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Newton polygons and curve gonalities

Wouter Castryck · Filip Cools

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Abstract We give a combinatorial upper bound for the gonality of a curve that is defined by a bivariate Laurent polynomial with given Newton polygon. We conjecture that this bound is generically attained, and provide proofs in a considerable number of special cases. One proof technique uses recent work of M. Baker on linear systems on graphs, by means of which we reduce our conjecture to a purely combinatorial statement.

Keywords Newton polygons · Algebraic curves · Gonality · Toric surfaces · Degenerations

1 Introduction

The most renowned birational invariant of an algebraic curve over $\mathbb{C}$ is its geometric genus. Although it enjoys the plastic description as the number of handles on the corresponding Riemann surface, some high-tech machinery is needed to give a rigorous definition. E.g., one nowadays approach is to define the geometric genus as the $\mathbb{C}$-dimension of the Riemann–Roch space associated to a canonical divisor K_C on C.

On the other hand, already at the end of the 19th century, H. Baker shared the following elementary interpretation. Let $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial defining a curve $U(f) \subset \mathbb{T}^2$, where $\mathbb{T}^2 = (\mathbb{C} \setminus \{0\})^2$ is the two-dimensional torus over $\mathbb{C}$. Let $\Delta(f)$ be the Newton polygon of f. It is an instance of a *lattice polygon*, by which we mean the convex hull in $\mathbb{R}^2$ of a finite subset of $\mathbb{Z}^2$.

W. Castryck (✉) · F. Cools
Departement Wiskunde, Afdeling Algebra, Katholieke Universiteit Leuven, Celestijnenlaan 200, 3001 Leuven (Heverlee), Belgium
e-mail: wouter.castryck@gmail.com

F. Cools
e-mail: filip.cools@wis.kuleuven.be

The *dimension* of a lattice polygon Δ is the minimal dimension of an affine subspace of $\mathbb{R}^2$ containing Δ. By the *interior* of Δ we mean the topological interior if Δ is two-dimensional, and the empty set if it is of strictly lower dimension. Points of $\mathbb{Z}^2$ will be called *lattice points*. Then:

Theorem 1 (Baker, 1893) *The geometric genus of $U(f)$ is at most the number of lattice points in the interior of $\Delta(f)$.*

Proof This can be found in [1]. See [6] for a more modern proof. □

Generically, Baker's bound is sharp.

Theorem 2 (Khovanskiĭ, 1977) *Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon. The set of irreducible Laurent polynomials $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ for which $\Delta(f) = \Delta$ and the bound in Theorem 1 is attained, is Zariski dense in the space of Laurent polynomials $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ for which $\Delta(f) \subset \Delta$.*

Proof See [20]. Khovanskiĭ actually proved something much stronger, which we will state in Sect. 5. □

Because of all this, one defines the *genus* of a lattice polygon as the number of lattice points in its interior.

The near-miraculous appearance of the interior lattice points of $\Delta(f)$ was secularized with the advent of tropical geometry. Loosely stated, by subdividing $\Delta(f)$ into triangles of area $1/2$ and taking the dual spine, one obtains a graph whose handles are in one-to-one correspondence with the interior lattice points of $\Delta(f)$. By considering the graph as a piece-wise linear limit of $U(f)$ in an appropriate way, one realizes that one has actually visualized the handles of the Riemann surface. This construction will be reviewed in Sects. 7 and 8 below.

The aim of this article is to give analogs of Theorems 1 and 2 for what is, arguably, the second most renowned birational invariant of an algebraic curve: its gonality. Our results will be partly conjectural. To state them, we need the following terminology. A *$\mathbb{Z}$-affine transformation* is a map $\mathbb{R}^2 \to \mathbb{R}^2 : x \mapsto Ax + b$ with $A \in \mathrm{GL}_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$. Two lattice polygons Δ, Δ' are called *equivalent* if there is a $\mathbb{Z}$-affine transformation φ such that $\varphi(\Delta) = \Delta'$ (notation: $\Delta \cong \Delta'$). The *lattice width* of a non-empty lattice polygon Δ is the smallest integer $s \geq 0$ such that there is a $\mathbb{Z}$-affine transformation φ for which $\varphi(\Delta)$ is contained in the horizontal strip

$$H_0^s = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq s\right\}.$$

It will be denoted by $\mathrm{lw}(\Delta)$. It is convenient to define $\mathrm{lw}(\emptyset) = -1$. If Δ is a lattice polygon, then for any integer $d \geq 0$, the polygon $d\Delta$ denotes the corresponding Minkowski multiple. We will denote the standard 2-simplex in $\mathbb{R}^2$ by Σ. Thus, $d\Sigma$ is the Newton polygon of a generic degree d polynomial. We use Υ to denote $\mathrm{Conv}\{(-1, -1), (1, 0), (0, 1)\}$.

Then our analogs of Theorems 1 and 2 read

Theorem 3 *The gonality of $U(f)$ is at most* $\mathrm{lw}(\Delta(f))$. *If $\Delta(f)$ is equivalent to $d\Sigma$ for some $d \geq 2$, or to 2Υ, then it is at most* $\mathrm{lw}(\Delta(f)) - 1$.

Conjecture 1 *Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon. The set of irreducible Laurent polynomials $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ for which $\Delta(f) = \Delta$ and the* (*sharpest applicable*) *bound in Theorem 3 is attained, is Zariski dense in the space of Laurent polynomials $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ for which $\Delta(f) \subset \Delta$.*

The article is organized as follows.

In Sect. 2, we prove Theorem 3.

In Sect. 3, we give a reformulation of Conjecture 1 that focuses on the convex hull of the interior lattice points of Δ, rather than on Δ itself. In doing so, the polygons $d\Sigma$ become ruled out as special instances.

In Sect. 4, we review how to associate a toric surface $\mathrm{Tor}(\Delta)$ to a lattice polygon Δ, and how, in general, $\mathrm{Tor}(\Delta(f))$ naturally appears as an ambient space for the complete non-singular model of $U(f)$.

In Sect. 5, we prove Conjecture 1 for all Δ for which $\mathrm{lw}(\Delta) \leq 4$ (including $\Delta \cong 2\Upsilon$), by analyzing the canonical image of $U(f)$. We briefly report on a computer experiment supporting Conjecture 1 for all lattice polygons up to genus 13, thereby relying on Green's canonical conjecture.

In Sect. 6, we see how previous results by Kawaguchi, Martens, and Namba prove Conjecture 1 in a considerable number of additional cases.

In Sect. 7, we review the process of degenerating a toric surface $\mathrm{Tor}(\Delta)$ according to a regular subdivision of Δ, and use this to deform a sufficiently generic $U(f)$ along with $\mathrm{Tor}(\Delta)$ into a union of irreducible curves. This yields a vast class of examples of socalled *strongly semi-stable arithmetic surfaces*.

In Sect. 8, we encode the combinatorial configuration of this union of irreducible curves in a graph, and we apply recent results due to M. Baker [2] to obtain a lower bound for the gonality of $U(f)$.

In Sect. 9 we conjecture that, in this way, one can always meet the upper bound of Theorem 3. This reduces Conjecture 1 to a purely combinatorial (albeit a priori stronger) statement. We prove this statement (and hence Conjecture 1) for an interesting class of lattice polygons, thereby partly confirming and partly extending the results of Sects. 5 and 6.

2 The lattice width as an upper bound

In this section, we prove Theorem 3.

Proof of Theorem 3 It is clear that if $\Delta(f)$ is contained in a horizontal strip

$$H_0^s = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq s\},$$

then the rational map

$$U(f) \to \mathbb{A}^1 : (x, y) \mapsto x$$

is of degree at most s. Now every $\mathbb{Z}$-affine transformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ acts on f as follows: if

$$f = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{ij}(x,y)^{(i,j)}, \quad \text{then} \quad f^\varphi = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} c_{ij}(x,y)^{\varphi(i,j)}$$

(where we use multi-index notation). It is clear that $\Delta(f^\varphi) = \varphi(\Delta(f))$ and that $U(f) \cong U(f^\varphi)$. The upper bound $\mathrm{lw}(\Delta)$ follows immediately.

Now suppose that $\Delta(f) \cong d\Sigma$ for some integer $d \geq 2$. Hence we can assume that $f \in \mathbb{C}[x,y]$ is a dense degree d polynomial, whose homogenization F with respect to a new variable z defines a curve $V(F)$ in $\mathbb{P}^2 = \mathrm{Proj}\,\mathbb{C}[x,y,z]$. A projective transformation takes us to a curve $V(F')$ containing the point $(0:1:0)$. Dehomogenizing F' with respect to z yields a polynomial $f' \in \mathbb{C}[x,y]$ whose Newton polygon is contained in

$$\mathrm{Conv}\{(0,0),(d,0),(1,d-1),(0,d-1)\}.$$

Thus, $\Delta(f')$ is of lattice width at most $d-1$. Hence the gonality of $U(f)$ is at most $d-1$. On the other hand, $\mathrm{lw}(d\Sigma) = d$ for all integers $d \geq 0$. Indeed, clearly $\mathrm{lw}(d\Sigma) \leq d$, and equality follows from the fact that each edge of $d\Sigma$ contains $d+1$ lattice points.

