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Möbius transform, moment-angle complexes and Halperin–Carlsson conjecture

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Abstract The motivation for this paper comes from the Halperin–Carlsson conjecture for (real) moment-angle complexes. We first give an algebraic combinatorics formula for the Möbius transform of an abstract simplicial complex K on $[m]=\{1,\dots,m\}$ in terms of the Betti numbers of the Stanley–Reisner face ring $\mathbf{k}(K)$ of K over a field $\mathbf{k}$. We then employ a way of compressing K to provide the lower bound on the sum of those Betti numbers using our formula. Next we consider a class of generalized moment-angle complexes $\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}$, including the moment-angle complex $\mathcal{Z}_K$ and the real moment-angle complex $\mathbb{R}\mathcal{Z}_K$ as special examples. We show that $H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k})$ has the same graded $\mathbf{k}$-module structure as $\mathrm{Tor}^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k})$. Finally we show that the Halperin–Carlsson conjecture holds for $\mathcal{Z}_K$ (resp. $\mathbb{R}\mathcal{Z}_K$) under the restriction of the natural T^m-action on $\mathcal{Z}_K$ (resp. $(\mathbb{Z}_2)^m$-action on $\mathbb{R}\mathcal{Z}_K$).

Keywords Möbius transform · Moment-angle complex · Halperin–Carlsson conjecture

1 Introduction

Throughout this paper, assume that m is a positive integer and $[m]=\{1,\dots,m\}$. Also, $\mathbf{k}_\ell$ denotes the field of characteristic ℓ and $\mathbf{k}$ denotes a field of arbitrary char-

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acteristic. Let

$$2^{[m]*} = \{f \mid f : 2^{[m]} \longrightarrow \mathbb{Z}/2\mathbb{Z} = \{0, 1\}\}$$

consist of all $\mathbb{Z}/2\mathbb{Z}$-valued functions on the power set $2^{[m]}$. Then $2^{[m]*}$ forms an algebra over $\mathbb{Z}/2\mathbb{Z}$ in the usual way, and it has a natural basis $\{\delta_a | a \in 2^{[m]}\}$ where δ_a is defined as follows: $\delta_a(b) = 1 \Longleftrightarrow b = a$. Given an element $f \in 2^{[m]*}$, the inverse image of f at 1 is called the *support* of f, denoted by $\operatorname{supp}(f)$. We say that f is *nice* if $\operatorname{supp}(f)$ is an abstract simplicial complex. Thus, we can identify all nice functions in $2^{[m]*}$ with all abstract simplicial subcomplexes in $2^{[m]}$. On $2^{[m]*}$, we then define a $\mathbb{Z}/2\mathbb{Z}$-valued Möbius transform $\mathcal{M} : 2^{[m]*} \longrightarrow 2^{[m]*}$ by the following way: for any $f \in 2^{[m]*}$ and $a \in 2^{[m]}$, $\mathcal{M}(f)(a) = \sum_{b \subseteq a} f(b)$.

Now suppose that $f \in 2^{[m]*}$ is nice such that $K = \operatorname{supp}(f)$ is an abstract simplicial complex on $[m]$. Let $\mathbf{k}(K)$ be the Stanley–Reisner face ring of K. The following result indicates an essential relationship between $\mathcal{M}(f)$ and the Betti numbers of $\mathbf{k}(K)$.

Theorem 1.1 (Algebraic combinatorics formula)

$$\mathcal{M}(f) = \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)} \delta_a$$

where h denotes the length of the minimal free resolution of $\mathbf{k}(K)$, *and* $\beta_{i,a}^{\mathbf{k}(K)}$ *denote the Betti numbers of* $\mathbf{k}(K)$ (*see Definition* 2.4).

The formula of Theorem 1.1 leads to the following inequality

$$\left|\operatorname{supp}\big(\mathcal{M}(f)\big)\right| \le \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)}.$$

See Corollary 3.2. Then we use an approach of compressing $\operatorname{supp}(f)$ to further analyze the lower bound on $|\operatorname{supp}(\mathcal{M}(f))|$, and the result is stated as follows.

Theorem 1.2 *There exists some* $a \in \operatorname{supp}(f)$ *such that*

$$\left|\operatorname{supp}\big(\mathcal{M}(f)\big)\right| \ge 2^{m-|a|}.$$

Remark 1 Since $a \in \operatorname{supp}(f)$, $|a| \le \dim K + 1$, so $|\operatorname{supp}(\mathcal{M}(f))| \ge 2^{m-|a|} \ge 2^{m-\dim K-1}$.

Next, associating with the Tor-algebra $\operatorname{Tor}^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K), \mathbf{k})$ of $\mathbf{k}(K)$, we study a class of generalized moment-angle complexes $\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}$, in which the moment-angle complex $\mathcal{Z}_K$ and the real moment-angle complex $\mathbb{R}\mathcal{Z}_K$ are contained, see Sect. 4.2 for the definition of $\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}$. We shall show that

Theorem 1.3 (Theorem 4.2) *$H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k})$ has the same graded $\mathbf{k}$-module structure as* $\mathrm{Tor}^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k})$.

Remark 2 Theorem 1.3 tells us that $\sum_i \dim_{\mathbf{k}} H^i(\mathcal{Z}_K;\mathbf{k}) = \sum_i \dim_{\mathbf{k}} H^i(\mathbb{R}\mathcal{Z}_K;\mathbf{k})$.

As a result, we can consider the Halperin–Carlsson conjecture in the category of (real) moment-angle complexes. It is well-known that $\mathcal{Z}_K$ (resp. $\mathbb{R}\mathcal{Z}_K$) naturally admits a T^m-action Φ (resp. $(\mathbb{Z}_2)^m$-action $\Phi_{\mathbb{R}}$).

Theorem 1.4 *Let H (resp. $H_{\mathbb{R}}$) be a rank r subtorus of T^m (resp. $(\mathbb{Z}_2)^m$). If the H-action (resp. $H_{\mathbb{R}}$-action) of Φ restricted to H (resp. $\Phi_{\mathbb{R}}$ restricted to $H_{\mathbb{R}}$) is free on $\mathcal{Z}_K$ (resp. $\mathbb{R}\mathcal{Z}_K$), then*

$$\sum_i \dim_{\mathbf{k}} H^i(\mathcal{Z}_K;\mathbf{k}) = \sum_i \dim_{\mathbf{k}} H^i(\mathbb{R}\mathcal{Z}_K;\mathbf{k}) \geq 2^r.$$

Corollary 1.5 *The Halperin–Carlsson conjecture holds for $\mathcal{Z}_K$ (resp. $\mathbb{R}\mathcal{Z}_K$) under the restriction of the T^m-action Φ (resp. the $(\mathbb{Z}_2)^m$-action $\Phi_{\mathbb{R}}$).*[1]

Remark 3 Following [16], the Halperin–Carlsson conjecture is stated as follows:

- Let X be a finite-dimensional paracompact Hausdorff space. If X admits a free action of a torus T^r (resp. a p-torus $(\mathbb{Z}_p)^r$, p prime) of rank r, then

$$\sum_i \dim_{\mathbf{k}_\ell} H^i(X;\mathbf{k}_\ell) \geq 2^r \tag{1.1}$$

  where ℓ is 0 (resp. p).

Historically, the above conjecture in the p-torus case originates from the work of P.A. Smith [17]. For the case of a p-torus $(\mathbb{Z}_p)^r$ freely acting on a finite CW-complex homotopic to $(S^n)^k$ this problem was suggested by P.E. Conner [10], and essential progress on it was made in [1, 7, 8, 18]. In the general case, the inequality (1.1) was conjectured by S. Halperin in [13] for the torus case, and by G. Carlsson in [9] for the p-torus case. So far, the conjecture is known to hold $r \leq 3$ in the torus and 2-torus cases and if $r \leq 2$ in the odd p-torus case (see [16]). Also, many authors contributed to the conjecture in many different aspects. For more details, see, e.g., [2, 3, 6, 15].

The paper is organized as follows. In Sect. 2 we study the basic structure of the algebra $2^{[m]*}$ and the basic properties of the $\mathbb{Z}/2\mathbb{Z}$-valued Möbius transform, and review the notions of Stanley–Reisner face rings and their Tor-algebras. Sections 3 and 4 are two main parts of this paper. We give the proof of the algebraic combinatorics formula and estimate the lower bound on $|\operatorname{supp}(\mathcal{M}(f))|$ in Sect. 3. In Sect. 4 we review the theorem of V.M. Buchstaber and T.E. Panov on the cohomology of $\mathcal{Z}_K$.

[1] T.E. Panov informs of us that using a different method, Yury Ustinovsky has also recently proved the Halperin's toral rank conjecture for the moment-angle complexes with the restriction of natural tori actions, see arXiv:0909.1053.

Then we prove Theorem 1.3 therein. Finally we finish the proof of Theorem 1.4 in Sect. 5.

2 Möbius transform and Stanley–Reisner face ring

2.1 An algebra over $\mathbb{Z}/2\mathbb{Z}$

Let $2^{[m]}$ denote the power set of $[m]$, which is the set of all subsets (including the empty set) of $[m]$. Then $2^{[m]}$ forms a poset with respect to inclusion $\subseteq$, and it is also a boolean algebra under the set operations of union, intersection and complement relative to $[m]$. Let

$$2^{[m]*} = \big\{ f \mid f : 2^{[m]} \longrightarrow \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \big\}.$$

Then $2^{[m]*}$ forms *an algebra over* $\mathbb{Z}/2\mathbb{Z}$, where the addition is defined by $(f + g)(a) = f(a) + g(a)$ and multiplication is defined by $(f \cdot g)(a) = f(a)g(a)$ for $a \in 2^{[m]}$. Given a function $f \in 2^{[m]*}$, define

$$\mathrm{supp}(f) := f^{-1}(1)$$

which is called the *support* of f.

