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On the existence of minimum cubature formulas for Gaussian measure on $\mathbb{R}^2$ of degree t supported by $[\frac{t}{4}]+1$ circles

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Abstract In this paper we prove that there exists no minimum cubature formula of degree $4k$ and $4k+2$ for Gaussian measure on $\mathbb{R}^2$ supported by $k+1$ circles for any positive integer k, except for two formulas of degree 4.

Keywords Cubature formula · Euclidean design · Gaussian design · Laguerre polynomial

1 Introduction

A pair (X, w) of a finite subset $X \subset \mathbb{R}^n$ and a positive weight function $w : X \longrightarrow \mathbb{R}_{>0}$ is called a cubature formula of degree t for the Gaussian measure on $\mathbb{R}^n$ if

$$\frac{1}{V_n}\int_{\mathbb{R}^n} f(\boldsymbol{x})e^{-\|\boldsymbol{x}\|^2}\,d\boldsymbol{x} = \sum_{\boldsymbol{x}\in X} w(\boldsymbol{x})f(\boldsymbol{x})$$

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for any polynomial $f(\boldsymbol{x})$ of degree at most t, where $V_n = \int_{\mathbb{R}^n} e^{-\|\boldsymbol{x}\|^2} d\boldsymbol{x}$. Let $\mathrm{Hom}_l(\mathbb{R}^n)$ be the vector space of homogeneous polynomials of degree l in n variables, and $\mathcal{P}_e(\mathbb{R}^n) = \sum_{l=0}^{e} \mathrm{Hom}_l(\mathbb{R}^n)$, $\mathcal{P}_e^*(\mathbb{R}^n) = \sum_{l=0, l\equiv e(2)}^{e} \mathrm{Hom}_l(\mathbb{R}^n)$. It is known that if (X, w) is such a cubature formula, then the following inequalities hold (see [4, 9–11], etc.):

$$|X| \geq \begin{cases} \dim(\mathcal{P}_e(\mathbb{R}^n)) & \text{if } t = 2e, \\ 2\dim(\mathcal{P}_e^*(\mathbb{R}^n)) - 1 & \text{if } t = 2e+1 \ (e = 2k),\ 0 \in X, \\ 2\dim(\mathcal{P}_e^*(\mathbb{R}^n)) & \begin{array}{l}\text{if } t = 2e+1 \ (e = 2k),\ 0 \notin X, \\ \text{or if } t = 2e+1 \ (e = 2k+1).\end{array} \end{cases}$$

Here, $\dim(\mathcal{P}_e(\mathbb{R}^n)) = \binom{n+e}{e}$ and $\dim(\mathcal{P}_e^*(\mathbb{R}^n)) = \sum_{i=0}^{[e/2]} \binom{n+e-2i-1}{e-2i}$. A cubature formula (X, w) is called a minimum formula if the equality holds for X in the inequalities given above. We note that the present definition of minimum seems to be different from the classical way in numerical analysis and related areas, where the minimum is often discussed only for cubature formula of degree $4k+1$ containing the origin (see, e.g., [9, 10]). This is also called a Gaussian tight t-design of $\mathbb{R}^n$.

A fundamental problem is the existence of Gaussian tight t-designs of $\mathbb{R}^n$ supported by $[\frac{t}{4}] + 1$ concentric spheres. "$[\frac{t}{4}] + 1$" is the minimum in the sense that if a Gaussian t-design exists, then the number of spheres over which the points are distributed must be at least $[\frac{t}{4}] + 1$; see, e.g., [4, 8]. The case $n = 2$ deserves a special attention. The first and second authors [1] proved that if there exists a Gaussian tight 4-design (X, w) on 2 concentric circles, then (X, w) is isomorphic to one of the following designs:

(a) $X = X_1 \cup \{0\}$, X_1 is a regular pentagon on the circle of radius $r_1 = \sqrt{2}$, $w(0) = \frac{1}{2}$, and $w(\boldsymbol{x}) = \frac{1}{10}$ for $\boldsymbol{x} \in X_1$.
(b) $X = X_1 \cup X_2$, X_1, and X_2 are regular triangles defined by

$$X_1 = \{(r_1\cos(i\theta), r_1\sin(i\theta)) \,|\, i = 0, 1, 2\},$$
$$X_2 = \left\{\left(r_2\cos\left(i\theta + \frac{\pi}{3}\right), r_2\sin\left(i\theta + \frac{\pi}{3}\right)\right) \,\middle|\, i = 0, 1, 2\right\},$$

where $\theta = \frac{2\pi}{3}$, $r_1 = \sqrt{3+\sqrt{5}}$, $r_2 = \sqrt{3-\sqrt{5}}$, $w(\boldsymbol{x}) = \frac{1}{6} - \frac{\sqrt{5}}{15}$ on X_1, and $w(\boldsymbol{x}) = \frac{1}{6} + \frac{\sqrt{5}}{15}$ on X_2.