Finally, suppose that $\Delta(f) \cong 2\Upsilon$. By Theorem 1, $U(f)$ has geometric genus at most 4, and it is classical that this implies the gonality to be at most 3, see e.g. [21]. On the other hand, $\mathrm{lw}(2\Upsilon) = 4$. Indeed, clearly $\mathrm{lw}(2\Upsilon) \leq 4$, and equality follows from the fact that the convex hull of the interior lattice points of 2Υ has itself an interior lattice point. This would be impossible if $\mathrm{lw}(2\Upsilon) \leq 3$. □

3 The interior lattice polygon

Our two exceptional cases $d\Sigma\,(d \geq 2)$ and 2Υ are of a very different kind. In the first case, one is able to clip off a vertex in such a way that it reduces the lattice width, without affecting the geometry of $U(f)$. For 2Υ, such a trick is impossible, since clipping off a vertex would necessarily mean reducing the number of interior lattice points. Hence this would affect the genus.

In this section we will deduce an equivalent formulation of Conjecture 1, in which the polygons $d\Sigma$ are no longer exceptional cases. This is done by focusing on the *interior lattice polygon*, rather than on the polygon itself. For any lattice polygon Δ, the interior lattice polygon $\Delta^{(1)}$ is defined as the convex hull of the interior lattice points of Δ. Somewhat dually, one can consider the *relaxed polygon*. That is, let Δ be a two-dimensional lattice polygon, and write it as a finite intersection of half-planes

$$\Delta = \bigcap_i \{(x,y) \in \mathbb{R}^2 \mid a_i x + b_i y \leq c_i\},$$

where $a_i, b_i, c_i \in \mathbb{Z}$ and $\gcd(a_i, b_i) = 1$ for all i. Then the relaxed polygon is defined as

$$\Delta^{(-1)} = \bigcap_i \{(x,y) \in \mathbb{R}^2 \mid a_i x + b_i y \leq c_i + 1\}.$$

Not every lattice polygon can be written as $\Delta^{(1)}$ for some larger lattice polygon Δ. Also, if Δ is a two-dimensional lattice polygon Δ, then $\Delta^{(-1)}$ need not be a lattice polygon: it may take vertices outside the lattice. The following statement connects and controls both phenomena.

Lemma 1 *Let Δ be a two-dimensional lattice polygon. Then $\Delta = \Gamma^{(1)}$ for a lattice polygon Γ if and only if $\Delta^{(-1)}$ is a lattice polygon. Moreover, if $\Delta^{(-1)}$ is a lattice polygon, then it is maximal (with respect to inclusion) among all lattice polygons Γ for which $\Gamma^{(1)} = \Delta$.*

Proof This is due to Koelman [22, Sect. 2.2]. Recently, this was rediscovered by Haase and Schicho [15, Lemmata 9 and 11]. □

The main result of this section is the following relationship between $\mathrm{lw}(\Delta)$ and $\mathrm{lw}(\Delta^{(1)})$. It was discovered independently (and almost simultaneously) by Lubbes and Schicho [25, Theorem 13].

Theorem 4 *Let Δ be a two-dimensional lattice polygon. Then*

$$\mathrm{lw}(\Delta) = \mathrm{lw}\big(\Delta^{(1)}\big) + 2,$$

unless $\Delta \cong d\Sigma$ for some integer $d \geq 2$, in which case

$$\mathrm{lw}(\Delta) = \mathrm{lw}\big(\Delta^{(1)}\big) + 3 = d.$$

Proof First, it is clear that $\Delta^{(1)}$ can be caught in a horizontal strip of width $\mathrm{lw}(\Delta) - 2$, from which

$$\mathrm{lw}\big(\Delta^{(1)}\big) \leq \mathrm{lw}(\Delta) - 2. \tag{1}$$

Second, in Sect. 2 we saw that $\mathrm{lw}(d\Sigma) = d$ for all integers $d \geq 0$. Third, we have $(d\Sigma)^{(1)} \cong (d-3)\Sigma$ for all integers $d \geq 3$, and that $(2\Sigma)^{(1)} = \emptyset$. So

$$\mathrm{lw}(d\Sigma) = \mathrm{lw}\big(d\Sigma^{(1)}\big) + 3 = d$$

for all $d \geq 2$. We therefore conclude that it suffices to prove: if the inequality in (1) is strict, then $\Delta \cong d\Sigma$ for some integer $d \geq 2$.

For technical reasons, we first get rid of the following cases.

- $\mathrm{lw}(\Delta^{(1)}) = -1$, i.e. Δ contains no interior lattice points. Then either $\Delta \cong 2\Sigma$ or Δ is a so-called Lawrence prism, see [22, Theorem 4.1.2] or the generalized statement of [4, Theorem 2.5]. Since Lawrence prisms have lattice width 1, the result follows.
- $\mathrm{lw}(\Delta^{(1)}) = 0$. Then Δ is a so-called elliptic or hyperelliptic lattice polygon. These have been classified in [22, Theorem 4.2.3 and Sect. 4.3], from which it follows that either $\Delta \cong 3\Sigma$, or $\mathrm{lw}(\Delta) = 2$.
- $\mathrm{lw}(\Delta^{(1)}) = 1$. Then $\Delta^{(1)}$ must be a Lawrence prism, and using Lemma 1 one concludes that either $\Delta \cong 4\Sigma$, or $\mathrm{lw}(\Delta) = 3$.
- $\Delta^{(1)}$ can be caught in a 3-by-3 lattice square. These cases can be exhaustively verified using Lemma 1.

To deal with the general case, we apply a $\mathbb{Z}$-affine transformation to catch $\Delta^{(1)}$ in the horizontal strip

$$H = H_0^{\mathrm{lw}(\Delta^{(1)})} = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le \mathrm{lw}(\Delta^{(1)})\}.$$

Since we assume (1) is strict, Δ must then contain at least one vertex v outside the strip

$$\{(x, y) \in \mathbb{R}^2 \mid -1 \le y \le \mathrm{lw}(\Delta^{(1)}) + 1\}.$$

We may assume that $v = (0, -k)$ with $k \ge 2$.

We will first prove that $k = 2$. Since Δ contains no interior lattice points on the line $y = -1$, it must intersect this line inside an interval $[\alpha, \alpha + 1]$ for some $\alpha \in \mathbb{Z}$. Let σ be the cone with top v, whose rays pass through $(\alpha, -1)$ and $(\alpha + 1, -1)$, respectively. Although Δ need (a priori) not be contained in σ, the part of Δ that lies on or above the line $y = -1$ must be. In particular, $\Delta^{(1)}$ will be contained in the open cone σ°. Modulo horizontally skewing and flipping if necessary, we can assume that

$$0 \le \alpha \le \left\lfloor \frac{k-2}{2} \right\rfloor. \tag{2}$$

Then $\Delta^{(1)} \subset \sigma^\circ \cap H$ is contained in the interior of the vertical strip

$$V = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{\alpha k}{k-1} \le x \le \frac{(\alpha+1)(\mathrm{lw}(\Delta^{(1)}) + k)}{k-1} \right\}$$

which has width

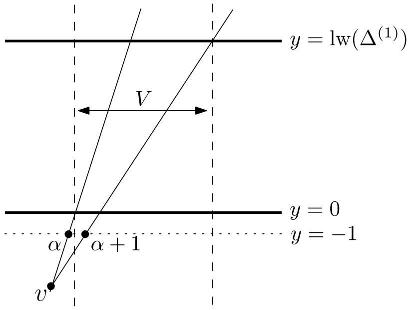
$$\frac{(\alpha+1)}{k-1} \mathrm{lw}(\Delta^{(1)}) + \frac{k}{k-1} \le \frac{\lfloor \frac{k}{2} \rfloor}{k-1} \mathrm{lw}(\Delta^{(1)}) + \frac{k}{k-1}.$$

By definition of the lattice width

$$\mathrm{lw}(\Delta^{(1)}) < \frac{\lfloor \frac{k}{2} \rfloor}{k-1} \mathrm{lw}(\Delta^{(1)}) + \frac{k}{k-1}.$$

This is impossible for $k \ge 3$ as soon as $\mathrm{lw}(\Delta^{(1)}) \ge 4$. If $\mathrm{lw}(\Delta^{(1)}) \in \{2, 3\}$, we find that $k \ge 3$ would cause $\Delta^{(1)} \subset H \cap V$ to be caught in a 3-by-3 square, a case covered in the list above.