Definition 2.1 For each $a \in 2^{[m]}$, the function $\delta_a \in 2^{[m]*}$ defined by

$$\delta_a(b) = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{otherwise} \end{cases}$$

is called *the a-function*. For each $i \in [m]$, the function $x_i \in 2^{[m]*}$ defined by

$$x_i(a) = 1 \quad \Leftrightarrow \quad i \in a$$

$\forall a \in 2^{[m]}$ is called *the ith coordinate function*.

Lemma 2.1 $\{\delta_a \mid a \in 2^{[m]}\}$ *forms a basis for* $2^{[m]*}$.

Proof This is because any $f \in 2^{[m]*}$ can be expressed as

$$f = \sum_{a \in 2^{[m]}} f(a)\delta_a = \sum_{a \in \mathrm{supp}(f)} \delta_a.$$

□

By $\underline{1}$ one denotes the constant function such that $\underline{1}(a) = 1$ for all a in $2^{[m]}$. Obviously, $\underline{1} = \sum_{a \in 2^{[m]}} \delta_a$. For each $a \in 2^{[m]}$, set

$$\mu_a := \begin{cases} \prod_{i \in a} x_i & \text{if } a \text{ is nonempty} \\ \underline{1} & \text{if } a \text{ is empty.} \end{cases}$$

Then it is easy to see that

Lemma 2.2 *Let $a, b \in 2^{[m]}$. Then $\mu_a(b) = 1 \Leftrightarrow a \subseteq b$.*

Definition 2.2 We say that $f \in 2^{[m]*}$ is *nice* if $\mathrm{supp}(f)$ is an abstract simplicial complex on vertex set $\bigcup_{a \in \mathrm{supp}(f)} a \subseteq [m]$. Note that an abstract simplicial complex K on a subset of $[m]$ is a collection of subsets in $[m]$ with the property that for each $a \in K$, all subsets (including the empty set) of a belong to K. Each $a \in K$ is called a *simplex* and has dimension $|a| - 1$. The dimension of K is defined as $\max_{a \in K}\{\dim a\}$.

It is easy to see that f is nice if and only if for each $a \in \mathrm{supp}(f)$, any subset $b \subseteq a$ has the property $f(b) = 1$.

Let $\mathcal{F}_{[m]} = \{f \in 2^{[m]*} | f \text{ is nice}\}$, and $\mathcal{K}_{[m]}$ the set of all abstract simplicial complexes on vertex set A where A runs over all possible subsets in $[m]$.

Proposition 2.1 *All functions of $\mathcal{F}_{[m]}$ bijectively correspond to all abstract simplicial complexes of $\mathcal{K}_{[m]}$.*

Proof Clearly, $f \mapsto \mathrm{supp}(f)$ gives a bijection $\mathcal{F}_{[m]} \longrightarrow \mathcal{K}_{[m]}$, whose inverse is $K \mapsto \sum_{a \in K} \delta_a$. □

2.2 Möbius transform

Based upon Proposition 2.1, we shall carry out our work from the viewpoint of functional analysis.

Definition 2.3 The map $\mathcal{M} : 2^{[m]*} \longrightarrow 2^{[m]*}$ given by the formula

$$\mathcal{M}(f)(a) = \sum_{b \subseteq a} f(b)$$

for all $f \in 2^{[m]*}$ and $a \in 2^{[m]}$ is called the *$\mathbb{Z}/2\mathbb{Z}$-valued Möbius transform*.

Lemma 2.3 *$\mathcal{M}$ is a linear transform such that $\mathcal{M}^2 = \mathrm{id}$. In particular,*

$$\mathcal{M}(\delta_a) = \mu_a \tag{2.1}$$

for any $a \in 2^{[m]}$. Consequently, $\mathcal{M}(\mu_a) = \delta_a$.

Proof The linearity of $\mathcal{M}$ is obvious. To check that $\mathcal{M}^2 = \mathrm{id}$, take $f \in 2^{[m]*}$, one has that for any $a \in 2^{[m]}$

$$\mathcal{M}^2(f)(a) = \sum_{b \subseteq a} \sum_{c \subseteq b} f(c) = \sum_{c \subseteq a} \sum_{b \in [c,a]} f(c) = f(a) + \sum_{c \subsetneq a} \sum_{b \in [c,a]} f(c). \tag{2.2}$$

For every term in the latter sum of (2.2), from $c \subsetneq a$ we see that $[c, a]$ is a boolean subalgebra of $2^{[m]}$ which has 2^k elements for some $k > 0$. So the sum $\sum_{b \in [c,a]} f(c) = 0$ in $\mathbb{Z}/2\mathbb{Z}$. Therefore $\mathcal{M}^2(f)(a) = f(a)$ for any $a \in 2^{[m]}$, so $\mathcal{M}^2(f) = f$ as desired. Equation (2.1) is a direct calculation by Lemma 2.2. □

As a consequence of Lemmas 2.1 and 2.3, one has

Corollary 2.1 $\{\mu_a | a \in 2^{[m]}\}$ *is also a basis of* $2^{[m]*}$.

Remark 4 By definition of $\mathcal{M}$, if $f(\emptyset) = 1$ then $\mathcal{M}(f)(\emptyset) = 1$.

In the next two subsections we shall review the Stanley–Reisner face rings and Tor-algebras. Our main reference is the book by E. Miller and B. Sturmfels [14].

2.3 Stanley–Reisner face ring

Now let $f \in \mathcal{F}_{[m]}$ be a nice function such that $K = \text{supp}(f) \in \mathcal{K}_{[m]}$ is an abstract simplicial complex on $[m]$.

Following the notions of [14], let $\mathbf{k}[\mathbf{v}] = \mathbf{k}[v_1, \dots, v_m]$ be the polynomial algebra over $\mathbf{k}$ on m indeterminates $\mathbf{v} = v_1, \dots, v_m$. Each monomial in $\mathbf{k}[\mathbf{v}]$ has the form of $\mathbf{v}^{\mathbf{a}} = v_1^{a_1} \cdots v_m^{a_m}$ for a vector $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{N}^m$ of nonnegative integers. Thus, $\mathbf{k}[\mathbf{v}]$ is $\mathbb{N}^m$-graded, i.e., $\mathbf{k}[\mathbf{v}]$ is a direct sum $\bigoplus_{\mathbf{a} \in \mathbb{N}^m} \mathbf{k}[\mathbf{v}]_{\mathbf{a}}$ with $\mathbf{k}[\mathbf{v}]_{\mathbf{a}} \cdot \mathbf{k}[\mathbf{v}]_{\mathbf{b}} = \mathbf{k}[\mathbf{v}]_{\mathbf{a}+\mathbf{b}}$ where $\mathbf{k}[\mathbf{v}]_{\mathbf{a}} = \mathbf{k}\{\mathbf{v}^{\mathbf{a}}\}$ is the vector space over $\mathbf{k}$, spanned by $\mathbf{v}^{\mathbf{a}}$. Generally, a $\mathbf{k}[\mathbf{v}]$-module M is $\mathbb{N}^m$*-graded* if $M = \bigoplus_{\mathbf{b} \in \mathbb{N}^m} M_{\mathbf{b}}$ and $\mathbf{v}^{\mathbf{a}} \cdot M_{\mathbf{b}} \subseteq M_{\mathbf{a}+\mathbf{b}}$. Given a vector $\mathbf{a} \in \mathbb{N}^m$, by $\mathbf{k}[\mathbf{v}](-\mathbf{a})$ one denotes the free $\mathbf{k}[\mathbf{v}]$-module generated in degree $\mathbf{a}$. So $\mathbf{k}[\mathbf{v}](-\mathbf{a})$ is isomorphic to the ideal $\langle \mathbf{v}^{\mathbf{a}} \rangle$ as $\mathbb{N}^m$-graded modules. Furthermore, a free $\mathbb{N}^m$-graded module of rank r is isomorphic to the direct sum $\mathbf{k}[\mathbf{v}](-\mathbf{a}_1) \oplus \cdots \oplus \mathbf{k}[\mathbf{v}](-\mathbf{a}_r)$ for some vectors $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{N}^m$.

A monomial $\mathbf{v}^{\mathbf{a}}$ in $\mathbf{k}[\mathbf{v}]$ is said to be *squarefree* if every coordinate of $\mathbf{a}$ is 0 or 1, i.e., $\mathbf{a} \in \{0, 1\}^m$ called a *squarefree vector*. Clearly, all elements in $2^{[m]}$ bijectively correspond to all vectors in $\{0, 1\}^m$ by mapping $\xi : a \in 2^{[m]} \longmapsto \mathbf{a} \in \{0, 1\}^m$, where $\mathbf{a}$ has entry 1 in the ith place when $i \in a$, and 0 in all other entries. With this understanding, for $a \in 2^{[m]}$, one may write $\mathbf{v}^a = \prod_{i \in a} v_i$. Then the *Stanley–Reisner ideal* of K is defined as $I_K = \langle \mathbf{v}^\tau | \tau \notin K \rangle$. Furthermore, the quotient ring

$$\mathbf{k}(K) = \mathbf{k}[\mathbf{v}]/I_K$$

is called the *Stanley–Reisner face ring*.

