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Addendum to: Olivier Schiffmann, “Drinfeld realization of the elliptic Hall algebra”

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Abstract In (J. Algebr. Comb. doi:10.1007/s10801-011-0302-8, 2011), O. Schiffmann gave a presentation of the Drinfeld double of the elliptic Hall algebra which is similar in spirit to Drinfeld's new realization of quantum affine algebras. Using this result together with a part of his proof, we can provide such a description for the elliptic Hall algebra.

We will use freely all the notations and the results of [1].

Let $\underline{\widetilde{\mathcal{E}}}^+$ be the algebra generated by the Fourier coefficients of the series $\mathbb{T}_1(z)$ and $\mathbb{T}_0^+(z)$ subject only to the relevant positive relations (5.1), (5.2), (5.3), (5.5) in [1]. To avoid any confusion with the generators of $\widetilde{\mathcal{E}}$, we denote the generators of $\underline{\widetilde{\mathcal{E}}}^+$ by $\mathfrak{u}_{1,d}, d \in \mathbb{Z}$ and $\Theta_{0,d}, d \geq 1$.

We denote by $\widetilde{\mathcal{E}}^{\pm}$ the *subalgebra* of $\widetilde{\mathcal{E}}$ generated by the positive (resp., negative) generators. Similarly for $\mathcal{E}^{\pm}$. Our goal is to prove that $\mathcal{E}^+$ is isomorphic to $\underline{\widetilde{\mathcal{E}}}^+$. The strategy is to go through their Drinfeld doubles. But first we need to define a coalgebra structure on $\underline{\widetilde{\mathcal{E}}}^+$.

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Lemma 1.1 *The map* $\Delta : \underline{\tilde{\mathcal{E}}}^+ \to \underline{\tilde{\mathcal{E}}}^+ \widehat{\otimes} \underline{\tilde{\mathcal{E}}}^+$ *given on generators by*

$$\Delta\big(\mathbb{T}_0^+(z)\big) = \mathbb{T}_0^+(z) \otimes \mathbb{T}_0^+(z),$$

$$\Delta\big(\mathbb{T}_1(z)\big) = \mathbb{T}_1(z) \otimes 1 + \mathbb{T}_0^+(z) \otimes \mathbb{T}_1(z)$$

is a well defined algebra map and makes $\underline{\tilde{\mathcal{E}}}^+$ *into a (topological) bialgebra.*

Proof We need to check that the map Δ respects all the relations between the generators of $\underline{\tilde{\mathcal{E}}}^+$. The relations (5.1), (5.2), (5.3) are an easy routine check. We are left to check the cubic relation (5.5). Using [1, Lemma 5.1], we only need to check the following relation:

$$\big[[\mathfrak{u}_{1,-1}, \mathfrak{u}_{1,1}], \mathfrak{u}_{1,0}\big] = 0.$$

Applying Δ, we obtain:

$$\big[[\mathfrak{u}_{1,-1}, \mathfrak{u}_{1,1}], \mathfrak{u}_{1,0}\big] \otimes 1 + E + \sum_{m,n,l \geq 0} \Theta_{0,m}\Theta_{0,n}\Theta_{0,l} \otimes \big[[\mathfrak{u}_{1,-1-m}, \mathfrak{u}_{1,1-n}], \mathfrak{u}_{1,-l}\big] \tag{1.1}$$

where $E \in \underline{\tilde{\mathcal{E}}}^+[1] \widehat{\otimes} \underline{\tilde{\mathcal{E}}}^+[2] + \underline{\tilde{\mathcal{E}}}^+[2] \widehat{\otimes} \underline{\tilde{\mathcal{E}}}^+[1]$.

The first term is 0 since it's exactly the cubic relation. We want to prove that E and the third term are also 0. Let us begin with E.

We will need to use the following easy lemma whose proof is omitted:

Lemma 1.2 *Let* A, B *be two algebras over a field. Suppose we have a morphism of algebras* $f : A \to B$. *Then* $\ker(f \otimes f) = A \otimes \ker(f) + \ker(f) \otimes A$.

The arguments of [1, Sect. 6.3] show that $\underline{\tilde{\mathcal{E}}}^+[\leq 2]$ and $\mathcal{E}^+[\leq 2]$ are isomorphic (through the canonical morphism). We apply the above lemma to this morphism **can** : $\underline{\tilde{\mathcal{E}}}^+ \to \mathcal{E}^+$ and we get in particular that

$$\underline{\tilde{\mathcal{E}}}^+[\leq 2] \otimes \underline{\tilde{\mathcal{E}}}^+[\leq 2] \to \mathcal{E}^+[\leq 2] \otimes \mathcal{E}^+[\leq 2]$$

is still an isomorphism.

