcal{E}_n(u)$ into a left $\mathcal{E}_n(u)$-module using $*$:

$$(xf)(y) := f(x^* y) \quad \text{for } x, y \in \mathcal{E}_n(u),\ f \in \mathcal{E}_n(u)'.$$

We need to consider the linear map

$$\epsilon : \mathcal{E}_n(u) \to K, \quad x \mapsto \mathrm{coeff}_{\mathrm{E}_\mathbf{n}}(x),$$

where $\mathrm{coeff}_{\mathrm{E}_\mathbf{n}}(x)$ is the coefficient of $E_\mathbf{n}$ when $x \in \mathcal{E}_n(u)$ is written in the basis elements $T_w E_A$ of G, see (3). Here by abuse of notation, we write $\mathbf{n}$ for the unique maximal set partition in $\mathcal{P}_n$. Its only block is $\mathbf{n}$.
With this we may construct a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{E}_n(u)$ by

$$\langle x, y \rangle = \epsilon(x^* y) \quad \text{for } x, y \in \mathcal{E}_n(u).$$

And then we finally obtain a homomorphism φ by the rule

$$\varphi : \mathcal{E}_n(u) \to \mathcal{E}_n(u)' : x \mapsto \big(y \mapsto \langle x, y \rangle\big).$$

Theorem 4 *With the above definitions, we get that φ is an isomorphism of left $\mathcal{E}_n(u)$-modules.*

Proof One first checks that the bilinear form satisfies

$$\langle xy, z \rangle = \langle y, x^* z \rangle \quad \text{for all } x, y, z \in \mathcal{E}_n(u),$$

which amounts to saying that φ is $\mathcal{E}_n(u)$-linear.

Since $\mathcal{E}_n(u)$ is finite dimensional, it is now enough to show that $\langle \cdot, \cdot \rangle$ is nondegenerate. For this we first observe that our construction of $\langle \cdot, \cdot \rangle$ is valid over $\mathcal{A}$ as well and hence also defines a bilinear form $\langle \cdot, \cdot \rangle_\mathcal{A}$ on $\mathcal{E}_n^\mathcal{A}(u)$. It is enough to show that $\langle \cdot, \cdot \rangle_\mathcal{A}$ is nondegenerate. Suppose $a \in \mathcal{E}_n^\mathcal{A}(u)$. Then as in the proof of Theorem 3 we can write it in the form $a = (u-1)^N a'$ where $a' = \sum_{g \in G} \lambda_g g$ and where $\lambda_g(1) \neq 0$ for at least one g. Letting $\pi : \mathcal{E}_n^\mathcal{A}(u) \to \mathcal{E}_n(1)$ be the specialization map we have $\pi(a') \neq 0$ since it was shown in the proof of that theorem that G is a basis of $\mathcal{E}_n^\mathcal{A}(1)$ as well.

Let us denote by $\langle \cdot, \cdot \rangle_1$ the bilinear form on $\mathcal{E}_n(1)$ constructed similarly to $\langle \cdot, \cdot \rangle$. Then we have that

$$\big\langle \pi(a), \pi(b) \big\rangle_1 = \langle a, b \rangle_\mathcal{A} \otimes_\mathcal{A} \mathbb{C} \quad \text{for all } a, b \in \mathcal{E}_n^\mathcal{A}(u)$$

since π is multiplicative and satisfies $\pi(a^*) = \pi(a)^*$. We are now reduced to proving that $\langle\cdot,\cdot\rangle_1$ is nondegenerate. Let us therefore consider an arbitrary $a = \sum_{w,A} \lambda_{w,A} E_A T_w \in \mathcal{E}_n(1)$, where $\lambda_{w,A} \in \mathbb{C}$. Let $A_0 \in \mathcal{P}_n$ be minimal subject to the condition that $\lambda_{w,A_0} \neq 0$ for some w. Take $z \in S_n$ with $\lambda_{z,A_0} \neq 0$ and define

$$b = E_{A_0} \prod_{A_0 \subsetneq A} (1 - E_A) T_z.$$

We claim that $\langle b, a\rangle_1 \neq 0$. Indeed, since $u = 1$ we have

$$b^* a = T_z^{-1} \prod_{A_0 \subsetneq A} (1 - E_A) E_{A_0} a.$$

Since A_0 was chosen minimal, there can be no cancellation of the coefficient of $E_{A_0} T_z$ in $E_{A_0} a$ which hence is λ_{z,A_0}. All E_A appearing in the expansion of $E_{A_0} a$ with respect to the basis $E_A T_w$ satisfy $A_0 \subseteq A$. Except for E_{A_0} they are all killed by $\prod_{A_0 \subsetneq A}(1 - E_A)$. By this we get

$$T_z^{-1} \prod_{A_0 \subsetneq A} (1 - E_A) E_{A_0} a = \lambda_{z,A_0} T_z^{-1} \prod_{A_0 \subsetneq A} (1 - E_A) E_{A_0} T_z.$$

