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A computational and combinatorial exposé of plethystic calculus

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Abstract In recent years, plethystic calculus has emerged as a powerful technical tool for studying symmetric polynomials. In particular, some striking recent advances in the theory of Macdonald polynomials have relied heavily on plethystic computations. The main purpose of this article is to give a detailed explanation of a method for finding combinatorial interpretations of many commonly occurring plethystic expressions, which utilizes expansions in terms of quasisymmetric functions. To aid newcomers to plethysm, we also provide a self-contained exposition of the fundamental computational rules underlying plethystic calculus. Although these rules are well-known, their proofs can be difficult to extract from the literature. Our treatment emphasizes concrete calculations and the central role played by evaluation homomorphisms arising from the universal mapping property for polynomial rings.

Keywords Plethysm · Symmetric functions · Quasisymmetric functions · LLT polynomials · Macdonald polynomials

1 Introduction

1.1 Plethysm

The *plethysm* $F[G]$ of a symmetric polynomial $F(x)$ with a symmetric polynomial $G(x)$ is essentially the polynomial obtained by substituting the monomials of $G(x)$

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for the variables of $F(x)$. This operation was introduced by Littlewood [31] in the study of group representation theory. This operation is often called "outer plethysm" to distinguish it from the operation of "inner plethysm" or Kronecker product of representations. One of the major open problems in the theory of symmetric functions and the representation theory of classical groups is to be able to compute the coefficients $a^{\nu}_{\lambda,\mu}$ in the expansion

$$s_{\lambda}[s_{\mu}] = \sum_{\nu} a^{\nu}_{\lambda,\mu} s_{\nu}(x) \tag{1}$$

where $s_{\lambda}, s_{\mu}, s_{\nu}$ denote the Schur functions corresponding to partitions λ, μ and ν, respectively. We note that Littlewood used the notation $\{\mu\} \otimes \{\lambda\}$ for $s_{\lambda}[s_{\mu}]$. The operation of plethysm arises naturally in both the representation theory of the general linear group $GL(n, \mathbb{C})$ and the symmetric group S_n. For example, if E is a finite dimensional vector space over a field of characteristic 0, we let $\bigwedge^{\lambda} E$ and $\bigwedge^{\mu} E$ denote the representation space for the irreducible representation of $GL(E)$ associated with the partitions λ and μ. Then the coefficient $a^{\nu}_{\lambda,\mu}$ gives the multiplicity of $\bigwedge^{\nu} E$ in the direct sum decomposition of $\bigwedge^{\lambda}(\bigwedge^{\mu} E)$. Similarly, if λ is a partition of n and μ is a partition of m, $s_{\lambda}[s_{\mu}]$ can be viewed as the Frobenius image of the character of the representation of S_{mn} which can be described as follows. Let A_{λ} be the irreducible S_n-module corresponding to λ, and let A_{μ} be the irreducible S_m-module corresponding to μ. The wreath product $S_m \wr S_n$, which is the normalizer of $S_n^m = S_n \times \cdots \times S_n$ in S_{mn}, acts on A_{λ} and on the mth tensor power $T^m(A_{\mu})$ and, hence, it also acts on $A_{\lambda} \otimes T^m(A_{\mu})$. Then $s_{\lambda}[s_{\mu}]$ is the Frobenius image of the character of the S_{mn}-module induced by $A_{\lambda} \otimes T^m(A_{\mu})$, see [34] or [27]. To date, there is no satisfactory combinatorial description of the coefficients $a^{\nu}_{\lambda,\mu}$. There are only a very few special cases of λ and μ, such as Littlewood's expansion [32] of $s_2[s_n]$ and $s_{(1^2)}[s_n]$, where we have explicit formulas for $a^{\nu}_{\lambda,\mu}$. However, there are a variety of algorithms to compute $a^{\nu}_{\lambda,\mu}$, see [1, 8–10, 26, 35, 42].

While the representation-theoretic motivation for the operation of plethysm is clear, one can more generally consider the operation of plethysm of any two symmetric functions $F[G]$. In particular, the plethysm operation can also be formulated in the more abstract setting of λ-rings. The notion of a λ-ring was introduced by Grothendieck [21] in the study of Chern classes. Later Atiyah [2] used λ-rings and the theory of representations of S_n over the complex numbers $\mathbb{C}$ to investigate operations in K-theory. Atiyah and Tall [3] and Knutson [28] showed that the Grothendieck representation ring $R = R(S_n)$ of the symmetric group S_n forms a special λ-ring with respect to exterior power. In fact, the Hopf ring of symmetric functions in countably many variables, see [19], is a free λ-ring on the elementary symmetric function e_1. Thus the graded Hopf ring R is also a free λ-ring on one generator $F^{-1}(e_1) = \eta_1$ where $F : R \to H$ is the Frobenius map defined in [3, 28]. Thus η_1 is the class of the trivial representation. In fact, the λ-ring structure on R can be derived from the plethysm operation, see [40]. This connection with λ-rings leads to an axiomatic presentation of plethysm as developed, for example, in [28].

Computations involving plethysm have many applications. For example, Hoffman [25] investigated inner and outer plethysms in the framework of τ-rings.

Wybourne [41] showed that there are many applications of plethysm in physics. The special case of the expansion of the plethysm $s_m[s_n]$ has applications in nineteenth-century invariant theory, see [11, 31]. Plethysm plays an important role in the theory of symmetric functions and Schubert polynomials. It can be used to unify many proofs of old identities and has been used to prove a host of new results. This is beautifully illustrated in Alain Lascoux's book [29].

1.2 Plethystic calculus

The subject of this paper is *plethystic calculus*, which is an extension of the plethysm operation to expressions of the form $F[G]$, where F is a symmetric function (or a formal limit of symmetric functions) and G is a formal power series or Laurent series. This plethystic calculus has been used extensively by researchers including Francois Bergeron, Nantel Bergeron, Adriano Garsia, Jim Haglund, Mark Haiman, and Glenn Tesler (among others) in an ongoing study of the Bergeron–Garsia nabla operator, diagonal harmonics modules, Macdonald polynomials, and related symmetric functions [4–6, 13–18, 22]. Plethystic calculus has become an indispensable computational tool for organizing and manipulating intricate relationships between symmetric functions.

Many commonly occurring symmetric functions (like skew Schur functions) have well-known combinatorial interpretations involving tableaux or similar structures. In contrast, it is not always easy to write down a combinatorial formula for a plethystic expression $F[G]$. The *primary goal* of this paper is to give a detailed explanation of a general technique for finding such combinatorial formulas for many choices of F and G. This technique, which was used in [23, 24] to study Macdonald polynomials, employs an extension of plethystic calculus in which F is allowed to be a quasisymmetric function. We will give a complete, self-contained description of this technique in Sect. 4, filling in many details that are only hinted at in [23, 24].

The second goal of this paper is to give a rigorous, detailed account of the algebraic foundations of the plethystic calculus using a minimum of technical machinery. We will supply complete derivations of many plethystic identities that are well-known, but whose proofs are difficult to extract from the literature. In particular, we will prove the plethystic addition formula for skew Schur functions as well as plethystic versions of the Cauchy identities. The starting point for our development of plethystic calculus is the fact that the power-sums p_n are algebraically independent elements that generate the ring of symmetric functions. Plethystic notation gives a concise way to define homomorphisms on this ring by specifying their effect on every p_n. It follows that each plethystic identity ultimately results from properties of the transition matrices between the power-sums and other bases of symmetric functions. We will see that this leads to elementary computational or combinatorial proofs of many fundamental plethystic identities, which avoid the more technical aspects of λ-rings and Hopf algebras. A slight disadvantage of this approach is that we must restrict ourselves to working over fields of characteristic zero.

This paper is organized as follows. Section 2 develops the basic algebraic properties of plethysm from scratch, using a definition based on power-sum symmetric functions and evaluation homomorphisms. Section 3 gives two proofs of the crucial addition formula for plethystic evaluation of skew Schur functions. Section 4 describes a method for finding combinatorial interpretations of a variety of plethystic expressions. In particular, we show how plethystic transforms of quasisymmetric functions provide combinatorial interpretations for plethystic evaluations of skew Schur functions and Lascoux–Leclerc–Thibon (LLT) polynomials. Section 5 derives the plethystic Cauchy formulas and illustrates their use by giving an application (due to Garsia) to the theory of Macdonald polynomials.

2 Plethysm and power-sums

Throughout this paper, let K denote a field of characteristic zero. This section defines the graded ring Λ of symmetric functions with coefficients in K, Littlewood's binary plethysm operation on Λ, and extended versions of this operation commonly called "plethystic notation." The theorems in this section are all well-known among specialists, although our proofs are more detailed and less technical than those found in the standard references [28, 34]. Our approach is based on the universal mapping properties of polynomial rings (reviewed below) and accords a central role to the power-sum symmetric functions.

2.1 Review of polynomial rings

Let $K[z_1,\dots,z_N]$ denote the polynomial ring in N variables with coefficients in K. This ring is a K-algebra satisfying the following *universal mapping property* (UMP): for every[1] K-algebra S and every N-tuple $(a_1,\dots,a_N)$ of elements of S, there exists a unique K-algebra homomorphism $\phi : K[z_1,\dots,z_N]\to S$ such that $\phi(z_i)=a_i$ for $1\le i\le N$. This homomorphism is given explicitly by

$$\phi\left(\sum_{\beta\in\mathbb{N}^N} c_\beta z^\beta\right)=\sum_\beta c_\beta a^\beta \quad (c_\beta\in K),$$

where we write $z^\beta=\prod_{i=1}^N z_i^{\beta_i}$ and $a^\beta=\prod_{i=1}^N a_i^{\beta_i}$. For $f\in K[z_1,\dots,z_N]$, we often write $f(a_1,\dots,a_N)$ to denote the element $\phi(f)\in S$. We call ϕ the *evaluation homomorphism determined by setting* $z_i=a_i$.

We will also need polynomial rings in countably many indeterminates. For $M<N$, we can view $K[z_1,\dots,z_M]$ as a subset of $K[z_1,\dots,z_N]$ in the natural way. Now define the set

$$R=K\bigl[\{z_i : i\ge 1\}\bigr]=\bigcup_{N=1}^{\infty} K[z_1,\dots,z_N].$$

With the obvious definitions of addition and multiplication, R becomes a K-algebra. Moreover, R satisfies the expected universal mapping property: for every K-algebra S and indexed family $\{a_i : i\ge 1\}\subseteq S$, there exists a unique K-algebra homomorphism $\phi : R\to S$ such that $\phi(z_i)=a_i$ for all $i\ge 1$. For more information on polynomial rings, see Chap. IV of [7].

[1] We assume throughout that all K-algebras under consideration are commutative, associative, and have a unit element. All K-algebra homomorphisms are assumed to preserve the unit element.

2.2 Symmetric polynomials, symmetric functions, and power-sums

Consider the polynomial ring $R_N = K[x_1, \ldots, x_N]$. By the universal mapping property, every permutation $w \in S_N$ induces a unique K-algebra endomorphism $\phi_w : R_N \to R_N$ such that $\phi_w(x_i) = x_{w(i)}$. By the uniqueness part of the UMP, we have $\phi_{v\circ w} = \phi_v \circ \phi_w$, because both sides are K-algebra homomorphisms on R_N that send x_i to $x_{v(w(i))}$ for $1 \le i \le N$. Thus, S_N acts on R_N by "permuting the variables." Define

$$\Lambda_N = \bigl\{ f \in R_N : \phi_w(f) = f \text{ for all } w \in S_N \bigr\},$$

which is a K-subalgebra of R_N. Elements of Λ_N are called *symmetric polynomials in N variables*. R_N and Λ_N become graded algebras by letting each x_i have degree 1.

The kth *power-sum symmetric polynomial in N variables* is $p_{k,N} = x_1^k + x_2^k + \cdots + x_N^k$. For an integer partition $\mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_s > 0)$, define $p_{\mu,N} = \prod_{j=1}^s p_{\mu_j,N}$. Let Par be the set of all integer partitions, and let Par(n) be the set of partitions of n. It is well-known that, for all $N \ge n$, the indexed set $\{p_{\mu,N} : \mu \in \text{Par}(n)\}$ is a K-basis for the subspace Λ_N^n consisting of all polynomials in Λ_N which are homogeneous of degree n [38, 39].

Next we define the ring of *abstract symmetric functions with coefficients in K* to be the polynomial ring

$$\Lambda = K\bigl[\{p_i : i \ge 1\}\bigr],$$

where the p_i's are indeterminates called *abstract power-sum symmetric functions*. We define a grading on Λ by letting $\deg(p_i) = i$. For every partition μ, define $p_\mu = \prod_{i\ge 1} p_{\mu_i}$. It follows from the very definition of polynomial rings that the set $\{p_\mu : \mu \in \text{Par}(n)\}$ is a K-basis for the subspace Λ^n consisting of all polynomials which are homogeneous of degree n, and the set $\{p_i : i \ge 1\}$ is algebraically independent over K. Note $\Lambda = \bigoplus_{n\ge 0} \Lambda^n$.

To relate abstract symmetric functions to concrete symmetric polynomials, introduce the evaluation homomorphisms $\text{ev}_N : \Lambda \to R_N$ such that $\text{ev}_N(p_i) = p_{i,N} = x_1^i + \cdots + x_N^i$. For $f \in \Lambda$, we will often write $f(x_1, \ldots, x_N)$ instead of $\text{ev}_N(f)$. One can show that for $N \ge n$, ev_N restricts to a vector space isomorphism of Λ^n onto Λ_N^n.

2.3 Littlewood's plethysm operation on Λ

We are about to define a binary operation $\bullet : \Lambda \times \Lambda \to \Lambda$ called *plethysm*, first introduced by Littlewood [32]. For $f, g \in \Lambda$, $f \bullet g$ (also denoted $f[g]$) is called the *plethystic substitution of g into f*. (Some authors write $f \circ g$ instead of $f \bullet g$. We reserve the open circle $\circ$ to denote composition of functions.) The results presented next are equivalent to those stated in [34, Sect. I.8], but we start from a different definition of plethysm that avoids the use of "fictitious variables."