Cools and Schmid [7] considered the degree $4k+1$ case in general and showed that there exists no Gaussian tight $(4k+1)$-design supported by $k+1$ concentric circles for any integer k with $k \geq 2$. In the case of $k = 1$, there exists a Gaussian tight 5-design (X, w) on 2 concentric circles, and it is isomorphic to the following design (see, e.g., [4, 8]):

(c) $X = X_1 \cup \{0\}$, X_1 is a regular hexagon on the circle of radius $r_1 = \sqrt{2}$, $w(0) = \frac{1}{2}$, and $w(\boldsymbol{x}) = \frac{1}{12}$ for $\boldsymbol{x} \in X_1$.

Following the work of Cools and Schmid, the third and fourth authors [8] considered the degree $4k+3$ case and proved that there exists no Gaussian tight t-design supported by $k+1$ concentric circles for $t=4k+1$ ($k\geq 2$) and $t=4k+3$ ($k\geq 1$).

The purpose of the present paper is to solve the even-degree case.

Theorem 1 *Let (X,w) be a Gaussian tight $2e$-design of $\mathbb{R}^2$ with $e\geq 2$. Suppose that X is supported by $[\frac{e}{2}]+1$ concentric circles. Then, $e=2$, and (X,w) is isomorphic to one of the two Gaussian tight 4-designs* (a), (b).

The proof of Theorem 1 is based on the property of Laguerre polynomials and uses some elementary identities in combinatorics. It is simple and so will be understood by many researchers. Also, the proof can be applied to the odd-degree case. By summarizing the results known so far, we obtain the following classification result as a corollary.

Corollary 1 *Let (X,w) be a Gaussian tight t-design of $\mathbb{R}^2$ with $t\geq 4$. Then, X is supported by $[\frac{t}{4}]+1$ concentric circles if and only if $t=4,5$. In particular, if $t=4$, then (X,w) is isomorphic to one of the designs* (a) *and* (b), *and if $t=5$, then (X,w) is isomorphic to the design* (c).

For more information on cubature formula, we refer to [3, 11].

2 Remarks on basic facts

Let (X,w) be a Gaussian tight t-design of $\mathbb{R}^n$. Let $\{r_1,r_2,\dots,r_p\}=\{\|\boldsymbol{x}\|\mid \boldsymbol{x}\in X\}$. We assume that $r_1>r_2>\cdots>r_p\geq 0$. Let $S_i=\{\boldsymbol{x}\in\mathbb{R}^n\mid \|\boldsymbol{x}\|=r_i\}$ be the sphere of radius r_i centered at the origin. We say that X is supported by p concentric spheres. Let $X_i=X\cap S_i$ for $i=1,\dots,p$. We note that a Gaussian tight t-design is a Euclidean t-design. We refer the readers to [2, 4, 5] about Euclidean (tight) t-designs. The following proposition is known (see Proposition 1.7 in [2] and Proposition 2.4.4 in [4]).

Proposition 1 *Let (X,w) be a Gaussian tight t-design of $\mathbb{R}^n$. Suppose that X is supported by p concentric spheres. Then the following hold.*

(1) $p\geq [\frac{t}{4}]+1$.
(2) *If $t\equiv 2$ or 3 mod 4, then $0\notin X$.*
(3) *(X,w) is a Euclidean tight t-design of $\mathbb{R}^n$.*

If $t=2e$, then Lemma 1.10 in [2] implies that the weight function w is constant on each X_i for $1\leq i\leq p$. For $t=2e+1$ (if e is even and $0\notin X$, we need an extra condition on X, such as $p=[\frac{t}{4}]+1$), Theorem 2.3.5 and Theorem 2.3.6 in [4] imply that X is antipodal. Then Lemma 1.7 in [5] implies that the weight function w is constant on each X_i for $1\leq i\leq p$. Proposition 1 suggests that it is important to study the case $p=[\frac{t}{4}]+1$. Then, as we mentioned above, the weight function w is constant on each X_i.