The coefficient of $E_{\mathbf{n}}$ in this expression is by Corollary 1 equal to the coefficient of $E_{\mathbf{n}}$ in

$$\lambda_{z,A_0} \prod_{A_0 \subsetneq A} (1 - E_A) E_{A_0}.$$

On the other hand, the coefficient of $E_{\mathbf{n}}$ in $\prod_{A_0 \subsetneq A}(1 - E_A) E_{A_0}$ is given by the Moebius function associated with the partial order $\subset$ on $\mathcal{P}_n$. It is equal to $(-1)^{k-1} k!$, where k is the number of blocks of A_0. Summing up we find that $\langle b, a\rangle_1 = (-1)^{k-1} \lambda_{z,A_0} k! \neq 0$ which proves the theorem. □

6 Classification of the irreducible representations

In this section we give the classification of the irreducible representations of $\mathcal{E}_n(u)$.

For M a left $\mathcal{E}_n(u)$-module we make its linear dual M' into a left $\mathcal{E}_n(u)$-module using the antiautomorphism $*$. If M is a simple $\mathcal{E}_n(u)$-module then any $m \in M \setminus \{0\}$ defines a surjection

$$\mathcal{E}_n(u) \to M, x \mapsto xm \quad \text{for } x \in \mathcal{E}_n(u).$$

By duality and by the last section, we then get an injection of M' into $\mathcal{E}_n(u)$. On the other hand, the canonical isomorphism $M \to M''$ is $\mathcal{E}_n(u)$-linear because $** = \mathrm{Id}$ and so we conclude that all simple $\mathcal{E}_n(u)$-modules appear as left ideals in $\mathcal{E}_n(u)$.

Let now I be a simple left ideal of $\mathcal{E}_n(u)$ and let $x_0 \in I \setminus \{0\}$. Since the tensor space $V^{\otimes n}$ is a faithful $\mathcal{E}_n(u)$-module, we find a $v \in V^{\otimes n}$ such that $x_0 v \neq 0$. But then the $\mathcal{E}_n(u)$-linear map

$$I \to V^{\otimes n}, \quad x \mapsto xv \quad \text{for } x \in I$$

is nonzero, and therefore injective. We conclude that all simple $\mathcal{E}_n(u)$-modules appear as submodules of $V^{\otimes n}$.

Consider a simple submodule M of $V^{\otimes n}$. Take $A_0 \subset \mathbf{n}$ maximal such that $E_{A_0}M \neq 0$. By Sect. 3, in the two extreme situations $A_0 = \emptyset$ or $A_0 = \mathbf{n}$ we can give a precise description of M, since in those cases M is a module for KS_n or $H_n(u)$. In other words, M is the pullback of a Specht module $S(\lambda)$ for KS_n or a Specht module $S_u(\lambda)$ for $H_n(u)$ as described in Sect. 3. The general case is going to be a mixture of these two cases as we shall explain in this section.

Let $\mathcal{L}_n$ be the set of tuples

$$\mathcal{L}_n = \big\{\big(\lambda^s, m_s, \mu^s\big) | s = 1, \dots, k\big\},$$

where λ^s is a partition, m_s a positive integer and μ^s a partition of m_s such that $\sum_s m_s|\lambda^s| = n$ and such that $\lambda^1 < \lambda^2 < \cdots < \lambda^k$ where $<$ is the total order on partitions defined above.

Suppose $\Lambda = (\lambda^s, m_s, \mu^s) \in \mathcal{L}_n$. We associate to it the vector $v_\Lambda \in V^{\otimes n}$ defined in the following way

$$v_\Lambda := v^1_{\lambda^1} \otimes v^2_{\lambda^1} \otimes \cdots \otimes v^{m_1+1}_{\lambda^2} \otimes v^{m_1+2}_{\lambda^2} \otimes \cdots \otimes v^l_{\lambda^k},$$

where $l := \sum_s m_s$ and where for any integer partition (even composition) $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ of m and any integer i we define $v^i_\mu \in V^{\otimes m}$ as follows

$$v^i_\mu := \big(v^i_1\big)^{\otimes \mu_1} \otimes \big(v^i_2\big)^{\otimes \mu_2} \otimes \cdots \otimes \big(v^i_r\big)^{\otimes \mu_r}.$$