We want the plethysm operation to satisfy the following three basic properties.

P1. For all $m, n \ge 1$, $p_m \bullet p_n = p_{mn}$.

P2. For all $m \ge 1$, let $L_m : \Lambda \to \Lambda$ be the map $L_m(g) = p_m \bullet g$ ("left plethysm by p_m"). Then L_m is a K-algebra homomorphism.

P3. For all $g \in \Lambda$, let $R_g : \Lambda \to \Lambda$ be the map $R_g(f) = f \bullet g$ ("right plethysm by g"). Then R_g is a K-algebra homomorphism.

Spelled out in more detail, property P2 says that for all $m \geq 1$, $g_1, g_2 \in \Lambda$, and $c \in K$,

$$p_m \bullet (g_1 + g_2) = p_m \bullet g_1 + p_m \bullet g_2, \qquad p_m \bullet (g_1 \cdot g_2) = (p_m \bullet g_1) \cdot (p_m \bullet g_2),$$
$$p_m \bullet c = c.$$

Property P3 says that for each $g, f_1, f_2 \in \Lambda$ and $c \in K$,

$$(f_1 + f_2) \bullet g = f_1 \bullet g + f_2 \bullet g, \qquad (f_1 \cdot f_2) \bullet g = (f_1 \bullet g) \cdot (f_2 \bullet g), \qquad c \bullet g = c.$$

Theorem 1 *There exists a unique binary operation $\bullet$ on Λ satisfying P1, P2, and P3.*

Proof Fix $m \geq 1$. By the UMP for polynomial rings, there is a unique K-algebra homomorphism $L_m : \Lambda \to \Lambda$ such that $L_m(p_n) = p_{mn}$ for all n. So, for each fixed $m \geq 1$, there is a unique way of defining $p_m \bullet g$ ($g \in \Lambda$) so that P1 and P2 hold. Now fix $g \in \Lambda$ and let m vary. Using the UMP again, we see that there is a unique K-algebra homomorphism $R_g : \Lambda \to \Lambda$ such that $R_g(p_m) = p_m \bullet g$. So there is a unique definition of $f \bullet g$ (for $f \in \Lambda$) satisfying P3. □

To compute $f \bullet g$ in practice, first express f in terms of the power-sum basis of Λ, say $f = \sum_\nu c_\nu p_\nu$. Then, thanks to P3,

$$f \bullet g = \sum_\nu c_\nu \prod_i (p_{\nu_i} \bullet g).$$

Second, write g in terms of the power-sum basis, say $g = \sum_\mu d_\mu p_\mu$. Then, by P2 and P1,

$$p_{\nu_i} \bullet g = \sum_\mu d_\mu \prod_j (p_{\nu_i} \bullet p_{\mu_j}) = \sum_\mu d_\mu \prod_j (p_{\nu_i \mu_j}).$$

We will see that the most basic plethystic identities all follow from the universal mapping properties for polynomial rings.

Theorem 2 (Plethysm vs. Substitution) *For all $g \in \Lambda$ and $m \geq 1$,*

$$(p_m \bullet g)(x_1, \ldots, x_N) = g\big(x_1^m, \ldots, x_N^m\big).$$

Proof Define evaluation homomorphisms $\mathrm{ev}_N : \Lambda \to K[x_1, \ldots, x_N]$ and $\pi_m : K[x_1, \ldots, x_N] \to K[x_1, \ldots, x_N]$ by setting $\mathrm{ev}_N(p_k) = x_1^k + \cdots + x_N^k$ and $\pi_m(x_i) = x_i^m$ for all $k, i \geq 1$. The theorem asserts that $\mathrm{ev}_N \circ L_{p_m} = \pi_m \circ \mathrm{ev}_N$. This is true because both sides are K-algebra homomorphisms with domain Λ that send p_k to $x_1^{mk} + \cdots + x_N^{mk}$. □

Theorem 3 (Centrality of p_n) *For all $h \in \Lambda$ and all $n \geq 1$, $p_n \bullet h = h \bullet p_n$.*

Proof Consider the K-algebra homomorphisms L_n and R_{p_n} on Λ. For all $m \geq 1$, property P1 shows that

$$L_n(p_m) = p_n \bullet p_m = p_{nm} = p_{mn} = p_m \bullet p_n = R_{p_n}(p_m).$$

By the uniqueness part of the UMP for polynomial rings, $L_n = R_{p_n}$. Applying these functions to h gives the result. □

Theorem 4 (Unit Element for Plethysm) *For all* $h \in \Lambda$, $p_1 \bullet h = h = h \bullet p_1$.

Proof By the last proposition with $n = 1$, we need only prove the first equality. By property P2, L_1 is the unique K-algebra homomorphism on Λ sending p_m to $p_1 \bullet p_m = p_m$ for all m. Since the identity map on Λ also sends p_m to p_m for all m, uniqueness shows that $L_1 = \mathrm{id}_\Lambda$. So $p_1 \bullet h = \mathrm{id}(h) = h$. □

Theorem 5 (Associativity of Plethysm) *For all* $f, g, h \in \Lambda$, $(f \bullet g) \bullet h = f \bullet (g \bullet h)$.

Proof *Step 1*: The result holds when $f = p_i$, $g = p_j$, and $h = p_k$. For in this case, repeated use of property P1 shows that both sides equal p_{ijk}.

Step 2: The result holds for $f = p_i$, $g = p_j$, and all $h \in \Lambda$. Since $f \bullet g = p_{ij}$, we must prove that $L_{ij} = L_i \circ L_j$. This holds since both sides are K-algebra homomorphisms of Λ that agree on all p_k's (by step 1).

Step 3: The result holds for $f = p_i$ and all $g, h \in \Lambda$. Here we must prove that $R_h \circ L_i = L_i \circ R_h$. By step 2, both sides are K-algebra homomorphisms of Λ having the same effect on every p_j. So they are equal by the UMP for Λ.

Step 4: The result holds for all $f, g, h \in \Lambda$. In this final step, we must show that $R_h \circ R_g = R_{g \bullet h}$. This holds since both sides are K-algebra homomorphisms of Λ that agree on all p_i's (by step 3). □

2.4 Plethystic notation

Classically, the plethysm $f \bullet g$ was only defined when f and g both belong to Λ. Researchers including Francois Bergeron, Nantel Bergeron, Adriano Garsia, Jim Haglund, Mark Haiman, and Glenn Tesler (among others) extended the idea of plethystic substitution to situations where $f \in \Lambda$ and g belongs to some larger K-algebra. This led to the development of a plethystic calculus that has become an enormously useful computational tool for proving results about Macdonald polynomials, the Bergeron–Garsia nabla operator, and related constructs [4–6, 13–18, 22].

We can define this extended version of plethysm by suitably modifying the axioms P1, P2, and P3. We take as initial data a K-algebra Z and, for each integer $m \geq 1$, a $\mathbb{Q}$-algebra homomorphism $L_m : Z \to Z$. (In many applications, Z contains Λ as a subalgebra, and $L_m|\Lambda$ is the usual map $p_n \mapsto p_{mn}$.) For every $g \in Z$, write $L_m(g) = p_m \bullet g$. By the UMP for Λ, we have for each $g \in Z$ a K-algebra homomorphism $R_g : \Lambda \to Z$ such that $R_g(p_m) = p_m \bullet g$. We now have an operation

$$\cdot[\cdot] : \Lambda \times Z \to Z \quad \text{given by } f[g] = R_g(f).$$

(Here we have reverted to the notation for plethysm now in common use, in which the second argument is enclosed by square brackets.) When $f = p_m$, we have $p_m[g] = R_g(p_m) = p_m \bullet g = L_m(g)$. To compute $f[g]$ for a general $f \in \Lambda$, express f as a K-linear combination of products of p_m's and then replace each p_m by $p_m[g] = L_m(g)$. The following example illustrates a typical application of this general setup.

Example 1 Let K be the field $\mathbb{Q}(q,t)$, and let z, w, y be some additional variables. Let $Z = \Lambda(z, w, y)$ be the fraction field of the polynomial ring $\Lambda[z, w, y]$, so that

$$Z \cong \mathbb{Q}[p_1, \ldots, p_n, \ldots](q, t, z, w, y).$$

Using the UMP's for polynomial rings and fraction fields, we see that for each $m \geq 1$, there is a unique $\mathbb{Q}$-algebra endomorphism of Z such that $p_n \mapsto p_{nm}$ for all n, $q \mapsto q^m$, $t \mapsto t^m$, $z \mapsto z^m$, $w \mapsto w^m$, and $y \mapsto y^m$. Informally, this means that we compute $p_m[g]$ by replacing every "variable" in g by its mth power (cf. Theorem 2). In this informal description, we are viewing q, t, z, w, y as "variables" (even though $q, t \in K$), and we think of p_n as an infinite sum $\sum_{i\geq 0} x_i^n$. Then the rule $p_n \mapsto p_{nm}$ arises by replacing each "variable" x_i by its mth power. To compute $f[g]$ for arbitrary $f \in \Lambda$, express f as a sum of products of power-sums and use the previous rule. Note that L_m is *not* a K-algebra homomorphism for $m > 1$, since $L_m(q) = q^m$.

Remark 1 Researchers are continually extending plethystic notation to cover successively more general situations. The key point to remember is that a plethystic expression of the form $f[A]$ always denotes the image of $f \in \Lambda$ under some K-algebra homomorphism ϕ_A of Λ, where ϕ_A is supposed to be determined in some natural way by the "plethystic alphabet" A. The following list gives some conventions that have been developed for converting certain alphabets A to the associated homomorphism ϕ_A.

- Writing $X = x_1 + x_2 + \cdots + x_N + \cdots$, we have by convention $f[X] = f$ (so plethystic substitution of X into f is just the identity homomorphism on Λ). This notation arises by analogy with the finite version $f[x_1 + \cdots + x_N] = \mathrm{ev}_N(f) = f(x_1, \ldots, x_N)$. Instead of $f[X]$, it would be more precise to write $f[p_1]$ (cf. Theorem 4).
- The ring Λ is also a Hopf algebra with a comultiplication map $\Delta : \Lambda \to \Lambda \otimes_K \Lambda$. This map is a K-algebra homomorphism defined (using the UMP) by setting $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$ for all $k \geq 1$. By convention, the plethystic notation $f[X + Y]$ denotes $\Delta(f)$. (Thus $f[X + Y]$ is really an abbreviation for $f[p_1 \otimes 1 + 1 \otimes p_1]$.) We will obtain a formula for $\Delta(s_{\lambda/\nu})$ in the next section.
- Let $\omega : \Lambda \to \Lambda$ be the usual involutory K-automorphism of Λ defined (via the UMP) by sending each p_k to $(-1)^{k-1} p_k$. The plethystic expressions $f[-^- X] = f[-\epsilon X]$ are used by some authors to stand for $\omega(f)$. By contrast, $f[-X] = f[-p_1]$ is the image of f under the homomorphism sending each p_k to $-p_k$ (the antipode map for the Hopf algebra Λ).

Theorem 6 (Negation Rule) *If $g \in \Lambda^n$ is homogeneous of degree n and A is any plethystic alphabet, then*

$$g[-A] = (-1)^n \big(\omega(g)\big)[A].$$

Proof Let ϕ_A be the K-algebra homomorphism with domain Λ determined by A. Since L_m is assumed to be a group homomorphism, we have $p_m[-A] = -p_m[A] = (-1)^m(-1)^{m-1}p_m[A] = (-1)^m(\omega(p_m))[A]$ for all m. Next, if $\mu = (\mu_1, \dots, \mu_s)$ is a partition of n,

$$\begin{aligned} p_\mu[-A] &= \prod_{i=1}^{s} p_{\mu_i}[-A] = \prod_{i=1}^{s}(-1)^{\mu_i}\big(\omega(p_{\mu_i})\big)[A] \\ &= (-1)^n \prod_{i=1}^{s} \phi_A\big(\omega(p_{\mu_i})\big) = (-1)^n\big(\omega(p_\mu)\big)[A], \end{aligned} \tag{2}$$

where the last step follows since ϕ_A and ω are ring homomorphisms. Finally, given $g \in \Lambda^n$, we can write $g = \sum_{\mu\in\mathrm{Par}(n)} c_\mu p_\mu$ for suitable $c_\mu \in K$. Then, by K-linearity,

$$\begin{aligned} g[-A] &= \sum_{\mu} c_\mu p_\mu[-A] = \sum_{\mu} c_\mu(-1)^n\big(\omega(p_\mu)\big)[A] \\ &= (-1)^n\omega\left(\sum_{\mu} c_\mu p_\mu\right)[A] = (-1)^n\big(\omega(g)\big)[A]. \end{aligned} \tag{3}$$

□

Theorem 7 (Monomial Substitution Rule) *In the context of Example 1, suppose A is a finite sum of* monic *monomials $M_1, \dots, M_N$ in Z. For any $g \in \Lambda$,*

$$g[A] = g(M_1, M_2, \dots, M_N),$$

where the right side denotes the image of $g(x_1, \dots, x_N) = \mathrm{ev}_N(g)$ under the evaluation homomorphism $E : K[x_1, \dots, x_N] \to Z$ that sends x_i to M_i for all i.

Proof This result follows immediately from the UMP, since ϕ_A and $E \circ \mathrm{ev}_N$ are K-algebra homomorphisms that have the same effect on every p_k. □

Example 2 For any symmetric function f,

$$f\big[2q + qt + 3t^4\big] = f\big[q + q + qt + t^4 + t^4 + t^4\big] = f\big(q, q, qt, t^4, t^4, t^4\big).$$

As another example, note that

$$p_{(2,2)}[3t] = p_{(2,2)}(t, t, t) = p_2(t, t, t)^2 = \big(t^2 + t^2 + t^2\big)^2 = 9t^4.$$

However, evaluating $p_{(2,2)}(x_1)$ at $x_1 = 3t$ gives

$$p_{(2,2)}(3t) = p_2(3t)^2 = \big((3t)^2\big)^2 = 81t^4 \neq p_{(2,2)}[3t].$$

This example shows that the monomials involved in the last theorem must be *monic*. It also shows that one must take care to distinguish the square plethystic brackets from ordinary round parentheses used to denote the image of a polynomial under an evaluation homomorphism.

