

Planar flows and quadratic relations over semirings

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Abstract Adapting Lindström’s well-known construction, we consider a wide class of functions which are generated by flows in a planar acyclic directed graph whose vertices (or edges) take weights in an arbitrary commutative semiring. We give a combinatorial description for the set of “universal” quadratic relations valid for such functions. Their specializations to particular semirings involve plenty of known quadratic relations for minors of matrices (e.g., Plücker relations) and the tropical counterparts of such relations. Also some applications and related topics are discussed.

Keywords Plücker relation · Dodgson condensation · Tropicalization · Semiring · Planar graph · Network flow · Lindström’s lemma · Schur function · Laurent phenomenon

1 Introduction

In this paper, we consider functions which take values in a commutative semiring and are generated by planar flows. Functions of this sort satisfy plenty of quadratic relations, and our goal is to describe a combinatorial method to reveal and prove such relations. One important class consists of quadratic relations of Plücker type.

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Recall some basic facts concerning Plücker algebra and Plücker coordinates. For a positive integer n , let $[n]$ denote the set $\{1, 2, \dots, n\}$. Consider the $n \times n$ matrix \mathbf{x} of indeterminates x_{ij} and the corresponding commutative polynomial ring $\mathbb{Z}[\mathbf{x}]$. Also consider the polynomial ring $\mathbb{Z}[\Delta]$ generated by variables Δ_S indexed by the subsets $S \subseteq [n]$. They are linked by the natural ring homomorphism $\psi : \mathbb{Z}[\Delta] \rightarrow \mathbb{Z}[\mathbf{x}]$ that brings each variable Δ_S to the flag minor polynomial for S , i.e., to the determinant of the submatrix \mathbf{x}_S formed by the column set S and the row set $\{1, \dots, |S|\}$ of \mathbf{x} . An important fact is that the ideal $\ker(\psi)$ of $\mathbb{Z}[\Delta]$ is generated by homogeneous quadrics, each being a certain integer combination of products $\Delta_S \Delta_{S'}$. They correspond to quadratic relations on the Plücker coordinates of an invertible $n \times n$ matrix (regarded as a point of the corresponding flag manifold embedded in an appropriate projective space); for a survey see, e.g., [13, Chap. 14].

There are many quadratic Plücker relations on flag minors of a matrix whose entries are assumed to belong to an arbitrary commutative ring \mathfrak{R} (the case $\mathfrak{R} = \mathbb{R}$ or \mathbb{C} is most popular). Let $f(S)$ denote the flag minor with a column set S in this matrix.

The simplest examples of Plücker relations involve triples: for any three elements $i < j < k$ in $[n]$ and any subset $X \subseteq [n] - \{i, j, k\}$, the flag minor function $f : 2^{[n]} \rightarrow \mathfrak{R}$ of an $n \times n$ matrix satisfies

$$f(Xik)f(Xj) - f(Xij)f(Xk) - f(Xjk)f(Xi) = 0, \tag{1.1}$$

where for brevity we write $Xi' \dots j'$ for $X \cup \{i', \dots, j'\}$. We call (1.1) the *AP3-relation* (abbreviating “algebraic Plücker relation with triples”). Another well-known special case (in particular, encountered in a characterization of Grassmannians) involves quadruples $i < j < k < \ell$; this is of the form

$$f(Xik)f(Xj\ell) - f(Xij)f(Xk\ell) - f(Xi\ell)f(Xjk) = 0. \tag{1.2}$$

A general algebraic Plücker relation on flag minors of a matrix can be written as

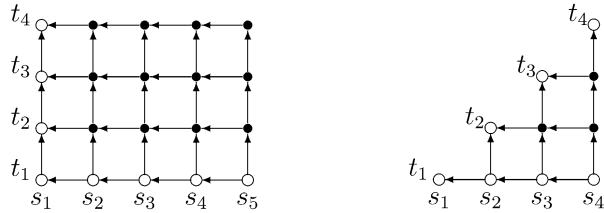
$$\sum_{A \in \mathcal{A}} f(X \cup A) f(X \cup (Y - A)) - \sum_{B \in \mathcal{B}} f(X \cup B) f(X \cup (Y - B)) = 0. \tag{1.3}$$

Here X and Y are disjoint subsets of $[n]$, and \mathcal{A} and \mathcal{B} are certain families of p -element subsets of Y , for some p .

In fact, an instance of (1.3) (such as (1.1) or (1.2)) represents a class of relations of “the same type”. More precisely, let $m := |Y|$ and define γ_Y to be the order preserving bijective map $[m] \rightarrow Y$, i.e., $\gamma_Y(i) < \gamma_Y(j)$ for $i < j$. This gives the families $\mathcal{A}_0, \mathcal{B}_0$ of p -element subsets of the initial interval $[m]$ such that $\mathcal{A} = \{\gamma_Y(C) : C \in \mathcal{A}_0\}$ and $\mathcal{B} = \{\gamma_Y(C) : C \in \mathcal{B}_0\}$. We call \mathcal{A}_0 the *pattern* of \mathcal{A} and write $\mathcal{A} = \gamma_Y(\mathcal{A}_0)$, and similarly for \mathcal{B}_0 and \mathcal{B} . When considering a class of functions $f : 2^{[n]} \rightarrow \mathcal{R}$ and speaking of (1.3) as a “universal” (or “stable”) relation, we require that (1.3) be valid for all functions within this class and depend only on m, p , and the patterns $\mathcal{A}_0, \mathcal{B}_0$, but not on X and Y . Namely, (1.3) should hold for any choice of disjoint $X, Y \subseteq [n]$ with $|Y| = m$ and for the corresponding families $\mathcal{A} := \gamma_Y(\mathcal{A}_0)$ and $\mathcal{B} := \gamma_Y(\mathcal{B}_0)$.

In particular, (1.3) turns into (1.1) when $m = 3, p = 2, \mathcal{A}_0 = \{13\}, \mathcal{B}_0 = \{12, 23\}$, and $Y = \{i, j, k\}$, and turns into (1.2) when $m = 4, p = 2, \mathcal{A}_0 = \{13\}, \mathcal{B}_0 = \{12, 14\}$, and $Y = \{i, j, k, \ell\}$.

Fig. 1 The grid $\Gamma_{5,4}$ (left) and the half-grid Γ_4^Δ (right)



An important fact established by Lindström [12] is that the minors of many matrices can be expressed in terms of flows in a planar graph. A certain flow model will play a key role in our description; next we specify the terminology and notation that we use. (A more general flow model yielding a generalization of Lindström’s result is given in [14, 15].)

By a *planar network* we mean a finite directed planar *acyclic* graph $G = (V, E)$ in which two subsets $S = \{s_1, \dots, s_n\}$ and $T = \{t_1, \dots, t_{n'}\}$ of vertices are distinguished, called the sets of *sources* and *sinks*, respectively. We assume that these vertices, also called *terminals*, lie in the *boundary* of a compact convex region in the plane, which we denote by O and sometimes conditionally call a “circumference”, and the remaining part of the graph lies inside O . The terminals appear in O in the cyclic order $s_n, \dots, s_1, t_1, \dots, t_{n'}$ clockwise (with possibly $s_1 = t_1$ or $s_n = t_{n'}$), and for convenience we say that the sources and sinks lie in the “lower” and “upper” halves of O , respectively, and that the indices in each set grow “from left to right”.

Two important particular cases are: the (square) *grid* $\Gamma_{n,n'}$ and the *half-grid* Γ_n^Δ , where the vertices in the former are the integer points $(i, j) \in \mathbb{R}^2$ with $1 \leq i \leq n, 1 \leq j \leq n'$, the vertices in the latter are the integer points (i, j) with $1 \leq j \leq i \leq n$, and the edges in both cases are all possible ordered pairs of the form $((i, j), (i - 1, j))$ or $((i, j), (i, j + 1))$. The sources are the points $s_i := (i, 1)$, whereas the sinks are the points $t_j := (1, j)$ in the former case, and the $t_j := (j, j)$ in the latter case. The graphs $\Gamma_{5,4}$ and Γ_4^Δ are illustrated in Fig. 1.

In what follows, the collection of pairs $(I \subseteq [n], I' \subseteq [n'])$ with equal sizes: $|I| = |I'|$, is denoted by $\mathcal{E}^{n,n'}$. By an $(I|I')$ -*flow* we mean a collection ϕ of *pairwise (vertex) disjoint directed paths* in G going from the source set $S_I := \{s_i : i \in I\}$ to the sink set $T_{I'} := \{t_j : j \in I'\}$. The set of $(I|I')$ -flows in G is denoted by $\Phi_{I|I'}^G$, or simply by $\Phi_{I|I'}$.

Let $w : V \rightarrow \mathfrak{R}$ be a *weighting* on the vertices of G (alternatively, one can consider a weighting on the edges; see the end of this section). We associate to w the function $f = f_w$ on $\mathcal{E}^{n,n'}$ defined by

$$f(I|I') := \sum_{\phi \in \Phi_{I|I'}} \prod_{v \in V_\phi} w(v), \quad (I, I') \in \mathcal{E}^{n,n'}, \tag{1.4}$$

where V_ϕ is the set of vertices occurring in a flow ϕ . (When G has no flow for some (I, I') , we set $f(I|I') := 0$.) We refer to f obtained in this way as an *algebraic flow-generated function*, or an *AFG-function* for short.

When an $(I|I')$ -flow ϕ enters the first $|I| =: k$ sinks (i.e., $I' = [k]$), we say that ϕ is a *flag flow* for I . Accordingly, we use the abbreviated notation Φ_I for $\Phi_{I|[k]}$, and

$f_w(I)$ for $f_w(I|[k])$. When we are interested in the flag case only, f_w is regarded as a function on the set $2^{[n]}$ of subsets of $[n]$.

Lindström [12] showed that if M is the $n' \times n$ matrix whose entries m_{ji} are defined as $\sum_{\phi \in \Phi_{[i][j]}} \prod_{v \in V_\phi} w(v)$, then for any $(I, I') \in \mathcal{E}^{n,n'}$, the minor of M with the column set I and the row set I' is equal to the value $f(I|I')$ as in (1.4). A converse property is valid for many matrices, in particular, for the totally nonnegative matrices (see [3]): the minors of such a matrix can be expressed as above via flows for some planar network and weighting. (Recall that a real matrix is called *totally nonnegative* (*totally positive*) if all minors in it are nonnegative (resp., positive).)

Another important application of the flow model concerns tropical analogues of the above quadratic relations. In this case, the flow-generated function $f = f_w$ determined by a weighting w on V is defined by

$$f(I|I') := \max_{\phi \in \Phi_{I|I'}} \left(\sum_{v \in V_\phi} w(v) \right), \quad (I, I') \in \mathcal{E}^{n,n'}. \tag{1.5}$$

Here w is assumed to take values in a totally ordered abelian group \mathcal{L} (usually one deals with $\mathcal{L} = \mathbb{R}$ or \mathbb{Z}). The expression for f in (1.5) is the tropicalization of that in (1.4), and f is said to be a *tropical flow-generated function*, or a *TFG-function*. Some appealing properties of such functions and related objects in the flag flow case are demonstrated in [4, 5] (where real-valued tropical functions are considered but everywhere \mathbb{R} can be replaced by \mathcal{L}). In particular, a TFG-function f satisfies the tropical analog of (1.1), or the *TP3-relation*: for $i < j < k$ and $X \subseteq [n] - \{i, j, k\}$,

$$f(Xik) + f(Xj) = \max\{f(Xij) + f(Xk), f(Xjk) + f(Xi)\}. \tag{1.6}$$

In this paper, we combine both cases, the algebraic and tropical ones, by considering functions taking values in an arbitrary *commutative semiring* \mathfrak{S} , a set equipped with two associative and commutative binary operations \oplus (addition) and \odot (multiplication) satisfying the distributive law $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$. Sometimes we will assume that \mathfrak{S} contains neutral elements $\underline{0}$ (for addition) and/or $\underline{1}$ (for multiplication). Two special cases are: (i) a commutative ring (in which case $\underline{0} \in \mathfrak{S}$ and each element has an additive inverse); (ii) a commutative semiring with division (in which case $\underline{1} \in \mathfrak{S}$ and each element has a multiplicative inverse). Examples of (ii) are: the set $\mathbb{R}_{>0}$ of positive reals (with $\oplus = +$ and $\odot = \cdot$), and the above-mentioned tropicalization of a totally ordered abelian group \mathcal{L} , denoted as $\mathcal{L}^{\text{trop}}$ (with $\oplus = \max$ and $\odot = +$). The set $2\mathbb{Z}_{>0}$ of positive even integers (with usual addition and multiplication) gives an example of a commutative semiring having neither $\underline{0}$ nor $\underline{1}$.