Example 2.1 If $K = 2^{[m]}$ then $\mathbf{k}(K) = \mathbf{k}[\mathbf{v}]$, and if $K = 2^{[m]} \setminus \{[m]\}$ then $\mathbf{k}(K) = \mathbf{k}[\mathbf{v}]/\langle \mathbf{v}^{[m]} \rangle$.

It is well-known that $\mathbf{k}(K)$ is a finitely generated graded $\mathbf{k}[\mathbf{v}]$-module. Hilbert's syzygy theorem tells us that there exists a free resolution of $\mathbf{k}(K)$ of length at most m. One knows from [14, Sect. 1.4] that $\mathbf{k}(K)$ is $\mathbb{N}^m$-graded and it has an $\mathbb{N}^m$-graded minimal free resolution as follows:

$$0 \longleftarrow \mathbf{k}(K) \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \longleftarrow \cdots \longleftarrow F_{h-1} \xleftarrow{\phi_h} F_h \longleftarrow 0 \tag{2.3}$$

where each homomorphism ϕ_i is $\mathbb{N}^m$-graded degree-preserving. Since each F_i is a free $\mathbb{N}^m$-graded $\mathbf{k}[\mathbf{v}]$-module, one may write $F_i = \bigoplus_{\mathbf{a} \in \mathbb{N}^m} \mathbf{k}[\mathbf{v}](-\mathbf{a})^{\beta_{i,\mathbf{a}}^{\mathbf{k}(K)}}$ where $\beta_{i,\mathbf{a}}^{\mathbf{k}(K)} \in \mathbb{N}$ (see also [14, Sect. 1.5]). By [14, Corollary 1.40], if $\mathbf{a} \in \mathbb{N}^m$ is not squarefree, then $\beta_{i,\mathbf{a}}^{\mathbf{k}(K)} = 0$ for all i. Thus, we only need to consider those $\beta_{i,\mathbf{a}}^{\mathbf{k}(K)}$ with

$\mathbf{a} \in \{0,1\}^m$. Throughout the following we shall write $\beta_{i,a}^{\mathbf{k}(K)} := \beta_{i,\mathbf{a}}^{\mathbf{k}(K)}$ where $a \in 2^{[m]}$ with $\xi(a) = \mathbf{a}$.

Definition 2.4 (Cf. [14, Definition 1.29]) The number $\beta_{i,a}^{\mathbf{k}(K)}$ is called the (i,a)*th Betti number* of $\mathbf{k}(K)$.

2.4 Tor-algebra of $\mathbf{k}(K)$

Applying the functor $\otimes_{\mathbf{k}[\mathbf{v}]}\mathbf{k}$ to the sequence (2.3), one may obtain the following chain complex of $\mathbb{N}^m$-graded $\mathbf{k}[\mathbf{v}]$-modules:

$$0 \longleftarrow F_0 \otimes_{\mathbf{k}[\mathbf{v}]} \mathbf{k} \xleftarrow{\phi_1'} F_1 \otimes_{\mathbf{k}[\mathbf{v}]} \mathbf{k} \longleftarrow \cdots \xleftarrow{\phi_h'} F_h \otimes_{\mathbf{k}[\mathbf{v}]} \mathbf{k} \longleftarrow 0.$$

Since the free resolution (2.3) is minimal, the differentials ϕ_i''s become zero homomorphisms. Then the ith homology module of the above chain complex is $\frac{\ker \phi_i'}{\mathrm{Im}\phi_{i+1}'} = F_i \otimes_{\mathbf{k}[\mathbf{v}]} \mathbf{k}$, denoted by $\mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k})$. Namely, $\mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k}) = F_i \otimes_{\mathbf{k}[\mathbf{v}]} \mathbf{k}$ so

$$\dim_{\mathbf{k}} \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}\big(\mathbf{k}(K),\mathbf{k}\big) = \mathrm{rank}\, F_i = \sum_{a \in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)}.$$

This also implies that for $\mathbf{a} \in \mathbb{N}^m$ with $\mathbf{a} \notin \{0,1\}^m$, $\mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k})_{\mathbf{a}} = 0$, and so $\mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k})$ can be decomposed into a direct sum

$$\bigoplus_{a \in 2^{[m]}} \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}\big(\mathbf{k}(K),\mathbf{k}\big)_a$$

with $\dim_{\mathbf{k}} \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k})_a = \beta_{i,a}^{\mathbf{k}(K)}$ (see also [14, Lemma 1.32]). Furthermore, one has that

$$\mathrm{Tor}^{\mathbf{k}[\mathbf{v}]}\big(\mathbf{k}(K),\mathbf{k}\big) = \bigoplus_{i=0}^{h} \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}\big(\mathbf{k}(K),\mathbf{k}\big) = \bigoplus_{i \in [0,h]\cap\mathbb{N}, a \in 2^{[m]}} \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}\big(\mathbf{k}(K),\mathbf{k}\big)_a$$

which is a bigraded $\mathbf{k}[\mathbf{v}]$-module. Combining with the above arguments, this gives

Proposition 2.2 $\sum_{i=0}^{h} \dim_{\mathbf{k}} \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k}) = \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)}$.

3 Möbius transform of abstract simplicial complexes and Betti numbers of face rings

3.1 An algebraic combinatorics formula

Following Sects. 2.3–2.4, we now investigate the essential relationship between the Möbius transform $\mathcal{M}(f)$ of f and the Betti numbers of the face ring $\mathbf{k}(K)$ of K.

Theorem 3.1 (Algebraic combinatorics formula)

$$\mathcal{M}(f) = \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)} \delta_a.$$

Proof For any $b \in 2^{[m]}$, the exact sequence (2.3) in degree b reads into

$$0 \longleftarrow \mathbf{k}^{D_b} \longleftarrow \mathbf{k}^{d_{b,0}} \longleftarrow \mathbf{k}^{d_{b,1}} \longleftarrow \mathbf{k}^{d_{b,2}} \longleftarrow \cdots \longleftarrow \mathbf{k}^{d_{b,h}} \longleftarrow 0$$

where $D_b = \dim_{\mathbf{k}} \mathbf{k}(K)_b$ and $d_{b,i} = \dim_{\mathbf{k}}(F_i)_b$. Since the above sequence is also exact, we have $D_b = \sum_{i=0}^{h} (-1)^i d_{i,b}$. An easy observation shows that $f(b) = \dim_{\mathbf{k}} \mathbf{k}(K)_b = D_b$, and $d_{b,i} = \sum_{a \subseteq b} \beta_{i,a}^{\mathbf{k}(K)}$ (this is induced from $F_i = \bigoplus_{\mathbf{a} \in \mathbb{N}^m} \mathbf{k}[\mathbf{v}](-\mathbf{a})^{\beta_{i,\mathbf{a}}^{\mathbf{k}(K)}}$).

Now let us work in integers modulo 2. We then have $D_b = \sum_{i=0}^{h} d_{i,b}$, and further

$$f(b) = \sum_{i=0}^{h} \sum_{a \subseteq b} \beta_{i,a}^{\mathbf{k}(K)} = \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)} \mu_a(b).$$

So

$$f = \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)} \mu_a.$$

Applying $\mathcal{M}$ to the above equality and noting that $\mathcal{M}(\mu_a) = \delta_a$, we arrive at the required formula. $\square$

Corollary 3.2 *Let $f \in 2^{[m]*}$ be a nice function such that $K = \operatorname{supp}(f) \in \mathcal{K}_{[m]}$ is an abstract simplicial complex on $[m]$. Then*

$$\left|\operatorname{supp}\bigl(\mathcal{M}(f)\bigr)\right| \leq \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)}.$$

Proof From the formula of Theorem 3.1, one has

$$\mathcal{M}(f) = \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)} \delta_a = \sum_{a \in 2^{[m]}} \left(\sum_{i=0}^{h} \beta_{i,a}^{\mathbf{k}(K)} \right) \delta_a$$

so for any $a \in \operatorname{supp}(\mathcal{M}(f))$, $\sum_{i=0}^{h} \beta_{i,a}^{\mathbf{k}(K)}$ must be odd and nonnegative, and then $\sum_{i=0}^{h} \beta_{i,a}^{\mathbf{k}(K)} \geq 1$. Therefore

$$\sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)} \geq \sum_{a \in \operatorname{supp}(\mathcal{M}(f))} \sum_{i=0}^{h} \beta_{i,a}^{\mathbf{k}(K)} \geq \sum_{a \in \operatorname{supp}(\mathcal{M}(f))} 1 = \left|\operatorname{supp}\bigl(\mathcal{M}(f)\bigr)\right|$$

as desired. $\square$

3.2 The estimation of the lower bound on $|\operatorname{supp}(\mathcal{M}(f))|$

We shall upbuild a method of compressing $\operatorname{supp}(f)$ to get the desired lower bound on $|\operatorname{supp}(\mathcal{M}(f))|$.

Definition 3.1 Fix $k \in [m]$. We say that $f \in \mathcal{F}_{[m]}$ is *kth extendable* if

(k-1) $f(\{k\}) = 1$;
(k-2) $\mathcal{M}(f) \cdot x_k \neq 0$ in $2^{[m]*}$.

The linear transformation $E_k : 2^{[m]*} \longrightarrow 2^{[m]*}$ determined by $\mu_a \mapsto \mu_{a\setminus\{k\}}$ is called the *kth compression-operator*. A function $f \in \mathcal{F}_{[m]}$ is said to be *extendable* if it is kth extendable for some $k \in [m]$; otherwise, f is said to be *non-extendable*.