Example 3 Let $f \in \Lambda$ and $\mu \in \mathrm{Par}(n)$. The *bi-exponent generator* of μ is

$$B_\mu = \sum_{(i,j):1\le j\le \mu_i} q^i t^j \in \mathbb{Q}(q,t).$$

Let $(i_1, j_1), \ldots, (i_n, j_n)$ be any ordering of the pairs (i, j) appearing in this sum. Treating q and t as variables, we see that $f[B_\mu]$ can be computed by evaluating $f(x_1, \ldots, x_n)$ at $x_k = q^{i_k} t^{j_k}$. This gives a concrete way of thinking about the plethysm $f[B_\mu]$, which occurs frequently in the theory of Macdonald polynomials.

3 Plethystic calculus and Schur functions

This section discusses the plethystic addition formula for simplifying $s_{\lambda/\nu}[A + B]$, where $s_{\lambda/\nu}$ denotes a skew Schur function. This addition formula is known from the theory of λ-rings (cf. [34, pp. 72, 74]), but we will provide two elementary proofs that make no use of λ-rings. We begin by reviewing the relevant combinatorial definitions.

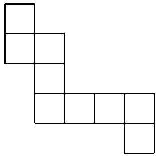
3.1 Review of Schur functions

Suppose $\nu = (\nu_i : i \ge 1)$ and $\lambda = (\lambda_i : i \ge 1)$ are integer partitions such that $\nu \subseteq \lambda$, i.e., $\nu_i \le \lambda_i$ for all i. The *skew diagram* λ/ν is the set of all points $(i, j) \in \mathbb{N}^+ \times \mathbb{N}^+$ satisfying $\nu_i < j \le \lambda_i$. A *semistandard tableau of shape* λ/ν *over the alphabet* $[N] = \{1, 2, \ldots, N\}$ is a function $T : \lambda/\nu \to [N]$ that weakly increases along rows and strictly increases up columns. More precisely, this means $T(i, j) \le T(i, j + 1)$ whenever (i, j) and $(i, j + 1)$ both lie in λ/ν; and $T(i, j) < T(i + 1, j)$ whenever (i, j) and $(i + 1, j)$ both lie in λ/ν. We write $\mathrm{SSYT}_N(\lambda/\nu)$ for the set of all such tableaux. The *weight monomial* of such a tableau is $\mathrm{wt}(T) = x^T = \prod_{(i,j)\in\lambda/\nu} x_{T(i,j)}$. The *skew Schur polynomial in N variables indexed by* λ/ν is

$$s_{\lambda/\nu,N} = \sum_{T\in \mathrm{SSYT}_N(\lambda/\nu)} x^T \in K[x_1, \ldots, x_N]. \tag{4}$$

One can prove that, for $N \ge n$, the set $\{s_{\lambda/0,N} : \lambda \in \mathrm{Par}(n)\}$ is a basis for the K-vector space Λ_N^n [38, 39].

Next we describe the expansion of skew Schur polynomials in terms of power-sum symmetric polynomials. For this we need a few more definitions. Suppose μ is a partition of n with a_i parts equal to i, for $1 \le i \le s$. Then z_μ is the integer $\prod_{i=1}^s (i^{a_i} a_i!)$. We note that $n!/z_\mu$ is the number of permutations of n objects with cycle type μ. A skew shape is a *ribbon* iff it consists of a connected sequence of squares that contains no 2×2 rectangle. A *k-ribbon* is a ribbon consisting of k squares. The *spin*, $\mathrm{spin}(S)$, of a ribbon S that occupies j rows is defined by $\mathrm{spin}(S) = j - 1$ and the *sign*, $\mathrm{sgn}(S)$, of S is defined by $\mathrm{sgn}(S) = (-1)^{j-1}$. For example, the skew shape $(5, 5, 2, 2, 1)/(4, 1, 1)$, whose diagram appears below, is a 9-ribbon of spin 4 and sign $+1$.



Now suppose λ/ν is a skew shape with n squares and α is a composition of n. A *rim hook tableau of shape λ/ν and type α* is a sequence of partitions $T = (\nu^0 \subseteq \nu^1 \subseteq \cdots)$ such that $\nu^0 = \nu$, $\nu^N = \lambda$ for all sufficiently large N, and ν^i/ν^{i-1} is an α_i-ribbon for all $i \geq 1$. T is most easily visualized by placing an i in each square of the α_i-ribbon ν^i/ν^{i-1}. For example, the following picture represents the rim hook tableau

$$T = \big((2,2,1),(5,3,1),(5,4,4,2),(5,4,4,4),(5,4,4,4),(5,5,5,5),(5,5,5,5,1)\big)$$

of shape $(5,5,5,5,1)/(2,2,1)$ and type $(4,6,2,0,3,1)$.

6				
2	2	3	3	5
	2	2	2	5
		1	2	5
		1	1	1

We define $\operatorname{spin}(T) = \sum_i \operatorname{spin}(\nu^i/\nu^{i-1})$ and $\operatorname{sgn}(T) = (-1)^{\operatorname{spin}(T)}$. Our example has spin 5 and sign -1. Define $\chi_\alpha^{\lambda/\nu} = \sum_T \operatorname{sgn}(T)$, where we sum over all rim hook tableaux T of shape λ/ν and type α. It can be shown that reordering the parts of α does not change $\chi_\alpha^{\lambda/\nu}$. Furthermore, we have the following formula for skew Schur polynomials:

$$s_{\lambda/\nu,N} = \sum_{\mu \in \operatorname{Par}(n)} z_\mu^{-1} \chi_\mu^{\lambda/\nu} p_{\mu,N} \quad \big(N \geq n = |\lambda/\nu|\big). \tag{5}$$

Keeping in mind the isomorphism $\Lambda_N^n \cong \Lambda^n$ (for $N \geq n$), we now define abstract *skew Schur functions* by setting

$$s_{\lambda/\nu} = \sum_{\mu \in \operatorname{Par}(n)} z_\mu^{-1} \chi_\mu^{\lambda/\nu} p_\mu \in \Lambda. \tag{6}$$

As special cases of this definition, we obtain the *complete symmetric functions* $h_n = s_{(n)/0}$ and the *elementary symmetric functions* $e_n = s_{(1^n)/0}$. Furthermore, we set $h_0 = e_0 = 1$, $h_\mu = \prod_{i\geq 1} h_{\mu_i}$, and $e_\mu = \prod_{i \geq 1} e_{\mu_i}$ (these are also special cases of skew Schur functions). It can be shown that, for $n \geq 0$, $\{e_\mu : \mu \in \operatorname{Par}(n)\}$ and $\{h_\mu : \mu \in \operatorname{Par}(n)\}$ are both bases for the K-vector space Λ^n. This is equivalent to the fact that $\{e_n : n \geq 1\}$ and $\{h_n : n \geq 1\}$ are algebraically independent over K. Thus, we can view the ring Λ as a polynomial ring in three different ways:

$$\Lambda = K[p_1, p_2, \ldots] \cong K[e_1, e_2, \ldots] \cong K[h_1, h_2, \ldots].$$

Example 4 Let us use (6) to show that $\omega(s_{\lambda/\nu}) = s_{\lambda'/\nu'}$, where the prime denotes conjugation. First, if T is a rim hook tableau of shape λ/ν, then the conjugate T' of T is a rim hook tableau of shape λ'/ν'. For example, if T is the rim hook tableau pictured above, then T' is pictured below.

1	5	5	5	
1	2	2	3	
1	1	2	3	
		2	2	
			2	6

Moreover, if R is a ribbon, then one easily checks that $\mathrm{spin}(R') = |R| - 1 - \mathrm{spin}(R)$ and $\mathrm{sgn}(R') = (-1)^{|R|-1}\,\mathrm{sgn}(R)$. Hence if T is a rim hook tableau of shape λ/ν and type $\mu \in \mathrm{Par}$, then T' is a rim hook tableau of shape λ'/ν' and type μ such that $\mathrm{sgn}(T') = (-1)^{|\lambda/\nu|-\ell(\mu)}\,\mathrm{sgn}(T)$, where $\ell(\mu)$ is the length of μ. Thus $\chi_\mu^{\lambda'/\nu'} = (-1)^{|\lambda/\nu|-\ell(\mu)}\chi_\mu^{\lambda/\nu}$ and

$$\begin{aligned}
\omega(s_{\lambda/\nu}) &= \sum_{\mu\in\mathrm{Par}(n)} z_\mu^{-1}\chi_\mu^{\lambda/\nu}\omega(p_\mu)\\
&= \sum_{\mu\in\mathrm{Par}(n)} z_\mu^{-1}\chi_\mu^{\lambda/\nu}(-1)^{|\lambda/\nu|-\ell(\mu)}p_\mu\\
&= \sum_{\mu\in\mathrm{Par}(n)} z_\mu^{-1}\chi_\mu^{\lambda'/\nu'}p_\mu\\
&= s_{\lambda'/\nu'}.
\end{aligned}$$

By the negation rule, we therefore have

$$s_{\lambda/\nu}[-A] = (-1)^{|\lambda/\nu|}s_{\lambda'/\nu'}[A].$$

In particular, for $\mu \in \mathrm{Par}(n)$, $e_\mu[-A] = (-1)^n h_\mu[A]$ and $h_\mu[-A] = (-1)^n e_\mu[A]$.

3.2 Plethystic addition formula

The "plethystic addition formula" is usually written

$$s_{\lambda/\nu}[A+B] = \sum_{\mu:\nu\subseteq\mu\subseteq\lambda} s_{\mu/\nu}[A]s_{\lambda/\mu}[B].$$

To emphasize the role played by homomorphisms and the UMP, we will rewrite this formula in the following somewhat more general form. Suppose D and E are any K-algebra homomorphisms from Λ into a K-algebra S. We write $D +_P E$ to denote the unique K-algebra homomorphism from Λ to S that sends p_k to $D(p_k) + E(p_k)$ for all $k \geq 1$.

Theorem 8 *For any skew shape λ/ν of size n, we have*

$$(D +_P E)(s_{\lambda/\nu}) = \sum_{\substack{\mu\in\mathrm{Par}\\ \nu\subseteq\mu\subseteq\lambda}} D(s_{\mu/\nu})E(s_{\lambda/\mu}). \tag{7}$$

We will now give a new computational proof of this theorem based on the combinatorial formula (6). A different, more abstract proof starting from (4) is given in Sect. 3.4. Expanding $s_{\lambda/\nu}$ in terms of power-sum symmetric functions and then using the definition of the homomorphism $D +_P E$, we have

$$(D +_P E)(s_{\lambda/\nu}) = \sum_{\gamma\in\mathrm{Par}(n)} \frac{\chi_\gamma^{\lambda/\nu}}{z_\gamma} \prod_{i=1}^{\ell(\gamma)} \big(D(p_{\gamma_i}) + E(p_{\gamma_i})\big). \tag{8}$$

We proceed to prove some lemmas analyzing different components of this formula. First some definitions: if α and β are compositions (or partitions) with $\ell(\alpha) = s$ and $\ell(\beta) = t$, the *concatenation* $\alpha|\beta$ is the composition $(\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t)$. For any composition α, let α^+ be the partition obtained by sorting the parts of α into decreasing order.

Lemma 1 *For $\gamma \in \mathrm{Par}$,*

$$\frac{1}{z_\gamma} \prod_{i=1}^{\ell(\gamma)} \big(D(p_{\gamma_i}) + E(p_{\gamma_i})\big) = \sum_{\substack{\alpha,\beta\in\mathrm{Par}\\ (\alpha|\beta)^+=\gamma}} \frac{D(p_\alpha)}{z_\alpha} \cdot \frac{E(p_\beta)}{z_\beta}. \tag{9}$$

Proof Let γ have c_j parts equal to j, for $1 \le j \le n$. The left side of (9) can then be written

$$\frac{1}{1^{c_1}2^{c_2}\cdots n^{c_n} c_1!c_2!\cdots c_n!} \prod_{j=1}^{n} \big(D(p_j) + E(p_j)\big)^{c_j}. \tag{10}$$

Expanding each factor $(D(p_j) + E(p_j))^{c_j}$ by the binomial theorem and rearranging terms, this becomes

$$\begin{aligned}
&\prod_{j=1}^{n} \sum_{a_j=0}^{c_j} \frac{D(p_j)^{a_j}}{a_j!j^{a_j}} \cdot \frac{E(p_j)^{c_j-a_j}}{(c_j-a_j)!j^{c_j-a_j}} \\
&\quad = \sum_{a_1=0}^{c_1} \cdots \sum_{a_n=0}^{c_n} \prod_{j=1}^{n} \frac{D(p_j)^{a_j}}{a_j!j^{a_j}} \prod_{j=1}^{n} \frac{E(p_j)^{c_j-a_j}}{(c_j-a_j)!j^{c_j-a_j}}.
\end{aligned} \tag{11}$$

Introduce new summation variables $\alpha, \beta \in \mathrm{Par}$ by letting α have a_j parts equal to j and β have $b_j = c_j - a_j$ parts equal to j for all j. The multiple sum over $a_1, \ldots, a_n$ becomes a sum over all partitions α and β such that $(\alpha|\beta)^+ = \gamma$. With this change of variables, the right side of (11) becomes the right side of (9), as desired. □

Lemma 2 *Suppose α is a composition of k, β is a composition of m, $\gamma = \alpha|\beta$, and λ/ν is a skew shape of size $k+m$. Then*

$$\chi_\gamma^{\lambda/\nu} = \sum_{\substack{\mu\in\mathrm{Par}\\ \nu\subseteq\mu\subseteq\lambda, |\mu/\nu|=k}} \chi_\alpha^{\mu/\nu}\chi_\beta^{\lambda/\mu}. \tag{12}$$

Proof A typical signed object counted by the left side looks like

$$T = \left(\nu^0 \subseteq \nu^1 \subseteq \cdots \subseteq \nu^s \subseteq \cdots \subseteq \nu^{s+t}\right)$$

where $\nu^0 = \nu$, $\nu^{s+t} = \lambda$, and ν^i/ν^{i-1} is a γ_i-ribbon. Map T to the triple (μ, T^1, T^2), where $\mu = \nu^s$, $T^1 = (\nu^0 \subseteq \cdots \subseteq \nu^s)$, and $T^2 = (\nu^s \subseteq \cdots \subseteq \nu^{s+t})$. This gives a bijection onto the set of signed objects enumerated by the right side of (12). Signs are preserved, since $\mathrm{spin}(T) = \mathrm{spin}(T^1) + \mathrm{spin}(T^2)$, so the lemma follows.