From now on let (X, w) denote a Gaussian tight t-design supported by $p = [\frac{t}{4}]+1$ concentric spheres. Let $k = [\frac{t}{4}]$ and $w(\boldsymbol{x}) = w_i$ for $\boldsymbol{x} \in X_i$ $(1 \le i \le k+1)$. Then we must have

$$\frac{V_n}{|S^{n-1}|}\sum_{i=1}^{k+1}|X_i|w_i r_i^{2j} = \int_0^\infty r^{2j} r^{n-1} e^{-r^2}\,dr, \quad 0 \le 2j \le t.$$

Therefore, w_i is determined uniquely by $r_1, \dots, r_p$.

3 Laguerre polynomials

Let (X, w) be a Gaussian tight $2e$-design of $\mathbb{R}^2$ supported by $[\frac{e}{2}] + 1$ concentric circles. Let $k = [\frac{e}{2}]$. Then we have

$$\begin{aligned}\sum_{\boldsymbol{x}\in X} w(\boldsymbol{x})\|\boldsymbol{x}\|^{2j} &= \frac{1}{\int_{\mathbb{R}^2} e^{-\|\boldsymbol{x}\|^2} d\boldsymbol{x}}\int_{\mathbb{R}^2}\|\boldsymbol{x}\|^{2j} e^{-\|\boldsymbol{x}\|^2}\,d\boldsymbol{x}\\ &= \frac{1}{\int_0^\infty re^{-r^2}dr}\int_0^\infty r^{2j} re^{-r^2}\,dr = \int_0^\infty y^j e^{-y}\,dy = j!\end{aligned}$$

for $j = 0, 1, \dots, e$. Let $\lambda_i = \sum_{\boldsymbol{x}\in X_i} w(\boldsymbol{x}) = |X_i|w_i$ and $R_i = r_i^2$ for $i = 1, 2, \dots, k+1$. Then we obtain the following quadrature formula of degree e for the weight function e^{-y} on the interval $[0, \infty)$ with the Christoffel numbers $\lambda_1, \dots, \lambda_{k+1}$:

$$\int_0^\infty y^j e^{-y}\,dy = \sum_{i=1}^{k+1}\lambda_i R_i^j.$$

Hence, we have

$$\sum_{i=1}^{k+1}\lambda_i R_i^j = j! \tag{1}$$

for $j = 0, 1, \dots, e$.

Let $P_l(x)$ be the orthogonal polynomial (Laguerre polynomial) of degree l for the weight function e^{-y} on $[0, \infty)$. It is well known that Laguerre polynomials are given by

$$P_l(x) = \sum_{i=0}^{l}\binom{l}{l-i}\frac{(-x)^i}{i!}$$

(see [6, 12], etc.). We have the following proposition.

Proposition 2 *Definitions and notation are as given above. Let* $F(x) = \prod_{i=1}^{k+1}(x - R_i)$. *Then the following hold.*

(1) *If* $e = 2k+1$, *then* $F(x) = c_{k+1}P_{k+1}(x)$.
(2) *If* $e = 2k$, *then* $F(x) = c_{k+1}P_{k+1}(x) + c_k P_k(x)$.

Here, $c_{k+1} = (-1)^{k+1}(k+1)!$, *and* c_k *is a real number.*

Proof There exist real numbers $c_0, \dots, c_{k+1}$ satisfying $F(x) = \sum_{l=0}^{k+1} c_l P_l(x)$. Since the coefficient of x^{k+1} in $P_{k+1}(x)$ is $(-1)^{k+1}\frac{1}{(k+1)!}$, we have $c_{k+1} = (-1)^{k+1}(k+1)!$. Moreover, for any $l \le e-k-1$, we have

$$c_l \int_0^\infty P_l(x)^2 e^{-y}\,dy = \int_0^\infty P_l(x)F(x)e^{-y}\,dy = \sum_{i=1}^{k+1} \lambda_i P_l(R_i)F(R_i) = 0.$$

Hence, $c_0 = c_1 = \cdots = c_{e-k-1} = 0$. Thus, if $e = 2k+1$, then $e-k-1 = k$, and we obtain

$$F(x) = c_{k+1}P_{k+1}(x).$$

On the other hand, if $e = 2k$, then $e-k-1 = k-1$, and we have

$$F(x) = c_{k+1}P_{k+1}(x) + c_k P_k(x). \qquad \square$$

For more information on orthogonal polynomials, please refer to [6, 12], etc.