We moreover associate to $\Lambda = (\lambda^s, m_s, \mu^s)$ the set partition $A_\Lambda \in \mathcal{P}_n$, that has blocks of consecutive numbers, the first m_1 blocks being of size $|\lambda^1|$, the next m_2 blocks of size $|\lambda^2|$ and so on. The blocks correspond to the factors of v_Λ that have equal upper indices. Note that it is possible that $|\lambda^1| = |\lambda^2|$ making the first $m_1 + m_2$ blocks of equal size and so on. Writing $A_\Lambda = (I_1, I_2, \dots, I_l)$ we set

$$S_\Lambda := S_{m_1} \times S_{m_2} \times \cdots \times S_{m_k},$$
$$H_\Lambda(u) := H_{I_1}(u) \otimes H_{I_2}(u) \otimes \cdots \otimes H_{I_l}(u).$$

Let ι_j be the group isomorphism from S_{m_j} to $1 \times \cdots \times S_{m_j} \times \cdots \times 1$ and also the algebra isomorphism from $H_{I_j}(u)$ to $1 \otimes \cdots \otimes H_{I_j}(u) \otimes \cdots \otimes 1$.

Corresponding to A_Λ there is an analogous block decomposition of the factors of $V^{\otimes n}$ and S_Λ acts on this by permutation of the blocks.

Let us illustrate this action on an example. Take $n = 6$, $k = 1$ and $\Lambda = (\lambda, 2, \mu)$ where $\lambda = (2, 1)$ and $\mu = (1, 1)$. Then $A_\Lambda = \{(1, 2, 3), (4, 5, 6)\}$ and S_Λ is the group of order two that permutes the two blocks, thus generated by $\sigma = (1, 4)(2, 5)(3, 6)$. In other words

$$v_\Lambda = v^1_1 \otimes v^1_1 \otimes v^1_2 \otimes v^2_1 \otimes v^2_1 \otimes v^2_2 \quad \text{and}$$
$$\sigma v_\Lambda = v^2_1 \otimes v^2_1 \otimes v^2_2 \otimes v^1_1 \otimes v^1_1 \otimes v^1_2.$$

In general, we have that

$$T_\sigma v_\Lambda = \sigma v_\Lambda \quad \text{for } \sigma \in S_\Lambda \tag{7}$$

since in a reduced expression $\sigma = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_N}$ the action of each σ_{i_j} and T_{i_j} on v_Λ will only involve distinct upper indices.

In the above example, we have $\sigma = \sigma_3\sigma_4\sigma_5\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3 \in S_\Lambda$ and hence

$$T_\sigma = T_3T_4T_5T_2T_3T_4T_1T_2T_3 \in \mathcal{E}_n(u).$$

Both σ and T_σ will move the first v_1^2 to the first position, then the second v_1^2 to the second position and finally v_2^2 to the third position.

We consider the row and column (anti)symmetrizer $r_{\mu^i}, c_{\mu^i} \in KS_{|\mu^i|}$ of the partitions μ^i as elements of $\mathcal{E}_n(u)$ by mapping each occurring σ to $T_{\iota_i(\sigma)}$. By Corollary 1, we then get that r_{μ^i} and c_{μ^i} commute with E_{A_Λ}.

We define $w_\Lambda := (r_{\mu^1}\otimes r_{\mu^2}\otimes\cdots\otimes r_{\mu^k})v_\Lambda$. It has the form $w_\Lambda := w_{\lambda_1}^{\mu_1}\otimes\cdots\otimes w_{\lambda_k}^{\mu_k}$ where we for general λ, μ define

$$w_\lambda^\mu := \sum_{\sigma\in r_\mu} v_\lambda^{\sigma(1)}\otimes\cdots\otimes v_\lambda^{\sigma(m)},$$

where $|\mu| = m$. We define the 'permutation module' as

$$M(\Lambda) := \mathcal{E}_n(u) = \mathcal{E}_n(u)w_\Lambda.$$

Define now

$$e_\Lambda := (c_{\mu^1}\otimes c_{\mu^2}\otimes\cdots\otimes c_{\mu^k})\big(c_{\lambda^1}(u)^{\otimes m_1}\otimes\cdots\otimes c_{\lambda^k}(u)^{\otimes m_k}\big)E_{A_\Lambda},$$

where $c_{\lambda^i}(u)$ is as in Sect. 4. Note that the three factors of e_Λ commute by the definitions and Corollary 1. We define the 'Specht module' as

$$S(\Lambda) := \mathcal{E}_n(u)e_\Lambda w_\Lambda \subset M(\Lambda).$$

Actually, the factor E_{A_Λ} could have been left out of e_Λ in the definition of the Specht module, since it commutes with $r_{\mu^1}\otimes r_{\mu^2}\otimes\cdots\otimes r_{\mu^k}$ and $E_{A_\Lambda}w_\Lambda = w_\Lambda$ by the next Lemma 8, but for later use we prefer to include it in e_Λ.