Since sorting the parts of γ does not change $\chi_\gamma^{\lambda/\nu}$, we conclude that

$$\chi_{(\alpha|\beta)^+}^{\lambda/\nu} = \sum_{\substack{\mu\in\mathrm{Par}\\ \nu\subseteq\mu\subseteq\lambda, |\mu/\nu|=|\alpha|}} \chi_\alpha^{\mu/\nu}\chi_\beta^{\lambda/\mu}. \tag{13}$$

Now we continue the proof of the main theorem. From (8) and (9), we get

$$(D +_P E)(s_{\lambda/\nu}) = \sum_{\gamma\in\mathrm{Par}(n)} \chi_\gamma^{\lambda/\nu} \sum_{\substack{\alpha,\beta\in\mathrm{Par}\\ (\alpha|\beta)^+=\gamma}} \frac{D(p_\alpha)}{z_\alpha}\cdot\frac{E(p_\beta)}{z_\beta}. \tag{14}$$

Moving $\chi_\gamma^{\lambda/\nu}$ inside the sum and using (13), this becomes

$$\sum_{\gamma\in\mathrm{Par}(n)} \sum_{\substack{\alpha,\beta\in\mathrm{Par}\\ (\alpha|\beta)^+=\gamma}} \sum_{\substack{\mu\in\mathrm{Par}\\ \nu\subseteq\mu\subseteq\lambda\\ |\mu/\nu|=|\alpha|}} \chi_\alpha^{\mu/\nu}\chi_\beta^{\lambda/\mu}\frac{D(p_\alpha)}{z_\alpha}\cdot\frac{E(p_\beta)}{z_\beta}. \tag{15}$$

Changing the order of summation (which allows us to eliminate the summation variable γ altogether) and regrouping, we obtain

$$\begin{aligned}
&\sum_{\substack{\mu\in\mathrm{Par}\\ \nu\subseteq\mu\subseteq\lambda}} \sum_{\alpha\in\mathrm{Par}(|\mu/\nu|)} \sum_{\beta\in\mathrm{Par}(|\lambda/\mu|)} \chi_\alpha^{\mu/\nu}\chi_\beta^{\lambda/\mu}\frac{D(p_\alpha)}{z_\alpha}\cdot\frac{E(p_\beta)}{z_\beta}\\
&\quad= \sum_{\substack{\mu\in\mathrm{Par}\\ \nu\subseteq\mu\subseteq\lambda}} \left(\sum_{\alpha\in\mathrm{Par}(|\mu/\nu|)} \frac{\chi_\alpha^{\mu/\nu}D(p_\alpha)}{z_\alpha}\right)\left(\sum_{\beta\in\mathrm{Par}(|\lambda/\mu|)} \frac{\chi_\beta^{\lambda/\mu}E(p_\beta)}{z_\beta}\right).
\end{aligned} \tag{16}$$

We now recognize the definitions of $D(s_{\mu/\nu})$ and $E(s_{\lambda/\mu})$ appearing in this formula. Thus we finally get the desired result

$$(D +_P E)(s_{\lambda/\nu}) = \sum_{\substack{\mu\in\mathrm{Par}\\ \nu\subseteq\mu\subseteq\lambda}} D(s_{\mu/\nu})E(s_{\lambda/\mu}).$$

□

3.3 Consequences of the addition formula

Recalling that $h_m = s_{(m)/0} = s_{(m+k)/(k)}$ and $e_m = s_{(1^m)/0} = s_{(1^{m+k})/(1^k)}$ for all m and k, we deduce the following formulas.

$$(D +_P E)(h_n) = \sum_{k=0}^{n} D(h_k)E(h_{n-k}), \qquad (D +_P E)(e_n) = \sum_{k=0}^{n} D(e_k)E(e_{n-k}).$$

In plethystic notation, our formulas read:

$$s_{\lambda/\nu}[A+B] = \sum_{\mu:\nu\subseteq\mu\subseteq\lambda} s_{\mu/\nu}[A]s_{\lambda/\mu}[B],$$

$$h_n[A+B] = \sum_{k=0}^{n} h_k[A]h_{n-k}[B],$$

$$e_n[A+B] = \sum_{k=0}^{n} e_k[A]e_{n-k}[B].$$

Combining the addition and negation formulas, we also obtain the *subtraction formulas*:

$$s_{\lambda/\nu}[A-B] = \sum_{\mu:\nu\subseteq\mu\subseteq\lambda} (-1)^{|\lambda/\mu|} s_{\mu/\nu}[A]s_{\lambda'/\mu'}[B],$$

$$h_n[A-B] = \sum_{k=0}^{n} (-1)^{n-k} h_k[A]e_{n-k}[B],$$

$$e_n[A-B] = \sum_{k=0}^{n} (-1)^{n-k} e_k[A]h_{n-k}[B].$$

Example 5 Using our rules, we can now interpret more plethystic expressions as encoding some rather unusual substitutions of variables. For instance,

$$e_4[2q+z-3t-w] = \sum_{k=0}^{4} (-1)^{4-k} e_k(q,q,z)h_{4-k}(t,t,t,w);$$

here we are using the subtraction rule with $A = 2q+z = q+q+z$ and $B = 3t+w = t+t+t+w$. Similarly,

$$h_4[x_1+x_2-y_1-y_2] = \sum_{k=0}^{4} (-1)^{4-k} h_k(x_1,x_2)e_{4-k}(y_1,y_2).$$

Example 6 Let us compute $s_\nu[1-u]$, for $\nu \in \mathrm{Par}$. First, assume $\nu = (a, 1^{n-a})$ is a hook partition. The subtraction formula and Theorem 7 give

$$s_\nu[1-u] = \sum_{\mu \subseteq \nu} (-1)^{n-|\mu|} \big(s_\mu(x_1)\big|_{x_1 \to 1}\big) \cdot \big(s_{\nu'/\mu'}(x_1)\big|_{x_1 \to u}\big).$$

By considering semistandard tableaux, one easily sees that $s_{\rho/\xi}(x_1)$ is zero unless ρ/ξ is a horizontal strip, in which case $s_{\rho/\xi}(x_1) = x_1^{|\rho/\xi|}$. Therefore, in the preceding formula, we only get a nonzero summand if both μ and ν'/μ' are horizontal strips. This occurs iff $\mu = (a)$ or $\mu = (a-1)$. Making the indicated substitutions for x_1, we conclude that

$$s_{(a,1^{n-a})}[1-u] = (-1)^{n-a} u^{n-a} + (-1)^{n-a+1} u^{n-a+1} = (-u)^{n-a}(1-u).$$

If $\nu \in \mathrm{Par}$ is not a hook shape, one easily sees that for all $\mu \subseteq \nu$, one of μ or ν'/μ' is not a horizontal strip. Thus, $s_\nu[1-u] = 0$ for all such ν. The calculations in this example will be generalized in Sect. 4 below.

As another corollary of our addition formula, note that the comultiplication map Δ has the form $D +_P E$ where $D, E : \Lambda \to \Lambda \otimes_K \Lambda$ are given by $D(p_k) = p_k \otimes 1$ and $E(p_k) = 1 \otimes p_k$ for all k. Therefore,

$$\Delta(s_{\lambda/\nu}) = \sum_{\mu : \nu \subseteq \mu \subseteq \lambda} s_{\mu/\nu} \otimes s_{\lambda/\mu}. \tag{17}$$

In fact, we can use the comultiplication map Δ to give an alternate proof of Theorem 8.

3.4 Alternate proof of Theorem 8

The following proof is essentially a translation into the language of Hopf algebras of the classical λ-ring approach to the plethystic addition formula.

Step 1. If λ/ν has size n and $M, N \geq n$, then

$$s_{\lambda/\nu}(x_1, \ldots, x_{M+N}) = \sum_{\mu : \nu \subseteq \mu \subseteq \lambda} s_{\mu/\nu}(x_1, \ldots, x_M) s_{\lambda/\mu}(x_{M+1}, \ldots, x_{M+N}).$$

This identity is combinatorially evident, since the occurrences of $1, \ldots, M$ in a tableau $T \in \mathrm{SSYT}_{M+N}(\lambda/\nu)$ form a tableau of some shape μ/ν, whereas the remaining entries in T must then constitute a tableau of shape λ/μ.

Step 2. We prove the comultiplication formula (17). Consider the diagram of K-algebra homomorphisms

$$\begin{array}{ccc} \Lambda & \xrightarrow{\ \Delta\ } & \Lambda \otimes_K \Lambda \\ \Big\downarrow{\scriptstyle \mathrm{ev}_{M+N}} & & \Big\downarrow{\scriptstyle \mathrm{ev}_M \otimes \mathrm{ev}'_N} \\ R_{M+N} & \xleftarrow{\ \pi\ } & R_M \otimes_K R'_N \end{array}$$

where $R_M = K[x_1, \ldots, x_M]$, ev_M sends every p_k to $p_{k,M} \in R_M$, R_{M+N} and ev_{M+N} are defined analogously, $R'_N = K[x_{M+1}, \ldots, x_{M+N}]$, ev'_N sends p_k to $\sum_{M<i\le M+N} x_i^k$, and π is the canonical isomorphism sending $f \otimes g$ to fg. By checking on the algebra generators p_k of Λ, we see that the diagram commutes. Step 1 says that

$$\mathrm{ev}_{M+N}(s_{\lambda/\nu}) = \pi\left(\mathrm{ev}_M \otimes \mathrm{ev}'_N\left(\sum_{\mu:\nu\subseteq\mu\subseteq\lambda} s_{\mu/\nu} \otimes s_{\lambda/\mu}\right)\right).$$

On the other hand, commutativity of the diagram means that

$$\mathrm{ev}_{M+N}(s_{\lambda/\nu}) = \pi\big(\mathrm{ev}_M \otimes \mathrm{ev}'_N\big(\Delta(s_{\lambda/\nu})\big)\big).$$

Comparing these equations and noting that π is an isomorphism, we get

$$\mathrm{ev}_M \otimes \mathrm{ev}'_N\left(\sum_{\mu:\nu\subseteq\mu\subseteq\lambda} s_{\mu/\nu} \otimes s_{\lambda/\mu}\right) = \mathrm{ev}_M \otimes \mathrm{ev}'_N\big(\Delta(s_{\lambda/\nu})\big).$$

The desired formula now follows since the restriction of $\mathrm{ev}_M \otimes \mathrm{ev}'_N$ to the graded component of $\Lambda \otimes_K \Lambda$ of degree $n = |\lambda/\nu|$ is a vector space isomorphism for $M, N \ge n$.

Step 3. With notation as in Theorem 8, consider the diagram of K-algebra homomorphisms

$$\begin{array}{ccc} \Lambda & \xrightarrow{\Delta} & \Lambda \otimes_K \Lambda \\ {\scriptstyle D+_P E}\downarrow & & \downarrow{\scriptstyle D\otimes E} \\ S & \xleftarrow{m_S} & S \otimes_K S \end{array}$$

where m_S is the map sending $f \otimes g$ to fg (this is a K-algebra homomorphism because S is commutative). This diagram commutes, as we see by checking on power-sums. It follows that

$$\begin{aligned} (D +_P E)(s_{\lambda/\nu}) &= m_S \circ (D \otimes E) \circ \Delta(s_{\lambda/\nu}) \\ &= m_S \circ (D \otimes E)\left(\sum_{\mu:\nu\subseteq\mu\subseteq\lambda} s_{\mu/\nu} \otimes s_{\lambda/\mu}\right) \quad \text{(by Step 2)} \\ &= m_S\left(\sum_{\mu:\nu\subseteq\mu\subseteq\lambda} D(s_{\mu/\nu}) \otimes E(s_{\lambda/\mu})\right) \\ &= \sum_{\mu:\nu\subseteq\mu\subseteq\lambda} D(s_{\mu/\nu}) E(s_{\lambda/\mu}). \end{aligned}$$

4 Combinatorial interpretations of plethystic expressions

This section presents a new general technique for finding interpretations of plethystic expressions as sums of signed, weighted combinatorial objects. This technique extends and systematizes a special case used in [23, 24] to provide combinatorial formulas for certain plethystic transformations of Macdonald polynomials. The key idea involves extending the plethystic calculus to apply to quasisymmetric functions, then using standardization bijections to find quasisymmetric function expansions for the symmetric functions under consideration. We begin by reviewing the necessary facts concerning quasisymmetric functions, standard tableaux, and standardization in Sects. 4.1, 4.2, and 4.3.

4.1 Quasisymmetric functions

For each $n \geq 1$, the space Q^n of *quasisymmetric functions of degree n over K* is defined to be a K-vector space of dimension 2^{n-1} with basis consisting of the symbols $L_{n,S}$ as S ranges over all subsets of $\{1, 2, \dots, n-1\}$. $L_{n,S}$ is called a *fundamental quasisymmetric function*. By the universal mapping property for the basis of a vector space, any function f mapping this basis into a K-vector space W uniquely extends by linearity to a K-linear transformation $T_f : Q^n \to W$. We remark that the space $Q = \bigoplus_n Q^n$ can be made into a graded ring and a Hopf algebra, but we will only need the vector space structure for our purposes here.

We can use the universal mapping property to define *fundamental quasisymmetric polynomials in N variables*. For each $N \geq n$, define an injective linear map $\mathrm{ev}_{N,n}$ from Q^n to $K[x_1, \dots, x_N]$ by mapping $L_{n,S}$ to

$$L_{n,S}(x_1, \dots, x_N) = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq N \\ i_k = i_{k+1} \Rightarrow k \notin S}} x_{i_1} x_{i_2} \cdots x_{i_n} \in K[x_1, \dots, x_N]$$

and extending by linearity (cf. [20, 38, 39, Chap. 7]). These polynomials "interpolate" between the homogeneous and elementary symmetric polynomials, in the sense that

$$L_{n,\emptyset}(x_1, \dots, x_N) = h_n(x_1, \dots, x_N) \quad \text{and}$$
$$L_{n,\{1,2,\dots,n-1\}}(x_1, \dots, x_N) = e_n(x_1, \dots, x_N).$$

The image $\mathrm{ev}_{N,n}(Q^n)$ is denoted Q^n_N and consists of the quasisymmetric polynomials in N variables that are homogeneous of degree n.