Extending (1.4) and (1.5), the flow-generated function $f = f_w$ determined by a weighting $w : V \rightarrow \mathfrak{S}$ is defined by

$$f(I|I') := \bigoplus_{\phi \in \Phi_{I|I'}} w(\phi), \quad (I, I') \in \mathcal{E}^{n,n'},$$

where $w(\phi)$ denotes the weight $\odot(w(v) : v \in V_\phi)$ of a flow ϕ . We call f an *FG-function* (abbreviating “flow-generated function”), and denote the set of these functions by $\mathbf{FG}_{n,n'}(\mathfrak{S})$.

Remark 1 Note that an $(I|I')$ -flow in G may not exist, making $f(I|I')$ undefined if \mathfrak{S} does not contain $\underline{0}$ (e.g., in the tropical case). To overcome this trouble, we may formally extend \mathfrak{S} , when needed, by adding an “extra neutral” element $*$, setting $* \oplus a = a$ and $* \odot a = *$ for all $a \in \mathfrak{S}$. In the extended semiring $\widehat{\mathfrak{S}}$, one defines $f(I|I') := *$ whenever $\Phi_{I|I'} = \emptyset$.

As before, we write $f(I)$ for $f(I|[|I|])$ in the flag flow case. Then a direct analogue of the general Plücker relation (1.3) for \mathfrak{S} is viewed as

$$\bigoplus_{A \in \mathcal{A}} (f(X \cup A) \odot f(X \cup (Y - A))) = \bigoplus_{B \in \mathcal{B}} (f(X \cup B) \odot f(X \cup (Y - B))). \tag{1.7}$$

Definition 1 Let $p < m \leq n$ and let $\mathcal{A}_0, \mathcal{B}_0$ be two families of p -element subsets of $[m]$. If (1.7) (with f determined as above) holds for any commutative semiring \mathfrak{S} , acyclic directed graph G , weighting w , disjoint subsets X and Y , and the families $\mathcal{A} := \gamma_Y(\mathcal{A}_0)$ and $\mathcal{B} := \gamma_Y(\mathcal{B}_0)$, then we call (1.7) a *stable quadratic relation of Plücker type*, or a *PSQ-relation* for short, and say that it is induced by the patterns $\mathcal{A}_0, \mathcal{B}_0$.

Note that in general we admit that \mathcal{A}_0 or \mathcal{B}_0 can contain multiple members. In other words, one may assume that for m, p fixed, the pairs of patterns inducing PSQ-relations constitute an abelian group under the operations $(\mathcal{A}_0, \mathcal{B}_0) + (\mathcal{A}'_0, \mathcal{B}'_0) := (\mathcal{A}_0 \sqcup \mathcal{A}'_0, \mathcal{B}_0 \sqcup \mathcal{B}'_0)$ and $(\mathcal{A}_0, \mathcal{B}_0) - (\mathcal{A}'_0, \mathcal{B}'_0) := (\mathcal{A}_0 \sqcup \mathcal{B}'_0, \mathcal{B}_0 \sqcup \mathcal{A}'_0)$, where \sqcup denotes the disjoint set union. For this reason, we will write $\mathcal{A}_0 \in \binom{[m]}{p}$ and $\mathcal{A} \in \binom{Y}{p}$ (with symbol \in rather than \subseteq), and similarly for \mathcal{B}_0 and \mathcal{B} .

Next, a Plücker relation on minors of a matrix deals with flag minors and is *homogeneous*, in the sense that the pairs of minor sizes in all products are the same. However, there are quadratic relations involving non-flag and non-homogeneous minors. One relation of this sort is expressed by Dodgson’s condensation formula [6]:

$$f(iX|i'X') f(Xk|X'k') = f(iXk|i'X'k') f(X|X') + f(iX|X'k') f(Xk|i'X'), \tag{1.8}$$

where $f(I|I')$ stands for the minor of a matrix with the column set I and the row set I' , $k - i = k' - i' > 0$, X is the interval $[i + 1..k - 1]$ (from $i + 1$ to $k - 1$) and X' is the interval $[i' + 1..k' - 1]$.

This inspires a study of a larger class of quadratic relations on flow-generated functions over commutative semirings. Now for \mathfrak{S}, G, w as before, we deal with the function $f = f_w$ on $\mathcal{E}^{n,n'}$, and consider disjoint $X, Y \subseteq [n]$ and disjoint $X', Y' \subseteq [n']$. An identity of our interest is of the form

$$\begin{aligned} & \bigoplus_{(A,A') \in \mathcal{A}} (f(XA|X'A') \odot f(X\bar{A}|X'\bar{A}')) \\ &= \bigoplus_{(B,B') \in \mathcal{B}} (f(XB|X'B') \odot f(X\bar{B}|X'\bar{B}')). \end{aligned} \tag{1.9}$$

Here, to simplify notation, we write KL for the union $K \cup L$ of disjoint sets K, L , denote the complement $Y - C$ of $C \subseteq Y$ by \bar{C} , and the complement $Y' - C'$ of

$C' \subseteq Y'$ by \overline{C}' . The families \mathcal{A}, \mathcal{B} consist of certain pairs $(C \subseteq Y, C' \subseteq Y')$ (admitting multiplicities). As before, we are interested in “universal” relations, and for this reason, consider the patterns $\mathcal{A}_0, \mathcal{B}_0$ formed by the pairs $(A_0 \subseteq [m], B_0 \subseteq [m'])$ such that $\mathcal{A} = \gamma_{Y, Y'}(\mathcal{A}_0)$ and $\mathcal{B} = \gamma_{Y, Y'}(\mathcal{B}_0)$, where $m := |Y|$, $m' := |Y'|$, and $\gamma_{Y, Y'}$ is the bi-component order preserving bijective map of $[m] \sqcup [m']$ to $Y \sqcup Y'$. Observe that (1.7) is a special case of (1.9) with $X' = \{1, 2, \dots, |X| + r\}$ and $Y' = \{|X| + r + 1, \dots, |X| + m - r\}$, where $r := \min\{p, m - p\}$.

Definition 2 When (1.9) holds for fixed $\mathcal{A}_0, \mathcal{B}_0$ as above and any corresponding $\mathfrak{S}, G, w, X, Y, X', Y'$ and the families $\mathcal{A} := \gamma_{Y, Y'}(\mathcal{A}_0)$ and $\mathcal{B} := \gamma_{Y, Y'}(\mathcal{B}_0)$, we call (1.9) a (general) *stable quadratic relation*, or an *SQ-relation*, and say that it is induced by the patterns $\mathcal{A}_0, \mathcal{B}_0$.

To distinguish between the general and Plücker cases, we will refer to \mathcal{A}, \mathcal{B} in Definition 2 as *2-families*, and to $\mathcal{A}_0, \mathcal{B}_0$ as *2-patterns*, whereas analogous objects in Definition 1 will be called *1-families* and *1-patterns*.

The goal of this paper is to describe a relatively simple combinatorial method of characterizing the patterns $\mathcal{A}_0, \mathcal{B}_0$ inducing SQ-relations (in particular, PSQ-relations). In fact, our method generalizes a flow rearranging approach used in [4] for proving the TP3-relation for TFG-functions. The method consists in reducing to a certain combinatorial problem, and as a consequence, provides an “efficient” procedure to recognize whether or not a pair \mathcal{A}, \mathcal{B} of 2-families yields an SQ-relation.

The main result obtained on this way is roughly as follows. We associate to a pair $(C \subseteq [m], C' \subseteq [m'])$ a certain set $\mathcal{M}(C, C')$ of perfect matchings on $[m] \sqcup [m']$. Given a pair $\mathcal{A}_0, \mathcal{B}_0$ of 2-patterns for m, m' , define $\mathcal{M}(\mathcal{A}_0)$ to be the collection of such matchings over all members of \mathcal{A}_0 (counting multiplicities), and define $\mathcal{M}(\mathcal{B}_0)$ in a similar way. We say that $\mathcal{A}_0, \mathcal{B}_0$ are *balanced* if the families $\mathcal{M}(\mathcal{A}_0)$ and $\mathcal{M}(\mathcal{B}_0)$ are equal, and show (Theorem 3.1) that

2-patterns $\mathcal{A}_0, \mathcal{B}_0$ induce an SQ-relation if and only if they are balanced.

Our approach to handling flows and reducing the problem to examining certain collections of matchings is close in essence to a lattice paths method elaborated in Fulmek and Kleber [9] and Fulmek [8] to generate quadratic identities on Schur functions. The latter method is based on the Gessel–Viennot interpretation [11] of semistandard Young tableaux by use of “flows” in a special directed graph, and [8, 9] give sufficient conditions on quadratic identities for Schur functions, formulated just in terms of relations on matchings.

The paper is organized as follows. Section 2 describes properties of certain pairs of flows, called *double flows*, which lie in the background of the method. Section 3 states the main result (Theorem 3.1) and proves sufficiency, claiming that all balanced families \mathcal{A}, \mathcal{B} give SQ-relations. Section 4 is devoted to illustrations of the method; it demonstrates a number of examples of SQ-relations, including rather wide classes (a majority concerns the flag flow case). Section 5 proves the other direction of Theorem 3.1, which is more intricate. Moreover, we show that if 2-patterns $\mathcal{A}_0, \mathcal{B}_0$ are not balanced, then for any corresponding X, Y, X', Y' , one can construct a planar network G with integer weights w such that the FG-function f_w violates relation (1.9).

As a consequence, for \mathcal{A}, \mathcal{B} fixed, validity of (1.9) for all commutative semirings \mathfrak{S} is equivalent to its validity for $\mathfrak{S} = \mathbb{Z}$. (This matches the so-called *transfer principle* for semirings; see, e.g., [1, Sect. 3].) Section 6 is devoted to applications to Schur functions. The final Sect. 7 contains a short discussion on nice additional properties (namely, the existence and an explicit construction of a so-called basis for $\mathbf{FG}_{n,n'}(\mathfrak{S})$, the Laurent phenomenon for FG-functions, and some others) in the case when \mathfrak{S} is a commutative semiring *with division*; this extends corresponding results from [4]. Here we also indicate one more sufficient condition on a matrix guaranteeing that the function of its minors is flow-generated (Proposition 7.2). The paper is concluded with a (rather routine) proof of the assertion that the function of minors of any $n' \times n$ matrix A over a commutative ring obeys all SQ-relations concerning n, n' (Proposition 7.3).

We have mentioned above that, instead of a weighting on the vertices of a graph G in question, one can consider a weighting on the edges. However, this does not affect the problem and our results in essence. When an edge e is endowed with a weight, one can split e into two edges in series and transfer the weight into the intermediate vertex, yielding an equivalent flow model (up to assigning the weight to each old vertex to be the “neutral element for multiplication”). Throughout the paper (except for Sect. 6), we prefer to deal with a weighting on vertices for technical reasons.