Lemma 8 *In the above setting we have that*

$$E_B w_\Lambda = \begin{cases} w_\Lambda & \text{if } B\subseteq A_\Lambda, \\ 0 & \text{otherwise.}\end{cases} \tag{8}$$

Proof If $B\subseteq A_\Lambda$ this is an immediate consequence of the definitions. If $B \not\subseteq A_\Lambda$ there are $i, j\in \mathbf{n}$ belonging to the same block of B and to different blocks of A_Λ, let these be $I_{\alpha(i)}$ and $I_{\alpha(j)}$. Since E_{ij} is a factor of E_B it is enough to show that $E_{ij}\sigma v_\Lambda = 0$ for $\sigma\in S_\Lambda$. But from formula (1) we have that

$$E_{ij} = T_{j-1}T_{j-2}\cdots T_{i+1}E_iT_{i+1}^{-1}\cdots T_{j-2}^{-1}T_{j-1}^{-1}.$$

Using it we can decompose E_{ij} from the right to the left in an element of $\iota_{\alpha(j)}(H_{I_{\alpha(j)}})$, followed by the product of the remaining T_k^{-1}, then E_i and finally the product of the T_k. The action of $\iota_j(H_{I_{\alpha(j)}})$ on σv_Λ produces a linear combination of basis elements v of $V^{\otimes n}$ where all appearing v are obtained from σv_Λ by permuting the factors corresponding to the block $I_{\alpha(i)}$. The upper indices of the factors of v are exactly as those of σv_Λ. The product of T_k^{-1} acts on each v by permuting the first factor of the $I_{\alpha(j)}$ block to the $i+1$st position, that is inside the $I_{\alpha(i)}$ block. But E_i acts as zero on this and the lemma follows. □

The main result of this section is the following theorem.

Theorem 5 *$S(\Lambda)$ is a simple module for $\mathcal{E}_n(u)$. The simple $\mathcal{E}_n(u)$-modules are classified by $S(\Lambda)$ for $\Lambda \in \mathcal{L}_n$.*

Proof Write for simplicity $A := A_\Lambda$.

Our first step is to show that $e_\Lambda M(\Lambda) = K e_\Lambda w_\Lambda$. For this we take $x \in \mathcal{E}_n(u)$ and first consider the element $E_A x w_\Lambda \in M(\Lambda)$.

We can write x as a linear combination of elements $E_B T_w$ from our basis G. By Corollary 2, $E_A E_B$ is equal to a E_C for C with $A \subseteq C$. By Lemma 8 and Corollary 1 we have that $E_C T_w w_\Lambda = T_w E_{w^{-1}C} w_\Lambda = 0$ unless $w^{-1}C = A$, since $A \subseteq C$. We may therefore assume that $B = A$ and $A = wA$ such that $E_A x$ is a linear combination of elements of the form $E_A T_w$ where T_w permutes the blocks of A of equal cardinality.

Thus, let $\overline{S_\Lambda} \le S_n$ be the subgroup consisting of the permutations of the blocks of A of equal cardinality. Note that $S_\Lambda \le \overline{S_\Lambda}$, the inclusion being strict in general. As in the case of S_Λ, the elements of $\overline{S_\Lambda}$ can be seen as elements of $\mathcal{E}_n(u)$, by the map $z \mapsto T_z$.

In this notation, if $E_A x w_\Lambda$ is nonzero it is a linear combination of elements of the form

$$T_z(T_{w_1} \otimes T_{w_2} \otimes \cdots \otimes T_{w_l}) w_\Lambda, \tag{9}$$

where $z \in \overline{S_\Lambda}$ and $T_{w_1} \otimes T_{w_2} \otimes \cdots \otimes T_{w_l} \in H_\Lambda(u)$ and where we used that E_A commutes with the other factors and $E_A w_\Lambda = w_\Lambda$. Since the upper indices of the $w^j_{\lambda^i}$ are distinct, T_z acts by permuting the T_{w_i}-factors.