4 Proof of the main theorem

Throughout this section we use the same notations $c_i, r_i, R_i, t, w, w_i, X, X_i, \lambda_i$ as in the previous sections. We assume that $r_1 > \cdots > r_{k+1}$ and so $R_1 > \cdots > R_{k+1}$.

Before giving the proof of Theorem 1, we state propositions.

Proposition 3 *If* $R_{k+1} \ne 0$, *then both* $\sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{r_i R_i^{k-1}}$ *and* $\sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{r_i R_i^{k}}$ *are nonzero. If* $R_{k+1} = 0$, *then* $\sum_{i=1}^{k} \frac{(-1)^{i-1}}{r_i R_i^{k-1}} \ne 0$.

Proof Assume that $R_{k+1} \ne 0$. If k is odd, then since $r_i R_i^{k-1} = r_i^{2k-1} > r_{i+1}^{2k-1} = r_{i+1}R_{i+1}^{k-1}$, we have

$$\sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{r_i R_i^{k-1}} = \left(\frac{1}{r_1 R_1^{k-1}} - \frac{1}{r_2 R_2^{k-1}}\right) + \cdots + \left(\frac{1}{r_k R_k^{k-1}} - \frac{1}{r_{k+1} R_{k+1}^{k-1}}\right) < 0.$$

If k is even, then

$$\sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{r_i R_i^{k-1}} = \frac{1}{r_1 R_1^{k-1}} + \left(-\frac{1}{r_2 R_2^{k-1}} + \frac{1}{r_3 R_3^{k-1}}\right) + \cdots + \left(-\frac{1}{r_k R_k^{k-1}} + \frac{1}{r_{k+1} R_{k+1}^{k-1}}\right) > 0,$$

which implies $\sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{r_i R_i^{k-1}} \neq 0$. The other assertions follow by the same argument as above. $\square$

Proposition 4 (i) *If* $t = 4k+2$, *then*

$$\sum_{1\le l_1<\cdots<l_s\le k+1} R_{l_1} R_{l_2} \cdots R_{l_s} = \frac{(k+1)!}{(k+1-s)!}\binom{k+1}{s}.$$

(ii) *If* $t = 4k$ *and* $0 \in X$, *then*

$$\prod_{i=1}^{k+1}(x - R_i) = x\sum_{j=0}^{k}(-1)^{k+j}(k+1)!\binom{k}{j}\frac{x^j}{(j+1)!}.$$

Moreover, for $s = 1, 2, \ldots, k$,

$$\sum_{1\le l_1<l_2<\cdots<l_s\le k} R_{l_1} R_{l_2} \cdots R_{l_s} = (k+1)!\binom{k}{k-s}\frac{1}{(k-s+1)!}.$$

(iii) *If* $t = 4k$ *and* $0 \notin X$, *then*

$$\prod_{i=1}^{k+1}(x - R_i) = (k+1)!\sum_{j=0}^{k+1}\binom{k+1}{k+1-j}\frac{(-1)^{k+1+j}x^j}{j!} + c_k\sum_{j=0}^{k}\binom{k}{k-j}\frac{(-1)^j x^j}{j!}.$$

Moreover, for $s = 1, 2, \ldots, k+1$,

$$\sum_{1\le l_1<\cdots<l_s\le k+1} R_{l_1} R_{l_2} \cdots R_{l_s}$$
$$= (k+1)!\binom{k+1}{s}\frac{1}{(k+1-s)!} + c_k\binom{k}{s-1}\frac{(-1)^{k+1}}{(k+1-s)!}.$$

Proof (i) Proposition 2(1) implies that $c_{k+1}P_{k+1}(x) = \prod_{i=1}^{k+1}(x - R_i)$ and $c_{k+1} = (-1)^{k+1}(k+1)!$. Comparing the coefficients at x^{k+1-s}, we obtain the result.