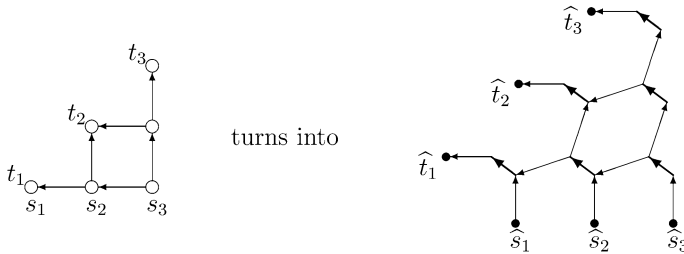
2 Flows and double flows

As before, let $G = (V, E)$ be an (acyclic) planar network with sources s_1, \dots, s_n and sinks $t_1, \dots, t_{n'}$. In this section, we describe ideas and tools behind the method of constructing 2-patterns $\mathcal{A}_0, \mathcal{B}_0$ that ensure validity of (1.9) for all flow-generated functions $f = f_w$ on $\mathcal{E}^{n,n'}$ determined by weightings $w : V \rightarrow \mathfrak{S}$, where \mathfrak{S} is an arbitrary commutative semiring.

First of all, we specify some terminology and notation. By a *path* in a digraph (directed graph) we mean a sequence $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ such that each e_i is an edge connecting vertices v_{i-1}, v_i . An edge e_i is called *forward* if it is directed from v_{i-1} to v_i , denoted as $e_i = (v_{i-1}, v_i)$, and *backward* otherwise (when $e_i = (v_i, v_{i-1})$). The path P is called *directed* if it has no backward edge, and *simple* if all vertices v_i are distinct. When $k > 0, v_0 = v_k$, and all v_1, \dots, v_k are distinct, P is called a *simple cycle*, or a *circuit*. The sets of vertices and edges of P are denoted by V_P and E_P , respectively.

Consider an $(I|I')$ -flow ϕ in G , where $(I, I') \in \mathcal{E}^{n,n'}$. It consists of pairwise disjoint directed paths going from the source set S_I to the sink set $T_{I'}$. Since G is acyclic, these paths are simple, and in view of the ordering of sources and sinks in the boundary O , the path in ϕ beginning at i th source in S_I enters i th sink in $T_{I'}$ (counting “from left to right”). Equivalently (when $s_1 \neq t_1$ and $s_n \neq t_{n'}$), we may think of ϕ as an induced subgraph of G satisfying: $\delta_\phi^{\text{out}}(s_i) = 1$ and $\delta_\phi^{\text{in}}(s_i) = 0$ if $i \in I$; $\delta_\phi^{\text{out}}(t_j) = 0$ and $\delta_\phi^{\text{in}}(t_j) = 1$ if $j \in I'$; and $\delta_\phi^{\text{out}}(v) = \delta_\phi^{\text{in}}(v) \in \{0, 1\}$ for the other vertices v of G . Here $\delta_\phi^{\text{out}}(v)$ (resp., $\delta_\phi^{\text{in}}(v)$) denotes the number of edges in ϕ leaving (resp., entering) a vertex v . Also we denote $\delta_\phi^{\text{out}}(v) + \delta_\phi^{\text{in}}(v)$ by $\delta_\phi(v)$.

Our approach is based on examining certain pairs of flows in G and rearranging them to form some other pairs. To simplify technical details, it is convenient to modify the original network G as follows. Let us split each vertex $v \in V$ into two vertices v', v'' (placing them in a small neighborhood of v in the plane) and connect them by edge $e_v = (v', v'')$, called a *split-edge*. Each edge (u, v) of G is replaced by an edge going from u'' to v' ; we call it an *ordinary* edge. Also for each $s_i \in S$, we add a new source \widehat{s}_i and the edge (\widehat{s}_i, s'_i) , and for each $t_j \in T$, add a new sink \widehat{t}_j and the edge (t''_j, \widehat{t}_j) ; we refer to such edges as *extra* ones. The picture illustrates the transformation for the half-grid Γ_3^Δ .



Note that the new (modified) graph is again acyclic, but it need not be planar in general (e.g., a local non-planarity arises when the original graph has a vertex v with four incident edges e_1, e_2, e_3, e_4 , in this order clockwise, such that e_1, e_3 enter and e_2, e_4 leave v); nevertheless, the latter fact will cause no trouble to us. We denote this graph by $\widehat{G} = (\widehat{V}, \widehat{E})$, and take $\widehat{S} := \{\widehat{s}_1, \dots, \widehat{s}_n\}$ and $\widehat{T} := \{\widehat{t}_1, \dots, \widehat{t}_{n'}\}$ as the sets of sources and sinks in it, respectively. As before, sources and sinks are also called *terminals*. Clearly, for any $i \in [n]$ and $j \in [n']$, there is a natural 1–1 correspondence between the directed paths from s_i to t_j in G and the ones from \widehat{s}_i to \widehat{t}_j in \widehat{G} . This is extended to a 1–1 correspondence between flows, and for $(I, I') \in \mathcal{E}^{n, n'}$, we keep notation $\Phi_{I|I'}$ for the set of flows in \widehat{G} going from $\widehat{S}_I := \{\widehat{s}_i : i \in I\}$ to $\widehat{T}_{I'} := \{\widehat{t}_j : j \in I'\}$. (When needed, a weighting w on the vertices v of the initial G is transferred to the split-edges of \widehat{G} , namely, by setting $w(e_v) := w(v)$. Then corresponding flows in both networks have equal weights, which are the \ominus -products of the weights of vertices or split-edges in the flows. This implies that the functions on $\mathcal{E}^{n, n'}$ generated by corresponding flows coincide.)

The digraph \widehat{G} possesses the following useful properties:

- (2.1) (a) Each non-terminal vertex is incident with exactly one split-edge, and if $e = (u, v)$ is a split-edge, then $\delta_G^{\text{out}}(u) = 1$ and $\delta_G^{\text{in}}(v) = 1$.
- (b) Each source (sink) has exactly one leaving edge and no entering edge (resp., one entering edge and no leaving edge).

Consider disjoint subsets $X, Y \subseteq [n]$ and disjoint subsets $X', Y' \subseteq [n']$. Let $m := |Y|$ and $m' := |Y'|$. Consider a pair $(A \subseteq Y, A' \subseteq Y')$ satisfying

$$|X| + |A| = |X'| + |A'| \quad \text{and} \quad |X| + |\overline{A}| = |X'| + |\overline{A'}|,$$

as before denoting by \overline{A} and $\overline{A'}$ the sets $Y - A$ and $Y' - A'$, respectively.

Remark 2 The above equalities are necessary for the existence of an $(XA|X'A')$ -flow and an $(X\bar{A}|X'\bar{A}')$ -flow (as before, we write XA for $X \cup A$, and so on). They imply

$$(i) 2|X| + |Y| = 2|X'| + |Y'| \quad \text{and} \quad (ii) |Y| - |Y'| = 2(|A| - |A'|). \quad (2.2)$$

We say that X, Y, X', Y' satisfying (i) are *consistent* and refer to a pair $(A \subseteq Y, A' \subseteq Y')$ satisfying (ii) as being *proper* for (Y, Y') . The set of proper pairs for (Y, Y') is denoted by $\Pi_{Y, Y'}$. For brevity, we write $\Pi_{m, m'}$ for $\Pi_{[m], [m']}$.

Consider an $(XA|X'A')$ -flow ϕ and an $(X\bar{A}|X'\bar{A}')$ -flow ϕ' in \widehat{G} ; we call the pair (ϕ, ϕ') a *double flow* for (A, A') . Our method will rely on two lemmas. Hereinafter we write $C \Delta D$ for the symmetric difference $(C - D) \cup (D - C)$ of sets C, D .

Lemma 2.1 $E_\phi \Delta E_{\phi'}$ is partitioned into the edge sets of pairwise disjoint circuits C_1, \dots, C_d (for some d) and simple paths P_1, \dots, P_p (with $p = \frac{1}{2}(m + m')$), where each P_i connects either \widehat{S}_A and $\widehat{S}_{\bar{A}}$, or \widehat{S}_A and $\widehat{T}_{A'}$, or $\widehat{S}_{\bar{A}}$ and $\widehat{T}_{\bar{A}'}$, or $\widehat{T}_{A'}$ and $\widehat{T}_{\bar{A}'}$. In each of these circuits and paths, the edges of ϕ and the edges of ϕ' have opposed directions (say, the former edges are forward and the latter ones are backward).

Proof Observe that a vertex v of \widehat{G} satisfies: (i) $\delta_\phi(v) = 1$ and $\delta_{\phi'}(v) = 0$ if $v \in \widehat{S}_A \cup \widehat{T}_{A'}$; (ii) $\delta_\phi(v) = 0$ and $\delta_{\phi'}(v) = 1$ if $v \in \widehat{S}_{\bar{A}} \cup \widehat{T}_{\bar{A}'}$; (iii) $\delta_\phi(v) = \delta_{\phi'}(v) = 1$ if $v \in \widehat{S}_X \cup \widehat{T}_{X'}$; and (iv) $\delta_\phi(v), \delta_{\phi'}(v) \in \{0, 2\}$ otherwise. This together with (2.1) implies that any vertex v is incident with 0, 1 or 2 edges in $E_\phi \Delta E_{\phi'}$, and the number of such edges is equal to 1 if and only if $v \in \widehat{S}_A \cup \widehat{S}_{\bar{A}} \cup \widehat{T}_{A'} \cup \widehat{T}_{\bar{A}'}$. (This is where we essentially use the transformation of G into \widehat{G} .) Hence the weakly connected components of the subgraph of \widehat{G} induced by $E_\phi \Delta E_{\phi'}$ are circuits, say, C_1, \dots, C_d , and simple paths P_1, \dots, P_p , each of the latter connecting two terminals in $\widehat{S}_A \cup \widehat{S}_{\bar{A}} \cup \widehat{T}_{A'} \cup \widehat{T}_{\bar{A}'}$.

Consider consecutive edges e, e' in a circuit C_i or a path P_j . If both e, e' belong to the same flow among ϕ, ϕ' , then, obviously, they have the same direction in this circuit/path. Suppose e, e' belong to different flows. In view of (2.1), the common vertex v of e, e' is not a terminal and it is incident with a split-edge e'' . Clearly, e'' belongs to both ϕ, ϕ' , and therefore $e'' \neq e, e'$. Then either both e, e' enter v or both leave v , so they are directed differently along the circuit/path containing them. This yields the second assertion in the lemma.

Finally, consider a path $P_j = (v_0, e_1, v_1, \dots, e_r, v_r)$ as above, and suppose that some of its ends, say, v_0 , belongs to \widehat{S}_A . Then the extra edge e_1 is contained in ϕ and leaves the source v_0 . If $v_r \in \widehat{S}_A$, then the extra edge e_r is in ϕ as well and leaves the source v_r ; so e_1, e_r are directed differently along P_j , contradicting the argument above. And if $v_r \in \widehat{T}_{\bar{A}'}$, then e_r belongs to ϕ' and enters the sink v_r ; so e_1, e_r have the same direction along P_j , again obtaining a contradiction. Thus, P_j connects \widehat{S}_A and $\widehat{S}_{\bar{A}} \cup \widehat{T}_{A'}$. Similarly, any path P_j neither has both ends in exactly one of $\widehat{S}_{\bar{A}}, \widehat{T}_{A'}, \widehat{T}_{\bar{A}'}$, nor connects $\widehat{S}_{\bar{A}}$ and $\widehat{T}_{A'}$. \square

Figure 2 illustrates an example of \widehat{G}, ϕ, ϕ' and indicates $E_\phi \sqcup E_{\phi'}$ and $E_\phi \Delta E_{\phi'}$.