Using the fact that the map **can** commutes with the coproduct, we get that $\mathbf{can} \otimes \mathbf{can}(E) = 0$. By the above isomorphism, we deduce that $E = 0$.[1]

Let us now deal with the cubic term. For any integers $m, n, l \in \mathbb{Z}$, we put

$$R(m,n,l) = \sum_{(m,n,l)} \big[[\mathfrak{u}_{1,-1+m}, \mathfrak{u}_{1,1+n}], \mathfrak{u}_{1,l}\big]$$

[1] It looks as if we cheated here because E lives only in a completion of the tensor product. However, each graded piece of E (remember that $\underline{\tilde{\mathcal{E}}}^+$ is $\mathbb{Z}^2$ graded) lives in an ordinary tensor product and hence we can apply the lemma.

where the sum is over all the six permutations of the triplet (m,n,l). So in order to prove that the third term of the relation (1.1) vanishes, it is enough to prove that $R(m,n,l)=0$ for any $m,n,l\in\mathbb{Z}$.

Observe first that $R(l,l,l)=0$ for any $l\in\mathbb{Z}$ since it is the cubic relation (5.6) from [1]. By symmetry, we can suppose that $l\leq m,n$. Applying the adjoint action of $\mathfrak{u}_{0,k-l}$ to the relation $R(l,l,l)=0$, we get that $R(k,l,l)=0$ for any $k\geq l$. So in particular $R(m,l,l)=0$. Now applying the adjoint action of $\mathfrak{u}_{0,n-l}$ to $R(m,l,l)=0$, we obtain $R(m,n,l)=0$ which is exactly what we wanted. □

In order to prove that the algebra $\underline{\tilde{\mathcal{E}}}^+$ embeds in its Drinfeld double (or that the Drinfeld double has a triangular decomposition), we need to define an inverse for the antipode.

The inverse will take values in a completion of $\underline{\tilde{\mathcal{E}}}^+$, and we are forced to define it in the following way:

$$S^{-1}\big(\mathbb{T}_0(z)\big)=\mathbb{T}_0(z)^{-1},$$

$$S^{-1}\big(\mathbb{T}_1(z)\big)=-\mathbb{T}_1(z)\mathbb{T}_0(z)^{-1}.$$

Lemma 1.3 *The inverse of the antipode is well defined, i.e., it satisfies the relations defining the algebra* $\underline{\tilde{\mathcal{E}}}^+$.

Proof The only difficulty is to check the cubic relation, the other ones being easy verifications. So what we want to prove is the following:

$$\begin{aligned}&\operatorname{Res}_{z,y,w}\big((zyw)^d(z+w)\big(y^2-zw\big)\mathbb{T}_1(w)\mathbb{T}_0(w)^{-1}\mathbb{T}_1(y)\mathbb{T}_0(y)^{-1}\\&\quad\times\mathbb{T}_1(z)\mathbb{T}_0(z)^{-1}\big)=0\end{aligned}$$

for all $d\in\mathbb{Z}$.

Using the commutation relations between the series $\mathbb{T}_0(z)$ and $\mathbb{T}_1(w)$, the above expression becomes:

$$\begin{aligned}&\operatorname{Res}_{z,y,w}\big((zyw)^d(z+w)\big(y^2-zw\big)\mathbb{T}_1(z)\mathbb{T}_1(y)\mathbb{T}_1(w)\mathbb{T}_0(z)^{-1}\mathbb{T}_0(y)^{-1}\\&\quad\times\mathbb{T}_0(w)^{-1}\big)=0\end{aligned}$$

for all $d\in\mathbb{Z}$.

Expliciting this last expression we obtain:

$$\begin{aligned}&\sum_{m,n,l\geq 0}(\mathfrak{u}_{1,d-1-m}\mathfrak{u}_{1,d-2-n}\mathfrak{u}_{1,d-l}-\mathfrak{u}_{1,d-2-m}\mathfrak{u}_{1,d-n}\mathfrak{u}_{1,d-1-l}\\&\quad+\mathfrak{u}_{1,d-m}\mathfrak{u}_{1,d-2-n}\mathfrak{u}_{1,d-1-l}-\mathfrak{u}_{1,d-1-m}\mathfrak{u}_{1,d-n}\mathfrak{u}_{d-2-l})\Theta'_{0,m}\Theta'_{0,n}\Theta'_{0,l}=0\end{aligned}\tag{1.2}$$

where $\Theta'_{0,n}$ are the coefficients of the series $\mathbb{T}_0(z)^{-1}$.