It will be useful to express the formula for $L_{n,S}(x_1, \dots, x_N)$ as an explicit sum of weighted combinatorial objects. Let $W(n, S, N)$ denote the set of all weakly increasing sequences $w = w_1 w_2 \cdots w_n$ with each $w_k \in \{1, 2, \dots, N\}$, such that $w_k = w_{k+1}$ implies $k \notin S$. Write $\mathrm{wt}(w) = \prod_{i=1}^{n} x_{w(i)}$. Then

$$L_{n,S}(x_1, \dots, x_N) = \sum_{w \in W(n,S,N)} \mathrm{wt}(w). \tag{18}$$

4.2 Standard tableaux, descents, and reading words

A tableau T with n cells is called *standard* iff $x^T = x_1x_2\cdots x_n$. Let $\mathrm{SYT}(\lambda/\nu)$ denote the set of standard tableaux of shape λ/ν. If T is a standard tableau with n cells, let $\mathrm{Des}(T)$ be the set of all labels $k < n$ such that $k+1$ appears in a higher row than k in T. For example, the following picture illustrates a standard tableau $T \in \mathrm{SYT}((4,3,3,2))$ with $\mathrm{Des}(T) = \{2,5,7,9\}$.

$$\begin{array}{|c|c|c|c|}
\hline 10 & 11 \\
\cline{1-3} 6 & 8 & 12 \\
\hline 3 & 4 & 7 \\
\hline 1 & 2 & 5 & 9 \\
\hline
\end{array}$$

The following equivalent description of $\mathrm{Des}(T)$ will also be needed. Given a skew shape λ/ν with n cells, totally order the cells of λ/ν by scanning the diagram row by row from top to bottom, reading each row from left to right. Call this the *reading order* of the cells of λ/ν. Given $T \in \mathrm{SYT}(\lambda/\nu)$, the *reading word* $\mathrm{rw}(T)$ is the list of entries of T obtained by traversing the cells in the reading order. For example, the standard tableau illustrated above has

$$\mathrm{rw}(T) = 10, 11, 6, 8, 12, 3, 4, 7, 1, 2, 5, 9.$$

One verifies immediately that $\mathrm{Des}(T)$ is the set of all $k < n$ such that $k+1$ appears earlier than k in $\mathrm{rw}(T)$. Equivalently, $\mathrm{Des}(T)$ is the descent set of the inverse of the permutation $\mathrm{rw}(T)$.

4.3 Standardization

There is a canonical method called *standardization* for converting a semistandard tableau to a standard tableau of the same shape. Given a semistandard tableau T of shape λ/ν, where $|\lambda/\nu| = n$, we produce a standard tableau $U = \mathrm{std}(T)$ by renumbering the cells of T with the integers $1, 2, \ldots, n$ (in this order) according to the following rules. First, smaller entries in T are relabeled before larger entries. Second, entries in T equal to a given integer i must form a horizontal strip; the entries in this strip are relabeled from left to right. The second rule may be equivalently stated: if two cells in T have the same label, then the cell that occurs earlier in the reading order is relabeled first. For example, the standardization of

$$T = \begin{array}{ccccc}
5 & & & & \\
2 & 4 & 6 & 7 & \\
 & 3 & 5 & 5 & \\
 & & 3 & 3 & 5 \\
 & & 2 & 2 & 3
\end{array} \quad \text{is} \quad U = \mathrm{std}(T) = \begin{array}{ccccc}
9 & & & & \\
1 & 8 & 13 & 14 & \\
 & 4 & 10 & 11 & \\
 & & 5 & 6 & 12 \\
 & & 2 & 3 & 7
\end{array}.$$

This simple combinatorial process has important ramifications for relating symmetric functions to quasisymmetric functions. Specifically, we have the following well-known result for expanding skew Schur polynomials in terms of fundamental quasisymmetric polynomials [38, 39, Chap. 7].

Theorem 9 *Let λ/ν be a skew shape with n cells. For all $N \geq n$,*

$$s_{\lambda/\nu}(x_1,\dots,x_N) = \sum_{U\in \mathrm{SYT}(\lambda/\nu)} L_{n,\mathrm{Des}(U)}(x_1,\dots,x_N).$$

Proof Recalling the combinatorial formulas for each side of the desired equation, we must show that

$$\sum_{T\in \mathrm{SSYT}_N(\lambda/\nu)} \mathrm{wt}(T) = \sum_{U\in \mathrm{SYT}(\lambda/\nu)} \sum_{w\in W(n,\mathrm{Des}(U),N)} \mathrm{wt}(w).$$

This equality is a consequence of the following bijection. Map each semistandard tableau $T \in \mathrm{SSYT}_N(\lambda/\nu)$ to the pair (U, w) where $U = \mathrm{std}(T)$ and $w = w_1w_2\cdots w_n$ is the multiset of labels appearing in T, sorted into weakly increasing order. For instance, the semistandard tableau T in our previous example maps to (U, w), where $U = \mathrm{std}(T)$ is displayed above and

$$w = 2, 2, 2, 3, 3, 3, 3, 4, 5, 5, 5, 5, 6, 7.$$

It is clear that $\mathrm{wt}(w) = \mathrm{wt}(T)$. To check that w does lie in $W(n, \mathrm{Des}(U), N)$, suppose $w_i = w_{i+1} = j$. Then the labels i and $i + 1$ in U appear in two cells c and c' that were both labeled j in T. By definition of standardization, c must precede c' in the reading order, and therefore $i + 1$ appears to the right of i in $\mathrm{rw}(U)$. So $i \notin \mathrm{Des}(U)$, completing the verification that $w \in W(n, \mathrm{Des}(U), N)$. Finally, the map $T \mapsto (U, w)$ is a bijection, since we can recover T from U and w by replacing each entry i in the standard tableau U by w_i, for $1 \leq i \leq n$. One checks (as above) that the condition $w \in W(n, \mathrm{Des}(U), N)$ ensures that the tableau T built from any pair (U, w) must be semistandard. □

Note that $\Lambda^n \cong \Lambda^n_N \subseteq Q^n_N \cong Q^n$ for $N \geq n$, where the isomorphisms arise from evaluation homomorphisms that are compatible for different choices of N. We can therefore view Λ^n as a subspace of Q^n in a canonical way, and in particular we have the following abstract version of the previous result:

$$s_{\lambda/\nu} = \sum_{U\in \mathrm{SYT}(\lambda/\nu)} L_{|\lambda/\nu|,\mathrm{Des}(U)}.$$

4.4 Combinatorial models for $s_{\lambda/\nu}[A]$

Next we describe combinatorial interpretations for plethystic transformations of skew Schur functions. These interpretations generalize the usual combinatorial formula (4) for $\mathrm{ev}_N(s_{\lambda/\nu}) = s_{\lambda/\nu}(x_1, x_2, \dots, x_N)$ as a sum of weighted semistandard tableaux of shape λ/ν.

To proceed, we need the concept of a *combinatorial alphabet*. This is a 4-tuple $\mathcal{A} = (A, \mathrm{sgn}, \mathrm{wt}, <)$, where A is a finite set of letters (often taken to be a subset of $\mathbb{Z}$); $\mathrm{sgn} : A \to \{+1, -1\}$ is a function specifying a sign for each letter in A; $\mathrm{wt} : A \to K[x_1, x_2, \dots, q, t, z, \dots]$ is a weight function assigning a *monic monomial* to each letter in A; and $<$ is a strict total ordering of A (which need not coincide with the usual ordering of $\mathbb{Z}$). Intuitively, combinatorial alphabets will aid us in interpreting plethystic substitutions of the form

$$f\Bigg[\sum_{a\in A}\operatorname{sgn}(a)\operatorname{wt}(a)\Bigg],$$

which we abbreviate to $f[\operatorname{wt}(\mathcal{A})]$. (Many of the results below extend to suitable infinite alphabets $\mathcal{A}$. We leave this extension to the interested reader.)

Define a *semistandard super-tableau of type $\mathcal{A}$ and shape λ/ν* to be a filling T of λ/ν by elements of A such that: (i) entries in T weakly increase (relative to the ordering $<$ of A) reading across rows and up columns; (ii) for each positive letter $i \in A$, the occurrences of i in T form a horizontal strip; (iii) for each negative letter $j \in A$, the occurrences of j in T form a vertical strip. More formally, writing $A = \{a_1 < a_2 < \cdots < a_p\}$, we can identify T with a sequence of partitions $(\mu^0 \subseteq \mu^1 \subseteq \cdots \subseteq \mu^p)$ such that $\mu^0 = \nu$, $\mu^p = \lambda$, μ^i/μ^{i-1} is a horizontal strip whenever $\operatorname{sgn}(a_i) = +1$, and μ^i/μ^{i-1} is a vertical strip whenever $\operatorname{sgn}(a_i) = -1$; here μ^i/μ^{i-1} consists of the cells containing a_i in T. Let $\operatorname{SSYT}_{\mathcal{A}}(\lambda/\nu)$ denote the set of all such super-tableaux. The signed weight of a super-tableau T is the product of the signs and weights of all letters appearing in T. Writing $T = (\mu^0 \subseteq \cdots \subseteq \mu^p)$ as above, we have $\operatorname{wt}(T) = \prod_{i=1}^{p}(\operatorname{sgn}(a_i)\operatorname{wt}(a_i))^{|\mu^i/\mu^{i-1}|}$.

Example 7 Suppose we wish to evaluate a plethystic expression of the form $s_{\lambda/\nu}[(1-q)(1-t)]$. Define a combinatorial alphabet $\mathcal{A}$ by setting $A = \{a < b < c < d\}$, $\operatorname{sgn}(a) = \operatorname{sgn}(d) = -1$, $\operatorname{sgn}(b) = \operatorname{sgn}(c) = +1$, $\operatorname{wt}(a) = q$, $\operatorname{wt}(b) = 1$, $\operatorname{wt}(c) = qt$, and $\operatorname{wt}(d) = t$. The following picture gives an example of a semistandard super-tableaux $T \in \operatorname{SSYT}_{\mathcal{A}}((5,4,4,1)/(2,1))$.

a				
a	c	c	d	
	a	b	d	
		a	b	b

Formally, we can write

$$T = \big((2,1), (3,2,1,1), (5,3,1,1), (5,3,3,1), (5,4,4,1)\big).$$

The signed weight of T is $(-q)^4 1^3 (qt)^2 (-t)^2 = +q^6t^4$. It will follow from the next theorem that $s_{(5,4,4,1)/(2,1)}[(1-q)(1-t)]$ is the sum of the weights all such semistandard super-tableaux of shape $(5,4,4,1)/(2,1)$.

Theorem 10 *Let $\mathcal{A} = (A, \operatorname{sgn}, \operatorname{wt}, <)$ be a combinatorial alphabet. For all partitions $\nu \subseteq \lambda$,*

$$s_{\lambda/\nu}[\operatorname{wt}(\mathcal{A})] = \sum_{T\in\operatorname{SSYT}_{\mathcal{A}}(\lambda/\nu)} \operatorname{wt}(T). \tag{19}$$

Proof As we will see, this result follows from the addition and negation formulas for skew Schur functions (Sect. 3). For any plethystic alphabets $A_1, \dots, A_p$, iteration of the plethystic addition formula gives

$$s_{\lambda/\nu}[A_1 + \cdots + A_p] = \sum_{\nu=\mu^0 \subseteq \mu^1 \subseteq \cdots \subseteq \mu^p = \lambda} \prod_{i=1}^{p} s_{\mu^i/\mu^{i-1}}[A_i]. \tag{20}$$

Write $A = \{a_1 < a_2 < \cdots < a_p\}$, and take A_i to be the single monomial $\operatorname{sgn}(a_i)\operatorname{wt}(a_i)$. In the case $\operatorname{sgn}(a_i) = +1$, it follows that

$$s_{\mu^i/\mu^{i-1}}[A_i] = s_{\mu^i/\mu^{i-1}}(x_1)|_{x_1=\operatorname{wt}(a_i)},$$

which is $\operatorname{wt}(a_i)^{|\mu^i/\mu^{i-1}|}$ if μ^i/μ^{i-1} is a horizontal strip, and zero otherwise. In the case $\operatorname{sgn}(a_i) = -1$, we get

$$s_{\mu^i/\mu^{i-1}}[A_i] = (-1)^{|\mu^i/\mu^{i-1}|} s_{(\mu^i)'/(\mu^{i-1})'}(x_1)|_{x_1=\operatorname{wt}(a_i)},$$

which is $(-\operatorname{wt}(a_i))^{|\mu^i/\mu^{i-1}|}$ if μ^i/μ^{i-1} is a vertical strip, and zero otherwise. Using these observations in (20), we see that the nonzero summands are indexed by the semistandard super-tableaux $T \in \mathrm{SSYT}_{\mathcal{A}}(\lambda/\nu)$, and the value of the summand for T is precisely $\operatorname{wt}(T)$. □

An important remark is that the combinatorial expression appearing on the right side of (19) depends on the total ordering $<$ of the alphabet A, but the left side of (19) does not depend on this total ordering (since addition in Z is commutative). We therefore obtain several different combinatorial interpretations for $s_{\lambda/\nu}[\operatorname{wt}(\mathcal{A})] = s_{\lambda/\nu}[\sum_{a\in A} \operatorname{sgn}(a)\operatorname{wt}(a)]$ by varying the total ordering imposed on A. This fact proves to be quite useful in certain applications [23].