Introducing the map $\epsilon_k : 2^{[m]} \longrightarrow 2^{[m]}$ defined by $a \mapsto a \cup \{k\}$, we derive the following formula for E_k.

Lemma 3.1 *For any $f \in 2^{[m]*}$ we have*

$$E_k(f) = f \circ \epsilon_k.$$

Proof It suffices to check that the formula $E_k(\mu_a) = \mu_a \circ \epsilon_k$ holds for each $a \in 2^{[m]}$. Indeed, take $b \in 2^{[m]}$, we have

$$E_k(\mu_a)(b) = 1 \quad \Leftrightarrow \quad a \setminus \{k\} \subseteq b \quad \Leftrightarrow \quad a \subseteq b \cup \{k\} \quad \Leftrightarrow \quad \mu_a\big(\epsilon_k(b)\big) = 1.$$

Therefore, $E_k(\mu_a) = \mu_a \circ \epsilon_k$ as desired. □

Remark 5 We see from Lemma 3.1 that E_k is exactly the star operator at k.

Proposition 3.1 *Fix $k \in [m]$. If $f \in \mathcal{F}_{[m]}$ satisfies $f(\{k\}) = 1$, then $E_k(f) \in \mathcal{F}_{[m]}$ and* $\operatorname{supp}(E_k(f)) \subseteq \operatorname{supp}(f)$.

Proof For any pair $a \subseteq b$ in $2^{[m]}$, we have $\epsilon_k(a) \subseteq \epsilon_k(b)$. So if $E_k(f)(b) = f(\epsilon_k(b)) = 1$, then $f(\epsilon_k(a)) = 1$ since $f \in \mathcal{F}_{[m]}$, and so $E_k(f)(a) = 1$. Also, $f(\{k\}) = 1$ implies that $E_k(f)(\emptyset) = f(\emptyset \cup \{k\}) = 1$. Thus, $E_k(f)$ is nice.

For any $a \in 2^{[m]}$, if $E_k(f)(a) = 1$ then by Lemma 3.1 $f(\epsilon_k(a)) = 1$, so $f(a) = 1$ since $a \subseteq \epsilon_k(a)$ and $f \in \mathcal{F}_{[m]}$. Hence, $\operatorname{supp}(E_k(f)) \subseteq \operatorname{supp}(f)$ as desired. □

Now let us look at the composition transformation $\mathcal{M} \circ E_k \circ \mathcal{M} =: \hat{E}_k$. For any $a \in 2^{[m]}$, one has

$$\hat{E}_k(\delta_a) = \mathcal{M} \circ E_k \circ \mathcal{M}(\delta_a) = \mathcal{M} \circ E_k(\mu_a) = \mathcal{M}(\mu_{a\setminus\{k\}}) = \delta_{a\setminus\{k\}}. \tag{3.1}$$

Note also that since $\mathcal{M}^2 = \mathrm{id}$, $\mathcal{M} \circ E_k = \hat{E}_k \circ \mathcal{M}$.

Lemma 3.2 *For any $g \in 2^{[m]*}$ and $k \in [m]$, we have $\hat{E}_k(g)x_k = 0$.*

Proof Write $g = \sum_{a\in \operatorname{supp}(g)} \delta_a$. Since $\hat{E}_k$ is linear and $\hat{E}_k(\delta_a) = \delta_{a\setminus\{k\}}$ for any $a \in 2^{[m]}$, it follows that $\hat{E}_k(g) = \sum_{a\in \operatorname{supp}(g)} \delta_{a\setminus\{k\}}$. Obviously, for any $a \in 2^{[m]}$, $\delta_{a\setminus\{k\}} x_k = 0$. Thus, $\hat{E}_k(g)x_k = 0$ as desired. □

Corollary 3.3 *Let $k \in [m]$. If $f \in 2^{[m]*}$ satisfies $\mathcal{M}(f)x_k \neq 0$, then $f \neq E_k(f)$.*

Proof Suppose that $f = E_k(f)$. Applying $\mathcal{M}$ to both sides, we get $\mathcal{M}(f) = \mathcal{M}(E_k(f)) = \hat{E}_k(\mathcal{M}(f))$. Write $g = \mathcal{M}(f)$. Then $g = \hat{E}_k(g)$. Multiplying by x_k on the two sides of $g = \hat{E}_k(g)$, we have $gx_k = \hat{E}_k(g)x_k$. Since $gx_k = \mathcal{M}(f)x_k \neq 0$, we have $\hat{E}_k(g)x_k \neq 0$, a contradiction by Lemma 3.2. □

Proposition 3.2 *Let $f \in 2^{[m]*}$. Then for each $k \in [m]$,*

$$\bigl|\operatorname{supp}\bigl(\hat{E}_k(f)\bigr)\bigr| \le \bigl|\operatorname{supp}(f)\bigr|.$$

Proof Let $A = \{a \in 2^{[m]} | k \notin a, a \in \operatorname{supp}(f)\}$ and $B = \{a \in 2^{[m]} | k \notin a, \epsilon_k(a) \in \operatorname{supp}(f)\}$. Then we have

$$f = \sum_{a\in\operatorname{supp}(f)} \delta_a = \sum_{\substack{a\in\operatorname{supp}(f)\\ k\notin a}} \delta_a + \sum_{\substack{a\in\operatorname{supp}(f)\\ k\in a}} \delta_a = \sum_{a\in A} \delta_a + \sum_{a\in B} \delta_{\epsilon_k(a)}$$

and by (3.1)

$$\hat{E}_k(f) = \sum_{a\in A} \delta_{a\setminus\{k\}} + \sum_{a\in B} \delta_{\epsilon_k(a)\setminus\{k\}} = \sum_{a\in A} \delta_a + \sum_{a\in B} \delta_a = \sum_{a\in A\triangle B} \delta_a$$

where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Now

$$\bigl|\operatorname{supp}\bigl(\hat{E}_k(f)\bigr)\bigr| = |A \triangle B| \le |A| + |B| = \bigl|\operatorname{supp}(f)\bigr|$$

as desired. □

Remark 6 Observe that for any $f \in \mathcal{F}_{[m]}$, whenever f is kth extendable for some $k \in [m]$, by Proposition 3.1 and Corollary 3.3 we obtain that $E_k(f) \in \mathcal{F}_{[m]}$ and $\operatorname{supp}(E_k(f)) \subsetneq \operatorname{supp}(f)$. In addition, since $(\mathcal{M} \circ E_k)(f) = (\hat{E}_k \circ \mathcal{M})(f)$, by Proposition 3.2 one has that $|\operatorname{supp}(\mathcal{M}(E_k(f)))| \le |\operatorname{supp}(\mathcal{M}(f))|$. We replace f with $E_k(f)$ and repeat the above process whenever possible, so as to get a sequence of functions in $\mathcal{F}_{[m]}$ with strictly decreasing support. This process must end after a finite number of steps, giving finally a $f_0 \in \mathcal{F}_{[m]}$ that is *non-extendable* with $\operatorname{supp}(f_0) \subseteq \operatorname{supp}(f)$ and $|\operatorname{supp}(\mathcal{M}(f_0))| \le |\operatorname{supp}(\mathcal{M}(f))|$. It remains to characterize such a non-extendable $f_0 \in \mathcal{F}_{[m]}$.

Proposition 3.3 *Let $f \in \mathcal{F}_{[m]}$. Then f is non-extendable if and only if there is some $a_0 \in 2^{[m]}$ such that $\operatorname{supp}(f) = 2^{a_0}$ (i.e., $f = \sum_{b\subseteq a_0} \delta_b$).*

Proof Suppose that f is non-extendable. Let $a_0 = \{k \in [m] | f(\{k\}) = 1\}$. If $a_0 = \emptyset$, obviously we have $f = \delta_\emptyset$. Assume that a_0 is non-empty. Given an element $b \in 2^{[m]}$, if $f(b)=1$, since $f\in\mathcal{F}_{[m]}$, then for any $k\in b$, $f(\{k\})=1$ so $k\in a_0$ and $b\subseteq a_0$. Since f is non-extendable, $\mathcal{M}(f)x_k=0$ for any $k\in a_0$. Then we see from $\mathcal{M}(f)=\sum_{b\in\mathrm{supp}(\mathcal{M}(f))}\delta_b$ that for any $b\in\mathrm{supp}(\mathcal{M}(f))$, $b\cap a_0=\emptyset$. Since $\mathcal{M}(f)(\emptyset)=1$, we have $\emptyset\in\mathrm{supp}(\mathcal{M}(f))$. Furthermore

$$\begin{aligned} f(a_0)=\mathcal{M}^2(f)(a_0)&=\mathcal{M}\Bigg(\sum_{b\in\mathrm{supp}(\mathcal{M}(f))}\delta_b\Bigg)(a_0)\\ &=\sum_{b\in\mathrm{supp}(\mathcal{M}(f))}\mu_b(a_0)=\mu_\emptyset(a_0)=1. \end{aligned}$$

Since $f\in\mathcal{F}_{[m]}$, it follows that for any subset $b\subseteq a_0$, $f(b)=1$. Therefore, for $b\in 2^{[m]}$

$$f(b)=1\quad\Leftrightarrow\quad b\subseteq a_0.$$

This implies that $\mathrm{supp}(f)=2^{a_0}=\{b\in 2^{[m]}|b\subseteq a_0\}$.