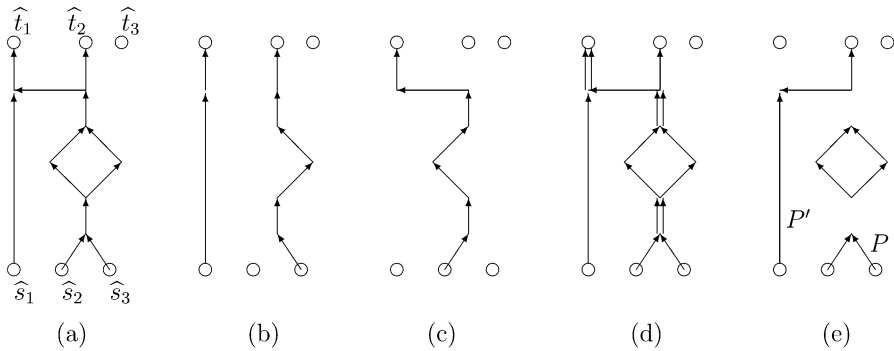


Fig. 2 (a) \widehat{G} ; (b) ϕ ; (c) ϕ' ; (d) $E_\phi \sqcup E_{\phi'}$; (e) $E_\phi \triangle E_{\phi'}$

Next we explain how to rearrange a double flow (ϕ, ϕ') for (A, A') so as to obtain a double flow for another pair $(B, B') \in \Pi_{Y, Y'}$. Let P_1, \dots, P_p be the paths as in Lemma 2.1, where $p = \frac{1}{2}(m + m')$. We denote the set of these paths by $\mathcal{P}(\phi, \phi')$. For a path $P \in \mathcal{P}(\phi, \phi')$, let $\pi(P)$ denote the pair of elements in $Y \sqcup Y'$ corresponding to the end vertices of P . We observe from Lemma 2.1 that $\pi(P)$ belongs to one of $A \times \bar{A}, A \times A', A' \times \bar{A}', \bar{A} \times \bar{A}'$ (considering $\pi(P)$ up to reversing). Define

$$M(\phi, \phi') := \{ \pi(P) : P \in \mathcal{P}(\phi, \phi') \}.$$

This set of pairs forms a *perfect matching* on $Y \sqcup Y'$ (i.e., each element of the latter set is contained in exactly one pair).

Lemma 2.2 *Choose an arbitrary subset $M_0 \subseteq M(\phi, \phi')$. Define $Z := \bigcup (\pi \in M_0) \cap Y$, $Z' := \bigcup (\pi \in M_0) \cap Y'$, $B := A \triangle Z$, and $B' := A' \triangle Z'$. Define $U := \bigcup (E_P : P \in \mathcal{P}(\phi, \phi'), \pi(P) \in M_0)$. Then there are a unique $(XB|X'B')$ -flow ψ and a unique $(X\bar{B}|X'\bar{B}')$ -flow ψ' such that $E_\psi = E_\phi \triangle U$ and $E_{\psi'} = E_{\phi'} \triangle U$. In particular, $E_\psi \sqcup E_{\psi'} = E_\phi \sqcup E_{\phi'}$.*

Proof By Lemma 2.1, each path $P \in \mathcal{P}(\phi, \phi')$ is a concatenation of directed paths Q_1, \dots, Q_r (considered up to reversing), where consecutive Q_j, Q_{j+1} are contained in different flows among ϕ, ϕ' and either both leave or both enter their common vertex. Therefore, exchanging the pieces Q_j in ϕ, ϕ' , we obtain an $(XC|X'C')$ -flow α and an $(X\bar{C}|X'\bar{C}')$ -flow α' such that $E_\alpha = E_\phi \triangle E_P$ and $E_{\alpha'} = E_{\phi'} \triangle E_P$, where $C := A \triangle (\pi \cap Y)$ and $C' := A' \triangle (\pi \cap Y')$.

Doing so for all $P \in \mathcal{P}(\phi, \phi')$ with $\pi(P) \in M_0$, we obtain flows ψ, ψ' satisfying the desired properties, taking into account that the paths in $\mathcal{P}(\phi, \phi')$ are pairwise disjoint. The uniqueness of ψ, ψ' is obvious. \square

Note that $M(\psi, \psi') = M(\phi, \phi')$ and $\mathcal{P}(\psi, \psi') = \mathcal{P}(\phi, \phi')$, and the transformation of ψ, ψ' by use of the paths in $\mathcal{P}(\psi, \psi')$ related to M_0 returns the flows ϕ, ϕ' .

Figure 3 illustrates flows ψ, ψ' created from ϕ, ϕ' in Fig. 2. Here the left fragment shows ψ, ψ' when the exchange is performed with respect to the (single) path P in

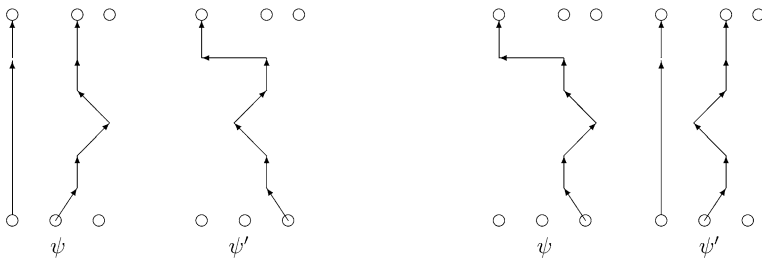


Fig. 3 Creating ψ, ψ' from ϕ, ϕ' in Fig. 2: by use of P (left); by use of P' (right)

$\mathcal{P}(\phi, \phi')$ connecting the sources \widehat{s}_2 and \widehat{s}_3 , and the right fragment shows those for the path P' connecting the source \widehat{s}_1 and the sink \widehat{t}_2 (see (e) in Fig. 2).

In the next section, we will use the fact that, although the modified graph \widehat{G} may not be planar, its subgraph $\phi \cup \phi'$ is planar.

To see this, consider a non-terminal vertex v in the initial graph G that belongs to both flows ϕ, ϕ' . Let a, a' be the edges of ϕ (concerning G) entering and leaving v , respectively, and let b, b' be similar edges for ϕ' . The only situation when the subgraph $\phi \cup \phi'$ (concerning \widehat{G}) is not locally planar in a small neighborhood of the split-edge e_v is that all a, a', b, b' are different and follow in this order (clockwise or counterclockwise) around v . We assert that this is not the case. Indeed, a, a' belong to a directed path P in \widehat{G} from a source \widehat{s}_i to a sink $\widehat{t}_{i'}$, and b, b' belong to a directed path Q from \widehat{s}_j to $\widehat{t}_{j'}$. From the facts that the initial graph G is planar and acyclic and that the edges a, a', b, b' occur in this order around v one can conclude that the paths P, Q can meet only at v . This implies that the terminals $s_i, t_{i'}, s_j, t_{j'}$ are different and follow in this cyclic order in the boundary O ; a contradiction. Thus, $\phi \cup \phi'$ is planar, as required.

Remark 3 In the definition of FG-functions, one can equivalently consider only the acyclic digraphs G having the additional property that each edge of G belongs to at least one directed path going from a source to a sink. Arguing as above, one easily shows that for any vertex v of such a G , the edge direction (to v or from v) changes at most twice when we move around v . Then the modified graph \widehat{G} is automatically planar, and so is $\phi \cup \phi'$.

3 Balanced families

In this section, we use the above observations and results to construct families involved in stable quadratic relations.

Consider the same objects as before: consistent sets X, Y, X', Y' and a proper pair (A, A') for (Y, Y') (obeying (2.2)), a double flow (ϕ, ϕ') for (A, A') , and the perfect matching $M = M(\phi, \phi')$ on $Y \sqcup Y'$, referring to the members of M as *couples*. We denote the set of double flows for (A, A') by $\mathcal{D}(A, A')$ (when X, Y, X', Y' are fixed).

We associate to (A, A') the set $\mathcal{M}(A, A')$ (or $\mathcal{M}_{Y, Y'}(A, A')$) of feasible perfect matchings M on $Y \sqcup Y'$ defined as follows. Let us think that the elements of Y and Y'

are placed, respectively, on the lower half and on the upper half of a circumference O , in the increasing order from left to right. Also let us call the elements (points) of $A \sqcup A'$ *white*, and the elements of $\bar{A} \sqcup \bar{A}'$ *black*. Then a perfect matching M on $Y \sqcup Y'$ is called *feasible* for (A, A') if:

- (3.1) (i) When both elements of a couple $\pi \in M$ lie either in the lower half of O or in the upper half of O , these elements have different colors.
- (ii) When the elements of π lie in different halves, these elements have the same color.
- (iii) M is planar, in the sense that the chords of O connecting the couples in M are pairwise not intersecting.

Observe that for $(\phi, \phi') \in \mathcal{D}(A, A')$, the matching $M = M(\phi, \phi')$ is feasible. This follows from Lemma 2.1, taking into account that the subgraph $\phi \cup \phi'$ of \widehat{G} is planar and that the paths in $\mathcal{P}(\phi, \phi')$ are pairwise disjoint. A priori any matching in $\mathcal{M}(A, A')$ may be expressed as $M(\phi, \phi')$ for some $(\phi, \phi') \in \mathcal{D}(A, A')$.

We refer to a triple (A, A', M) with $(A, A') \in \Pi_{Y, Y'}$ and $M \in \mathcal{M}(A, A')$ as a *configuration*. For a 2-family $\mathcal{A} \subseteq \Pi_{Y, Y'}$, we define $\mathcal{K}(\mathcal{A})$ to be the family of all configurations (A, A', M) (with possible multiplicities) arising when (A, A') runs over \mathcal{A} .

The *exchange operation* applied to a configuration (A, A', M) and to a chosen subset $M_0 \subseteq M$ makes the pair (B, B') defined by $B := A \Delta (\bigcup(\pi \in M_0) \cap Y)$ and $B' := A \Delta (\bigcup(\pi \in M_0) \cap Y')$; in other words, we change the colors of both elements in each couple $\pi \in M_0$ (cf. Lemma 2.2). Then M becomes a feasible matching for (B, B') , and the exchange operation applied to the configuration (B, B', M) and the same M_0 returns (A, A') .

Definition We say that two 2-families $\mathcal{A}, \mathcal{B} \subseteq \Pi_{Y, Y'}$ are *balanced* if there exists a bijection between $\mathcal{K}(\mathcal{A})$ and $\mathcal{K}(\mathcal{B})$ such that the corresponding configurations (A, A', M) and (B, B', M') have the same matching: $M = M'$. (We rely on the simple fact that for any two configurations (A, A', M) and (B, B', M) , the pair (B, B') can be obtained from (A, A') by the exchange operation w.r.t. some $M_0 \subseteq M$.) Equivalently, \mathcal{A}, \mathcal{B} are balanced if for each planar perfect matching M on $Y \sqcup Y'$, the number of times M occurs in sets $\mathcal{M}_{Y, Y'}(A, A')$ among $(A, A') \in \mathcal{A}$ is equal to a similar number for sets $\mathcal{M}_{Y, Y'}(B, B')$ among $(B, B') \in \mathcal{B}$. This can be written as

$$\mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{B}),$$

where for $\mathcal{C} \subseteq \Pi_{Y, Y'}$, $\mathcal{M}(\mathcal{C})$ denotes the family consisting of matchings M taken with the multiplicities equal to the number of $(C, C') \in \mathcal{C}$ such that $M \in \mathcal{M}(C, C')$. Clearly, \mathcal{A}, \mathcal{B} are balanced if and only if their 2-patterns $\mathcal{A}_0, \mathcal{B}_0$ are balanced.