Next, note that v is the only vertex of Δ on the line $y = -2$. Indeed, along with a vertex of $\Delta^{(1)}$ on the line $y = 0$, two such vertices would span a triangle which by Pick's theorem would need to contain a lattice point on the line $y = -1$. But this would be an interior lattice point of Δ: a contradiction. As a consequence, v is the only lattice point of Δ lying strictly under the line $y = -1$, hence $\Delta \subset \sigma$. By (2), we can assume that σ has top $(0, -2)$, and that its rays pass through $(0, -1)$ and $(1, -1)$, respectively.



The next step is to prove that Δ cannot have a vertex lying strictly above the line $y = \mathrm{lw}(\Delta^{(1)}) + 1$. Suppose that there is such a vertex w. By symmetry of arguments, it must be unique and lying on the line $y = \mathrm{lw}(\Delta^{(1)}) + 2$. Since w must be contained in σ, its x-coordinate must be among $0, \dots, \mathrm{lw}(\Delta^{(1)}) + 4$. As before, there must exist an integer β such that Δ is contained in the cone τ with top w, whose rays pass through $(\beta, \mathrm{lw}(\Delta^{(1)}) + 1)$ and $(\beta + 1, \mathrm{lw}(\Delta^{(1)}) + 1)$. Using that τ must contain v, a case-by-case analysis shows that $\sigma \cap \tau$ is too small for the lattice width of $\Delta \subset \sigma \cap \tau$ to exceed $\mathrm{lw}(\Delta^{(1)}) + 2$, a contradiction with the assumed strict inequality in (1).

Overall, we obtained that Δ must be contained in the triangle spanned by $(0, -2)$, $(0, \mathrm{lw}(\Delta^{(1)}) + 1)$ and $(\mathrm{lw}(\Delta^{(1)}) + 3, \mathrm{lw}(\Delta^{(1)}) + 1)$, which is a copy of $((\mathrm{lw}(\Delta^{(1)}) + 3)\Sigma$. If any of these three points does not appear as a vertex, Δ can be seen to have lattice width at most $\mathrm{lw}(\Delta^{(1)}) + 2$, a contradiction with the strict inequality in (1). Hence Δ must be the full triangle. □

Note that Theorem 4 yields an algorithm for recursively computing $\mathrm{lw}(\Delta)$.

We can now rephrase Conjecture 1 as follows.

Conjecture 1 (Equivalent formulation) *Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon, and let $S \subset \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be the set of irreducible Laurent polynomials for which $\Delta(f) = \Delta$ and*

- *$U(f)$ has gonality* $\mathrm{lw}(\Delta^{(1)}) + 2$ *if $\Delta \ncong 2\Upsilon$,*
- *$U(f)$ has gonality* 3 *if $\Delta \cong 2\Upsilon$.*

Then S is Zariski dense in the space of Laurent polynomials $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ for which $\Delta(f) \subset \Delta$.

Proof of equivalence This follows directly from Theorem 4. □

4 Toric surfaces as ambient spaces

We give a brief, notation-fixing overview of the geometry of toric surfaces. In Sect. 7, the material below will be put in a bigger framework.

Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon. Let S be the set of lattice points of Δ. Then we have an injective morphism

$$\phi : \mathbb{T}^2 \hookrightarrow \mathbb{P}^{|S|-1} : (x, y) \mapsto \left(x^i y^j\right)_{(i,j)\in S}.$$

The Zariski closure of the image is by definition the *toric surface* $\mathrm{Tor}(\Delta)$. If $X_{i,j}$ denotes the projective coordinate of $\mathbb{P}^{|S|-1}$ corresponding to $(i, j) \in S$, then all binomials of the form

$$\prod_{k=1}^{n} X_{i_k,j_k} - \prod_{k=1}^{n} X_{i'_k,j'_k}$$

for which

$$\sum_{k=1}^{n} (i_k, j_k) = \sum_{k=1}^{n} \left(i'_k, j'_k\right)$$

are zero on $\mathrm{Tor}(\Delta)$, and in fact these generate the homogeneous ideal of $\mathrm{Tor}(\Delta)$. In practice, it suffices to consider relations of degree $n \le 3$, and even $n \le 2$ if $\#(\partial\Delta \cap \mathbb{Z}^2) > 3$ by a result of Koelman [23].

The faces $\tau \subset \Delta$ (vertices, edges, and Δ itself) naturally partition $\mathrm{Tor}(\Delta)$ into sets

$$O(\tau) = \left\{(\alpha_{i,j})_{(i,j)\in S} \in \mathrm{Tor}(\Delta) \mid \alpha_{i,j} \neq 0 \text{ if and only if } (i, j) \in \tau\right\},$$

which are called the *toric orbits*. Note that $O(\Delta) = \phi(\mathbb{T}^2)$. More generally, one has $O(\tau) \cong \mathbb{T}^{\dim\tau}$. One can show that $\mathrm{Tor}(\Delta)$ is non-singular, except possibly at the zero-dimensional toric orbits.

Write $f = \sum_{(i,j)\in S} c_{i,j} x^i y^j$. Then $\phi(U(f))$ satisfies

$$\sum_{(i,j)\in S} c_{i,j} X_{i,j},$$

so it embeds into a hyperplane section of $\mathrm{Tor}(\Delta)$. More generally, if $\Delta = d\Delta'$ for an integer $d \ge 1$ and a lattice polygon Δ', then $U(f)$ can be embedded in a degree d hypersurface section of $\mathrm{Tor}(\Delta')$. Generically, this hyperplane/hypersurface section will be a complete non-singular model of $U(f)$. A sufficient condition is that f is *non-degenerate with respect to its Newton polygon* Δ, meaning that for each face $\tau \subset \Delta(f)$ (vertices, edges, and $\Delta(f)$ itself), the system

$$f_\tau = x\frac{\partial f_\tau}{\partial x} = y\frac{\partial f_\tau}{\partial y} = 0$$

has no solutions in $\mathbb{T}^2$. Here, f_τ is obtained from f by only considering those terms whose exponent vector is contained in τ. Geometrically, non-degeneracy can be rephrased as follows: the Zariski closure of $\phi(U(f))$ has no singular points in $O(\Delta)$, intersects the one-dimensional toric orbits transversally, and does not contain the zero-dimensional toric orbits. This is indeed a generic condition, since non-degeneracy can be rephrased in terms of the non-vanishing of a certain integral polynomial expression in the coefficients $c_{i,j}$, realized as a product of principal A-discriminants in the sense of [13]. See [8, Sect. 2] for some additional details.

5 Polygons of small lattice width

In this section we prove Conjecture 1 for lattice polygons Δ satisfying $\mathrm{lw}(\Delta) \leq 4$. By the results of Sect. 3, it suffices to do this for two-dimensional lattice polygons Δ for which $\mathrm{lw}(\Delta^{(1)}) \leq 2$.

Conjecture 1 is automatic in case $\mathrm{lw}(\Delta^{(1)}) = -1$, since by definition, genus 0 curves have gonality 1. Next, Theorem 4 immediately implies Conjecture 1 for polygons Δ for which $\mathrm{lw}(\Delta^{(1)}) = 0$. Indeed, by Theorem 2, our curve $U(f)$ will generically have genus at least 1, hence gonality at least 2. But by Theorem 4, either $\mathrm{lw}(\Delta) = 2$ or $\Delta \cong 3\Sigma$, and the statement follows from Theorem 3.

In order to extend this to a proof for the cases where $\mathrm{lw}(\Delta^{(1)}) \in \{-1, 0, 1, 2\}$, including $\Delta \cong 2\Upsilon$, we need the following refined version of Theorem 2.

Theorem 2 (Khovanskiĭ, 1977, refined formulation) *Let $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be an irreducible Laurent polynomial that is non-degenerate with respect to its Newton polygon $\Delta(f)$. Then there exists a canonical divisor $K_{\Delta(f)}$ on (the complete non-singular model of) $U(f)$ for which a basis of the Riemann–Roch space $\mathcal{L}(K_{\Delta(f)})$ is given by*

$$\left\{x^i y^j \mid (i, j) \in \Delta^{(1)} \cap \mathbb{Z}^2\right\}.$$

In this, the function field $\mathbb{C}(U(f))$ is understood to be identified with the fraction field of $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]/(f)$.

In particular, this says that the canonical image of $U(f)$ is contained in $\mathrm{Tor}(\Delta^{(1)})$. The following observation is due to Koelman.

Lemma 2 *Let $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta(f)$. Suppose that $\Delta(f)$ is of genus $g \geq 2$. Then $U(f)$ is hyperelliptic if and only if $\Delta(f)^{(1)}$ is one-dimensional.*

Proof See [22, Lemma 3.2.9]. An alternative proof was given in [8, Lemma 5.1] and uses the above reformulation of Theorem 2: the function field of the canonical image is $\mathbb{C}(x, y)$ if and only if $\Delta^{(1)}$ is two-dimensional. □

As a corollary, we obtain a proof of Conjecture 1 in case $\mathrm{lw}(\Delta^{(1)}) = 1$. Indeed, the above ensures that $U(f)$ generically defines a curve of gonality at least 3. But by Theorem 4, either $\mathrm{lw}(\Delta) = 3$ or $\Delta \cong 4\Sigma$, and the statement follows from Theorem 3. In an entirely similar way, we obtain a proof for the case $\Delta \cong 2\Upsilon$. Again, all of this can be turned into an 'if and only if' statement.