(ii) Let $F(x) = \prod_{i=1}^{k+1}(x - R_i)$. By the assumption $0 \in X$, we have $F(x) = x\prod_{i=1}^{k}(x - R_i)$. Proposition 2(2) implies $F(x) = c_{k+1}P_{k+1} + c_k P_k$. Since $F(0) = 0$ and $P_l(0) = 1$ for any l, we must have $c_{k+1} + c_k = 0$. This implies

$$F(x) = c_{k+1}\big(P_{k+1}(x) - P_k(x)\big).$$

By definition we have

$$P_{k+1}(x) - P_k(x) = \sum_{j=0}^{k+1}\binom{k+1}{k+1-j}\frac{(-x)^j}{j!} - \sum_{j=0}^{k}\binom{k}{k-j}\frac{(-x)^j}{j!}$$
$$= -x\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{x^j}{(j+1)!}.$$

Since $c_{k+1}=(-1)^{k+1}(k+1)!$, we obtain the former assertion. Comparing the coefficients at the terms of both sides of this equality, we obtain the latter assertion.

(iii) Let $F(x)=\prod_{i=1}^{k+1}(x-R_i)$. Proposition 2(2) implies $F(x)=c_{k+1}P_{k+1}+c_kP_k$. The result then follows by the same arguments as in the proof of (ii). □

We often use the following notation in the proof of the main theorem:

$$A_i=\begin{cases}(k+1)!+\sum_{s=1}^{k}(-1)^s(k+1-s)!\sum_{\substack{1\le l_1<\cdots<l_s\le k+1\\ l_1,\dots,l_s\ne i}}R_{l_1}\cdots R_{l_s},\\ \quad t=4k+2 \text{ or } t=4k,\ 0\notin X,\\ (k+1)!+\sum_{s=1}^{k-1}(-1)^s(k+1-s)!\sum_{\substack{1\le l_1<l_2<\cdots<l_s\le k\\ l_1,\dots,l_s\ne i}}R_{l_1}R_{l_2}\cdots R_{l_s},\\ \quad t=4k,\ 0\in X.\end{cases}$$

We are now ready to show Theorem 1.

Proof of Theorem 1 The proof consists of three steps.

Case $t=4k+2$ $(e=2k+1)$

In this case, X does not contain 0. Hence, $R_1>R_2>\cdots>R_k>R_{k+1}>0$.

Equations (1) for $j=1,\dots,k+1$ imply

$$\lambda_i=\frac{(-1)^{i-1}A_i}{R_i\prod_{l=1}^{i-1}(R_l-R_i)\prod_{l=i+1}^{k+1}(R_i-R_l)}. \tag{2}$$

On the other hand, Theorem 3.1.7(2) in [4] gives

$$\frac{\lambda_i}{\lambda_1}=\frac{w_i}{w_1}=\frac{r_1}{r_i}\frac{R_1^{k+1}}{R_i^{k+1}}\frac{\prod_{l=2}^{i-1}(R_1-R_l)\prod_{l=i+1}^{k+1}(R_1-R_l)}{\prod_{l=2}^{i-1}(R_l-R_i)\prod_{l=i+1}^{k+1}(R_i-R_l)} \tag{3}$$

for $i=2,3,\dots,k+1$. (Note that there is a typo in the formula of Theorem 3.1.7(2) in [4]: Since $0\notin X$, $p-1$ in the formula must be replaced by p.)

Since $\lambda_1\ne 0$, (2) implies $A_1\ne 0$. Hence, (2) and (3) imply

$$(-1)^{i-1}\frac{r_1R_1^k}{r_iR_i^k}A_1=A_i \tag{4}$$

for any $1\le i\le k+1$. Then by taking the sum of both sides of (4) over $i=1,\dots,k+1$ we obtain

$$\begin{aligned}&r_1R_1^kA_1\sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{r_iR_i^k}\\ &\quad=(k+1)(k+1)!+\sum_{s=1}^{k}(-1)^s(k+1-s)!\sum_{i=1}^{k+1}\sum_{\substack{1\le l_1<\cdots<l_s\le k+1\\ l_1,\dots,l_s\ne i}}R_{l_1}\cdots R_{l_s}\end{aligned}$$

$$= (k+1)(k+1)! + \sum_{s=1}^{k}(-1)^s(k+1-s)!(k+1-s) \sum_{1\le l_1<\cdots<l_s\le k+1} R_{l_1}\cdots R_{l_s}. \tag{5}$$