Our main result is the following

Theorem 3.1 *Let $\mathcal{A}_0, \mathcal{B}_0 \in \Pi_{m, m'}$. The following statements are equivalent:*

- (i) Expression (1.9) is a stable quadratic relation, where $\mathcal{A} = \gamma_{Y, Y'}(\mathcal{A}_0)$ and $\mathcal{B} = \gamma_{Y, Y'}(\mathcal{B}_0)$.
- (ii) $\mathcal{A}_0, \mathcal{B}_0$ are balanced.

Part (i)⇒(ii) of this theorem will be shown in Sect. 5. In its turn, part (ii)⇒(i) can be immediately proved by relying on the lemmas from the previous section.

Proposition 3.2 *Let $\mathcal{A}_0, \mathcal{B}_0 \in \Pi_{m,m'}$ be balanced. Then identity (1.9) holds for any consistent sets X, Y, X', Y' (corresponding to n, n', m, m' as above; cf. (2.2)(i)), the families $\mathcal{A} = \gamma_{Y,Y'}(\mathcal{A}_0)$ and $\mathcal{B} = \gamma_{Y,Y'}(\mathcal{B}_0)$, and any FG-function f on $\mathcal{E}^{n,n'}$ (corresponding to arbitrary G, w, \mathfrak{S} as above).*

Proof For corresponding $G, w, \mathfrak{S}, X, Y, X', Y'$, consider the FG-function $f = f_w$ on $\mathcal{E}^{n,n'}$. The summand corresponding to $(A, A') \in \mathcal{A}$ in the L.H.S. of (1.9) can be expressed via double flows as follows:

$$\begin{aligned}
 & f(XA|X'A') \odot f(X\bar{A}|X'\bar{A}') \\
 &= \left(\bigoplus_{\phi \in \Phi_{XA|X'A'}} w(\phi) \right) \odot \left(\bigoplus_{\phi' \in \Phi_{X\bar{A}|X'\bar{A}'}} w(\phi') \right) \\
 &= \bigoplus_{(\phi, \phi') \in \mathcal{D}(A, A')} w(\phi) \odot w(\phi') \\
 &= \bigoplus_{M \in \mathcal{M}(A, A')} \bigoplus_{(\phi, \phi') \in \mathcal{D}(A, A') : M(\phi, \phi') = M} w(\phi) \odot w(\phi'). \tag{3.2}
 \end{aligned}$$

The summand corresponding to $(B, B') \in \mathcal{B}$ in the L.H.S. of (1.9) is expressed similarly.

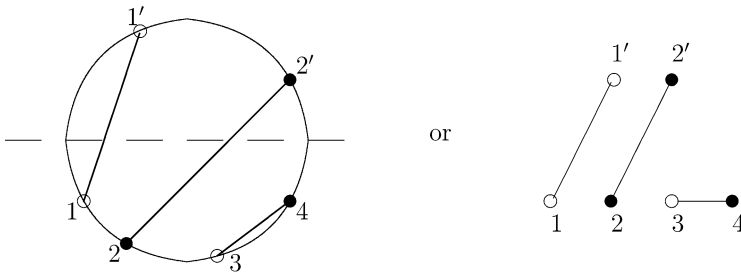
Consider a configuration $(A, A', M) \in \mathcal{K}(\mathcal{A})$ and suppose (ϕ, ϕ') is a double flow for (A, A') such that $M(\phi, \phi') = M$ (if it exists). Since \mathcal{A}, \mathcal{B} are balanced, (A, A', M) is bijective to some configuration (B, B', M) in \mathcal{B} . Since M is a feasible matching for both (A, A') and (B, B') , one can see from conditions (3.1)(i), (ii) that (B, B') is obtained from (A, A') by the exchange operation w.r.t. some $M_0 \subseteq M$. Then transforming (ϕ, ϕ') by use of the paths $P \in \mathcal{P}(\phi, \phi')$ with $\pi(P) \in M_0$, as described in Lemma 2.2, we obtain a double flow (ψ, ψ') for (B, B') such that $E_\psi \sqcup E_{\psi'} = E_\phi \sqcup E_{\phi'}$, and therefore $w(\psi) \odot w(\psi') = w(\phi) \odot w(\phi')$. Moreover, $(\phi, \phi') \mapsto (\psi, \psi')$ gives a bijection between all double flows involved in the configurations in $\mathcal{K}(\mathcal{A})$ and those in $\mathcal{K}(\mathcal{B})$. Now the desired equality (1.9) follows by considering the last term in (3.2). □

The rest of this section is devoted to additional conventions and illustrations.

Let M be a planar perfect matching on $Y \sqcup Y'$. Sometimes it will be convenient to assume that all couples $\pi \in M$ are ordered: if π consists of elements i, j , we may write $\pi = ij$ if: either (a) $i, j \in Y$ and $i < j$, or (b) $i, j \in Y'$ and $i < j$, or (c) $i \in Y$ and $j \in Y'$. We call a couple π in these cases *lower horizontal*, *upper horizontal*, and *vertical*, respectively. The subsets of such couples in M are denoted by M^{lh} , M^{uh} , and M^{vert} , respectively. When $\pi = ij$ is horizontal, we denote the interval $\{i, i + 1, \dots, j\}$ by $[\pi]$. The fact that M is planar (cf. (3.1)(iii)) implies that

(3.3) the set M^{lh} is *nested*, which means that for any $\pi, \pi' \in M^{\text{lh}}$, the intervals $[\pi]$ and $[\pi']$ are either disjoint or one includes the other; also for each $\pi \in M^{\text{lh}}$, all elements of Y within $[\pi]$ are covered by couples in M^{lh} ; similar properties hold for M^{uh} and Y' .

A proper pair $(A, A') \in \Pi_{Y, Y'}$ along with a feasible matching M for it can be illustrated by use of either a *circular* diagram or a *two-level* diagram; the couples in the former are connected by straight-line segments, and those in the latter by straight-line segments or by arcs. See the picture where $Y = \{1, 2, 3, 4\}$, $Y' = \{1', 2'\}$, $A = \{1, 3\}$, $A' = \{1'\}$, $M^{\text{lh}} = \{34\}$, $M^{\text{uh}} = \emptyset$, and $M^{\text{vert}} = \{11', 22'\}$.



Recall that in the flag flow case we deal with 1-patterns on $[m]$ and 1-families on $Y \subseteq [n]$ (with $|Y| = m$), which are formed by p -element subsets in these sets (cf. Definition 1 in the Introduction). They are equivalent, respectively, to 2-patterns on $([m], [m'])$ and 2-families on (Y, Y') , where $|Y'| = m' = |p - (m - p)|$. Theorem 3.1 implies the following criterion on Plücker type relations.

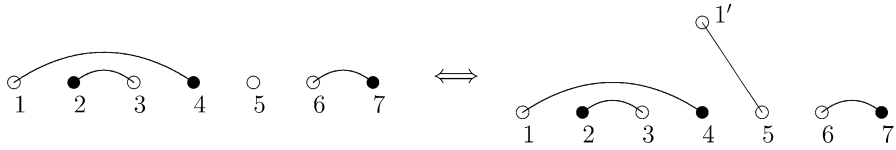
Corollary 3.3 *Let $\mathcal{A}_0, \mathcal{B}_0 \in \binom{[m]}{p}$. Then (1.7) is a PSQ-relation (where $\mathcal{A} = \gamma_Y(\mathcal{A}_0)$ and $\mathcal{B} = \gamma_{Y'}(\mathcal{B}_0)$) if and only if the 1-patterns $\mathcal{A}_0, \mathcal{B}_0$ are balanced.*

Here the notion of *balanced* 1-families \mathcal{A}, \mathcal{B} (in particular, 1-patterns) comes from the one given for 2-families and is specified as follows. Let $q := m - p$ and assume, w.l.o.g., that $p \geq q$. A *feasible matching* for a set $A \in \binom{Y}{p}$ (or for the partition (A, \bar{A}) of Y) is now defined to be a set \tilde{M} of pairs (couples) in Y such that

- (3.4) (i) $|\tilde{M}| = q$, the couples in \tilde{M} are mutually disjoint, and $|\pi \cap A| = |\pi \cap \bar{A}| = 1$ for each $\pi \in \tilde{M}$.
- (ii) \tilde{M} is *nested*, and for each $\pi \in \tilde{M}$, all elements of $[\pi]$ are covered by \tilde{M} ;

cf. (3.3). In other words, \tilde{M} is just the set M^{lh} in the corresponding planar perfect matching M for $Y \sqcup Y'$. In view of $|A| = p$, $|\bar{A}| = q$ and $|Y'| = p - q$, we have $M^{\text{uh}} = \emptyset$ and $|M^{\text{vert}}| = p - q$. In particular, the elements of Y' are colored white, provided that the elements of A and \bar{A} are white and black, respectively.

For illustrations in the flag case, we will use *flat* (or *one-level*) *diagrams*. An example of such a diagram and its corresponding two-level diagram are drawn in the picture; here $Y = [7]$, $A = \{1, 3, 5, 6\}$, $\tilde{M} = \{14, 23, 67\}$, and $Y' = \{1'\}$.



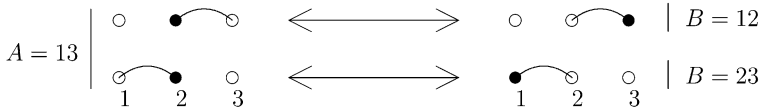
4 Examples of stable quadratic relations

In this section, we illustrate the method described in the previous section by demonstrating several classes of stable quadratic relations on FG-functions. According to Proposition 3.2, once we are able to show that one or another pair of families \mathcal{A}, \mathcal{B} is balanced, we can declare that relation (1.9) involving these families is stable.

As before, when visualizing a proper pair $(C \subseteq [m], C' \subseteq [m'])$ (i.e., satisfying (2.2)(ii)), we will refer to the elements of C and C' as white, and to the elements of their complements $\bar{C} = [m] - C$ and $\bar{C}' = [m'] - C'$ as black.

Most of examples below (namely, those in items 1–5) concern PSQ-relations for flag-flow-determined functions $f : 2^{[n]} \rightarrow \mathfrak{S}$. In these cases, we deal with 1-patterns $\mathcal{A}_0, \mathcal{B}_0 \subseteq \binom{[m]}{p}$ for some $p < m$ and set $q := m - p$. Also, considering one or another white-black partition (C, \bar{C}) of $[m]$ (with $|C| = p$) and a feasible matching M for it, we illustrate the configuration (C, M) by a flat diagram (introduced in the end of the previous section). The set of feasible matchings for (C, \bar{C}) is denoted by $\mathcal{M}(C)$.

1. When $m = 3$ and $p = 2$, the collection $\binom{[m]}{p}$ consists of three 2-element sets C , namely, 12, 13, 23, and their complements \bar{C} are the 1-element sets 3, 2, 1, respectively. Since $q = 1$, a feasible matching consists of a unique couple. The sets 12 and 23 admit only one feasible matching each, namely, $\mathcal{M}(12) = \{\{23\}\}$ and $\mathcal{M}(23) = \{\{12\}\}$, whereas 13 has two feasible matchings, namely, $\mathcal{M}(13) = \{\{12\}, \{23\}\}$. Therefore, the 1-patterns $\mathcal{A}_0 := \{13\}$ and $\mathcal{B}_0 := \{12, 23\}$ are balanced. The corresponding configurations and bijection are illustrated in the picture.