We need to show that $z \in S_\Lambda$ and therefore consider the action on $c_{\lambda^1}(u)^{\otimes m_1} \otimes \cdots \otimes c_{\lambda^k}(u)^{\otimes m_k}$ on (9). Let from this $\lambda^1, \lambda^2, \ldots, \lambda^t$ be the partitions with $|\lambda^i| = |\lambda^1| = |I_1|$. Note that in general $t \ge m_1$. Since the λ^i are ordered increasingly, we get by lemma 7 that the product is nonzero only if each factor $c_{\lambda^k}(u)$ of $c_{\lambda^1}(u)^{\otimes m_1} \otimes \cdots \otimes c_{\lambda^t}(u)^{\otimes m_t}$ acts in a $T_{w_a} v^{\sigma(a)}_{\lambda^k}$-factor of (9), i.e. a factor with the same λ^k appearing as index. This argument extends to the other factors of $c_{\lambda^1}(u)^{\otimes m_1} \otimes \cdots \otimes c_{\lambda^k}(u)^{\otimes m_k}$ and so we may assume that $z \in S_\Lambda$ as claimed.

After this preparation, we can show the claim about $e_\Lambda M(\Lambda)$. We take $x \in \mathcal{E}_n(u)$ and consider $e_\Lambda x w_\Lambda$. By the above, it is a linear combination of elements of the form

$$(c_{\mu^1} \otimes \cdots \otimes c_{\mu^k}) T_z \big(c_{\lambda^1}(u) \otimes \cdots \otimes c_{\lambda^l}(u)\big)(T_{w_1} \otimes \cdots \otimes T_{w_l}) w_\Lambda,$$

where $T_{w_1} \otimes T_{w_2} \otimes \cdots \otimes T_{w_l} \in H_\Lambda(u)$ and where $z \in S_\Lambda$ such that T_z commutes with $c_{\lambda^1}(u) \otimes \cdots \otimes c_{\lambda^l}(u)$. We now use the formulas (5), (6) and the definition of w_Λ to rewrite this as

$$\begin{aligned}
&C_1(c_{\mu^1}\otimes\cdots\otimes c_{\mu^k})T_z\big(s_{\lambda^1}(u)\otimes\cdots\otimes s_{\lambda^l}(u)\big)w_\Lambda\\
&\quad=C_2(c_{\mu^1}\otimes\cdots\otimes c_{\mu^k})T_z\big(s_{\lambda^1}(u)\otimes\cdots\otimes s_{\lambda^l}(u)\big)(r_{\mu^1}\otimes\cdots\otimes r_{\mu^k})w_\Lambda\\
&\quad=C_2(c_{\mu^1}\otimes\cdots\otimes c_{\mu^k})T_z(r_{\mu^1}\otimes\cdots\otimes r_{\mu^k})\big(s_{\lambda^1}(u)\otimes\cdots\otimes s_{\lambda^l}(u)\big)w_\Lambda\\
&\quad=C_3(s_{\mu^1}\otimes\cdots\otimes s_{\mu^k})\big(s_{\lambda^1}(u)\otimes\cdots\otimes s_{\lambda^l}(u)\big)w_\Lambda\\
&\quad=C_4(c_{\mu^1}\otimes\cdots\otimes c_{\mu^k})\big(c_{\lambda^1}(u)\otimes\cdots\otimes c_{\lambda^l}(u)\big)w_\Lambda=C_4e_\Lambda w_\Lambda,
\end{aligned}$$

where the $C_i\in K$ are constants and where we used that $r_{\mu^1}\otimes\cdots\otimes r_{\mu^k}$ commutes with $c_{\lambda^1}(u)\otimes\cdots\otimes c_{\lambda^l}(u)$ and $r_{\lambda^1}(u)\otimes\cdots\otimes r_{\lambda^l}(u)$ since r_{μ^1} permutes over equal factors $c_{\lambda^1}(u)$ etc. For $z=1$ all the constants are nonzero since the Young symmetrizers $s_\lambda(u)$ and s_μ are idempotents up to nonzero scalars and we have then finally proved that $e_\Lambda M(\Lambda)=Ke_\Lambda w_\Lambda$, as claimed. Since $S(\Lambda)\subseteq M(\Lambda)$ we also have $e_\Lambda S(\Lambda)\subseteq Ke_\Lambda w_\Lambda$.

We now proceed to prove that $S(\Lambda)$ is a simple module for $\mathcal{E}_n(u)$. We do it by setting up of version of James's submodule theorem [10]. Assume therefore $N\subset S(\Lambda)$ is a submodule. If $e_\Lambda N\neq 0$, we have by the above that $e_\Lambda N$ is a scalar multiple of $e_\Lambda w_\Lambda$ and so $N=S(\Lambda)$.