Lemma 3 *Let $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be non-degenerate with respect to its Newton polygon $\Delta(f)$, which we assume to be two-dimensional. Then $U(f)$ is trigonal if and only if*

$$\mathrm{lw}\big(\Delta(f)^{(1)}\big) = 1 \quad \textit{or} \quad \Delta(f) \cong 2\Upsilon.$$

Proof It remains to prove the 'only if' part. Parts of the following reasoning already appeared in an unpublished addendum to [8]. Suppose that $U(f)$ is trigonal. Then by Petri's theorem, the intersection of all quadrics containing the canonical image is a rational normal scroll $S \subset \mathbb{P}^{g-1}$, which is a surface of sectional genus 0. On the other hand, by Theorem 2, $U(f)$ is canonically embedded in $\mathrm{Tor}(\Delta(f)^{(1)})$. An earlier mentioned result by Koelman [23] states that $\mathrm{Tor}(\Delta(f)^{(1)})$ is generated by quadrics as soon as $\Delta(f)^{(1)}$ contains at least 4 lattice points on the boundary. So:

- If $\partial\Delta(f)^{(1)} \geq 4$, then we must have $\mathrm{Tor}(\Delta(f)^{(1)}) = S$. Since S is a surface of sectional genus zero, $\Delta(f)^{(1)}$ cannot have any interior lattice points. In particular, either $\mathrm{lw}(\Delta(f)^{(1)}) = 1$, or $\Delta(f)^{(1)} \cong 2\Sigma$. The latter is impossible, however, since then $U(f)$ would be isomorphic to a plane quintic, which has gonality 4 by a result of Namba [29]—see also Theorem 6 below.
- If $\partial\Delta(f)^{(1)} = 3$ and $\Delta(f)^{(1)}$ contains an interior lattice point, then using Lemma 1, it is an easy exercise to show that $\Delta(f)^{(1)} \cong \Upsilon$, hence $\Delta(f) \cong 2\Upsilon$.

This concludes the proof. □

Note that the above proof gives a prudent indication of the exceptionality of 2Υ.

Again, similarly to the foregoing cases, we can use this to obtain a proof of Conjecture 1 in case $\mathrm{lw}(\Delta^{(1)}) = 2$. We conclude:

Theorem 5 *Conjecture 1 is true for all lattice polygons Δ for which* $\mathrm{lw}(\Delta^{(1)}) \leq 2$.

To continue this type of iteration, we would need if-and-only-if statements for $\mathrm{lw}(\Delta^{(1)}) = 2, 3, 4, \ldots$, similar to Lemmas 2 and 3. In pursuing this, one naturally bumps into Green's canonical conjecture [14], which is an unproven generalization of Petri's theorem. It states that the Clifford index of $U(f)$ is the smallest integer p for which the canonical ideal of $U(f)$ does not satisfy *property* N_p. The latter is a certain non-vanishing property of the Betti numbers appearing in a minimal free resolution of the ideal—see [10, Chap. 9] for an introduction. We were not able to unveil a connection between property N_p and the combinatorics of the Newton polygon.

However, in an attempt to discover such a connection, we have carried out the following experiment, which provides evidence for Conjecture 1 up to genus 13. This being ongoing research, we will be concise here. Up to equivalence, we have enumerated all two-dimensional lattice polygons $\Delta^{(1)}$ that are interior to a bigger lattice polygon Δ and that contain between 3 and 13 lattice points. There are 176 such polygons. For each of these, we have picked a 'generic' Laurent polynomial with Newton polygon $\Delta^{(1)(-1)}$. For each such polynomial, we have computed the Betti table of the corresponding canonical ideal. We have worked over the finite field $\mathbb{F}_{10007}$ to speed up the computation; this is not expected to influence the outcome. Our most notable observations thus far are:

- in each of these 176 cases, Green's canonical conjecture was consistent with Conjecture 1;

- the following table is missing in Schreyer's conjectured list of Betti tables appearing in genus 10 [31, Sect. 6]:

1	.	.	.	.	.	.	.	.
.	28	105	168	154	70	6	.	.
.	.	6	70	154	168	105	28	.
.	.	.	.	.	.	.	.	1

(it appeared for the six $\Delta^{(1)}$ for which $\#(\Delta^{(1)(1)} \cap \mathbb{Z}^2) = 2$).

All computations were carried out using MAGMA [5].

6 Reinterpretation of some previous results

In this section we give additional support for Conjecture 1 by reinterpreting some previously obtained results.

Theorem 6 (Namba, 1979) *Conjecture 1 holds if* $\Delta \cong d\Sigma$ *for some integer* $d \geq 0$.

Proof If $\Delta \cong d\Sigma$, then a generic Δ-supported Laurent polynomial $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ defines a smooth curve of degree d. A result of Namba [29] states that such curves have gonality $d - 1$. □

The above can be generalized to the case where $\Delta^{(1)} \cong d\Sigma$ for some integer $d \geq 0$ (corresponding to smooth plane curves with, possibly, some prescribed behavior at the coordinate points).

Theorem 7 (Martens, 1996) *Let* $a, b \geq 1$ *and* $k \geq 0$ *be integers. Then Conjecture 1 holds if*

$$\Delta \cong \mathrm{Conv}\big\{(0,0), (a+bk, 0), (a,b), (0,b)\big\}.$$

Proof Note that $\mathrm{Tor}(\Delta) \cong H_k$, the Hirzebruch surface of invariant k. If $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ is non-degenerate with respect to its Newton polygon Δ, then $U(f)$ will embed smoothly in H_k. Martens [28] proved that the gonality is then computed by a ruling of H_k. The ruling is unique if $k \geq 1$ and is given by vertical projection, which is of degree $b = \mathrm{lw}(\Delta)$. If $k = 0$, then there are two rulings, namely horizontal projection and vertical projection. These are of degree a and b, respectively, and since $\mathrm{lw}(\Delta) = \min\{a, b\}$, the result follows. □

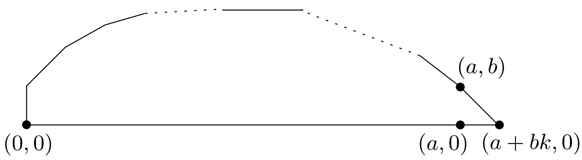
Note that the case $k = 0$, corresponding to rectangular polygons, already follows from older work of Schreyer [30], who studied the gonality of curves in $\mathrm{Tor}(\Delta) \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Again, Theorem 7 can be adapted to the case where actually

$$\Delta^{(1)} \cong \mathrm{Conv}\big\{(0,0), (a+bk, 0), (a,b), (0,b)\big\}.$$

Recently, Martens' result on Hirzebruch surfaces was generalized to a certain result on their blow-ups. This gives the following result, which subsumes Theorem 7.

Theorem 8 (Kawaguchi, 2008, 2010) *Let $a, b \geq 1$ and $k \geq 0$ be integers. Let $C \subset \mathbb{R}^2$ be the graph of a concave, continuous, piece-wise linear function $f : [0, a+bk] \to \mathbb{R}^+$ with $f(0) > 0$, $f(a) = b$, and $kf(a+bk) = 0$, such that its segments have lattice points as endpoints, there is at least one horizontal segment, and f is linear on $[a, a+bk]$. Then Conjecture 1 holds for the convex hull of $C \cup \{(0,0), (a,0)\}$.*



Proof Same proof, now using [18, 19] instead of [28]. □

We end with a very particular case. To us, it is nevertheless important because it proves Conjecture 1 in a situation where the combinatorial properties are somewhat harder to grasp.

Theorem 9 (Martens, 1980) *Conjecture 1 holds if $\Delta \cong 3\Upsilon$.*

Proof $\mathrm{Tor}(3\Upsilon) \cong \mathrm{Tor}(\Upsilon)$ can be realized in $\mathbb{P}^3 = \mathrm{Proj}\, \mathbb{C}[x, y, z, w]$ as the zero locus of $xyz - w^3$. An irreducible Laurent polynomial f that is supported on 3Υ then defines another cubic in $\mathbb{C}[x, y, z, w]$, the intersection of whose zero locus with $\mathrm{Tor}(\Upsilon)$ is birationally equivalent to $U(f)$. Generically, this intersection will be smooth. It is well-known that smooth complete intersections of two cubics in $\mathbb{P}^3$ have gonality 6. See [11, 27]. □

7 Toric degenerations

The material in this section is inspired by [32, Sect. 2.3]. It partly extends Sect. 4, but we will be slightly more concise here.