4.5 Plethystic calculus for quasisymmetric functions

We are now ready to define a version of plethysm that applies to quasisymmetric functions. Let $\mathcal{A} = (A, \operatorname{sgn}, \operatorname{wt}, <)$ be a combinatorial alphabet. Write $A^+ = \{a \in A : \operatorname{sgn}(a) = +1\}$ and $A^- = \{a \in A : \operatorname{sgn}(a) = -1\}$. For $n \geq 1$ and $S \subseteq \{1, 2, \dots, n-1\}$, define $W(n, S, \mathcal{A})$ to be the set of all words $w = w_1 w_2 \cdots w_n$ such that: $w_k \in A$ for all k; $w_1 \leq w_2 \leq \cdots \leq w_n$ (relative to the given ordering of A); for all $k < n$, $w_k = w_{k+1} \in A^+$ implies $k \notin S$; and for all $k < n$, $w_k = w_{k+1} \in A^-$ implies $k \in S$. For $w \in W(n, S, \mathcal{A})$, let $\operatorname{wt}(w) = \prod_{i=1}^n \operatorname{sgn}(w_i)\operatorname{wt}(w_i)$. Define

$$L_{n,S}^{\mathcal{A}} = \sum_{w \in W(n,S,\mathcal{A})} \operatorname{wt}(w). \tag{21}$$

(This definition is motivated by the analogy between (4) and (19) on the one hand, and (18) and (21) on the other hand.) Finally, we define a linear map $E^{\mathcal{A}}$ with domain Q^n by setting $E^{\mathcal{A}}(L_{n,S}) = L_{n,S}^{\mathcal{A}}$ and extending by linearity. Thus if $f = \sum_S c_S L_{n,S}$

is any quasisymmetric function of degree n, the plethystic transform of f relative to $\mathcal{A}$ is given by

$$E^{\mathcal{A}}(f) = \sum_S c_S L^{\mathcal{A}}_{n,S}.$$

Example 8 Let $A = \{a, b, c\}$, $\text{sgn}(a) = \text{sgn}(b) = +1$, $\text{sgn}(c) = -1$, $\text{wt}(a) = x$, $\text{wt}(b) = y$, $\text{wt}(c) = z$. Define two total orderings on A by letting $a <_1 b <_1 c$ and $c <_2 b <_2 a$. Let $\mathcal{A}_1 = (A, \text{sgn}, \text{wt}, <_1)$ and $\mathcal{A}_2 = (A, \text{sgn}, \text{wt}, <_2)$. On one hand, $W(3, \{1\}, \mathcal{A}_1) = \{abc, abb\}$ and

$$L^{\mathcal{A}_1}_{3,\{1\}} = -xyz + xy^2.$$

On the other hand, $W(3, \{1\}, \mathcal{A}_2) = \{baa, cba, ccb, cca, cbb, caa\}$ and

$$L^{\mathcal{A}_2}_{3,\{1\}} = x^2y - xyz + yz^2 + xz^2 - y^2z - x^2z.$$

This example shows that the value of the plethystic transformation of a quasisymmetric function *can* depend on the total ordering of the alphabet A.

The main result we want to prove is that this new plethysm operation on Q^n agrees with the previous notion of plethysm on the subspace Λ^n of symmetric functions. In particular, upon restriction to this subspace, the answers we get do *not* depend on the chosen total ordering of the alphabet. The next lemma provides the key link between the two kinds of plethysm.

Lemma 3 *Suppose* $\mathcal{A} = (A, \text{sgn}, \text{wt}, <)$ *is a combinatorial alphabet and* λ/ν *is a skew shape with n cells. Then*

$$E^{\mathcal{A}}(s_{\lambda/\nu}) = s_{\lambda/\nu}\big[\text{wt}(\mathcal{A})\big].$$

Proof Recall that $s_{\lambda/\nu} = \sum_{U \in \text{SYT}(\lambda/\nu)} L_{n,\text{Des}(U)}$. Invoking the definition of $E^{\mathcal{A}}$ and Theorem 10, we are reduced to proving the combinatorial identity

$$\sum_{U \in \text{SYT}(\lambda/\nu)} \sum_{w \in W(n,\text{Des}(U),\mathcal{A})} \text{wt}(w) = \sum_{T \in \text{SSYT}_{\mathcal{A}}(\lambda/\nu)} \text{wt}(T).$$

This follows from a standardization argument entirely analogous to the one in Sect. 4.3. Given $T \in \text{SSYT}_{\mathcal{A}}(\lambda/\nu)$, we relabel the entries of T using the integers $1, 2, \ldots, n$ via the following *standardization rules*.

– *Rule 1*: Smaller labels in T (relative to the ordering $<$ of $\mathcal{A}$) are relabeled before larger labels.
– *Rule 2*: If two cells in T contain the same *positive* letter, then the cell that occurs earlier in the reading order is relabeled first.
– *Rule 3*: If two cells in T contain the same *negative* letter, then the cell that occurs later in the reading order is relabeled first.

Using the definition of $\mathrm{SSYT}_{\mathcal{A}}(\lambda/\nu)$, one checks that applying these rules to T produces a uniquely determined standard tableau $U \in \mathrm{SYT}(\lambda/\nu)$. Define $w = w_1 \cdots w_n$ by letting $w_i \in A$ be the label in T that was replaced by the label i in U. Clearly, the passage from T to the pair (U, w) is weight-preserving. By Rule 1, we have $w_1 \le w_2 \le \cdots \le w_n$. By Rule 2, $w_k = w_{k+1} \in A^+$ implies $k \notin \mathrm{Des}(U)$. By Rule 3, $w_k = w_{k+1} \in A^-$ implies $k \in \mathrm{Des}(U)$. It follows that $w \in W(n, \mathrm{Des}(U), \mathcal{A})$, as required. Conversely, given any such pair (U, w), one recovers the tableau $T \in \mathrm{SSYT}_{\mathcal{A}}(\lambda/\nu)$ by replacing the integer i in U by the letter w_i for each i. □

Example 9 Continuing Example 7, the tableau

$$T = \begin{array}{ccccc} a & & & & \\ a & c & c & d & \\ & a & b & d & \\ & & a & b & b \end{array}$$

is mapped to the pair (U, w), where U is the standard tableau

$$\begin{array}{ccccc} 4 & & & & \\ 3 & 8 & 9 & 11 & \\ & 2 & 5 & 10 & \\ & & 1 & 6 & 7 \end{array}$$

and $w = aaaabbbccdd$. The reader should confirm that $\mathrm{Des}(U) = \{1, 2, 3, 7, 10\}$ and $w \in W(11, \mathrm{Des}(U), \mathcal{A})$.

Theorem 11 *Suppose $\mathcal{A} = (A, \mathrm{sgn}, \mathrm{wt}, <)$ is a combinatorial alphabet and $f \in \Lambda^n \subseteq Q^n$. Then*

$$f\big[\mathrm{wt}(\mathcal{A})\big] = E^{\mathcal{A}}(f).$$

In particular, the right side does not depend on the given ordering of A.

Proof By the lemma, the two linear maps $(f \mapsto f[\mathrm{wt}(\mathcal{A})] : f \in \Lambda^n)$ and $(f \mapsto E^{\mathcal{A}}(f) : f \in \Lambda^n)$ have the same effect on every Schur function. Since the Schur functions form a basis for Λ^n, the two maps must be equal. The last part of the theorem follows since the ordinary plethysm $f[\mathrm{wt}(\mathcal{A})]$ does not depend on the ordering of A. □

4.6 Application: plethystic evaluation of LLT polynomials

The result in Theorem 11 leads to the following strategy for finding combinatorial interpretations for plethystic expressions of the form $f[U]$, where $f \in \Lambda^n$:

(a) Define a combinatorial alphabet $\mathcal{A}$ such that $\mathrm{wt}(\mathcal{A}) = U$.
(b) Express f in terms of the fundamental quasisymmetric functions $L_{n,S}$.
(c) Use the expression in (b) to compute $E^{\mathcal{A}}(f)$.

Step (a) is usually easy—just write U as a sum of signed, monic monomials. In many situations, step (b) can be accomplished by a straightforward combinatorial standardization argument. In this case, an analogous "superized" version of the same standardization argument will often suffice to achieve step (c). We have already seen an example of this analogy in the case $f = s_{\lambda/\nu}$ (compare the proof of Theorem 9 in Sect. 4.3 with the proof of Lemma 3 in Sect. 4.5). One technical point: we must prove that f does lie in Λ^n (not merely Q^n), if this is not obvious from the definition of f. This point is important since quasisymmetric plethysm depends on the total ordering of the alphabet, whereas symmetric plethysm does not.

The strategy just outlined was used in [23] to give combinatorial interpretations of plethystically transformed Macdonald polynomials. We now give another illustration of the same strategy by deriving combinatorial formulas for plethystically transformed Lascoux–Leclerc–Thibon (LLT) polynomials [30].

First we give a combinatorial definition of the LLT polynomials. Let

$$\Gamma = \left(\lambda^1/\nu^1, \ldots, \lambda^s/\nu^s\right)$$

be an ordered list of skew shapes consisting of n total squares. Let $\mathrm{SSYT}_N(\Gamma)$ denote the set of all lists $\mathbf{T} = (T_1, \ldots, T_s)$ such that $T_k \in \mathrm{SSYT}_N(\lambda^k/\nu^k)$ for all k. We write $\mathrm{wt}(\mathbf{T}) = \prod_k \mathrm{wt}(T_k)$ to keep track of which labels appear in $\mathbf{T}$. Furthermore, we assign an additional weight $\mathrm{dinv}(\mathbf{T})$ to $\mathbf{T}$ as follows. If $c = (i, j)$ is a cell in λ^k/ν^k for any k, we say that c belongs to the *diagonal* $d(c) = j - i$. Suppose (c_1, c_2) is a pair of cells in $\mathbf{T}$ such that c_1 is labeled u and c_2 is labeled v. These cells constitute a *diagonal inversion* of $\mathbf{T}$ iff $c_1 \in \lambda^k/\nu^k$ and $c_2 \in \lambda^l/\nu^l$ for some $k < l$; and either $d(c_1) = d(c_2)$ and $u > v$, or else $d(c_1) = d(c_2) - 1$ and $v > u$. Let $\mathrm{dinv}(\mathbf{T})$ be the total number of diagonal inversions in $\mathbf{T}$. Finally, define

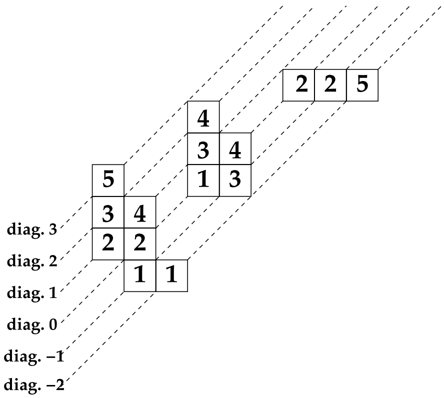
$$\mathrm{LLT}_\Gamma(x_1, \ldots, x_N) = \sum_{\mathbf{T} \in \mathrm{SSYT}_N(\Gamma)} q^{\mathrm{dinv}(\mathbf{T})}\, \mathrm{wt}(\mathbf{T}) \in \mathbb{Q}(q)[x_1, \ldots, x_N].$$

It is true but non-obvious [23] that $\mathrm{LLT}_\Gamma(x_1, \ldots, x_N)$ is a symmetric polynomial in the x_i's; we shall not prove this fact here. Taking $N \geq n$ and applying ev_N^{-1}, we obtain an abstract symmetric function $\mathrm{LLT}_\Gamma \in \Lambda^n$.

Example 10 Let $\Gamma = ((3, 2, 2, 1)/(1), (2, 2, 1), (3))$. Figure 1 shows an element $\mathbf{T} \in \mathrm{SSYT}_N(\Gamma)$, where $N \geq 5$. It is convenient to align the tableaux diagonally, as shown here, so that diagonal inversions are more readily computed. We have $\mathrm{dinv}(\mathbf{T}) = 10$ and $\mathrm{wt}(T) = x_1^3 x_2^4 x_3^3 x_4^3 x_5^2$.

Following our strategy, the next task is to use standardization to discover the expansion of LLT polynomials in terms of fundamental quasisymmetric functions. Given a tuple Γ with n cells total, let $\mathrm{SYT}(\Gamma)$ be the set of all $\mathbf{U} \in \mathrm{SSYT}_n(\Gamma)$ in which the labels $1, 2, \ldots, n$ appear once each; elements of $\mathrm{SYT}(\Gamma)$ are called *standard*. Define the *reading order* of the cells of Γ by traversing each diagonal in turn from southwest to northeast, working from higher diagonals to lower diagonals. Define the *reading word* $\mathrm{rw}(\mathbf{U})$ of a standard object to be the list of labels encountered when the cells of $\mathbf{U}$ are scanned in the reading order. Define $\mathrm{Des}(\mathbf{U})$ to be the set of all $k < n$ such that $k + 1$ appears before k in $\mathrm{rw}(\mathbf{U})$.