In order to treat the other case $e_\Lambda N=0$, we define a bilinear form on $V^{\otimes n}$ by setting

$$\big\langle v_{i_1}^{j_1}\otimes\cdots\otimes v_{i_n}^{j_n},v_{i'_1}^{j'_1}\otimes\cdots\otimes v_{i'_n}^{j'_n}\big\rangle=v^{\bar{i}}\delta_{\bar{i}=\overline{i'},\bar{j}=\overline{j'}}$$

and extending linearly, where we write $\bar{i}=(i_1,i_2,\dots,i_n)$ and similarly for $\overline{i'},\bar{j},\overline{j'}$. The power $v^{\bar{i}}$ is defined as follows. Order $v_{i_1}^{j_1}\otimes\cdots\otimes v_{i_n}^{j_n}$ by first moving all factors $v_{i_k}^{j_k}$ with minimal upper indices to the left of $v_{i_1}^{j_1}\otimes\cdots\otimes v_{i_n}^{j_n}$ but maintaining their relative position, then moving the factors $v_{i_k}^{j_k}$ with second smallest upper indices to the positions just to the right of the first ones and so on. This gives a permutation $\sigma\in S_n$ such that $\sigma(v_{i_1}^{j_1}\otimes\cdots\otimes v_{i_n}^{j_n})$ has increasing upper indices, let these be $f(1),f(2),\dots,f(m)$ without repetitions. We then find compositions $\tau_i,i=1,\dots,m$ and minimal coset representations $w_i\in S_{|I_{\tau_i}|}/S_{\tau_i}$ such that

$$\sigma\big(v_{i_1}^{j_1}\otimes\cdots\otimes v_{i_n}^{j_n}\big)=w_1v_{\tau^1}^{f(1)}\otimes w_2v_{\tau^2}^{f(2)}\otimes\cdots\otimes w_mv_{\tau^m}^{f(m)}$$

and define $v^{\bar{i}}:=v^{\sum l(w_i)}$.

This bilinear form is modeled on the one for the tensor space module for Hecke algebras [4], and inherits from it the following invariance property

$$\langle xv,w\rangle=\langle v,x^*w\rangle\quad\text{for all }x\in\mathcal{E}_n(u),v,w\in V^{\otimes n},$$

where $*$ is as in Sect. 4. We have that

$$c_\lambda^*=c_\lambda,\qquad r_\lambda^*=r_\lambda,\qquad c_\lambda(u)^*=c_\lambda(u),\qquad r_\lambda(u)^*=r_\lambda(u),$$

where we used that $*$ is an antiautomorphism to show for instance that $T_{w_\lambda^{-1}}y_{\lambda'}T_{w_\lambda}^*=T_{w_\lambda^{-1}}y_{\lambda'}T_{w_\lambda}$. Since the factors of e_Λ commute, we also have that

$$e_\Lambda^*=e_\Lambda.$$

We are now in position to finish the treatment of the case $e_\Lambda N = 0$. We have

$$0 = \big\langle e_\Lambda N, M(\Lambda)\big\rangle = \big\langle N, e_\Lambda M(\Lambda)\big\rangle = \langle N, e_\Lambda w_\Lambda\rangle,$$

which implies that $\langle N, S(\Lambda)\rangle = 0$ that is $N \subset S(\Lambda)^\perp$. Since u is generic, we have that $\langle e_\Lambda w_\Lambda, e_\Lambda w_\Lambda\rangle \neq 0$ and therefore $S(\Lambda) \cap S(\Lambda)^\perp = 0$. This gives a contradiction unless $N = 0$. We have therefore proved that $S(\Lambda)$ is simple.

We next prove that different choices of parameters give different modules $S(\Lambda)$. Take Λ as before and suppose $\Upsilon = ((\nu^t), (n_t), (\tau^t)) \in \mathcal{L}_n$ such that $S(\Lambda) \cong S(\Upsilon)$. The element $A \in \mathcal{P}_n$ associated with $S(\Lambda)$ is maximal with respect to having blocks of consecutive numbers such that $E_A S(\Lambda) \neq 0$. Hence, if $B \in \mathcal{P}_n$ is the element associated with $S(\Upsilon)$, we have that $A = B$. But then (λ^s) and (ν^t) must be partitions of the same numbers, corresponding to the block sizes of A, or B. Both $c_{\lambda^1}(u) \otimes \cdots \otimes c_{\lambda^l}(u)$ and $c_{\nu^1}(u) \otimes \cdots \otimes c_{\nu^l}(u)$ act nontrivially in $E_A S(\Lambda)$ and hence by Lemma 7 we have $\lambda^i \leq \nu^i$ and $\lambda^i \geq \nu^i$ that is $\lambda^i = \nu^i$. Similarly, we get $(\mu^s) = (\tau^t)$. This proves the claim.