We first recall, in an arbitrary dimension $n \geq 1$, the construction of a *toric variety* $\mathrm{Tor}(\Delta)$ associated to a *lattice polytope* $\Delta \subset \mathbb{R}^n$, i.e. the convex hull of a finite number of points of $\mathbb{Z}^n$. We will assume throughout that Δ is *very ample*. The latter is a technical notion for which we refer to [7] (it guarantees that $\mathrm{Tor}(\Delta)$, as constructed below, is isomorphic to the abstract toric variety associated to the normal fan $\Sigma(\Delta)$ of Δ, see below), but we note that very-ampleness is implied by the existence of a subdivision into unimodular simplices. This is automatic if $n \leq 2$. Define $S := \Delta \cap \mathbb{Z}^n$ and consider the map ϕ sending a point $(x_1, \ldots, x_n)$ in the n-dimensional torus $\mathbb{T}^n$ over $\mathbb{C}$ to the point in $\mathbb{P}^{|S|-1}$ with projective coordinates $(x_1^{i_1} \cdot \cdots \cdot x_n^{i_n})_{(i_1,\ldots,i_n) \in S}$. The Zariski closure of the image of ϕ is the toric variety $\mathrm{Tor}(\Delta)$. If $X_{i_1,\ldots,i_n}$ denotes the projective coordinate of $\mathbb{P}^{|S|-1}$ corresponding to $(i_1, \dots, i_n) \in S$, then all binomials of the form

$$\prod_{k=1}^{s} X_{i_1^{(k)},\dots,i_n^{(k)}} - \prod_{k=1}^{s} X_{j_1^{(k)},\dots,j_n^{(k)}}$$

for which

$$\sum_{k=1}^{s} \big(i_1^{(k)}, \dots, i_n^{(k)}\big) = \sum_{k=1}^{s} \big(j_1^{(k)}, \dots, j_n^{(k)}\big)$$

are zero on $\operatorname{Tor}(\Delta)$. These binomials generate the homogeneous ideal of $\operatorname{Tor}(\Delta)$. As in Sect. 4, the faces $\tau \subset \Delta$ naturally decompose $\operatorname{Tor}(\Delta)$ in a disjoint union of toric orbits $O(\tau) \cong \mathbb{T}^{\dim \tau}$. One has $\phi(\mathbb{T}^n) = O(\Delta)$, and one can show that $\operatorname{Tor}(\Delta)$ is a normal variety.

Now let $\Delta \in \mathbb{R}^2$ be a two-dimensional lattice polygon. Let $\Delta_1, \dots, \Delta_r$ be a *regular subdivision* of Δ, i.e. a collection of two-dimensional lattice polygons for which there is an upper-convex piece-wise linear function $v : \Delta \to \mathbb{R}$ such that $\Delta_1, \dots, \Delta_r$ are the maximal closed subsets on which v is linear. If such a v exists, we may assume that $v(i, j) \in \mathbb{Z}$ for each $(i, j) \in \Delta \cap \mathbb{Z}^2$—see [7, Proposition 1.69(i)]. Let δ be an integer such that δ is strictly greater than each of these $v(i, j)$'s and let $\tilde{\Delta}$ be the convex hull in $\mathbb{R}^3$ of all the points $(i, j, v(i, j))$ and (i, j, δ) with $(i, j) \in \Delta \cap \mathbb{Z}^2$. The latter is easily seen to be very ample. For $\ell \in \{1, \dots, r\}$, we let $\tilde{\Delta}_\ell$ be the face

$$\big\{(i, j, v(i, j)) \mid (i, j) \in \Delta_\ell\big\} \subset \tilde{\Delta}.$$

Let $\tilde{S} = \tilde{\Delta} \cap \mathbb{Z}^3$ and consider the toric threefold $Y = \operatorname{Tor}(\tilde{\Delta})$ in $\mathbb{P}^{|\tilde{S}|-1}$, along with the corresponding monomial map $\tilde{\phi} : \mathbb{T}^3 \hookrightarrow \mathbb{P}^{|\tilde{S}|-1}$. There is a natural fibration

$$p : Y \to \mathbb{P}^1 : P = (X_{i,j,k})_{(i,j,k)\in\tilde{S}} \mapsto p(P) = (X_{i,j,k+1} : X_{i,j,k})$$

where i, j, k are chosen so that $X_{i,j,k}$ and $X_{i,j,k+1}$ are not both zero. The image $p(P)$ is independent of this choice, and for $(x, y, t) \in \mathbb{T}^3$ one has $p(\tilde{\phi}(x, y, t)) = (t : 1)$. The fiber $Y_\infty := p^{-1}(1 : 0)$ is equal to the copy of $\operatorname{Tor}(\Delta)$ contained in the linear subspace V of $\mathbb{P}^{|\tilde{S}|-1}$, defined by $X_{i,j,k} = 0$ for all $(i, j, k) \in \tilde{S}$ with $k < \delta$ (i.e., Y_∞ is the toric orbit associated to the top face of $\tilde{\Delta}$). If $t \in \mathbb{C} \setminus \{0\}$, the restriction of the projection

$$\pi : \mathbb{P}^{|\tilde{S}|-1} \to V : (X_{i,j,k})_{(i,j,k)\in\tilde{S}} \mapsto (X_{i,j,\delta})_{(i,j)\in\Delta\cap\mathbb{Z}^2}$$

to $Y_t := p^{-1}(t : 1)$ is an isomorphism between Y_t and Y_∞. On the other hand, the fiber $Y_0 := p^{-1}(0 : 1)$ is equal to $\bigcup_{\ell=1}^{r} \operatorname{Tor}(\tilde{\Delta}_\ell) \cong \bigcup_{\ell=1}^{r} \operatorname{Tor}(\Delta_\ell)$. So we get a degeneration of the toric surface $\operatorname{Tor}(\Delta)$ to $\bigcup_{\ell=1}^{r} \operatorname{Tor}(\Delta_\ell)$.

Let $R = \mathbb{C}[t]$ be equipped with the natural t-adic valuation $\operatorname{val} : R \setminus \{0\} \to \mathbb{Z}$. Let

$$f_t = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} a_{i,j}(t) x^i y^j$$

be a Laurent polynomial with coefficients in R that is supported on Δ, such that, when considered as a trivariate polynomial over $\mathbb{C}$, it is supported on $\tilde{\Delta}$. In particular, for all $(i,j) \in \Delta \cap \mathbb{Z}^2$ one has $\operatorname{val} a_{i,j}(t) \geq v(i,j)$. Define $c_{i,j} = (a_{i,j}(t) \cdot t^{-v(i,j)})|_{t=0}$. We make two assumptions about f_t:

- f_t is non-degenerate with respect to Δ (when considered as a Laurent polynomial over the field of Puiseux series $\mathbb{C}\{\{t\}\}$): this will be referred to as the *non-degeneracy* of f_t;
- the Laurent polynomials

$$\sum_{(i,j)\in\Delta_\ell\cap\mathbb{Z}^2} c_{i,j}x^i y^j \in \mathbb{C}[x^{\pm1}, y^{\pm1}]$$

(for $\ell = 1, \dots, r$) are non-degenerate with respect to the respective Δ_ℓ: this will be referred to as the *local non-degeneracy* of f_t.

Now, when considered as an element of $\mathbb{C}[t, x, y]$, our Laurent polynomial f_t defines a hyperplane section X of Y. For $t \in \mathbb{C} \setminus \{0\}$, the fiber $X_t := X \cap Y_t$ is equal to the intersection of Y_t with the hyperplane defined by

$$\sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} a_{i,j}(t) X_{i,j,\delta} = 0.$$

The fiber $X_0 := X \cap Y_0$ of X is equal to

$$X \cap \bigcup_{\ell=1}^{r} \operatorname{Tor}(\tilde{\Delta}_\ell) = \bigcup_{\ell=1}^{r} \bigl(X \cap \operatorname{Tor}(\tilde{\Delta}_\ell)\bigr),$$

where $X^{(\ell)} := X \cap \operatorname{Tor}(\tilde{\Delta}_\ell)$ is the intersection of $\operatorname{Tor}(\tilde{\Delta}_\ell)$ with the hyperplane

$$\sum_{(i,j)\in\Delta_\ell\cap\mathbb{Z}^2} c_{i,j} X_{i,j,v(i,j)} = 0.$$

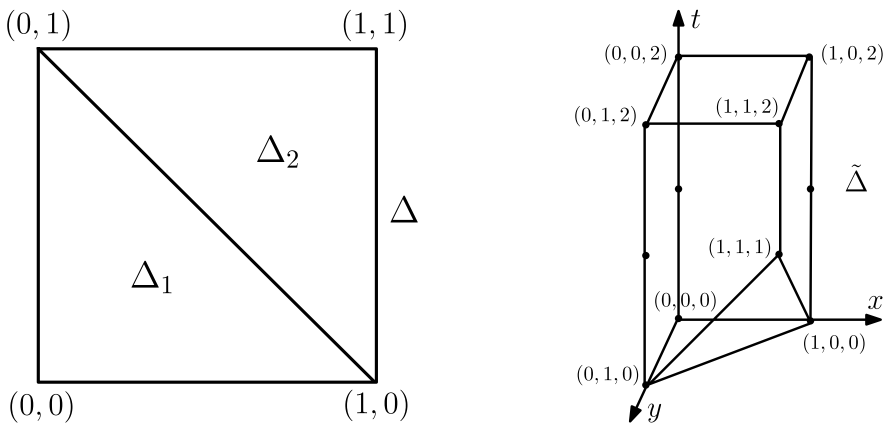
A pair $X^{(\ell)}, X^{(m)}$ intersects if and only if the polygons $\tilde{\Delta}_\ell$ and $\tilde{\Delta}_m$ have an edge in common, and if so, the intersection is defined by the hyperplane section