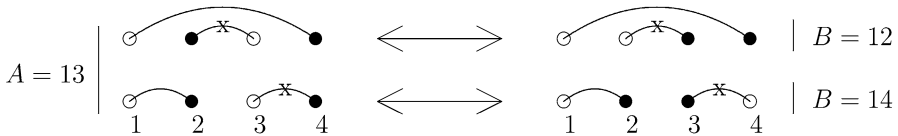


This gives rise to the *P3-relation* (generalizing AP3- and TP3-relations (1.1), (1.6)): for a triple $i < j < k$ (forming Y) and $X \subseteq [n] - \{i, j, k\}$,

$$f(Xik) \odot f(Xj) = (f(Xij) \odot f(Xk)) \oplus (f(Xjk) \odot f(Xi)). \tag{4.1}$$

2. Let $p = q = 2$. Take the 1-patterns $\mathcal{A}_0 := \{13\}$ and $\mathcal{B}_0 := \{12, 14\}$ in $\binom{[4]}{2}$. One can see that each of 12 and 14 admits a unique feasible matching: $\mathcal{M}(12) = \{\{14, 23\}\}$

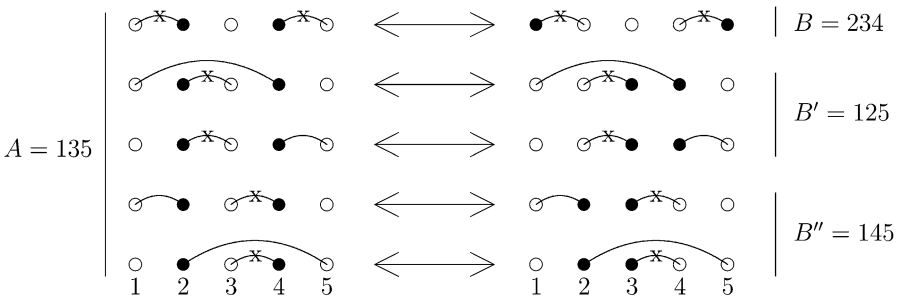
and $\mathcal{M}(14) = \{\{12, 34\}\}$, whereas $\mathcal{M}(13)$ consists of two feasible matchings: just those $\{14, 23\}$ and $\{12, 34\}$. Thus, $\mathcal{A}_0, \mathcal{B}_0$ are balanced; see the picture where the couples (arcs) involved in the corresponding exchange operations are marked with crosses.



As a consequence, we obtain the *P4-relation* (generalizing (1.2) and its tropical counterpart): for $i < j < k < \ell$ and $X \subseteq [n] - \{i, j, k, \ell\}$,

$$f(Xik) \odot f(Xjl) = (f(Xij) \odot f(Xkl)) \oplus (f(Xil) \odot f(Xjk)). \tag{4.2}$$

3. As one more illustration of the method, let us consider one particular case for $m = 5$ and $p = 3$. Put $\mathcal{A}_0 := \{135\}$ and $\mathcal{B}_0 := \{234, 125, 145\}$. One can check that $\mathcal{M}(234) = \{\{12, 45\}\}$, $\mathcal{M}(125) = \{\{14, 23\}, \{23, 45\}\}$, $\mathcal{M}(145) = \{\{12, 34\}, \{25, 34\}\}$, and that $\mathcal{M}(135)$ consists just of the five matchings occurring in those three collections. Therefore, $\mathcal{A}_0, \mathcal{B}_0$ are balanced. The corresponding configurations and bijection are shown in the picture.



This implies a particular PSQ-relation on quintuples: for $i < j < k < \ell < r$ and $X \subseteq [n] - \{i, j, k, \ell, r\}$,

$$f(Xikr) \odot f(Xjl) = (f(Xjkl) \odot f(Xir)) \oplus (f(Xijr) \odot f(Xkl)) \oplus (f(Xilr) \odot f(Xjk)).$$

4. The next illustration concerns a wide class of balanced 1-patterns for $m > p \geq m - p =: q$; it includes the 1-patterns indicated in items 1 and 2 as special cases.

The 1-pattern \mathcal{A}_0 contains a distinguished set $A_0 \in \binom{[m]}{p}$. Fix a nonempty subset $Z \subseteq \overline{A_0}$ and consider the collection

$$\mathcal{C} := \{C \subset [m] : |C| = p, C \cap \overline{A_0} = Z\}. \tag{4.3}$$

For a subset $C \subseteq [n]$, we will denote by $\Sigma(C)$ the number $\sum(i \in C)$. Now define

$$\begin{aligned} \mathcal{A}_0 &:= \{A_0\} \cup \{A \in \mathcal{C} : \Sigma(A) - \Sigma(A_0) + |Z| \text{ is odd}\} \quad \text{and} \\ \mathcal{B}_0 &:= \{B \in \mathcal{C} : \Sigma(B) - \Sigma(A_0) + |Z| \text{ is even}\}. \end{aligned} \tag{4.4}$$

In particular, $\mathcal{A}_0 \cap \mathcal{B}_0 = \emptyset$. We assert the following:

Lemma 4.1 *The pair $\mathcal{A}_0, \mathcal{B}_0$ in (4.4) is balanced.*

Proof Consider $C \in \mathcal{C}$ and $M \in \mathcal{M}(C)$. We describe a rule which associates to (C, M) another configuration (D, M) , aiming to obtain the desired bijection between $\mathcal{K}(\mathcal{A}_0)$ and $\mathcal{K}(\mathcal{B}_0)$.

Since M is feasible and $p \geq q$, we have $|M| = q = |\overline{A_0}|$. This implies that exactly one of the two cases is possible: (i) there is a couple $\pi \in M$ with both elements in A_0 ; and (ii) each $\pi \in M$ satisfies $|\pi \cap A_0| = 1$ (whence $|\pi \cap \overline{A_0}| = 1$ and M covers $\overline{A_0}$).

In case (i), take the couple $\pi = ij \in M$ ($i < j$) such that $i, j \in A_0$ and i is minimum under this property. Let $D := C \Delta \pi$. Then $D \cap \overline{A_0} = C \cap \overline{A_0} = Z$, whence $D \in \mathcal{C}$. Also the interval $[\pi]$ is partitioned into couples (cf. (3.4)(ii)), implying that $j - i$ is odd. Hence the numbers $\Sigma(C) - \Sigma(A_0) + |Z|$ and $\Sigma(D) - \Sigma(A_0) + |Z|$ have different parity, and therefore, C, D belong to different collections among $\mathcal{A}_0, \mathcal{B}_0$. Obviously, M is a feasible matching for D , the couple π satisfies the above minimality property for D , and applying to D the exchange operation w.r.t. π returns C . We associate the configurations (C, M) and (D, M) to each other.

In case (ii), each couple of M has one element in A_0 and the other in $\overline{A_0}$. Let M_Z be the set of $\pi \in M$ such that $\pi \cap Z \neq \emptyset$ and let $Q := \bigcup(\pi \in M_Z)$. Then the set $D := C \Delta Q$ satisfies $|D| = p$ and $D \cap \overline{A_0} = \emptyset$. This means that $D = A_0$. Also $M \in \mathcal{M}(A_0)$ and $C = A_0 \Delta Q$. Since $j - i$ is odd for each $ij \in M$, the numbers $\Sigma(C) - \Sigma(A_0)$ and Z have the same parity. So $C \in \mathcal{B}_0$. We associate (C, M) and (A_0, M) to each other. Conversely, for each $M' \in \mathcal{M}(A_0)$, let $C' := A_0 \Delta Q'$, where $Q' := \bigcup(\pi \in M' : \pi \cap Z \neq \emptyset)$. Then $Q' \cap \overline{A_0} = Z$, $C' \in \mathcal{B}_0$, and the above construction associates (C', M') with (A_0, M') .

This gives the desired bijection between $\mathcal{K}(\mathcal{A}_0)$ and $\mathcal{K}(\mathcal{B}_0)$. □

Remark 4 (i) Consider $m = 3, p = 2, A_0 := 12$, and $Z := \{3\}$. Then the collection \mathcal{C} as in (4.3) consists of the sets 13, 23, and we have $\mathcal{A}_0 = \{12, 23\}$ and $\mathcal{B}_0 = \{13\}$. These 1-patterns correspond to those in item 1. (ii) When $m = 4, p = 2, A_0 := 12$, and $Z := \{3\}$, we obtain $\mathcal{A}_0 = \{12, 23\}$ and $\mathcal{B}_0 = \{13\}$. They generate the same PSQ-relations as the balanced 1-patterns $\{13\}, \{12, 14\}$ in item 2.

The 1-patterns $\mathcal{A}_0, \mathcal{B}_0$ as in (4.4) give rise to the following PSQ-relation on FG-functions f ; for brevity, it is exposed when \mathfrak{S} is a ring. Let $I, J \subseteq [n]$ and $|I| \geq |J|$. Fix $Z \subseteq J - I$. Then

$$f(I)f(J) = \sum_K (-1)^{a + \text{Inv}((I-K) \cup Z, (J-Z) \cup K)} f((I - K) \cup Z) f((J - Z) \cup K), \tag{4.5}$$

where the sum is over all $K \subseteq I - J$ with $|K| = |Z|$; $\text{Inv}(I', J')$ denotes the number of pairs $(i, j) \in I' \times J'$ with $i > j$ (inversions); and $a := |Z| + \text{Inv}(I - J, J - I)$. In this case, one should set $X := I \cap J, Y := I \Delta J, m := |Y|, p := |I - J|$, and $A_0 := I - J$.

Relations similar to (4.5) (but possibly written in a different form) appear in a characterization of flag manifolds $\text{Fl} = \text{Fl}^{d_1, \dots, d_r}(\mathbb{C}^n)$, where $d_1 < \dots < d_r \leq n$; cf. [10, Chap. 9]. In this case one should take all subsets $I, J \subseteq [n]$ and $Z \subseteq J - I$ with $|I| = d_i, |J| = d_j, i \geq j$; then (4.5) generate the ideal of polynomials with zero values on Fl canonically embedded in the corresponding product of projective spaces.

5. One more representable class of balanced 1-patterns for $p < m \leq n$ with $p \geq m - p =: q$ is obtained by slightly modifying the previous construction.

Fix a set $Z \subseteq [m]$ with $0 < |Z| \leq q - 1$ and a subset $Z' \subseteq Z$. Form the collection

$$\mathcal{C} := \{C \subseteq [m] : |C| = p, C \cap Z = Z'\}.$$

Partition \mathcal{C} into two 1-patterns

$$\mathcal{A}_0 := \{A \in \mathcal{C} : \Sigma(A) \text{ odd}\} \quad \text{and} \quad \mathcal{B}_0 := \{A \in \mathcal{C} : \Sigma(A) \text{ even}\}. \tag{4.6}$$

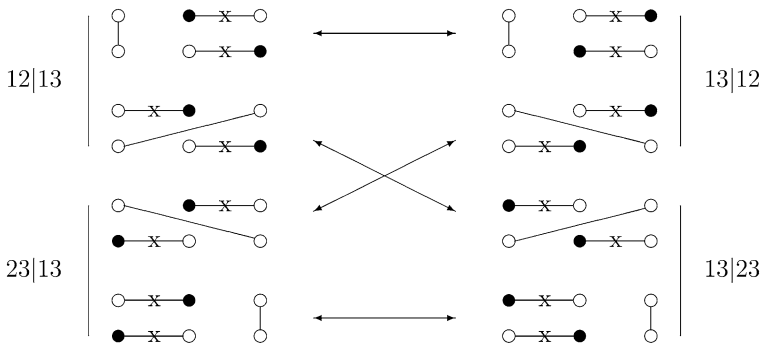
Lemma 4.2 *The pair $\mathcal{A}_0, \mathcal{B}_0$ in (4.6) is balanced.*

Proof Let $C \in \mathcal{C}$ and $M \in \mathcal{M}(C)$. Since $|M| = q > |Z|$, there exists a couple $\pi = ij \in M$ ($i < j$) with both elements in $[m] - Z$; take such a π so that i be minimum. Form $D := C \Delta \pi$. Then C, D belong to different 1-patterns among $\mathcal{A}_0, \mathcal{B}_0$ (since $j - i$ is odd), and we can associate (C, M) and (D, M) to each other, taking into account that $M \in \mathcal{M}(D)$ and that the choice of π depends only on M . □

This lemma gives rise to the corresponding class of PSQ-relations; we omit it here. (In fact, such relations can be derived from those in item 4 when \mathfrak{S} is a ring.)