Conversely, suppose that $f=\sum_{b\subseteq a_0}\delta_b$ for some $a_0\in[m]$. If $a_0=[m]$, then $f=\underline{1}=\mu_\emptyset$ so $\mathcal{M}(f)=\delta_\emptyset$. Moreover, for any $k\in a_0$, $\mathcal{M}(f)x_k=0$ so f is non-extendable. If $a_0=\emptyset$, obviously f is non-extendable. Assume that $a_0\neq[m],\emptyset$. Then an easy argument shows that

$$f=\prod_{i\in[m]\setminus a_0}(\underline{1}+x_i)=\sum_{b\subseteq[m]\setminus a_0}\mu_b.$$

Applying $\mathcal{M}$ to the above equality, it follows that $\mathcal{M}(f)=\sum_{b\subseteq[m]\setminus a_0}\delta_b$. Now for any $k\in a_0$ and any $b\subseteq[m]\setminus a_0$, we have $\delta_b x_k=0$ so $\mathcal{M}(f)x_k=0$. This means that f is also non-extendable. □

From the proof of Proposition 3.3, we easily see that

Corollary 3.4 *Let $a\in 2^{[m]}$. Then $f=\sum_{b\subseteq a}\delta_b$ if and only if $\mathcal{M}(f)=\sum_{b\subseteq[m]\setminus a}\delta_b$ (i.e., $\mathrm{supp}(f)=2^a$ if and only if $\mathrm{supp}(\mathcal{M}(f))=2^{[m]\setminus a}$). In this case, $|\mathrm{supp}(\mathcal{M}(f))|=2^{m-|a|}$.*

We now summarize the above arguments as follows.

Theorem 3.5 *For any $f\in\mathcal{F}_{[m]}$, there exists some $a\in\mathrm{supp}(f)$ such that*

$$\big|\mathrm{supp}\big(\mathcal{M}(f)\big)\big|\geq 2^{m-|a|}.$$

Remark 7 The interested readers are invited to see a simple fact that $f\in\mathcal{F}_{[m]}$ can be compressed by compression-operators into a non-extendable f_0 with $\mathrm{supp}(f_0)=2^{a_0}$ if and only if a_0 is a maximal element in $\mathrm{supp}(f)$ as a poset. This result will not be used later in this article.

4 Moment-angle complexes and their cohomologies

Let K be an abstract simplicial complex on vertex set $[m]$. Let (X, W) be a pair of topological spaces with $W \subset X$. Following [5, Construction 6.38], for each simplex σ in K, set

$$B_\sigma(X, W) = \prod_{i=1}^{m} A_i$$

such that

$$A_i = \begin{cases} X & \text{if } i \in \sigma, \\ W & \text{if } i \in [m] \setminus \sigma. \end{cases}$$

Then one can define the following subspace of the product space X^m:

$$K(X, W) = \bigcup_{\sigma \in K} B_\sigma(X, W) \subset X^m.$$

4.1 Moment-angle complexes

When the pair (X, W) is chosen as (D^2, S^1),

$$\mathcal{Z}_K := K\big(D^2, S^1\big) \subset \big(D^2\big)^m$$

is called the *moment-angle complex* on K where $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is the unit disk in $\mathbb{C}$, and $S^1 = \partial D^2$. Since $(D^2)^m \subset \mathbb{C}^m$ is invariant under the standard action of T^m on $\mathbb{C}^m$ given by

$$\big((g_1, \dots, g_m), (z_1, \dots, z_m)\big) \longmapsto (g_1 z_1, \dots, g_m z_m),$$

$(D^2)^m$ admits a natural T^m-action whose orbit space is the unit cube $I^m \subset \mathbb{R}^m_{\geq 0}$. The action $T^m \curvearrowright (D^2)^m$ then induces a canonical T^m-action Φ on $\mathcal{Z}_K$.

When the pair (X, W) is chosen as (D^1, S^0),

$$\mathbb{R}\mathcal{Z}_K := K\big(D^1, S^0\big) \subset \big(D^1\big)^m$$

is called the *real moment-angle complex* on K where $D^1 = \{x \in \mathbb{R} \mid |x| \leq 1\} = [-1, 1]$ is the unit disk in $\mathbb{R}$, and $S^0 = \partial D^1 = \{\pm 1\}$. Similarly, $(D^1)^m \subset \mathbb{R}^m$ is invariant under the standard action of $(\mathbb{Z}_2)^m$ on $\mathbb{R}^m$ given by

$$\big((g_1, \dots, g_m), (x_1, \dots, x_m)\big) \longmapsto (g_1 x_1, \dots, g_m x_m).$$

Thus $(D^1)^m$ admits a natural $(\mathbb{Z}_2)^m$-action whose orbit space is also the unit cube $I^m \subset \mathbb{R}^m_{\geq 0}$, where $\mathbb{Z}_2 = \{-1, 1\}$ is the group with respect to multiplication. Furthermore, the action $(\mathbb{Z}_2)^m \curvearrowright (D^1)^m$ also induces a canonical $(\mathbb{Z}_2)^m$-action $\Phi_{\mathbb{R}}$ on $\mathbb{R}\mathcal{Z}_K$.

Let P_K be the cone on the barycentric subdivision of K. Since the cone on the barycentric subdivision of a k-simplex is combinatorially equivalent to the standard subdivision of a $(k+1)$-cube, P_K is naturally a cubical complex and it is decomposed into cubes indexed by the simplices of K. Then one knows from [5] and [11] that both T^m-action Φ on $\mathcal{Z}_K$ and $(\mathbb{Z}_2)^m$-action $\Phi_{\mathbb{R}}$ on $\mathbb{R}\mathcal{Z}_K$ have the same orbit space P_K.

Example 4.1 When $K = 2^{[m]}$, $\mathcal{Z}_K = (D^2)^m$ and $\mathbb{R}\mathcal{Z}_K = (D^1)^m$. When $K = 2^{[m]} \setminus \{[m]\}$, $\mathcal{Z}_K = S^{2m-1}$ and $\mathbb{R}\mathcal{Z}_K = S^{m-1}$.

Remark 8 In general, $\mathcal{Z}_K$ and $\mathbb{R}\mathcal{Z}_K$ are not manifolds. However, if K is a simplicial sphere, then both $\mathcal{Z}_K$ and $\mathbb{R}\mathcal{Z}_K$ are closed manifolds (see [5, Lemma 6.13]).

4.2 Cohomology

V.M. Buchstaber and T.E. Panov in [5, Theorem 7.6] have calculated the cohomology of $\mathcal{Z}_K$ (see also [15, Theorem 4.7]). Their result is stated as follows.

Theorem 4.1 (Buchstaber–Panov) *As* $\mathbf{k}$-*algebras*,

$$H^*(\mathcal{Z}_K; \mathbf{k}) \cong \operatorname{Tor}^{\mathbf{k}[\mathbf{v}]}\big(\mathbf{k}(K), \mathbf{k}\big)$$

where $\mathbf{k}(K) = \mathbf{k}[\mathbf{v}]/I_K = \mathbf{k}[v_1, \dots, v_m]/I_K$ *with* $\deg v_i = 2$.

Here we calculate the cohomologies of a class of generalized moment-angle complexes. For this, we begin with the notion of the generalized moment-angle complex, due to N. Strickland, cf. [4] and [12]. Given an abstract simplicial complex K on $[m]$, let $(\underline{X}, \underline{W}) = \{(X_i, W_i)\}_{i=1}^m$ be m pairs of CW-complexes with $W_i \subset X_i$. Then the *generalized moment-angle complex* is defined as follows:

$$K(\underline{X}, \underline{W}) = \bigcup_{\sigma \in K} B_\sigma(\underline{X}, \underline{W}) \subset \prod_{i=1}^m X_i$$

where $B_\sigma(\underline{X}, \underline{W}) = \prod_{i=1}^m H_i$ and

$$H_i = \begin{cases} X_i & \text{if } i \in \sigma, \\ W_i & \text{if } i \in [m] \setminus \sigma. \end{cases}$$

Now take $(\underline{X}, \underline{W}) = (\underline{\mathbb{D}}, \underline{\mathbb{S}}) = \{(\mathbb{D}_i, \mathbb{S}_i)\}_{i=1}^m$ with each CW-complex pair $(\mathbb{D}_i, \mathbb{S}_i)$ subject to the following conditions:

(1) $\mathbb{D}_i$ is acyclic, that is, $\widetilde{H}_j(\mathbb{D}_i) = 0$ for any j.
(2) There exists a unique κ_i such that $\widetilde{H}_{\kappa_i}(\mathbb{S}_i) = \mathbb{Z}$ and $\widetilde{H}_j(\mathbb{S}_i) = 0$ for any $j \neq \kappa_i$.