Fig. 1 An object **T** counted by an LLT polynomial



Just as before, we can use a *standardization bijection* $\mathbf{T} \mapsto (\mathbf{U}, w)$ to prove the identity

$$\sum_{\mathbf{T}\in \mathrm{SSYT}_N(\Gamma)} q^{\mathrm{dinv}(\mathbf{T})}\,\mathrm{wt}(\mathbf{T}) = \sum_{\mathbf{U}\in \mathrm{SYT}(\Gamma)} q^{\mathrm{dinv}(\mathbf{U})} \sum_{w\in W(n,\mathrm{Des}(\mathbf{U}),N)} \mathrm{wt}(w). \tag{22}$$

Given $\mathbf{T}$, relabel the cells of $\mathbf{T}$ with the integers $1, 2, \ldots, n$ such that cells with smaller labels in $\mathbf{T}$ get relabeled first and the set of cells with a given label in $\mathbf{T}$ are relabeled according to the reading order. This relabeling defines the standardization $\mathbf{U} = \mathrm{std}(\mathbf{T})$; as usual, we let w be the list of all labels in $\mathbf{T}$ written in weakly increasing order. It follows easily from the definitions of dinv and standardization that $w \in W(n, \mathrm{Des}(\mathbf{U}), N)$ and $\mathrm{dinv}(\mathbf{U}) = \mathrm{dinv}(\mathbf{T})$; furthermore, it is clear that $\mathrm{wt}(w) = \mathrm{wt}(\mathbf{T})$. So (22) holds. Recalling the relevant definitions and applying ev_N^{-1}, we obtain the desired abstract expansion

$$\mathrm{LLT}_\Gamma = \sum_{\mathbf{U}\in \mathrm{SYT}(\Gamma)} q^{\mathrm{dinv}(\mathbf{U})} L_{n,\mathrm{Des}(\mathbf{U})}. \tag{23}$$

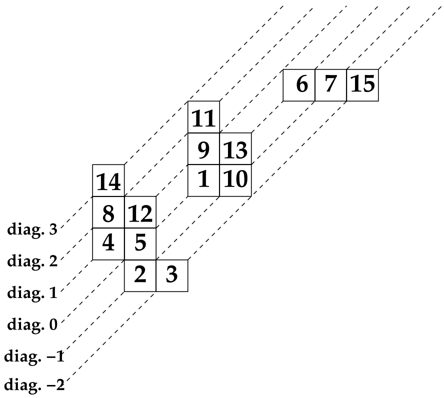
Example 11 The object $\mathbf{T}$ shown in Fig. 1 standardizes to give the standard object $\mathbf{U}$ shown in Fig. 2. In this case, $w = 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5$;

$$\mathrm{rw}(\mathbf{U}) = 14, 8, 11, 4, 12, 9, 5, 1, 13, 6, 2, 10, 7, 3, 15;$$

and $\mathrm{Des}(\mathbf{U}) = \{3, 7, 10, 13\}$. The reader should confirm that $\mathrm{dinv}(\mathbf{U}) = \mathrm{dinv}(\mathbf{T})$ and that $w \in W(15, \mathrm{Des}(\mathbf{U}), 5)$.

We can now compute plethysms of the form $\mathrm{LLT}_\Gamma[Z]$. Let $\mathcal{A} = (A, \mathrm{sgn}, \mathrm{wt}, <)$ be a combinatorial alphabet such that $\mathrm{wt}(\mathcal{A}) = Z$. Define $\mathrm{SSYT}_{\mathcal{A}}(\Gamma)$ to be the set of all tuples $\mathbf{T} = (T_1, \ldots, T_s)$ such that $T_k \in \mathrm{SSYT}_{\mathcal{A}}(\lambda^k/\nu^k)$ for all k. To standardize such an object, relabel the cells in $\mathbf{T}$ with the integers $1, 2, \ldots, n$ so that earlier letters of A (relative to $<$) are relabeled first, equal positive letters of A are relabeled in the reading order, equal negative letters of A are relabeled in reverse reading order. The

Fig. 2 Standardization of **T**



resulting object $\mathbf{U} = \text{std}(\mathbf{T})$ is easily seen to be standard. Furthermore, letting $w = w_1 w_2 \cdots w_n$ be the list of labels in **T** in weakly increasing order, the standardization rules show immediately that $w \in W(n, \text{Des}(\mathbf{U}), \mathcal{A})$. We conclude that

$$\sum_{\mathbf{T}\in \text{SSYT}_{\mathcal{A}}(\Gamma)} q^{\text{dinv}(\mathbf{T})} \text{wt}(\mathbf{T}) = \sum_{\mathbf{U}\in \text{SYT}(\Gamma)} q^{\text{dinv}(\mathbf{U})} \sum_{w\in W(n,\text{Des}(\mathbf{U}),\mathcal{A})} \text{wt}(w),$$

where (by definition) $\text{dinv}(\mathbf{T}) = \text{dinv}(\text{std}(\mathbf{T}))$ and $\text{wt}(\mathbf{T})$ is the product of the signs and weights of all entries of **T**. Comparing this expansion to (23), and recalling the formula for $L^{\mathcal{A}}_{n,\text{Des}(\mathbf{U})}$, we have proved:

$$\text{LLT}_{\Gamma}[Z] = \text{LLT}_{\Gamma}\big[\text{wt}(\mathcal{A})\big] = \sum_{\mathbf{T}\in \text{SSYT}_{\mathcal{A}}(\Gamma)} q^{\text{dinv}(\mathbf{T})} \text{wt}(\mathbf{T}).$$

5 Plethystic calculus and dual bases

Some of the most useful plethystic identities are *Cauchy formulas* for simplifying expressions of the form $h_n[AB]$ or $e_n[AB]$ [28, pp. 38, 43]. Before proving these, we review the non-plethystic versions of the Cauchy identities, in which dual bases of Λ play a key role.

5.1 Review of dual bases

We define the *Hall inner product* on the vector space Λ by setting $\langle p_\lambda, p_\mu\rangle = \delta_{\lambda,\mu} z_\mu$ and extending by bilinearity. Here, $\delta_{\lambda,\mu}$ is 1 for $\lambda = \mu$ and 0 otherwise. Relative to this inner product, the basis $\{p_\mu : \mu \in \text{Par}(n)\}$ of Λ^n is dual to the basis $\{p_\mu / z_\mu : \mu \in \text{Par}(n)\}$, whereas the basis $\{s_\mu : \mu \in \text{Par}(n)\}$ is self-dual (orthonormal). Furthermore, the h_μ's are dual to the *monomial basis* m_μ, and the e_μ's are dual to the *forgotten basis* f_μ. One easily sees that the involution ω is an isometry relative to this scalar product: $\langle \omega(f), \omega(g)\rangle = \langle f, g\rangle$ for all $f, g \in \Lambda$.

Suppose $B=\{B_\mu:\mu\in\mathrm{Par}(n)\}$ and $C=\{C_\mu:\mu\in\mathrm{Par}(n)\}$ are two bases of Λ^n. It can be shown that B and C are dual bases (relative to the Hall scalar product) iff the following Cauchy identity holds:

$$\sum_{\mu\in\mathrm{Par}(n)} B_\mu(x_1,\dots,x_M)C_\mu(y_1,\dots,y_N)=\prod_{i=1}^{M}\prod_{j=1}^{N}\frac{1}{1-x_iy_j}\bigg|_{\text{terms of degree 2n}}$$
$$(M,N\geq n).$$

We note that the right side of this identity can also be written

$$h_n(z_1,z_2,\dots,z_{MN})|_{z_1\to x_1y_1,\dots,z_{MN}\to x_My_N};$$

here we are using the UMP for the polynomial ring $K[z_1,\dots,z_{MN}]$. There is also a *dual Cauchy identity*, which states that B and C are dual bases iff

$$\begin{aligned}\sum_{\mu\in\mathrm{Par}(n)} &B_\mu(x_1,\dots,x_M)\big[\omega(C_\mu)\big](y_1,\dots,y_N)\\ &=\prod_{i=1}^{M}\prod_{j=1}^{N}(1+x_iy_j)\bigg|_{\text{terms of degree 2n}} \qquad (M,N\geq n)\\ &=e_n(z_1,\dots,z_{MN})|_{z_1\to x_1y_1,\dots,z_{MN}\to x_My_N}.\end{aligned}$$

For many choices of the bases B_μ and C_μ, these polynomial identities have nice combinatorial or algebraic proofs. For example, when $B_\mu=C_\mu=s_\mu$, the Cauchy identities follow from the RSK algorithm [37–39].

5.2 Plethystic Cauchy identities

As before, we find it convenient to phrase the "plethystic Cauchy identities" in a slightly more general form involving ring homomorphisms. Let S be a K-algebra and D, E two K-algebra homomorphisms of Λ into S. By the UMP, there is a unique K-algebra homomorphism $D*_P E$ of Λ into S sending p_k to $D(p_k)E(p_k)$ for all $k\geq 1$.

Theorem 12 *If* $B=\{B_\mu:\mu\in\mathrm{Par}(n)\}$ *and* $C=\{C_\mu:\mu\in\mathrm{Par}(n)\}$ *are dual bases for* Λ^n, *then*

$$(D*_P E)(h_n)=\sum_{\mu\in\mathrm{Par}(n)} D(B_\mu)E(C_\mu), \tag{24}$$

$$(D*_P E)(e_n)=\sum_{\mu\in\mathrm{Par}(n)} D(B_\mu)E\big(\omega(C_\mu)\big). \tag{25}$$

Proof We adapt the usual proof of the classical Cauchy identities [34, Sect. I.4] to incorporate the homomorphisms D and E. There exist unique scalars $b_{\mu,\nu},c_{\mu,\nu}\in K$

so that

$$B_\mu=\sum_{\nu\in\mathrm{Par}(n)} b_{\mu,\nu}p_\nu,\qquad C_\mu=\sum_{\nu\in\mathrm{Par}(n)} c_{\mu,\nu}(p_\nu/z_\nu).$$

Let $U=(b_{\mu,\nu})_{\mu,\nu\in\mathrm{Par}(n)}$ and $V=(c_{\mu,\nu})_{\mu,\nu\in\mathrm{Par}(n)}$ denote matrices obtained from these scalars by arranging the partitions of n in any fixed order. Since B and C are dual bases, a short calculation with inner products reveals that $UV^T=I$. This condition is equivalent to $V^TU=I$ and hence to $U^TV=I$. We therefore have

$$\sum_{\mu\in\mathrm{Par}(n)} b_{\mu,\nu}c_{\mu,\xi}=\delta_{\nu,\xi}\quad \big(\nu,\xi\in\mathrm{Par}(n)\big).$$

Now, using K-linearity of D and E, compute

$$\begin{aligned}\sum_{\mu\in\mathrm{Par}(n)} D(B_\mu)E(C_\mu)&=\sum_{\mu\in\mathrm{Par}(n)} D\Bigg(\sum_{\nu\in\mathrm{Par}(n)} b_{\mu,\nu}p_\nu\Bigg)E\Bigg(\sum_{\xi\in\mathrm{Par}(n)} c_{\mu,\xi}p_\xi/z_\xi\Bigg)\\&=\sum_{\nu\in\mathrm{Par}(n)}\sum_{\xi\in\mathrm{Par}(n)}\frac{D(p_\nu)E(p_\xi)}{z_\xi}\Bigg(\sum_{\mu\in\mathrm{Par}(n)} b_{\mu,\nu}c_{\mu,\xi}\Bigg)\\&=\sum_{\nu\in\mathrm{Par}(n)}\frac{D(p_\nu)E(p_\nu)}{z_\nu}=\sum_{\nu\in\mathrm{Par}(n)}\frac{(D*_P E)(p_\nu)}{z_\nu}.\end{aligned}$$

Since $h_n=\sum_\nu p_\nu/z_\nu$ (as follows from (6)), we see that the last expression is just $(D*_P E)(h_n)$.

To deduce (25) from (24), we first observe the identity

$$(D*_P E)\circ\omega=D*_P(E\circ\omega)=(D\circ\omega)*_P E. \tag{26}$$

This follows by the universal mapping property, since all three homomorphisms send p_k to $(-1)^{k-1}D(p_k)E(p_k)\in S$. Now compute

$$\begin{aligned}(D*_P E)(e_n)&=\big((D*_P E)\circ\omega\big)(h_n)=\big(D*_P(E\circ\omega)\big)(h_n)\\&=\sum_{\mu\in\mathrm{Par}(n)} D(B_\mu)E\big(\omega(C_\mu)\big),\end{aligned}$$

where the last step is an application of (24) to the homomorphisms D and $E\circ\omega$. □

Some useful particular cases of formulas (24) and (25), written in plethystic notation, are as follows:

$$\begin{aligned}h_n[AB]&=\sum_{\mu\in\mathrm{Par}(n)} p_\mu[A]p_\mu[B]/z_\mu=\sum_{\mu\in\mathrm{Par}(n)} h_\mu[A]m_\mu[B]=\sum_{\mu\in\mathrm{Par}(n)} m_\mu[A]h_\mu[B]\\&=\sum_{\mu\in\mathrm{Par}(n)} e_\mu[A]f_\mu[B]=\sum_{\mu\in\mathrm{Par}(n)} f_\mu[A]e_\mu[B]=\sum_{\mu\in\mathrm{Par}(n)} s_\mu[A]s_\mu[B],\end{aligned}$$

$$e_n[AB]=\sum_{\mu\in\mathrm{Par}(n)}\frac{(-1)^{n-\ell(\mu)}p_\mu[A]p_\mu[B]}{z_\mu}=\sum_{\mu\in\mathrm{Par}(n)} h_\mu[A]f_\mu[B]$$

$$= \sum_{\mu \in \mathrm{Par}(n)} m_\mu[A] e_\mu[B]$$

$$= \sum_{\mu \in \mathrm{Par}(n)} e_\mu[A] m_\mu[B] = \sum_{\mu \in \mathrm{Par}(n)} f_\mu[A] h_\mu[B] = \sum_{\mu \in \mathrm{Par}(n)} s_\mu[A] s_{\mu'}[B].$$

One can prove a converse to Theorem 12, in which the duality of the bases B and C can be deduced if (24) holds for sufficiently many homomorphisms D and E. We will not give a precise statement or proof of this converse, since it will not be needed in the sequel.

5.3 The Ω operator

In [16], Garsia et al. encode various symmetric function "kernels" via plethystic substitutions of the form $\Omega[A]$, where

$$\Omega = \prod_{i \geq 1} \frac{1}{1 - x_i} = \sum_{n=0}^{\infty} h_n = \exp\left(\sum_{k \geq 1} p_k / k\right).$$

Some care must be exercised here since, as remarked in [34, Chap. I, Sect. 2], Ω is *not an element of the ring* $\Lambda = \bigoplus_{n \geq 0} \Lambda^n$. Rather, Ω is an element of the larger ring $\hat{\Lambda}$, which is the F-module $\prod_{n \geq 0} \Lambda^n$ endowed with the natural multiplication.

To make sense of the symbol $\Omega[A]$, we need to assume that the associated homomorphism $\phi_A : \Lambda \to S$ takes its values in a topological ring S, e.g., a ring of formal power series. Then we define

$$\Omega[A] = \lim_{N \to \infty} \sum_{n=0}^{N} h_n[A] = \lim_{N \to \infty} \sum_{n=0}^{N} \phi_A(h_n) \in S,$$

provided that this limit exists in S. (See [7] for a discussion of limits in rings of formal power series.) For example, consider the alphabet $A = 0$. We have $\phi_A(h_0) = \phi_A(1) = 1$ and $\phi_A(h_n) = 0$ for all $n > 0$. Therefore, $\Omega[0] = 1$.