Then Proposition 4(i) and (5) imply

$$\begin{aligned}
&r_1R_1^kA_1\sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{r_iR_i^k}\\
&\quad= (k+1)(k+1)!\\
&\qquad+\sum_{s=1}^{k}(-1)^s(k+1-s)!(k+1-s)(k+1)!\binom{k+1}{s}\frac{1}{(k+1-s)!}\\
&\quad= (k+1)(k+1)! + (k+1)(k+1)!\sum_{s=1}^{k}(-1)^s\binom{k}{s}\\
&\quad= (k+1)(k+1)!\sum_{s=0}^{k}(-1)^s\binom{k}{s} = 0.
\end{aligned}$$

Since $A_1 \neq 0$, we must have

$$\sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{r_iR_i^k} = 0.$$

This is a contradiction by Proposition 3.

Case $t = 4k$ $(e = 2k)$ *and* $0 \in X$
Let $R_{k+1} = 0$. Then (1) implies

$$\sum_{i=1}^{k}\lambda_iR_i^j = j! \tag{6}$$

for $1 \le j \le e$ $(= 2k)$.

If $k = 1$, then $\lambda_1R_1 = 1$ and $\lambda_1R_1^2 = 2$ imply $R_1 = 2$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{2}$. Hence, $w(\boldsymbol{x}) = \frac{1}{10}$ and $w(0) = \frac{1}{2}$. In this case, $X\backslash\{0\}$ is a spherical tight 4-design and a regular pentagon. This gives the Gaussian tight 4-design given in Corollary 1(1).

Next we assume that $k \ge 2$. Using (6) for $j = 2, 3, \ldots, k+1$, we obtain

$$\lambda_i = \frac{(-1)^{i-1}A_i}{R_i^2\prod_{l=1}^{i-1}(R_l-R_i)\prod_{l=i+1}^{k}(R_i-R_l)}$$

for $i=1,2,\dots,k$. On the other hand, Theorem 3.1.4(2) in [4] implies

$$\lambda_i=\frac{r_1R_1^{k+1}}{r_iR_i^{k+1}}\frac{\prod_{j=2}^{i-1}(R_1-R_j)\prod_{j=i+1}^{k}(R_1-R_j)}{\prod_{j=2}^{i-1}(R_j-R_i)\prod_{j=i+1}^{k}(R_i-R_j)}\lambda_1$$

for $i=2,\dots,k$. Then a similar argument given before implies that for $i=1,2,\dots k$,

$$(-1)^{i-1}\frac{r_1R_1^{k-1}}{r_iR_i^{k-1}}A_1=A_i. \tag{7}$$

Then, taking the sum of both sides of (7) over $i=1,2,\dots,k$, we obtain

$$\begin{aligned}
&A_1r_1R_1^{k-1}\sum_{i=1}^{k}\frac{(-1)^{i-1}}{r_iR_i^{k-1}}\\
&\quad=k(k+1)!+\sum_{s=1}^{k-1}(-1)^s(k-s+1)!\sum_{i=1}^{k}\sum_{\substack{1\le l_1<\cdots<l_s\le k\\ l_1,l_2,\dots,l_s\ne i}}R_{l_1}R_{l_2}\cdots R_{l_s}\\
&\quad=k(k+1)!+\sum_{s=1}^{k-1}(-1)^s(k-s+1)!(k-s)\sum_{1\le l_1<\cdots<l_s\le k}R_{l_1}R_{l_2}\cdots R_{l_s}.
\end{aligned} \tag{8}$$

Then Proposition 4(ii) and (8) imply

$$\begin{aligned}
&A_1r_1R_1^{k-1}\sum_{i=1}^{k}\frac{(-1)^{i-1}}{r_iR_i^{k-1}}\\
&\quad=k(k+1)!+\sum_{s=1}^{k-1}(-1)^s(k-s+1)!(k-s)(k+1)!\binom{k}{k-s}\frac{1}{(k-s+1)!}\\
&\quad=k(k+1)!\sum_{s=0}^{k-1}(-1)^s\binom{k-1}{s}.
\end{aligned}$$

Since $k\ge 2$, then $k(k+1)!\sum_{s=0}^{k-1}(-1)^s\binom{k-1}{s}=0$. Since $A_1\ne 0$, this implies

$$\sum_{i=1}^{k}\frac{(-1)^{i-1}}{r_iR_i^{k-1}}=0.$$

This is a contradiction by Proposition 3.