In the rest of this section, we give simple examples of balanced families in the non-flag case. Now we deal with disjoint sets $X, Y \subseteq [n]$ and disjoint sets $X', Y' \subseteq [n']$, denote $m := |Y|$ and $m' := |Y'|$, and consider 2-patterns formed by proper pairs for $([m], [m'])$ (cf. (2.2)). Corresponding matchings will be illustrated by use of *two-level diagrams* (see the end of Sect. 3) in which the white/black elements of $[m]$ (resp., $[m']$) are disposed in the lower (resp. upper) horizontal line.

6. The picture below shows an example of balanced homogeneous 2-patterns $\mathcal{A}_0, \mathcal{B}_0$.



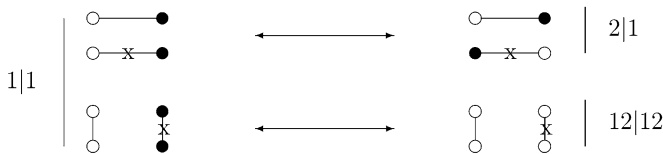
Here $m = m' = 3$, \mathcal{A}_0 consists of the pairs 12|13 and 23|13, and \mathcal{B}_0 consists of the pairs 13|12 and 13|23 (indicated by light circles); we write $a|b$ for (a, b) . The feasible matchings are indicated by line segments, and the couples involved in the corresponding exchange operations are marked with crosses.

These 2-patterns give rise to the SQ-relation

$$\begin{aligned}
 & (f(Xij|X'i'k') \odot f(Xk|X'j')) \oplus (f(Xjk|X'i'k') \odot f(Xi|X'j')) \\
 & = (f(Xik|X'i'j') \odot f(Xj|X'k')) \oplus (f(Xik|X'j'k') \odot f(Xj|X'i')),
 \end{aligned}$$

where $i < j < k$ and $i' < j' < k'$ (this is rather trivial for minors of a matrix).

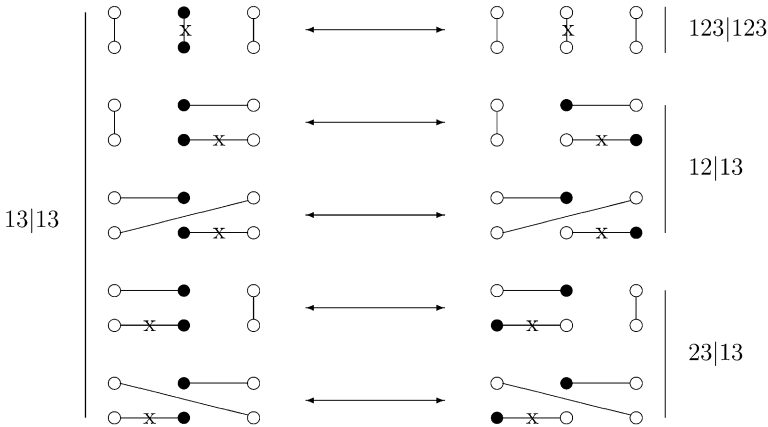
7. One of the simplest examples of balanced non-homogeneous 2-patterns is formed by $\mathcal{A}_0 = \{1|1\}$ and $\mathcal{B}_0 = \{2|1, 12|12\}$, in the case $m = m' = 2$. See the picture:



This gives the following SQ-relation similar to Dodgson’s condensation formula for minors of a matrix (cf. (1.8)): for $i < k$ and $i' < k'$,

$$\begin{aligned}
 & f(Xi|X'i') \odot f(Xk|X'k') \\
 & = (f(Xik|X'i'k') \odot f(X|X')) \oplus (f(Xk|X'i') \odot f(Xi|X'k')). \quad (4.7)
 \end{aligned}$$

Another simple non-homogeneous example is analogous to the row decomposition (by row 2) of the determinant of a 3×3 matrix. Here $m = m' = 3$, $\mathcal{A}_0 = \{13|13\}$ and $\mathcal{B}_0 = \{12|13, 23|13, 123|123\}$; see the picture:



5 Necessity of the balancedness

This section is devoted to the other direction in Theorem 3.1. Moreover, we show a sharper property. It says that if a pair $\mathcal{A}_0, \mathcal{B}_0$ is not balanced, then for any choice of appropriate consistent sets X, Y, X', Y' and for $\mathfrak{S} := \mathbb{Z}$, there exist (and can be explicitly constructed) a planar network and a weighting such that the corresponding flow-generated function f violates relation (1.9).

As before, for subsets $C \subseteq Y$ and $C' \subseteq Y'$, we write \bar{C} for $Y - C$, and \bar{C}' for $Y' - C'$, and call (C, C') a proper pair for (Y, Y') if it satisfies (2.2)(ii).

Theorem 5.1 Fix disjoint sets $X, Y \subseteq [n]$ and disjoint sets $X', Y' \subseteq [n']$ satisfying (2.2)(i). Let $\mathcal{A}, \mathcal{B} \in \Pi_{Y, Y'}$. Suppose that \mathcal{A}, \mathcal{B} are not balanced. Then (1.9) does not hold for some (G, w) and $\mathfrak{S} = \mathbb{Z}$. More precisely, there exists a planar network $G = (V, E)$ with n sources and n' sinks such that for the all-unit weighting $w \equiv 1$ on V , the flow-generated function $f = f_w$ on $\mathcal{E}^{n, n'}$ gives

$$\sum_{(A, A') \in \mathcal{A}} f(XA|X'A') f(X\bar{A}|X'\bar{A}') \neq \sum_{(B, B') \in \mathcal{B}} f(XB|X'B') f(X\bar{B}|X'\bar{B}'). \tag{5.1}$$

Proof Since \mathcal{A}, \mathcal{B} are not balanced, there exists a planar perfect matching M on $Y \sqcup Y'$ such that

$$|\mathcal{A}_M| \neq |\mathcal{B}_M|, \tag{5.2}$$

where \mathcal{A}_M denotes the set of pairs $(A, A') \in \mathcal{A}$ having M as a feasible matching: $M \in \mathcal{M}(A, A')$, and similarly for \mathcal{B} .

We fix such an M , and our aim is to construct a planar network $G = (V, E)$ that satisfies the following properties: for each proper pair (C, C') for (Y, Y') ,

- (P1) If $M \in \mathcal{M}(C, C')$, then G has a unique $(XC|X'C')$ -flow and a unique $(X\bar{C}|X'\bar{C}')$ -flow, i.e., $|\Phi_{XC|X'C'}| = |\Phi_{X\bar{C}|X'\bar{C}'}| = 1$.
- (P2) If $M \notin \mathcal{M}(C, C')$, then at least one of $\Phi_{XC|X'C'}$ and $\Phi_{X\bar{C}|X'\bar{C}'}$ is empty.

Assuming that such a G does exist, assign the weight $w(v) := 1$ to each vertex v . By (P1) and (P2), for the function $f = f_w$ and a proper pair $(C, C') \subseteq (Y, Y')$, each of the values $f(XC|X'C')$ and $f(X\bar{C}|X'\bar{C}')$ is equal to one if $M \in \mathcal{M}(C, C')$, and at least one of them is zero otherwise. This implies

$$\sum_{(A,A') \in \mathcal{A}} f(XA|X'A') f(X\bar{A}|X'\bar{A}') = |\mathcal{A}_M|,$$

$$\sum_{(B,B') \in \mathcal{B}} f(XB|X'B') f(X\bar{B}|X'\bar{B}') = |\mathcal{B}_M|,$$

and now the required inequality (5.1) follows from (5.2).

It suffices to construct the desired network G in the case $n = |X| + |Y|$ and $n' = |X'| + |Y'|$ (for we can add a source s_i for $i \in [n] - (X \cup Y)$ (if exists) as an isolated vertex, and can do similarly for sinks). So we may assume, w.l.o.g., that X, Y form a partition of $[n]$, and X', Y' do that of $[n']$.

We first describe the construction when $X = X' = \emptyset$, which is the crucial special case. Subsequently we will explain that this construction can be easily extended to arbitrary X, X' .

Thus, we deal with $n = |Y| = |Y'|$ sources s_1, \dots, s_n and n sinks t_1, \dots, t_n . As usual, the sources (sinks) lie in the lower (resp., upper) half of a circumference O in the plane, and their indices grow from left to right. The other vertices of G lie inside O . All edges will be represented by directed straight-line segments. The graph G is constructed in five steps.

Step 1 For each couple $\pi \in M$, we draw the segment between corresponding terminals, denoted by L_π . Namely: (a) if $\pi = ij \in M^{\text{lh}}$, L_π connects the sources s_i, s_j (a lower horizontal segment); (b) if $\pi = ij \in M^{\text{uh}}$, L_π connects the sinks t_i, t_j (an upper horizontal segment); and (c) if $\pi = ij \in M^{\text{vert}}$, L_π connects the source s_i and sink t_j (a vertical segment). In case (c), we direct L_π from s_i to t_j . These segments are pairwise disjoint (since M is planar).

Step 2 For each $\pi = ij \in M^{\text{lh}}$, the lower horizontal segment L_π is transformed into a graph whose vertices are s_i, s_j and $j - i$ distinct points in the interior of the segment. The edges are the $j - i + 1$ subsegments connecting consecutive vertices. We distinguish between two sorts of vertices, called *odd* and *even* ones, so that s_i, s_j are regarded as odd, and the odd and even vertices alternate along L_π . Each edge is directed from the odd to even vertex. So L_π becomes a path with alternating edge directions, and its end vertices s_i, s_j have leaving edges.

Each upper horizontal segment $L_{\pi=ij}$ is transformed into a path in a similar fashion, but now we direct each edge from the even to odd vertex. So the end vertices t_i, t_j of L_π are odd and have entering edges.

Step 3 The horizontal segments (regarded as graphs) are connected by additional edges. To define them, let us say that a couple $\pi = ij \in M^{\text{lh}}$ is a *predecessor* of another couple $\pi' \in M^{\text{lh}}$ if $[\pi] \supset [\pi']$. If, in addition, there is no $\pi'' \in M^{\text{lh}}$ between π and π' (i.e. $[\pi] \supset [\pi'' \supset [\pi']$), π is called the *immediate predecessor* of π' . Accordingly, π' is called a *successor* of π in the former case, and an *immediate successor* in the latter case. A couple is *maximal (minimal)* if it has no predecessor (resp., no successor). The set of successors (immediate successors) of π is denoted by $\text{Succ}(\pi)$ (resp., by $\text{ISucc}(\pi)$); moreover, we order the couples in $\text{ISucc}(\pi)$, say, $\pi_1 = i_1 j_1, \dots, \pi_r = i_r j_r$, so that $j_d < i_{d+1}$ for $d = 1, \dots, r - 1$. It is easy to see that $i_1 = i + 1, j_r = j - 1$ and $i_{d+1} = j_d + 1$ for each d (when $r \geq 1$).