Then our objective is to calculate the cohomology of

$$\mathcal{Z}_K^{(\underline{\mathbb{D}}, \underline{\mathbb{S}})} := K(\underline{\mathbb{D}}, \underline{\mathbb{S}}) = \bigcup_{\sigma \in K} B_\sigma(\underline{\mathbb{D}}, \underline{\mathbb{S}}) \subset \prod_{i=1}^m \mathbb{D}_i.$$

First, for each $i \in [m]$, it follows immediately from the long exact sequence of $(\mathbb{D}_i, \mathbb{S}_i)$ that

$$0 = \widetilde{H}^{\kappa_i}(\mathbb{D}_i; \mathbf{k}) \longrightarrow \widetilde{H}^{\kappa_i}(\mathbb{S}_i; \mathbf{k}) \xrightarrow{\cong} \widetilde{H}^{\kappa_i+1}(\mathbb{D}_i, \mathbb{S}_i; \mathbf{k}) \longrightarrow \widetilde{H}^{\kappa_i+1}(\mathbb{D}_i; \mathbf{k}) = 0.$$

On the cellular cochain level, one has the following short exact sequence

$$0 \longrightarrow C^*(\mathbb{D}_i, \mathbb{S}_i; \mathbf{k}) \xrightarrow{j^*} C^*(\mathbb{D}_i; \mathbf{k}) \xrightarrow{i^*} C^*(\mathbb{S}_i; \mathbf{k}) \longrightarrow 0$$

where each $C^k(\mathbb{D}_i, \mathbb{S}_i; \mathbf{k})$ can be considered as a subgroup of $C^k(\mathbb{D}_i; \mathbf{k})$, so j^* is an inclusion. By the zig-zag lemma, one can choose a κ_i-cochain x_i of $C^{\kappa_i}(\mathbb{D}_i; \mathbf{k})$ such that

- $i^*(x_i)$ represents a generator of $\widetilde{H}^{\kappa_i}(\mathbb{S}_i; \mathbf{k})$.
- $dx_i \in \ker i^* = \operatorname{Im} j^*$ so $j^*(dx_i) = dx_i$ may be regarded as a cocycle in $C^{\kappa_i+1}(\mathbb{D}_i, \mathbb{S}_i; \mathbf{k})$ since j^* is an inclusion, where d is the coboundary operator of $C^*(\mathbb{D}_i; \mathbf{k})$. Thus, the cohomological class of dx_i generates $\widetilde{H}^{\kappa_i+1}(\mathbb{D}_i, \mathbb{S}_i; \mathbf{k})$.

Write $x_i^{(1)} = x_i$ and $x_i^{(2)} = dx_i$, and let $x_i^{(0)}$ denote the constant 0-cochain 1 in $C^0(\mathbb{D}_i; \mathbf{k})$. Obviously, $x_i^{(0)}$, $x_i^{(1)}$ and $x_i^{(2)}$ are linearly independent in $C^*(\mathbb{D}_i; \mathbf{k})$ as a $\mathbf{k}$-vector space.

Now let us work in the cellular cochain complex $C^*(\prod_{i=1}^m \mathbb{D}_i; \mathbf{k})$ of the product space $\prod_{i=1}^m \mathbb{D}_i$. Let Ω^* be the vector subspace of $C^*(\prod_{i=1}^m \mathbb{D}_i; \mathbf{k})$ spanned by the following cross products

$$x_1^{(k_1)} \times \cdots \times x_m^{(k_m)}, \quad k_i \in \{0, 1, 2\}.$$

An easy observation shows that Ω^* is a cochain subcomplex of $C^*(\prod_{i=1}^m \mathbb{D}_i; \mathbf{k})$, and $\{x_1^{(k_1)} \times \cdots \times x_m^{(k_m)} \mid k_i \in \{0, 1, 2\}\}$ forms a basis of Ω^* as a vector space over $\mathbf{k}$ since $x_i^{(0)}$, $x_i^{(1)}$ and $x_i^{(2)}$ are linearly independent in $C^*(\mathbb{D}_i; \mathbf{k})$. For convenience, we write each basis element $x_1^{(k_1)} \times \cdots \times x_m^{(k_m)}$ of Ω^* in the following form:

$$\mathbf{x}^{(\tau,\sigma)}$$

where $\mathbf{x} = x_1^{(k_1)}, \ldots, x_m^{(k_m)}$, $\tau = \{i \mid k_i = 1\}$ and $\sigma = \{i \mid k_i = 2\}$. In particular, if $\tau = \sigma = \emptyset$, then $\mathbf{x}^{(\emptyset,\emptyset)} = x_1^{(0)} \times \cdots \times x_m^{(0)}$. Thus, Ω^* can be expressed as

$$\Omega^* = \operatorname{Span}\left\{\mathbf{x}^{(\tau,\sigma)} \middle| \tau, \sigma \subseteq [m] \text{ with } \tau \cap \sigma = \emptyset\right\}.$$

Next by Φ_K we denote the composition

$$\Omega^* \hookrightarrow C^*\left(\prod_{i=1}^m \mathbb{D}_i; \mathbf{k}\right) \xrightarrow{l^*} C^*\left(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}; \mathbf{k}\right)$$

where the latter map l^* is induced by the inclusion $l : \mathcal{Z}_K^{(\mathbb{D},\mathbb{S})} \hookrightarrow \prod_{i=1}^m \mathbb{D}_i$, and is surjective. Set

$$S_K = \operatorname{Span}\left\{\mathbf{x}^{(\tau,\sigma)} \in \Omega^* \middle| \sigma \notin K\right\}.$$

Clearly S_K is a cochain subcomplex of Ω^*.

Lemma 4.1 *$S_K \subseteq \ker \Phi_K$. Furthermore, Φ_K induces a cochain map $\Omega^*/S_K \longrightarrow C^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}; \mathbf{k})$, also denoted by Φ_K.*

Proof Let $\mathbf{x}^{(\tau,\sigma)}$ be a basis element in $S_K \subset C^*(\prod_{i=1}^m \mathbb{D}_i;\mathbf{k})$. For any product cell $e = e_1 \times \cdots \times e_m \subset \mathcal{Z}_K^{(\mathbb{D},\mathbb{S})} \subseteq \prod_{i=1}^m \mathbb{D}_i$, there must be some $\sigma' \in K$ such that $e \subset B_{\sigma'}(\underline{\mathbb{D}},\underline{\mathbb{S}})$, where each e_i can represent a generator in the cellular chain group $C_{\dim e_i}(\mathbb{D}_i;\mathbf{k})$. In addition, it is easy to see that e can also be regarded as a generator of the cellular chain complex $C_*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k}) \xhookrightarrow{l_*} C_*(\prod_{i=1}^m \mathbb{D}_i;\mathbf{k})$ where l_* is the inclusion induced by $l : \mathcal{Z}_K^{(\mathbb{D},\mathbb{S})} \hookrightarrow \prod_{i=1}^m \mathbb{D}_i$. Since $\sigma \notin K$, σ is non-empty. Moreover, there is some $i_0 \in \sigma \setminus \sigma'$ such that $e_{i_0} \subset \mathbb{S}_{i_0} \subset \mathbb{D}_{i_0}$ and the factor $x_{i_0}^{(2)} \in C^{\kappa_{i_0}+1}(\mathbb{D}_{i_0},\mathbb{S}_{i_0};\mathbf{k}) \subset C^{\kappa_{i_0}+1}(\mathbb{D}_{i_0};\mathbf{k})$ in $\mathbf{x}^{(\tau,\sigma)}$, together yielding that $\langle x_{i_0}^{(2)}, e_{i_0}\rangle = 0$. Therefore, $\langle \mathbf{x}^{(\tau,\sigma)}, l_*(e)\rangle = \langle \mathbf{x}^{(\tau,\sigma)}, e\rangle = 0$ by the definition of cross product. Furthermore, we see that the value of $\Phi_K(\mathbf{x}^{(\tau,\sigma)})$ on e is

$$\big\langle \Phi_K\big(\mathbf{x}^{(\tau,\sigma)}\big), e\big\rangle = \big\langle \mathbf{x}^{(\tau,\sigma)} \circ l_*,\ e\big\rangle = \big\langle \mathbf{x}^{(\tau,\sigma)}, l_*(e)\big\rangle = 0$$

so $\Phi_K(\mathbf{x}^{(\tau,\sigma)}) = 0$ in $C^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k})$, as desired. □

By $\Omega^*(K)$ we denote the quotient Ω^*/S_K. Let L be a subcomplex of K. Then we obtain a pair $(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}, \mathcal{Z}_L^{(\mathbb{D},\mathbb{S})})$ of CW-complexes. Now since $S_K \subseteq S_L$, we have a short exact sequence

$$0 \longrightarrow \ker\pi^* \longrightarrow \Omega^*(K) \xrightarrow{\pi^*} \Omega^*(L) \longrightarrow 0 \tag{4.1}$$

where π^* is induced by the natural inclusion $\pi : S_K \hookrightarrow S_L$. By $\Omega^*(K,L)$ we denote the kernel $\ker\pi^*$. It is easy to see that two cochain maps $\Phi_K : \Omega^*(K) \longrightarrow C^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k})$ and $\Phi_L : \Omega^*(L) \longrightarrow C^*(\mathcal{Z}_L^{(\mathbb{D},\mathbb{S})};\mathbf{k})$ give a cochain map $\Phi_{(K,L)} : \Omega^*(K,L) \longrightarrow C^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}, \mathcal{Z}_L^{(\mathbb{D},\mathbb{S})};\mathbf{k})$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \Omega^*(K,L) \arrow[r] \arrow[d, "\Phi_{(K,L)}"'] & \Omega^*(K) \arrow[r, "\pi^*"] \arrow[d, "\Phi_K"'] & \Omega^*(L) \arrow[r] \arrow[d, "\Phi_L"] & 0 \\
0 \arrow[r] & C^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}, \mathcal{Z}_L^{(\mathbb{D},\mathbb{S})};\mathbf{k}) \arrow[r] & C^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k}) \arrow[r] & C^*(\mathcal{Z}_L^{(\mathbb{D},\mathbb{S})};\mathbf{k}) \arrow[r] & 0.
\end{tikzcd}$$

Furthermore, we may obtain a homomorphism between two long exact cohomology sequences given by two short exact sequences above.