$$\sum_{(i,j,k)\in\tilde{\Delta}_\ell\cap\tilde{\Delta}_m\cap\mathbb{Z}^3} c_{i,j} X_{i,j,k} = 0$$

of $\operatorname{Tor}(\tilde{\Delta}_\ell) \cap \operatorname{Tor}(\tilde{\Delta}_m) = \operatorname{Tor}(\tilde{\Delta}_\ell \cap \tilde{\Delta}_m)$. By local non-degeneracy, the curves $X^{(\ell)}$ are smooth. Moreover, $X^{(\ell)}$ and $X^{(m)}$ intersect $\operatorname{Tor}(\tilde{\Delta}_\ell \cap \tilde{\Delta}_m)$ transversally in the same points. Hence they intersect each other transversally. The number of intersection points is equal to the number of lattice points in $\tilde{\Delta}_\ell \cap \tilde{\Delta}_m$ minus one. For instance, if $\tilde{\Delta}_\ell \cap \tilde{\Delta}_m$ is a line segment without lattice points in its interior, then the curve $\operatorname{Tor}(\tilde{\Delta}_\ell \cap \tilde{\Delta}_m)$ is a projective line and $X^{(\ell)} \cap X^{(m)}$ is a point on this line.

What we have actually constructed is a *strongly semi-stable arithmetic surface* over $\mathbb{C}[[t]]$ (see [2, Sect. 1.1] for this terminology). Indeed, consider the restriction of p to $X^{\text{fin}} := X \setminus X_\infty (= X \setminus p^{-1}(1:0))$. This gives X^{fin} the structure of a scheme over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$. Define $\mathfrak{X} := X^{\text{fin}} \otimes \mathbb{C}[[t]]$ as a scheme over $\mathbb{C}[[t]]$. It is proper and flat, and its generic fiber $\mathfrak{X} \otimes \mathbb{C}\{\{t\}\}$ is precisely the Zariski-closed embedding of $U(f_t)$ in $\operatorname{Tor}(\Delta)$, as described in Sect. 4 (with $\mathbb{C}$ replaced by $\mathbb{C}\{\{t\}\}$). By non-degeneracy, this is a smooth curve, hence $\mathfrak{X}$ is an arithmetic surface. On the other hand, the special fiber $\mathfrak{X} \otimes \mathbb{C}$ is precisely the reducible curve X_0 having $X^{(1)}, \ldots, X^{(r)}$ as its components. By local non-degeneracy, these components are smooth and intersect each other transversally. Hence the reduction is strongly semi-stable.

Example Let $f_t = 1 + x + y + txy$ and let Δ, Δ_1, Δ_2 and $\tilde{\Delta}$ be as depicted below:



and

$$v : (x, y) \in \Delta \mapsto \begin{cases} 0 & \text{if } (x, y) \in \Delta_1 \\ x + y - 1 & \text{if } (x, y) \in \Delta_2. \end{cases}$$

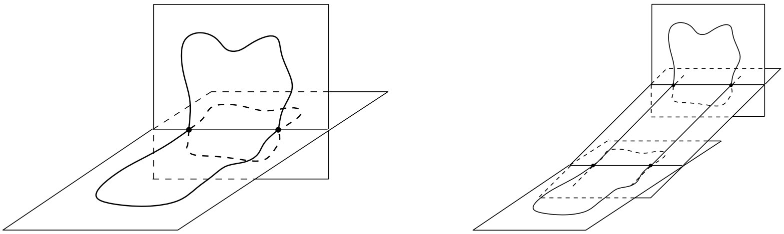
The toric threefold $Y = \operatorname{Tor}(\tilde{\Delta})$ lies in $\mathbb{P}^{10}$. For $t \in \mathbb{C} \setminus \{0\}$, the fiber Y_t is isomorphic to $\operatorname{Tor}(\Delta) = \mathbb{P}^1 \times \mathbb{P}^1$ and its special fiber Y_0 is the union of two planes $\operatorname{Tor}(\tilde{\Delta}_1)$ and $\operatorname{Tor}(\tilde{\Delta}_2)$ (respectively, with projective coordinates $(X_{0,0,0} : X_{1,0,0} : X_{0,1,0})$ and $(X_{1,0,0} : X_{0,1,0} : X_{1,1,1})$) that intersect each other in the line $\operatorname{Tor}(\tilde{\Delta}_1 \cap \tilde{\Delta}_2)$ (with projective coordinates $(X_{1,0,0} : X_{0,1,0})$). The special fiber X_0 is the union of two lines, namely $X^{(1)} \subset \operatorname{Tor}(\tilde{\Delta}_1)$ with equation $X_{0,0,0} + X_{1,0,0} + X_{0,1,0} = 0$ and $X^{(2)} \subset \operatorname{Tor}(\tilde{\Delta}_2)$ with equation $X_{1,0,0} + X_{0,1,0} + X_{1,1,1} = 0$, which intersect in the point $(1 : -1) \in \operatorname{Tor}(\tilde{\Delta}_1 \cap \tilde{\Delta}_2)$.

For the application that we have in mind, our strongly semi-stable arithmetic surface $\mathfrak{X}$ is supposed to be *regular*, which in our case is equivalent to saying that the singular locus of X does not meet X_0. In general, this is not satisfied. However, by local non-degeneracy, the singularities at X_0 are entirely related to the fact that the ambient space $Y = \operatorname{Tor}(\tilde{\Delta})$ is itself singular at Y_0, and a *toric resolution* automatically resolves the singularities of X at X_0. We give a brief sketch, in which we assume some additional background concerning toric varieties $\operatorname{Tor}(\Sigma)$ constructed from fans

Σ. For an account on such abstract toric varieties and toric resolutions, see [9]. For more details on resolving non-degenerate hypersurface singularities, see [24].

Let $\Sigma(\tilde{\Delta})$ be the normal fan of $\tilde{\Delta}$. One can always find a subdivision Σ' of $\Sigma(\tilde{\Delta})$ such that the induced birational morphism $\rho : \text{Tor}(\Sigma') \to \text{Tor}(\Sigma(\tilde{\Delta})) \cong \text{Tor}(\tilde{\Delta})$ is a resolution of singularities. Write $Y' = \text{Tor}(\Sigma')$ and let $X' \subset Y'$ be the strict transform of X under ρ. The morphism $p' = p \circ \rho$ yields a fibration $Y' \to \mathbb{P}^1$. One can then redo the argument and obtain an arithmetic surface $\mathfrak{X}'$ over $\mathbb{C}[[t]]$, which is still strongly semi-stable, but which is moreover regular. The generic fibers of $\mathfrak{X}$ and $\mathfrak{X}'$ are isomorphic, because $\rho|_{X'}$ is an isomorphism on $p'^{-1}(V)$ for an open subset V of $\mathbb{P}^1$. On the other hand, the special fiber of $\mathfrak{X}'$ differs from the special fiber of $\mathfrak{X}$. To see how the latter modifies under toric resolutions, it suffices to analyze what happens when we subdivide a two-dimensional cone.

First, we consider cones $\sigma^2_{\ell,m}$ spanned by rays σ^1_ℓ and σ^1_m that correspond to adjacent lower facets $\tilde{\Delta}_\ell$ and $\tilde{\Delta}_m$ of $\tilde{\Delta}$. Then $\sigma^2_{\ell,m}$ corresponds to the edge $\tilde{\Delta}_\ell \cap \tilde{\Delta}_m$, and the introduction of a new ray boils down to blowing up Y in $\text{Tor}(\tilde{\Delta}_\ell \cap \tilde{\Delta}_m)$. This separates the curves $X^{(\ell)}$ and $X^{(m)}$, and each intersection point becomes replaced by an exceptional curve intersecting $X^{(\ell)}$ and $X^{(m)}$ transversally. This exceptional curve is contained in the strict transform of X and hence belongs to the special fiber of our new arithmetic surface. All intersections remain transversal. More generally, if k rays are added to $\sigma_{\ell,m}$, then each intersection point becomes replaced by a chain of k transversally intersecting exceptional curves.