It remains to be shown that any simple module L is of the form $S(\Lambda)$ for some $\Lambda \in \mathcal{L}_n$. We saw in the remarks preceding the theorem, that it can be assumed that $L \subset V^{\otimes n}$. Choose $A = \{I_1, \dots, I_l\} \in \mathcal{P}_n$ maximal with respect to having blocks of consecutive numbers and $E_A L \neq 0$. For $\sigma \in S_n$, the map $\varphi^\sigma : V^\otimes \to V^\otimes$ given by

$$\varphi^\sigma : v_{i_1}^{j_1} \otimes \cdots \otimes v_{i_n}^{j_n} \to v_{i_1}^{\sigma(j_1)} \otimes \cdots \otimes v_{i_n}^{\sigma(j_n)}$$

is an $\mathcal{E}_n(u)$-linear isomorphism and replacing L by $\varphi_\sigma L$ for an appropriately chosen σ we may assume that $|I_i| \leq |I_{i+1}|$ for all i. We have now that $E_A L$ is a module for the tensor product $H_{I_1}(u) \otimes \cdots \otimes H_{I_l}(u)$. Choose for each I_i a partition λ_i of $|I_i|$ such that the product $c_{\lambda^1}(u) \otimes c_{\lambda^2}(u) \otimes \cdots \otimes c_{\lambda^l}(u)$ acts nontrivially in $E_A L$. Choose next partitions μ^i such that $s_{\mu^1} \otimes s_{\mu^2} \otimes \cdots \otimes s_{\mu^k}$ acts nontrivially in $(c_{\lambda^1}(u) \otimes \cdots \otimes c_{\lambda^l}(u))E_A L$. The data so collected give rise to a Λ with $S(\Lambda) = \mathcal{E}_n(u) e_\Lambda w_\Lambda \subset L$. But since L is simple, the inclusion must be an equality. With this we have finally proved all statements of the theorem. $\square$

Let us work out some low-dimensional cases. For $n = 2$ we have the following possibilities for Λ:

$$\big(\lambda^1, m_1, \mu^1\big) = \Big(\ydiagram{2}, 1, \ydiagram{1}\Big), \qquad \big(\lambda^1, m_1, \mu^1\big) = \Big(\ydiagram{1,1}, 1, \ydiagram{1}\Big),$$

$$\big(\lambda^1, m_1, \mu^1\big) = \Big(\ydiagram{1}, 2, \ydiagram{2}\Big), \qquad \big(\lambda^1, m_1, \mu^1\big) = \Big(\ydiagram{1}, 2, \ydiagram{1,1}\Big).$$

They all give rise to irreducible representations of dimension one. The first two are the one-dimensional representations of $H_2(u)$. By our construction the third is given by $v_1^1 \otimes v_1^2 + v_1^2 \otimes v_1^1$ and the last by $v_1^1 \otimes v_1^2 - v_1^2 \otimes v_1^1$. They correspond to the trivial and the sign representation of $K S_2$. The square sum of the dimensions is 4, which is also the dimension of $\mathcal{E}_2(u)$.

For $n = 3$ we first write down the multiplicity free possibilities of Λ, i.e. those having $m_s = 1$ and so $\mu_s = \ydiagram{1}$ for all s. They are

$$(\lambda^1) = \left(\ydiagram{3}\right), \qquad (\lambda^1) = \left(\ydiagram{2,1}\right), \qquad (\lambda^1) = \left(\ydiagram{1,1,1}\right),$$

$$(\lambda^1, \lambda^2) = \left(\ydiagram{1}, \ydiagram{2}\right), \qquad (\lambda^1, \lambda^2) = \left(\ydiagram{1}, \ydiagram{1,1}\right).$$

The first three of these are the Specht modules for $H_3(u)$, their dimensions are respectively 1, 2 and 1. The fourth is given by the vector $v_1^1 \otimes v_1^2 \otimes v_1^2$ and the last by the vector $v_1^1 \otimes (v_1^2 \otimes v_2^2 - u^{-1} v_2^2 \otimes v_1^2)$, according to our construction. In both cases, one gets dimension three.