Proposition 4.1 *For any $K \in \mathcal{K}_{[m]}$, Φ_K induces an isomorphism*

$$H^*\big(\Omega^*(K);\mathbf{k}\big) \xrightarrow{\cong} H^*\big(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k}\big)$$

as graded $\mathbf{k}$*-modules.*

Proof First observe that for $K = \{\emptyset\}$, $\Omega^*(K)$ is spanned by $\{\mathbf{x}^{(\tau,\emptyset)} | \tau \subseteq [m]\}$ with zero coboundary operator. On the other hand, if $K = \{\emptyset\}$ then $\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})} = \prod_{i=1}^m \mathbb{S}_i$. By the Künneth formula, the above set is not only a set but also a basis of $H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k})$

as a graded $\mathbf{k}$-module (if we view the elements of the set as cohomological classes). Thus, clearly Φ_K induces an isomorphism in this case.

Next we proceed inductively by considering a pair of abstract simplicial complexes (K, L) where $K = L \sqcup \{\sigma_0\}$ for some simplex σ_0 (which is a maximal element of K as a poset). Hence $(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}, \mathcal{Z}_L^{(\mathbb{D},\mathbb{S})})$ is a pair of CW-complexes, which has by excision the same cohomology as $(\mathcal{Z}_{2^{\sigma_0}}^{(\mathbb{D},\mathbb{S})}, \mathcal{Z}_{2^{\sigma_0}\setminus\{\sigma_0\}}^{(\mathbb{D},\mathbb{S})})$. This pair $(\mathcal{Z}_{2^{\sigma_0}}^{(\mathbb{D},\mathbb{S})}, \mathcal{Z}_{2^{\sigma_0}\setminus\{\sigma_0\}}^{(\mathbb{D},\mathbb{S})})$ is in turn homeomorphic to

$$\prod_{i\in[m]\setminus\sigma_0} \mathbb{S}_i \times \Bigl(\prod_{i\in\sigma_0} \mathbb{D}_i, A\Bigl(\prod_{i\in\sigma_0} \mathbb{D}_i\Bigr)\Bigr)$$

where $A(\prod_{i\in\sigma_0} \mathbb{D}_i) = (\mathbb{S}_{i_1} \times \mathbb{D}_{i_2} \times \cdots \times \mathbb{D}_{i_s}) \cup \cdots \cup (\mathbb{D}_{i_1} \times \cdots \times \mathbb{D}_{i_{s-1}} \times \mathbb{S}_{i_s})$ with $\sigma_0 = \{i_1, \dots, i_s | i_1 < \cdots < i_s\}$. By relative Künneth formula, its cohomology with $\mathbf{k}$ coefficients is isomorphic to

$$\mathrm{Span}\bigl\{\mathbf{x}^{(\tau,\sigma_0)} \bigm| \tau \subseteq [m] \text{ with } \tau \cap \sigma_0 = \emptyset\bigr\}$$

as graded $\mathbf{k}$-modules. On the other hand, we see easily from the short exact sequence (4.1) that $\Omega^*(K, L) = \ker \pi^*$ is exactly equal to the cochain complex

$$\mathrm{Span}\bigl\{\mathbf{x}^{(\tau,\sigma_0)} \bigm| \tau \subseteq [m] \text{ with } \tau \cap \sigma_0 = \emptyset\bigr\}$$

with zero coboundary operator. It then follows that $\Phi_{(K,L)}$ induces an isomorphism

$$H^*\bigl(\Omega^*(K, L); \mathbf{k}\bigr) \xrightarrow{\cong} H^*\bigl(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}, \mathcal{Z}_L^{(\mathbb{D},\mathbb{S})}; \mathbf{k}\bigr)$$

as graded $\mathbf{k}$-modules. Inductively, now we may assume that Φ_L induces an isomorphism $H^*(\Omega^*(L); \mathbf{k}) \longrightarrow H^*(\mathcal{Z}_L^{(\mathbb{D},\mathbb{S})}; \mathbf{k})$ as graded $\mathbf{k}$-modules. Hence we may conclude that the same holds for $H^*(\Omega^*(K); \mathbf{k}) \longrightarrow H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}; \mathbf{k})$ by the five-lemma. This completes the induction and the proof of Proposition 4.1. □

Now let us return to study the complex $(\Omega^*(K), \underline{d})$. First, we may impose a $\{0, 1\}^m$-graded (or $2^{[m]}$-graded) structure on $\Omega^*(K)$, by defining for $a \in 2^{[m]}$

$$\Omega^*(K)_a := \mathrm{Span}\bigl\{\mathbf{x}^{(\tau,\sigma)} \bigm| \tau \subseteq [m], \sigma \in K \text{ with } \tau \cup \sigma = a, \tau \cap \sigma = \emptyset\bigr\}.$$

Then, clearly $\Omega^*(K) = \bigoplus_{a\in 2^{[m]}} \Omega^*(K)_a$. Furthermore, given a basis element $\mathbf{x}^{(\tau,\sigma)} \in \Omega^*(K)_a$ with $\tau = a \setminus \sigma$, by a direct calculation we have

$$\underline{d}\bigl(\mathbf{x}^{(a\setminus\sigma,\sigma)}\bigr) = \sum_{\substack{k\in a\setminus\sigma \\ \sigma\cup\{k\}\in K}} \epsilon_k \mathbf{x}^{(a\setminus(\sigma\cup\{k\}),\sigma\cup\{k\})}$$

which still belongs to $\Omega^*(K)_a$, where $\epsilon_k = \pm 1$. So $(\Omega^*(K)_a, \underline{d})$ has also a cochain complex structure. This means that $\Omega^*(K)$ is a bigraded $\mathbf{k}$-module. Also, clearly the basis of $\Omega^*(K)_a$ is indexed by $K|_a$ where $K|_a = \{\sigma \in K | \sigma \subseteq a\}$.

Lemma 4.2 *For each $a \in 2^{[m]}$, $(\Omega^*(K)_a, \underline{d})$ is isomorphic to the coaugmented cochain complex $(C^*(K|_a; \mathbf{k}), d')$ as cochain complexes. Furthermore, $H^*(\Omega^*(K)_a; \mathbf{k}) \cong \widetilde{H}^*(K|_a; \mathbf{k})$ as graded $\mathbf{k}$-modules.*

Lemma 4.2 is a (dualized) consequence of the following general result.

Lemma 4.3 *Let K be an abstract simplicial complex on a finite set. Let $V(K)$ be a vector space over $\mathbf{k}$ with a K-indexed basis $\{v_\sigma | \sigma \in K\}$, and let $\iota : V(K) \longrightarrow V(K)$ be a linear map such that $\iota^2 = 0$ and $\iota(v_\sigma) = \sum_{k\in\sigma} \varepsilon_k v_{\sigma\setminus\{k\}}$ where $\varepsilon_k = \pm 1$. Then there is an isomorphism $f : V(K) \longrightarrow C_*(K; \mathbf{k})$ as $\mathbf{k}$-vector spaces with form $f : v_\sigma \longmapsto \varepsilon_\sigma \sigma$ such that $f \circ \iota = \partial \circ f$, where $\varepsilon_\sigma = \pm 1$ and $C_*(K; \mathbf{k})$ is the ordinary chain complex over $\mathbf{k}$ of K with the boundary operator ∂.*

Proof We proceed inductively. For $K = \{\emptyset\}$, $V(K) = \mathrm{Span}\{v_\emptyset\} \cong \mathbf{k}$ with $\iota = 0$ and $C_*(K; \mathbf{k}) = \mathrm{Span}\{\emptyset\} \cong \mathbf{k}$ with $\partial = 0$, so clearly we have such an f. Now for an arbitrary $K \neq \{\emptyset\}$, take a maximal element σ_0 of K (as a poset) so that $L = K \setminus \{\sigma_0\}$ is a subcomplex of K. The subspace $V(K)|_L = \mathrm{Span}\{v_\sigma | \sigma \in L\}$ is invariant under ι. So we can apply induction hypothesis to $(V(K)|_L, \iota)$, yielding an isomorphism $f_0 : V(K)|_L \longrightarrow C_*(L; \mathbf{k})$ by $v_\sigma \longmapsto \varepsilon_\sigma \sigma$ such that $f_0 \circ \iota = \partial \circ f_0$. Now observe that $\iota(v_{\sigma_0}) = \sum_{k\in\sigma_0} \varepsilon_k v_{\sigma_0\setminus\{k\}} \in V(K)|_L$, so $f_0(\iota(v_{\sigma_0})) = \sum_{k\in\sigma_0} \varepsilon_k \varepsilon_{\sigma_0\setminus\{k\}}(\sigma_0 \setminus \{k\})$ which is in the chain group $C_{|\sigma_0|-2}(2^{\sigma_0}; \mathbf{k}) \subset C_{|\sigma_0|-2}(L; \mathbf{k})$, and $(\partial \circ f_0)(\iota(v_{\sigma_0})) = (f_0 \circ \iota)(\iota(v_{\sigma_0})) = f_0(\iota^2(v_{\sigma_0})) = 0$, i.e., $f_0(\iota(v_{\sigma_0})) \in \ker\partial$. Since $C_*(2^{\sigma_0}; \mathbf{k})$ is acyclic and $C_{|\sigma_0|-1}(2^{\sigma_0}; \mathbf{k}) = \mathrm{Span}\{\sigma_0\}$, we have $f_0(\iota(v_{\sigma_0})) = \partial(n\sigma_0)$ for some $n \in \mathbf{k}$. However, $\partial(n\sigma_0) = n\partial(\sigma_0)$ so $n\partial(\sigma_0) = \sum_{k\in\sigma_0} \varepsilon_k \varepsilon_{\sigma_0\setminus\{k\}}(\sigma_0 \setminus \{k\})$. This forces n to be ± 1. We can then extend f_0 to $f : V(K) \longrightarrow C_*(K; \mathbf{k})$ by defining $v_{\sigma_0} \longmapsto n\sigma_0$, so that we have

$$f\big(\iota(v_{\sigma_0})\big) = f_0\big(\iota(v_{\sigma_0})\big) = \partial(n\sigma_0) = \partial\big(f(v_{\sigma_0})\big).$$