Next, consider cones σ^2_ℓ spanned by a ray σ^1_ℓ that corresponds to a lower facet $\tilde{\Delta}_\ell$ and a ray that corresponds to an adjacent vertical facet $\tilde{\Gamma}$. Then the introduction of a new ray boils down to blowing up Y in $\text{Tor}(\tilde{\Delta}_\ell \cap \tilde{\Gamma})$. Each point of intersection of $X^{(\ell)}$ with $\text{Tor}(\tilde{\Delta}_\ell \cap \tilde{\Gamma})$ becomes equipped with an emanating exceptional curve. More generally, if k rays are added to σ^2_ℓ, then the point becomes equipped with an emanating chain of k transversally intersecting exceptional curves.

8 The gonality of the dual graph

Linear systems on graphs and on metric graphs have been introduced by Baker and Norine in [2, 3]. It turns out that these linear systems obey properties that are analogous to those of linear systems on algebraic curves, such as the Riemann–Roch Theorem. Moreover, the Specialization Lemma [2, Lemma 2.8] can be used to transport results on metric graphs to algebraic curves or vice versa. In this section, we will briefly overview the basic notions of linear systems on metric graphs and use the Specialization Lemma to obtain a lower bound for curve gonalities.

Let $G=(V(G),E(G))$ be a connected graph without loops. The *metric graph* Γ associated to G is the compact connected metric space where each edge is identified with a line segment [0, 1]. A *divisor* $D=a_1v_1+\cdots+a_sv_s$ on Γ is an element of the free Abelian group generated by the points of the metric graph Γ. The *degree* of D is the sum $a_1+\cdots+a_s$ of its coefficients and D is *effective* if and only if each coefficient a_i is nonnegative. Let $\phi:\Gamma\to\mathbb{R}$ be a continuous map such that the restriction of ϕ to an edge of Γ is piece-wise linear with integer slopes and only finitely many pieces. If $v\in\Gamma$, write $\operatorname{ord}_v(\phi)$ to denote the sum of the incoming slopes of ϕ at v. Note that $\operatorname{ord}_v(\phi)$ is nonzero for only finitely many points $v\in\Gamma$, so we can consider the divisor $\operatorname{div}(\phi)=\sum_{v\in\Gamma}(\operatorname{ord}_v\phi)\cdot v$. We say that two divisors D and D' on Γ are *equivalent*, and denote this by $D\sim D'$, if and only if $D'-D=\operatorname{div}(\phi)$ for some ϕ. The *complete linear system* $|D|$ of a divisor D is the set of all effective divisors D' that are equivalent to D. The *rank* $r(D)$ of the linear system $|D|$ is defined as follows. We have $r(D)=-1$ if and only if $|D|=\emptyset$ and $r(D)\geq r$ if and only if $|D-E|\neq\emptyset$ for all effective divisors E on Γ of degree r. For instance, $r(D)=1$ if and only if $|D-P|\neq\emptyset$ for each point $P\in\Gamma$ and there exist points $P_1,P_2\in\Gamma$ (not necessarily distinct) such that $|D-P_1-P_2|=\emptyset$. The *gonality* of Γ is the minimal degree of a divisor on Γ having rank one.

For a lattice polygon $\Delta\subset\mathbb{R}^2$ and a regular subdivision $\Delta_1,\dots,\Delta_r\subset\Delta$, let $G=G(\Delta_1,\dots,\Delta_r)$ be the graph with vertex set $V(G)=\{v_1,\dots,v_r\}$ such that the number of edges between the vertices v_ℓ and v_m is equal to the number of lattice points of $\Delta_\ell\cap\Delta_m$ minus one. Let $\Gamma=\Gamma(\Delta_1,\dots,\Delta_r)$ be the metric graph associated to G.

Theorem 10 *Let $\Delta\subset\mathbb{R}^2$ be a two-dimensional lattice polygon and let $\Delta_1,\dots,\Delta_r$ be a regular subdivision of Δ. Let S be the set of irreducible Laurent polynomials $f\in\mathbb{C}[x^{\pm1},y^{\pm1}]$ for which $\Delta(f)=\Delta$ and the gonality of $U(f)$ is at least the gonality of the metric graph $\Gamma(\Delta_1,\dots,\Delta_r)$. Then S is Zariski dense in the space of Laurent polynomials $f\in\mathbb{C}[x^{\pm1},y^{\pm1}]$ for which $\Delta(f)\subset\Delta$.*

Proof Let $R=\mathbb{C}[t]$. We construct a Laurent polynomial $f_t\in R[x^{\pm1},y^{\pm1}]$ to which we can apply the machinery of Sect. 7. Let $v:\Delta\to\mathbb{R}$ be a un upper-convex piece-wise linear function realizing the subdivision $\Delta_1,\dots,\Delta_r$ such that $v(\Delta\cap\mathbb{Z}^2)\subset\mathbb{Z}$.

First, let

$$g_t=\sum_{(i,j)\in\Delta\cap\mathbb{Z}^2}c_{i,j}x^iy^j\in\mathbb{C}\big[x^{\pm1},y^{\pm1}\big]$$

be such that each $g_{t,\ell}:=\sum_{(i,j)\in\Delta_\ell\cap\mathbb{Z}^2}c_{i,j}x^iy^j$ is non-degenerate with respect to its Newton polygon Δ_ℓ. This is possible, because each non-degeneracy condition is generically satisfied, and it will guarantee the local non-degeneracy of f_t below. Second, we consider the polynomial

$$h_t=\sum_{(i,j)\in\Delta\cap\mathbb{Z}^2}c_{i,j}t^{v(i,j)}x^iy^j\in\mathbb{C}[t]\big[x^{\pm1},y^{\pm1}\big].$$

Now since $\mathbb{C}[t]$ is infinite and since non-degeneracy is generically satisfied, there does exist a Laurent polynomial

$$h'_t = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} a_{i,j}(t)x^i y^j \in \mathbb{C}[t]\big[x^{\pm 1}, y^{\pm 1}\big]$$

that is non-degenerate with respect to its Newton polygon Δ (when considered as a Laurent polynomial with coefficients in $\mathbb{C}\{\{t\}\}$). But then all but finitely many among the Laurent polynomials

$$h_t + \lambda\big(h'_t - h_t\big), \quad \lambda \in \mathbb{C}[t]$$

must be non-degenerate with respect to their Newton polygon Δ: indeed, this spans a line in coefficient space which is not strictly contained in the degenerate locus. By taking a λ with high t-adic valuation, we end up with a Laurent polynomial f_t that is non-degenerate with respect to its Newton polygon Δ, such that the t-adically leading terms of the coefficients are the same as in h_t.

Then by letting δ be an integer that is strictly bigger than the valuation of each of the coefficients of f_t, and by constructing $\tilde{\Delta}$ accordingly, we can follow Sect. 7 and end up with a (possibly non-regular) strongly semi-stable arithmetic surface $\mathfrak{X}$ over $\mathbb{C}[[t]]$. The dual graph of the special fiber $\mathfrak{X} \otimes \mathbb{C} = X^{(1)} \cup \cdots \cup X^{(r)}$ is equal to $G(\Delta_1, \ldots, \Delta_r)$. Indeed, each vertex v_ℓ corresponds to a curve $X^{(\ell)}$ and each edge $e = (v_\ell, v_m)$ corresponds to an intersection point of $X^{(\ell)}$ and $X^{(m)}$. Let $\Gamma = \Gamma(\Delta_1, \ldots, \Delta_r)$ be the associated metric graph. Now let $\mathfrak{X}'$ be a *regular* strongly semi-stable arithmetic surface, obtained from a subdivision of $\Sigma(\tilde{\Delta})$. By refining the subdivision if necessary, we may assume that each two-dimensional cone of $\Sigma(\tilde{\Delta})$ becomes subdivided by an equal amount of rays (say k). Then the dual graph of $\mathfrak{X}' \otimes \mathbb{C}$ is obtained from $G(\Delta_1, \ldots, \Delta_r)$ by introducing k new vertices on each edge, and by attaching to certain vertices an emanating linear graph. Denote it by G' and let Γ' be the associated metric graph. Then the gonalities of Γ and Γ' are the same. Indeed, removing the emanating linear graphs from Γ' clearly does not affect the gonality, and the remaining graph is a mere rescaling of Γ (by a factor $k+1$).

Then [2, Corollary 3.2] implies that the gonality of $U(f_t)$ over $\mathbb{C}\{\{t\}\}$ is at least the gonality of Γ', hence it is at least the gonality of Γ. To be precise, in [2], the results are stated using the $\mathbb{Q}$-graph $\Gamma'_{\mathbb{Q}}$ (i.e. only the rational points on the edges are considered), but the gonality of a metric graph Γ' is equal to the gonality of its corresponding $\mathbb{Q}$-graph $\Gamma'_{\mathbb{Q}}$. Indeed, by [2, Corrolary 1.5], a $\mathbb{Q}$-divisor has rank one on Γ' if and only if it has rank one on $\Gamma'_{\mathbb{Q}}$, so the gonality of Γ' is at least the gonality of $\Gamma'_{\mathbb{Q}}$. On the other hand, using the rational approximation argument from [12], it follows that the gonality of Γ' is at most the gonality of $\Gamma'_{\mathbb{Q}}$.