Now in order to prove that this last relation holds in $\underline{\tilde{\mathcal{E}}}^+$, it is enough to prove that for any $d \in \mathbb{Z}$ we have

$$\sum_{(m,n,l)} (\mathfrak{u}_{1,d-1-m}\mathfrak{u}_{1,d-2-n}\mathfrak{u}_{1,d-l} - \mathfrak{u}_{1,d-2-m}\mathfrak{u}_{1,d-n}\mathfrak{u}_{1,d-1-l} \\ + \mathfrak{u}_{1,d-m}\mathfrak{u}_{1,d-2-n}\mathfrak{u}_{1,d-1-l} - \mathfrak{u}_{1,d-1-m}\mathfrak{u}_{1,d-n}\mathfrak{u}_{d-2-l}) = 0$$

where $m, n, l \geq 0$ and the sum is over the permutations of the triplet (m, n, l).

This last relation can be proved in the same way as we did for $R(m, n, l)$ starting from the cubic relation:

$$\mathfrak{u}_{1,N-1}\mathfrak{u}_{1,N-2}\mathfrak{u}_{1,N} - \mathfrak{u}_{1,N-2}\mathfrak{u}_{1,N}\mathfrak{u}_{1,N-1} + \mathfrak{u}_{1,N}\mathfrak{u}_{1,N-2}\mathfrak{u}_{1,N-1} \\ - \mathfrak{u}_{1,N-1}\mathfrak{u}_{1,N}\mathfrak{u}_{N-2} = 0$$

for $N \in \mathbb{Z}$ small enough and applying the adjoint actions of $\mathfrak{u}_{0,d-m-N}$, $\mathfrak{u}_{0,d-n-N}$, and $\mathfrak{u}_{0,d-l-N}$.

This finishes the proof that S^{-1} is well defined on $\underline{\tilde{\mathcal{E}}}^+$. □

In [1], it is proved that $\tilde{\mathcal{E}}^+$ is isomorphic to $\mathcal{E}^+$. It follows that there is a natural surjective morphism $\pi : \underline{\tilde{\mathcal{E}}}^+ \to \tilde{\mathcal{E}}^+ \simeq \mathcal{E}^+$ and therefore a natural surjective morphism on the Drinfeld doubles:

$$D\underline{\tilde{\mathcal{E}}}^+ \to D\mathcal{E}^+ \simeq \mathcal{E} \simeq \tilde{\mathcal{E}}.$$

If the natural map $\tilde{\mathcal{E}} \to D\underline{\tilde{\mathcal{E}}}^+$ is well defined then since the composition

$$\tilde{\mathcal{E}} \to D\underline{\tilde{\mathcal{E}}}^+ \to \tilde{\mathcal{E}}$$

is the identity (because all the morphisms are the obvious ones) we obtain that

$$\tilde{\mathcal{E}}^+ \simeq \underline{\tilde{\mathcal{E}}}^+,$$

which is what we wanted.

To prove that the natural morphism $\tilde{\mathcal{E}} \to D\underline{\tilde{\mathcal{E}}}^+$ is well defined, we need to check that the relations (5.1)–(5.5) are satisfied in $D\underline{\tilde{\mathcal{E}}}^+$. It is clear that (5.1), (5.3), (5.5), and (5.2) ($\epsilon_1 = \epsilon_2$) are satisfied since they involve only the positive (resp., negative) part. We need to deal with (5.2) ($\epsilon_1 = -\epsilon_2$) and (5.4). We claim that they are implied by Drinfeld's relations in the double. This is an easy verification.

Putting all together, we have:

Theorem 1.4 *The elliptic Hall algebra $\mathcal{E}^+$ is isomorphic to the algebra generated by the Fourier coefficients of $\mathbb{T}_1(z)$ and $\mathbb{T}_0^+(z)$ subject to the relations*:

$$\mathbb{T}_0^+(z)\mathbb{T}_0^+(w) = \mathbb{T}_0^+(w)\mathbb{T}_0^+(z),$$

$$\chi_1(z,w)\mathbb{T}_0^+(z)\mathbb{T}_1(w) = \chi_{-1}(z,w)\mathbb{T}_1(w)\mathbb{T}_0^+(z),$$

$$\chi_1(z,w)\mathbb{T}_1(z)\mathbb{T}_1(w) = \chi_{-1}(z,w)\mathbb{T}_1(w)\mathbb{T}_1(z),$$

$$\mathrm{Res}_{z,y,w}\big[(zyw)^m(z+w)\big(y^2-zw\big)\mathbb{T}_1(z)\mathbb{T}_1(y)\mathbb{T}_1(w)\big] = 0, \quad \forall m \in \mathbb{Z}.$$

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References

1. Schiffmann, O.: Drinfeld realization of elliptic Hall algebra. J. Algebr. Comb. (2011), doi:10.1007/s10801-011-0302-8