Allowing multiplicities, we have the following possibilities:

$$(\lambda^1, m_1, \mu^1) = \left(\ydiagram{1}, 3, \ydiagram{1,1,1}\right), \qquad (\lambda^1, m_1, \mu^1) = \left(\ydiagram{1}, 3, \ydiagram{2,1}\right),$$

and

$$(\lambda^1, m_1, \mu^1) = \left(\ydiagram{1}, 3, \ydiagram{1,1,1}\right).$$

We get the Specht modules of KS_3 of dimensions 1, 2 and 1.

The square sum of all the dimensions is 30, in accordance with the dimension of $\mathcal{E}_3(u)$. We have thus proved that $\mathcal{E}_n(u)$ is semisimple for $n = 2$ and $n = 3$.

The classification of the simple modules for $n = 2$ and $n = 3$ has also been done in [1] with a different method.

7 Questions

The results of the paper raise a number of questions.

There is a canonical inclusion $\mathcal{E}_n(u) \subset \mathcal{E}_{n+1}(u)$ which at diagram level is given by adding a through line to the right of a diagram element from $\mathcal{E}_n(u)$. It gives rise to restriction and induction functors res and ind, that should obey a branching rule for the decomposition of res $S(\Lambda)$. Our first question is to give a description of it. Apart from the independent interest in such a branching rule, one possible application would be to obtain a dimension formula for $S(\Lambda)$.

We do not know what the general branching rule looks like, but using the above calculations, we can at least explain the cases $n = 2, 3$, corresponding to res $S(\Lambda)$ for $\Lambda \in \mathcal{P}_2$ and $\Lambda \in \mathcal{P}_3$, These cases are rather easy, since one only needs consider $n = 3$, $\Lambda = (\lambda^s, m_s, \mu^s)$, $m_s = 1$ and μ^s trivial and

$$(\lambda^1, \lambda^2) = \left(\ydiagram{1}, \ydiagram{2}\right), \qquad (\lambda^1, \lambda^2) = \left(\ydiagram{1}, \ydiagram{1,1}\right)$$

because, as we saw above, all other choices of Λ give Specht modules that are pull-backs of Specht modules of the symmetric group or of the Hecke algebra and therefore obey the usual branching rule. For both of them, the restriction contains the trivial and the sign module for KS_2 corresponding to the third and fourth Specht modules for $\mathcal{E}_2(u)$ in the above description. But the first of them moreover contains the trivial module for $\mathcal{H}_2(u)$ corresponding to the first Specht module of the classification, whereas the second contains the nontrivial one-dimensional module for $\mathcal{H}_2(u)$ corresponding to the fourth module of the classification. The question is now how to generalize this to higher n.

The paper treated the representation theory of $\mathcal{E}_n(u)$ for u generic, where one expects $\mathcal{E}_n(u)$ to be semisimple, as observed above for $n = 2, 3$. It is therefore natural to ask for a formal proof of semisimplicity beyond the cases $n = 2, 3$. If one had an explicit formula for the dimension of $S(\Lambda)$ it would be natural to try to generalize the above proof for $n = 2, 3$. On the other hand, in view of the nondegeneracy of the form defined in Sect. 5 and Wenzl's treatment of the Brauer algebra in [22], an attractive alternative approach to proving semisimplicity of $\mathcal{E}_n(u)$ would be to look for an analog of the Jones basic construction [8] in the setting, using the embedding $\mathcal{E}_n(u) \subset \mathcal{E}_{n+1}(u)$.

As already mentioned in Sect. 2, it is possible to define a specialized algebra $\mathcal{E}_n(u_0)$, for example by choosing u_0 to be an lth root of unity. This should be a nonsemisimple algebra. A natural first step into the representation theory of this specialized algebra is to show that $\mathcal{E}_n(u)$ is a cellular algebra in the sense of [5]. We firmly believe that this indeed is the case, but also think that a new set of tools would be needed to establish it. In this paper, the tensor module was a crucial ingredient in our determination of the rank of $\mathcal{E}_n(u)$ and so for the completeness of the paper we found it most natural to construct the Specht modules inside it.

Finally, the tensor module itself raises the question of determining its endomorphism algebra $\mathrm{End}_{\mathcal{E}_n(u)}(V^{\otimes n})$ and setting up an analog of Schur–Weyl duality. Given the result of the paper, $\mathrm{End}_{\mathcal{E}_n(u)}(V^{\otimes n})$ should be an interesting combination of quantum groups and symmetric groups/Hecke algebras.

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