Case $t=4k$ ($e=2k$) and $0\notin X$
If $k=1$, then (X,w) is a Gaussian tight 4-design of $\mathbb{R}^2$. Hence, X_1 and X_2 are regular triangles. Theorem 3.1.5(2) in [4] implies $\frac{\lambda_1}{\lambda_2}=(\frac{r_2}{r_1})^3$. This, together with

$\sum_{i=1}^{2} \lambda_i R_i = 1$ and $\sum_{i=1}^{2} \lambda_i R_i^2 = 2$, implies $R_1 = 3 + \sqrt{5}$, $R_2 = 3 - \sqrt{5}$, $\lambda_1 = \frac{1}{2} - \frac{1}{\sqrt{5}} = 3w_1$, and $\lambda_2 = \frac{1}{2} + \frac{1}{\sqrt{5}} = 3w_2$. This is the example given in Corollary 1(2).

Next we assume that $k \geq 2$. Using (1) for $j = 1, 2, \ldots, k+1$, we obtain

$$\lambda_i = \frac{(-1)^{i-1} A_i}{R_i \prod_{l=1}^{i-1}(R_l - R_i) \prod_{l=i+1}^{k+1}(R_i - R_l)} \tag{9}$$

for $i = 1, 2, \ldots, k+1$. On the other hand, Theorem 3.1.5(2) in [4] implies

$$\lambda_i = \frac{r_1 R_1^k}{r_i R_i^k} \frac{\prod_{j=2}^{i-1}(R_1 - R_j) \prod_{j=i+1}^{k+1}(R_1 - R_j)}{\prod_{j=2}^{i-1}(R_j - R_i) \prod_{j=i+1}^{k+1}(R_i - R_j)} \lambda_1. \tag{10}$$

(Note that there is a typo in the formula of Theorem 3.1.5(2) in [4]: $p - 1$ in the formula must be replaced by p, since $0 \notin X$ in this case.)

Then (9) and (10) imply

$$(-1)^{i-1} \frac{r_1 R_1^{k-1}}{r_i R_i^{k-1}} A_1 = A_i$$

for $i = 1, 2, \ldots, k+1$. Then, taking the sum over $i = 1, 2, \ldots, k+1$, we obtain

$$\begin{aligned} &r_1 R_1^{k-1} A_1 \sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{r_i R_i^{k-1}} \\ &\quad = (k+1)(k+1)! + \sum_{s=1}^{k} (-1)^s (k+1-s)!(k+1-s) \sum_{1 \leq l_1 < \cdots < l_s \leq k+1} R_{l_1} R_{l_2} \cdots R_{l_s}. \end{aligned} \tag{11}$$

Then Proposition 4(iii) and (11) imply

$$\begin{aligned} &r_1 R_1^{k-1} A_1 \sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{r_i R_i^{k-1}} \\ &\quad = (k+1)(k+1)! + \sum_{s=1}^{k} (-1)^s (k+1-s)!(k+1-s) \\ &\qquad \times \left((k+1)! \binom{k+1}{s} \frac{1}{(k+1-s)!} + c_k \binom{k}{s-1} \frac{(-1)^{k+1}}{(k+1-s)!} \right) \\ &\quad = (k+1)(k+1)! + \sum_{s=1}^{k} (-1)^s (k+1-s)(k+1)! \binom{k+1}{s} \\ &\qquad + c_k \sum_{s=1}^{k} (-1)^s (k+1-s) \binom{k}{s-1} (-1)^{k+1} \end{aligned}$$

$$= (k+1)(k+1)! \sum_{s=0}^{k} (-1)^s \binom{k}{s} + (-1)^k k c_k \sum_{s=0}^{k-1} (-1)^s \binom{k-1}{s}$$
$$= (-1)^k k c_k \sum_{s=0}^{k-1} (-1)^s \binom{k-1}{s}.$$

Since $k \geq 2$ and $A_1 \neq 0$, we have

$$\sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{r_i R_i^{k-1}} = 0.$$

This is a contradiction by Proposition 3, which completes the proof of the main theorem. □

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