The most important property of Ω is that it converts sums to products:

$$\Omega[A + B] = \Omega[A]\Omega[B]$$

for all alphabets A and B such that both sides are defined. To prove this, use the addition formula to compute

$$\Omega[A + B] = \sum_{n \geq 0} h_n[A + B] = \sum_{n \geq 0} \sum_{k=0}^{n} h_k[A] h_{n-k}[B]$$

$$= \sum_{k \geq 0} \sum_{j \geq 0} h_k[A] h_j[B]$$

$$= \left(\sum_{k \geq 0} h_k[A]\right) \cdot \left(\sum_{j \geq 0} h_j[B]\right) = \Omega[A]\Omega[B].$$

Setting $B=-A$, we deduce the *negation formula*

$$\Omega[-A]=1/\Omega[A].$$

Suppose m is a monic monomial in S unequal to 1. Using Theorem 7, we find that

$$\Omega[m]=\sum_{n\geq 0}h_n[m]=\sum_{n\geq 0}m^n=1/(1-m)\in S.$$

For example,

$$\Omega[x_iy_j]=\frac{1}{1-x_iy_j},\qquad \Omega[tx_iy_j]=\frac{1}{1-tx_iy_j},\qquad \Omega[q^kx_iy_j]=\frac{1}{1-q^kx_iy_j}.$$

Using the addition formula for Ω, we immediately deduce the following finite product expansions (where $X_n=x_1+\cdots+x_n$ and $Y_m=y_1+\cdots+y_m$):

$$\Omega[X_nY_m]=\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1}{1-x_iy_j},\qquad \Omega\big[X_nY_m(1-t)\big]=\prod_{i=1}^{n}\prod_{j=1}^{m}\frac{1-tx_iy_j}{1-x_iy_j}.$$

To extend these formulas to infinite products, we need one more limiting operation. Suppose $A=\sum_{n\geq 1}A_n$ is an "infinite sum of alphabets." We define

$$\Omega[A]=\lim_{N\to\infty}\Omega[A_1+\cdots+A_N]$$

provided that this limit exists in S. In this situation, we have $\Omega[A]=\prod_{n\geq 1}\Omega[A_n]$, since

$$\Omega[A]=\lim_{N\to\infty}\Omega[A_1+\cdots+A_N]=\lim_{N\to\infty}\prod_{n=1}^{N}\Omega[A_n]=\prod_{n=1}^{\infty}\Omega[A_n].$$

Let $X=x_1+x_2+\cdots=p_1\otimes 1\in\Lambda\otimes_K\Lambda$ and $Y=y_1+y_2+\cdots=1\otimes p_1\in\Lambda\otimes_K\Lambda$. Then

$$\Omega[XY]=\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\frac{1}{1-x_iy_j},$$

$$\Omega\big[XY(1-t)\big]=\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\frac{1-tx_iy_j}{1-x_iy_j},$$

$$\Omega\left[XY\frac{1-t}{1-q}\right]=\prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\prod_{k=0}^{\infty}\frac{1-tq^kx_iy_j}{1-q^kx_iy_j}.$$

The last formula follows by writing $\frac{1}{1-q}=\sum_{k\geq 0}q^k$ and using the addition rule for Ω.

5.4 Application to operators in Macdonald theory

We end this section by giving an example of how plethystic notation can be used in the theory of Macdonald polynomials. Macdonald [33] introduced an operator δ_1 that is defined as follows. Given a polynomial $P(x_1,\dots,x_n)$, let

$$T_q^{(s)}P(x_1,\dots,x_n)=P(x_1,\dots,x_{s-1},qx_s,x_{s+1},\dots,x_n) \tag{27}$$

and

$$\begin{aligned}\delta_1 P(x_1,\dots,x_n)&=\sum_{s=1}^n\prod_{\substack{i=1\\ i\neq s}}^n\frac{tx_s-x_i}{x_s-x_i}T_q^{(s)}P(x_1,\dots,x_n)\\&=\sum_{s=1}^n\frac{T_t^{(s)}\Delta(x_1,\dots,x_n)}{\Delta(x_1,\dots,x_n)}T_q^{(s)}P(x_1,\dots,x_n)\end{aligned} \tag{28}$$

where $\Delta(x_1,\dots,x_n)=\prod_{1\le i<j\le n}(x_i-x_j)$ is the Vandermonde determinant. Macdonald proved the following facts about the operator δ_1. Given a composition $p=(p_1,\dots,p_n)$ with n parts, let $\omega_p(q,t)=\sum_{i=1}^n t^{n-i}q^{p_i}$. Recall that p^+ is the partition obtained by sorting the parts of p into decreasing order. Macdonald proved that for all partitions λ of n,

$$\delta_1\big(m_\lambda(x_1,\dots,x_n)\big)=\omega_\lambda(q,t)m_\lambda(x_1,\dots,x_n)+\sum_{\mu<_D\lambda}m_\mu\sum_{\substack{p=(p_1,\dots,p_n):\\ p^+=\lambda}}\varepsilon_\mu(p)\omega_p(q,t), \tag{29}$$

where $\varepsilon_\mu(p)=\mathrm{sgn}(\sigma)$ if there is a permutation σ such that $p_{\sigma_i}+n-i=\mu_i+n-i$ for $i=1,\dots,n$ and $\varepsilon_\mu(p)=0$ if there is no such σ, and $<_D$ denotes the dominance ordering on partitions. It immediately follows that for all partitions λ of n,

$$\delta_1\big(s_\lambda(x_1,\dots,x_n)\big)=\omega_\lambda(q,t)s_\lambda(x_1,\dots,x_n)+\sum_{\mu<_D\lambda}d_{\lambda,\mu}(q,t)s_\mu(x_1,\dots,x_n) \tag{30}$$

for some polynomials $d_{\lambda,\mu}(q,t)$. In fact, Macdonald proved that if

$$z_\lambda(x)=\prod_{\mu\in\mathrm{Par}(n)-\{\lambda\}}\frac{x-\omega_\mu(q,t)}{\omega_\lambda(q,t)-\omega_\mu(q,t)}, \tag{31}$$

then the Macdonald polynomial $P_\lambda(x_1,\dots,x_n;q,t)$ is given by

$$P_\lambda(x_1,\dots,x_n;q,t)=z_\lambda(\delta_1)s_\lambda(x_1,\dots,x_n). \tag{32}$$

Garsia and Haiman [15] gave a plethystic interpretation for the operator δ_1 which they used to prove a number of fundamental results about the q,t-Catalan numbers, and which was later used by Remmel [36] to develop the combinatorics of δ_1 applied to Schur functions. That is, letting $X_n = x_1 + \cdots + x_n$, Garsia and Haiman proved that for any symmetric polynomial P,

$$\delta_1\big(P[X_n]\big) = \frac{P[X_n]}{1-t} + \frac{t^n}{t-1} P\left[X_n + \frac{q-1}{tz}\right] \Omega\big[z(t-1)X_n\big]\Big|_{z^0}. \tag{33}$$

More generally, note that in plethystic notation $T_q^{(i)}(P[X_n]) = P[X_n + (q-1)x_i]$. Garsia [12] introduced the following family of Macdonald-like operators

$$D_1^{(k)} = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{tx_i - x_j}{x_i - x_j} x_i^k T_q^{(i)} \tag{34}$$

and proved that for any symmetric polynomial P,

$$D_1^{(k)}\big(P[X_n]\big) = \chi(k=0)\frac{P[X_n]}{1-t} + \frac{t^{n-k}}{t-1} P\left[X_n + \frac{q-1}{tz}\right] \Omega\big[z(t-1)X_n\big]\Big|_{z^k}. \tag{35}$$

We now present Garsia's proof [12] of (35) because it shows the remarkable power of the plethystic calculus. Both sides of (35) are linear in P, so it suffices to verify the formula when P is a Schur symmetric function. Letting $Y = u(y_1 + \cdots + y_n + \cdots) = 1 \otimes p_1 u$, we have

$$\begin{aligned} \Omega[X_n Y] &= \sum_{N\geq 0} h_N[X_n Y] = \sum_{N\geq 0} \sum_{\lambda\in \mathrm{Par}(N)} s_\lambda[X_n] s_\lambda[Y] \\ &= \sum_{N\geq 0} u^N \left(\sum_{\lambda\in\mathrm{Par}(N)} s_\lambda(x_1,\dots,x_n) \otimes s_\lambda \right) \in (\Lambda_n \otimes_K \Lambda)\big[[u]\big]. \end{aligned} \tag{36}$$

Using $\Omega[X_n Y]$ as the generating function for the Schur polynomials $s_\lambda[X_n]$, we can establish (35) for all Schur functions by showing that

$$\begin{aligned} D_1^{(k)}\big(\Omega[X_n Y]\big) &= \chi(k=0)\frac{\Omega[X_n Y]}{1-t} + \frac{t^{n-k}}{t-1} \Omega\left[\left(X_n + \frac{q-1}{tz}\right)Y\right] \\ &\quad \times \Omega\big[z(t-1)X_n\big]\Big|_{z^k} \\ &= \chi(k=0)\frac{\Omega[X_n Y]}{1-t} + \frac{t^{n-k}}{t-1} \Omega[X_n Y] \Omega\left[\frac{q-1}{tz} Y\right] \\ &\quad \times \Omega\big[z(t-1)X_n\big]\Big|_{z^k}. \end{aligned} \tag{37}$$

(More precisely, the left side is the image of $\Omega[X_n Y]$ under the linear map on $(\Lambda_n \otimes_K \Lambda)[[u]]$ that applies $D_1^{(k)} \otimes 1$ to each coefficient of u^N; the desired result for Schur polynomials will follow from (37) by taking the coefficient of u^N and using the linear independence of $\{1 \otimes s_\lambda : \lambda \in \mathrm{Par}(N)\}$.) Recalling that $T_q^{(i)}(P[X_n]) = P[X_n + (q-1)x_i]$, we see from the definition of $D_1^{(k)}$ that

$$D_1^{(k)}\big(\Omega[X_nY]\big) = \sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{tx_i - x_j}{x_i - x_j} x_i^k \Omega\big[\big(X_n + (q-1)x_i\big)Y\big]$$

or, equivalently,

$$\frac{D_1^{(k)}(\Omega[X_nY])}{\Omega[X_nY]} = \sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{tx_i - x_j}{x_i - x_j} x_i^k \Omega\big[(q-1)x_iY\big]. \tag{38}$$

Next, the definition of Ω gives

$$\Omega\big[(q-1)x_iY\big] = \sum_{m\geq 0} h_m\big[(q-1)Y\big]x_i^m. \tag{39}$$

Substituting (39) into (38), we obtain

$$\frac{D_1^{(k)}(\Omega[X_nY])}{\Omega[X_nY]} = \sum_{m\geq 0} h_m\big[(q-1)Y\big] \sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{tx_i - x_j}{x_i - x_j} x_i^{k+m}. \tag{40}$$

Observe that $\Omega[(t-1)X_nz] = \prod_{i=1}^n \frac{1-zx_i}{1-ztx_i}$ has a partial fraction expansion of the form

$$\Omega\big[(t-1)X_nz\big] = \frac{1}{t^n} + \sum_{i=1}^{n} A_i(\mathbf{x},t)\frac{1}{1-ztx_i} = t^{-n} + \sum_{i=1}^{n}\sum_{m\geq 0} A_i(\mathbf{x},t)z^m t^m x_i^m. \tag{41}$$

Fixing i, multiplying both sides of (41) by $1 - ztx_i$, and taking the limit as $z \to \frac{1}{tx_i}$ gives

$$A_i(\mathbf{x},t) = \frac{t-1}{t^n} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{tx_i - x_j}{x_i - x_j}. \tag{42}$$

Substituting (42) into (41) and taking the coefficient of z^m on both sides, we obtain

$$\sum_{i=1}^{n} \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{tx_i - x_j}{x_i - x_j} x_i^m = \frac{1}{t^m}\left(\frac{t^n\Omega[(t-1)X_nz]}{t-1} - \frac{1}{t-1}\right)\bigg|_{z^m}. \tag{43}$$

One can then use (43) with m replaced by $m+k$ to show that (40) becomes

$$\frac{D_1^{(k)}(\Omega[X_nY])}{\Omega[X_nY]} = \sum_{m\geq 0} h_m\big[(q-1)Y\big]\frac{1}{t^{m+k}}\left(\frac{t^n\Omega[(t-1)X_nz]}{t-1} - \frac{1}{t-1}\right)\bigg|_{z^{m+k}}$$

$$= \sum_{m\geq 0} h_m\left[\frac{(q-1)Y}{tz}\right](zt)^m \frac{1}{t^{m+k}}\left(\frac{t^n\Omega[(t-1)X_n z]}{t-1} - \frac{1}{t-1}\right)\bigg|_{z^{m+k}}$$

$$= \sum_{m\geq 0} h_m\left[\frac{(q-1)Y}{tz}\right]\frac{1}{t^k}\left(\frac{t^n\Omega[(t-1)X_n z]}{t-1} - \frac{1}{t-1}\right)\bigg|_{z^k}. \tag{44}$$

Observing that the term $\frac{1}{t-1}$ inside the big parentheses does not contribute anything if $k > 0$, we see that

$$\frac{D_1^{(k)}(\Omega[X_nY])}{\Omega[X_nY]} = \chi(k=0)\frac{1}{1-t} + \frac{t^{n-k}}{t-1}\Omega\big[(t-1)X_n z\big]\bigg(\sum_{m\geq 0} h_m\Big[\frac{(q-1)}{tz}Y\Big]\bigg)\bigg|_{z^k}$$

$$= \chi(k=0)\frac{1}{1-t} + \frac{t^{n-k}}{t-1}\Omega\big[(t-1)X_n z\big]\Omega\left[\frac{(q-1)}{tz}Y\right]\bigg|_{z^k}. \tag{45}$$

Multiplying both sides of (45) by $\Omega[X_nY]$ gives (37), which is what we needed to prove.

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