Hence $f \circ \iota = \partial \circ f$ in $V(K)$. The induction step is finished, proving the lemma. $\square$

The famous Hochster formula tells us (see [14, Corollary 5.12]) that for each $a \in 2^{[m]}$,

$$\widetilde{H}^{|a|-i-1}(K|_a; \mathbf{k}) \cong \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}\big(\mathbf{k}(K), \mathbf{k}\big)_a.$$

We know by Lemma 4.2 that each class of $\widetilde{H}^{|a|-i-1}(K|_a; \mathbf{k})$ may be understood as one of $H^*(\Omega^*(K)_a; \mathbf{k})$, represented by a linear combination of the elements of the form $\mathbf{x}^{(a\setminus\sigma,\sigma)} \in \Omega^*(K)_a$ with $|\sigma| = |a| - i$; so by Proposition 4.1 it corresponds to a cohomological class of degree $|\sigma| + \sum_{k\in a} \kappa_k = -i + \sum_{k\in a}(\kappa_k + 1)$ in $H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}; \mathbf{k})$. To sum up, it follows that for each $n \geq 0$,

$$H^n\big(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})}; \mathbf{k}\big) \cong \bigoplus_{\substack{a\in 2^{[m]} \\ -i+\sum_{k\in a}(\kappa_k+1)=n}} \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}\big(\mathbf{k}(K), \mathbf{k}\big)_a.$$

Combining with all arguments above, we conclude that

Theorem 4.2 *As graded* $\mathbf{k}$*-modules*,

$$H^*\big(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k}\big) \cong \mathrm{Tor}^{\mathbf{k}[\mathbf{v}]}\big(\mathbf{k}(K),\mathbf{k}\big).$$

Together with Proposition 2.2 and Theorem 4.2, we obtain that

Corollary 4.3 $\sum_i \dim_{\mathbf{k}} H^i(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k}) = \sum_{i=0}^h \sum_{a\in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)}$.

Remark 9 It should be pointed out that here we merely determine the $\mathbf{k}$-module structure of $H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k})$. Of course, this is enough for our purpose in this paper. Observe that if there are two $i,j\in[m]$ with $i\neq j$ such that κ_i and κ_j are even, then for $x_i^{(2)}, x_j^{(2)} \in \Omega^*(K)$, $x_i^{(2)} \times x_j^{(2)} = -x_j^{(2)} \times x_i^{(2)}$. This means that in this case, if $\mathbf{k}$ is not a field of characteristic 2, then $H^*(\Omega^*(K);\mathbf{k})$ cannot be isomorphic to $\mathrm{Tor}^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k})$ as $\mathbf{k}$-algebras since $\mathbf{k}(K)$ is a commutative ring. Even when $\mathbf{k}$ is a field of characteristic 2, there is still some nuance preventing us from simply extending the ring structure result (4.1) of Buchstaber and Panov to the case of, say $\mathbb{R}\mathcal{Z}_K$; Indeed, in this case $x_i^{(1)}$ would be a 0-cochain, which satisfies $x_i^{(1)} \cup x_i^{(1)} = x_i^{(1)}$, whereas in the cases when $\kappa_i > 0$, $x_i^{(1)} \cup x_i^{(1)}$ would be instead zero element in $H^*(\mathbb{S}_i;\mathbf{k})$. Nevertheless, our calculation of the module structure actually represents any cohomological class in $H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k})$ as a sum of $\mathbf{x}^{(\tau,\sigma)}$'s via the isomorphism $H^*(\Omega^*(K);\mathbf{k}) \cong H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k})$, from which we may also figure out the cohomological equivalence relation amongst such sums; since the cup product of pairs of these elements is clear, in a certain sense we should have also determined the ring structure of $H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k})$. In other words, let $\mathbf{k}(K) = \mathbf{k}[\mathbf{v}]/I_K = \mathbf{k}[v_1,\dots,v_m]/I_K$ be the Stanley–Reisner face ring of K with $\deg v_i = \kappa_i + 1$. Then it should be reasonable to *conjecture* that the following results hold:

- If all κ_i's are odd, then $H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k}) \cong \mathrm{Tor}^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k})$ as $\mathbf{k}$-algebras.
- If $\kappa_i > 0$ for any $i\in[m]$, then $H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k}_2) \cong \mathrm{Tor}^{\mathbf{k}_2[\mathbf{v}]}(\mathbf{k}_2(K),\mathbf{k}_2)$ as $\mathbf{k}_2$-algebras.
- In general, $H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k}_2) \cong H[H^*(\prod_{i=1}^m \mathbb{S}_i;\mathbf{k}_2) \otimes_{\mathbf{k}_2[\mathbf{v}]} \mathbf{k}_2(K)]$ as $\mathbf{k}_2$-algebras.

5 Application to the free actions on $\mathcal{Z}_K$ and $\mathbb{R}\mathcal{Z}_K$

First we prove a useful lemma.

Lemma 5.1 *Let* $K\in\mathcal{K}_{[m]}$ *be an abstract simplicial complex on vertex set* $[m]$, *and let* H (*resp.* $H_{\mathbb{R}}$) *be a rank* r *subtorus of* T^m (*resp.* $(\mathbb{Z}_2)^m$). *If the restricted action of* Φ *to* H (*resp.* $\Phi_{\mathbb{R}}$ *to* $H_{\mathbb{R}}$) *is free on* $\mathcal{Z}_K$ (*resp.* $\mathbb{R}\mathcal{Z}_K$), *then* $r\le m-\dim K-1$.

Proof It is well-known that the restricted action of Φ to H (resp. $\Phi_{\mathbb{R}}$ to $H_{\mathbb{R}}$) is free on $\mathcal{Z}_K$ (resp. $\mathbb{R}\mathcal{Z}_K$) if and only if for any point z (resp. x) of $\mathcal{Z}_K$ (resp. $\mathbb{R}\mathcal{Z}_K$), $H\cap G_z$ (resp. $H_{\mathbb{R}}\cap G_x$) is trivial, where G_z (resp. G_x) is the isotropy subgroup at z (resp. x) of the T^m-action Φ (resp. the $(\mathbb{Z}_2)^m$-action $\Phi_{\mathbb{R}}$). Suppose that $r >$

$m - \dim K - 1$. Take $a \in K$ with $|a| = \dim K + 1$. Without the loss of generality, assume that $a = \{1, \dots, |a|\}$. Then we see that $\mathcal{Z}_K$ (resp. $\mathbb{R}\mathcal{Z}_K$) contains the point of the form $z = (0, \dots, 0, z_{|a|+1}, \dots, z_m)$ (resp. $x = (0, \dots, 0, x_{|a|+1}, \dots, x_m)$). It is easy to see that the isotropy subgroup G_z (resp. G_x) has rank at least $|a|$, so the intersection $H \cap G_z$ (resp. $H_{\mathbb{R}} \cap G_x$) cannot be trivial. This contradiction means that r must be equal to or less than $m - \dim K - 1$. □

Now let us use the preceding results to complete the proof of Theorem 1.4.

Proof Theorem 1.4 Let $f = \sum_{a \in K} \delta_a \in \mathcal{F}_{[m]}$ such that $\mathrm{supp}(f) = K$. If $f = \underline{1}$ (i.e., $K = 2^{[m]}$), then $\mathcal{Z}_K = (D^2)^m$ (resp. $\mathbb{R}\mathcal{Z}_K = (D^1)^m$). However, any properly nontrivial subtorus of T^m (resp. $(\mathbb{Z}_2)^m$) cannot freely act on $(D^2)^m$ (resp. $(D^1)^m$) since the point $(0, \dots, 0)$ is always a fixed point. Thus we may assume that $f \neq \underline{1}$. By Theorem 3.5, there exists some $a \in 2^{[m]}$ with $a \neq [m]$ such that $a \in \mathrm{supp}(f) = K$ and $|\mathrm{supp}(\mathcal{M}(f))| \geq 2^{m-n}$ where $n = |a|$. Since $a \in K$, we have $n \leq \dim K + 1$. So by Lemma 5.1 it follows that $n \leq m - r$ and $r \leq m - n$. Combining with Theorem 3.5 and Corollaries 3.2 and 4.3 together gives

$$2^r \leq 2^{m-n} \leq \left|\mathrm{supp}\big(\mathcal{M}(f)\big)\right| \leq \sum_i \dim_{\mathbf{k}} H^i(\mathcal{Z}_K; \mathbf{k}) = \sum_i \dim_{\mathbf{k}} H^i(\mathbb{R}\mathcal{Z}_K; \mathbf{k})$$

as desired. □

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