Because $\mathbb{C} \cong \mathbb{C}\{\{t\}\}$, there exists a Laurent polynomial $f' \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ such that the gonality of $U(f')$ over $\mathbb{C}$ is equal to the gonality of $U(f_t)$ over $\mathbb{C}\{\{t\}\}$. Thus the gonality of $U(f')$ is bounded from below by the gonality of Γ. To conclude the proof, one can either analyze the degree of freedom in the construction of f_t above, or apply the semi-continuity lemma below. □

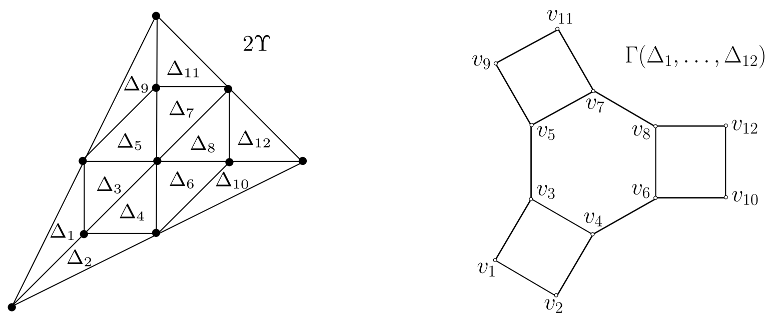
Lemma 4 (Semi-continuity) *Let Δ be a lattice polygon. Let $M_\Delta \subset \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ be the set of Laurent polynomials that are non-degenerate with respect to their Newton polygon Δ (seen as a quasi-affine variety in coefficient space). Then the map $M_\Delta \to \mathbb{Z}$ sending f to the gonality of $U(f)$ is lower semi-continuous.*

Proof Let g be the genus of Δ and let $\mathcal{M}_g$ be the moduli space of curves of genus g. It is well-known that the map $\mathcal{M}_g \to \mathbb{Z}$ sending a curve to its gonality is lower semi-continuous—see e.g. [17, Proposition 3.4]. By the flatness of the family of curves parameterized by M_Δ, we are given a unique morphism $M_\Delta \to \mathcal{M}_g$ sending f to the isomorphism class of $U(f)$. See [8, Sect. 2] for more details. Since M_Δ is irreducible, the result follows. □

We expect that Theorem 10 is sharp, in the following sense:

Conjecture 2 *Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon. Then there is a regular subdivision $\Delta_1, \dots, \Delta_r$ of Δ such that the set of irreducible Laurent polynomials $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ for which $\Delta(f) = \Delta$ and the gonality of $U(f)$ is equal to the gonality of the metric graph $\Gamma(\Delta_1, \dots, \Delta_r)$, is Zariski dense in the space of Laurent polynomials $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ with $\Delta(f) \subset \Delta$.*

The above conjecture is true for $\Delta = 2\Upsilon$, since the metric graph Γ corresponding to the subdivision $\Delta_1, \dots, \Delta_{12}$ of 2Υ (see the picture below) has gonality equal to 3. For instance, the divisor $v_1 + v_2 + v_3$ has rank one.



9 A purely combinatorial conjecture

Conjectures 1 and 2 can be combined to a purely combinatorial statement.

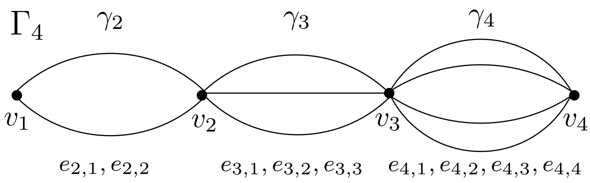
Conjecture 3 *Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon. Then there exists a regular subdivision $\Delta_1, \dots, \Delta_r$ of Δ such that the gonality of the metric graph $\Gamma(\Delta_1, \dots, \Delta_r)$ is equal to $\mathrm{lw}(\Delta^{(1)}) + 2$ if $\Delta \ncong 2\Upsilon$, and to 3 if $\Delta \cong 2\Upsilon$.*

We will prove this conjecture for a particular family of lattice polygons (see Theorem 11). For this, we need to study the gonality of a certain metric graph. When dealing with linear systems on metric graphs, it is often convenient to view an effective divisor $D = a_1 v_1 + \cdots + a_s v_s$ as a chip configuration on Γ where a stack of a_i chips is placed on the point v_i of Γ. We will use the chip terminology, the notion of reduced divisors [16, Theorem 10] and Dhar's burning algorithm [26, Sect. 2] in the following proof.

Lemma 5 *Let $r \geq 1$ be an integer and let G_r be the graph defined by $V(G_r) = \{v_1, \dots, v_r\}$ and*

$$E(G_r) = \{e_{i,j} = (v_{i-1}, v_i) \mid i = 2, \dots, r; j = 1, \dots, i\},$$

where the latter should be seen as a multiset. Then its corresponding metric graph Γ_r has gonality equal to r.

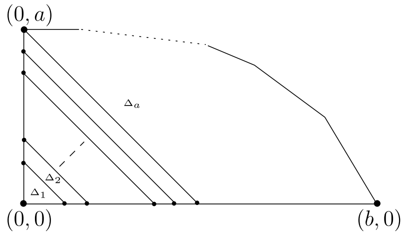


Proof Since the divisor $v_1 + \cdots + v_r$ on Γ has rank one, the gonality of Γ is at most equal to r. Suppose D is an effective divisor on Γ with $\deg(D) < r$ and rank at least one. We may assume that D is v_1-reduced, hence D has at least one chip at v_1. Let γ_i be obtained by taking the union of the i edges $e_{i,1}, \dots, e_{i,i}$ between v_{i-1} and v_i and excluding the vertex v_{i-1}. Since $\Gamma = \{v_1\} \cup \gamma_2 \cup \cdots \cup \gamma_r$, the pigeonhole principle implies that at least one of the subsets $\gamma_2, \dots, \gamma_r$ of Γ does not contain a chip of D. Let i be the maximal index for which γ_i does not contain a chip of D. If we perform Dhar's burning algorithm to reduce D with respect to v_i, the chips of D on the subset $\gamma_{i+1} \cup \cdots \cup \gamma_r$ will not move, since D is v_1-reduced and hence fire from v_1 will pass through v_i. So we need that at some point chips must move along γ_i, since D has rank at least one. If a chip moves along one of the edges of γ_i, this must also be the case for the other edges. Indeed, otherwise we can find a cycle in $\gamma_i \cup \{v_{i-1}\} \subset \Gamma$ such that chips on it only move in one direction, which cannot happen inside a linear system. We conclude that D must have at least i chips in $\{v_1\} \cup \gamma_2 \cup \cdots \cup \gamma_{i-1}$. Since $\gamma_{i+1}, \dots, \gamma_r$ contain at least one chip of D, the total amount of chips or the degree of D is at least $i + (r - i) = r$, a contradiction. □

Theorem 11 *Let a, b be integers with $2 \leq a \leq b$. Let $C \subset \mathbb{R}^2$ be the graph of a concave, continuous, piece-wise linear function $f : [0, b] \to \mathbb{R}^+$ with $f(0) = a \geq f(1)$ and $f(b) = 0$ such that its segments have lattice points as end points. Then Conjecture 3 holds for the convex hull Δ of C with $\{(0, 0)\}$.*

Proof Consider the regular subdivision $\Delta_1, \dots, \Delta_a$ of Δ where

$$\Delta_i = \begin{cases} \text{Conv}\{(0,0), (2,0), (0,2)\} & \text{if } i = 1, \\ \text{Conv}\{(i,0), (i+1,0), (0,i+1), (0,i)\} & \text{if } i = 2, \dots, a-1, \\ \Delta \setminus \text{Conv}\{(0,0), (a,0), (0,a)\} & \text{if } i = a. \end{cases}$$



If $\Delta \neq a\Sigma$, then $\Gamma(\Delta_1, \ldots, \Delta_a) = \Gamma_a$, which by Lemma 5 has gonality equal to a. On the other hand, $\mathrm{lw}(\Delta^{(1)}) + 2 = \mathrm{lw}(\Delta)$ is equal to a (it cannot be strictly less than a because Δ has two adjacent edges containing at least $a + 1$ lattice points).

If $\Delta = a\Sigma$, then $\Delta_a = \emptyset$ and $\Gamma(\Delta_1, \ldots, \Delta_{a-1}) = \Gamma_{a-1}$, which has gonality equal to $a - 1 = \mathrm{lw}(\Delta^{(1)}) + 2$. □

This gives a new proof of Theorems 6 and 7, and parts of Theorem 8. But it also deals with various new cases, including triangles of the form $\mathrm{Conv}\{(0, 0), (b, 0), (0, a)\}$.

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