Note that each path L_{π_d} has exactly $(j_d - i_d + 1)/2$ even vertices. Also L_π has exactly $(j - i - 1)/2$ nonterminal odd vertices v (i.e., $v \neq s_i, s_j$). Then

$$\frac{1}{2} \sum_{d=1}^r (j_d - i_d + 1) = \frac{1}{2} (j_r - i_1 + 1) = \frac{1}{2} (j - i - 1),$$

yielding the equality

$$|\text{V}^{\text{odd}}(\pi)| = \sum (|\text{V}^{\text{even}}(\pi')| : \pi' \in \text{ISucc}(\pi)),$$

where $\text{V}^{\text{odd}}(\pi'')$ ($\text{V}^{\text{even}}(\pi'')$) denotes the set of *nonterminal* odd vertices (resp., of even vertices) in a path $L_{\pi''}$. Observe that within the circle O^* surrounded by O , the region confined by the segments for $\{\pi\} \cup \text{ISucc}(\pi)$ is convex and does not meet any other segment for M . Ordering the vertices in each of the two equally sized sets $W(\pi) := \bigcup (\text{V}^{\text{even}}(\pi') : \pi' \in \text{ISucc}(\pi))$ and $\text{V}^{\text{odd}}(\pi)$ from left to right, we draw a directed edge (segment) from each vertex of the former to the corresponding vertex of the latter. These edges are pairwise disjoint; we call them *lower bridges* for π . See the picture where $\pi = 16$ and the dark and light circles indicate even and nonterminal odd vertices in L_π , respectively.



A similar edge set is constructed for each non-minimal upper couple $\pi \in M^{\text{uh}}$, connecting the set $\text{V}^{\text{odd}}(\pi)$ of nonterminal odd vertices in L_π and the set $W(\pi)$ of even vertices in $L_{\pi'}$ among $\pi' \in \text{ISucc}(\pi)$. The only difference is that such edges, called *upper bridges* for π , are now directed from odd to even vertices.

The following observation is useful:

$$\text{For each } \pi \in M^{\text{lh}}, \quad |\text{V}^{\text{even}}(\pi)| = |\{\pi\} \cup \text{Succ}(\pi)|, \quad \text{and similarly for } \pi \in M^{\text{uh}}. \tag{5.3}$$

(This follows from the equality $|\text{V}^{\text{even}}(\pi)| = (j - i + 1)/2$, where $\pi = ij$, and the fact that the successors of π form a perfect matching on $\{i + 1, \dots, j - 1\}$, implying $|\text{Succ}(\pi)| = (j - i - 1)/2$.)

Step 4 Let M_{\max}^{lh} (M_{\max}^{uh}) be the set of maximal couples in M^{lh} (resp., M^{uh}). The segments of couples in $M_{\max}^{\text{lh}} \cup M_{\max}^{\text{uh}}$ confine a convex region Ω within the circle O^* . Consider the sets $Q := \bigcup(\text{V}^{\text{even}}(\pi) : \pi \in M_{\max}^{\text{lh}})$ and $Q' := \bigcup(\text{V}^{\text{even}}(\pi) : \pi \in M_{\max}^{\text{uh}})$; we order the vertices in each of them from left to right. Property (5.3) and the equalities $|Y| = |Y'| = 2|M|$ lead to the following relations:

$$|Q| = |M^{\text{lh}}| = \frac{1}{2}(|Y| - |M^{\text{vert}}|) = \frac{1}{2}(|Y'| - |M^{\text{vert}}|) = |M^{\text{uh}}| = |Q'|. \tag{5.4}$$

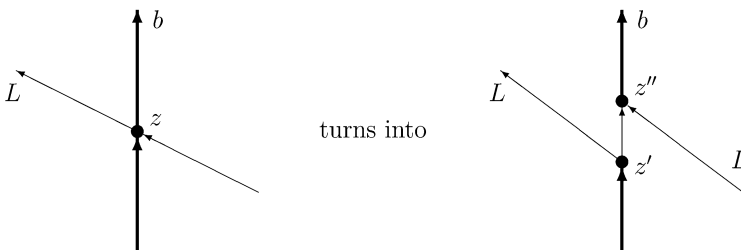
So $|Q| = |Q'|$. We draw a directed edge from each vertex of the sequence Q to the corresponding vertex of Q' . These (pairwise non-crossing) edges are called *middle bridges*.

Step 5 When $M^{\text{vert}} \neq \emptyset$, the graph G' constructed during the previous steps need not be planar since some middle bridges may intersect vertical segments. The final step transforms G' within small neighborhoods of such intersection points.

More precisely, for $\pi = ij \in M^{\text{vert}}$, the (directed) vertical segment L_π goes from the source s_i to the sink t_j and lies in the convex region Ω (defined above). The set B_π of edges of G' intersecting L_π consists of some middle bridges. (One can see that $i - j$ is even and that $B_\pi = \emptyset$ if $i = j$.) Let $z_{\pi,b}$ denote the intersection point of L_π and $b \in B_\pi$. Also for a middle bridge b , we denote by R_b the set of vertical segments intersecting b .

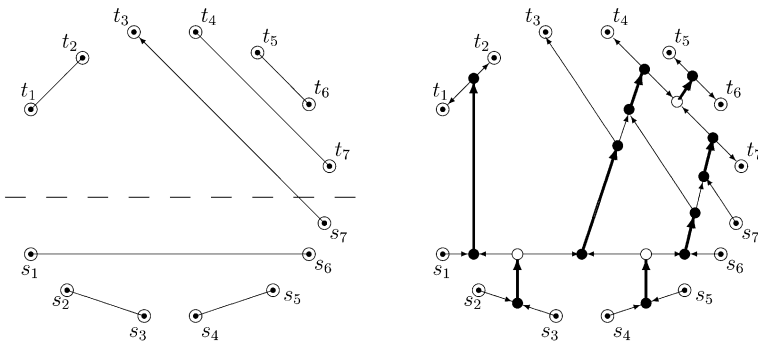
To transform G' into the desired graph G , we first turn each vertical segment $L_{\pi=ij}$ into the directed path (going from s_i to t_j) whose inner vertices are the points $z_{\pi,b}$ for $b \in B_\pi$, and similarly turn each middle bridge b (directed “upwards”) into the directed path whose inner vertices are the points $z_{\pi,b}$ for $L_\pi \in R_b$. Next we iteratively modify the graph as follows. At each iteration, choose a vertex $z = z_{\pi,b}$ in the current graph, split z into two vertices z' and z'' , and connect them by edge $e_{\pi,b}$ from z' to z'' , called the *extra edge* generated by π, b .

Geometrically, we choose z', z'' to be two points in the segment b within a small neighborhood of z so that z' lies below z'' . Then b (regarded as path) is modified in a natural way: if b is of the form $\dots, e, z, \tilde{e}, \dots$ (where e and \tilde{e} are the edges in b entering and leaving z , respectively), then we make e enter z' and make \tilde{e} leave z'' ; this turns b into the directed path $\dots, e, z', e_{\pi,b}, z'', \tilde{e}, \dots$. The local transformation of the path $L := L_\pi$ at z is different: if L is of the form $\dots, e, z, \tilde{e}, \dots$, we make e entering z'' , and \tilde{e} leaving z' , obtaining the non-directed path $\dots, e, z'', e_{\pi,b}, z', \tilde{e}, \dots$ (in which $e_{\pi,b}$ has the backward direction). Geometrically, L turns into a zigzag-shaped line. The transformation at $z = z_{\pi,b}$ is illustrated in the picture:



Eventually we obtain the desired graph G . We refer to the edges of G generated by vertical segments (resp., middle bridges) of G' and different from extra edges as v -edges (resp., b -edges). Thus, under the transformation $G' \mapsto G$, each middle bridge b turns into a directed path with $|R_b| + 1$ b -edges and $|R_b|$ extra edges which alternate. In its turn, each vertical segment $L = L_\pi$ turns into a “zigzag” path with $|B_\pi| + 1$ v -edges and $|B_\pi|$ extra edges; these edges alternate and are, respectively, the forward and backward edges in the path.

We will distinguish between two sorts of edges in G , referring to the lower and upper bridges and b -edges as *thick* edges, and to the remaining edges as *thin* ones. The picture below illustrates the construction for an instance of M . Here $n = 7$, $M^{\text{lh}} = \{16, 23, 45\}$, $M^{\text{uh}} = \{12, 47, 56\}$, and $M^{\text{vert}} = \{73\}$, and the left fragment shows the segment representation of M after Step 1. The graph G is drawn in the right fragment where the dark circles indicate even vertices and those formed by splitting, the light circles indicate nonterminal odd vertices, and the thin and thick edges are as defined above.



Note that the obtained G is acyclic (as all edges not contained in “horizontal segments” are “directed upwards”). Also we will take advantages from the following features of G which can be seen from the above construction:

- (5.5) (i) Each source has one leaving edge and no entering edge, whereas each sink has one entering edge and no leaving edge.
- (ii) Each inner (i.e., nonterminal) vertex is of degree 3, and it has either two thin entering edges and one thick leaving edge, or two thin leaving edges and one thick entering edge.
- (iii) The connected components of the subgraph of G induced by the thin edges correspond to the lower horizontal paths L_π for $\pi \in M^{\text{lh}}$, the upper horizontal paths L_π for $\pi \in M^{\text{uh}}$, and the (straight or zigzag) paths L_π for $\pi \in M^{\text{vert}}$, each of these paths having alternately directed edges.

It will be convenient to represent each thin path L_π as the union of two matchings (one being formed by the forward edges, and the other by the backward edges), denoted by N_π^1, N_π^2 . Also we denote the set of thick edges entering (leaving) vertices of L_π by Z_π^{in} (resp. Z_π^{out}). In particular, Z_π^{in} is the set of lower bridges for π when $\pi \in M^{\text{lh}}$, and Z_π^{out} is the set of upper bridges for π when $\pi \in M^{\text{uh}}$.

We assert that G satisfies properties (P1) and (P2) for the given M . To show this, we consider a proper pair (C, C') for (Y, Y') and argue as follows.

(a) Suppose $M \in \mathcal{M}(C, C')$. Take the subgraph F of G induced by the edge set U consisting of all thick edges and the following thin edges. For each $\pi \in M^{lh}$, U includes exactly one of the matchings N_π^1, N_π^2 in L_π , namely, the one containing the edge leaving the source s_i , where i is the element of $\pi \cap C$ (which is unique since M is feasible for (C, C')). Similarly, for each $\pi \in M^{uh}$, U includes the matching in L_π that contains the edge entering the sink t_j , where $\{j\} = \pi \cap C'$. And for each $\pi = ij \in M^{vert}$, if $i \in C$ (and therefore, $j \in C'$), then U includes the matching in L_π covering both s_i and t_j (which is formed by v -edges), whereas if $i \notin C$ (and $j \notin C'$), then U includes the matching formed by extra edges (which may be empty).

Using (5.5) and the fact that G is acyclic, one can conclude that F consists of pairwise disjoint directed paths going from S_C to $T_{C'}$, i.e., F is a $(C|C')$ -flow in G . Acting similarly w.r.t. \bar{C} and \bar{C}' , we construct a $(\bar{C}|\bar{C}')$ -flow F' in G .

(b) Next we show that in case $M \in \mathcal{M}(C, C')$ the flows F and F' as above are unique. Consider an arbitrary flow \tilde{F} from some sources to some sinks in G . From (5.5) it easily follows that for each $\pi \in M$,

- (5.6) (i) If $\tilde{F} \cap L_\pi$ is a matching N_π^α , $\alpha \in \{1, 2\}$, then \tilde{F} contains both Z_π^{in}, Z_π^{out} .
- (ii) Conversely, if \tilde{F} contains a set $Z \in \{Z_\pi^{in}, Z_\pi^{out}\}$ and if $Z \neq \emptyset$, then $\tilde{F} \cap L_\pi$ is exactly one of N_π^1, N_π^2 (regarding these objects as edge sets).

We explain that (5.6) determines \tilde{F} uniquely if \tilde{F} is a $(C|C')$ -flow. Indeed, from the construction of G it easily follows that there is an ordering $\pi(1), \dots, \pi(m)$ ($m = |Y|$) of the couples in M such that for $k = 1, \dots, m$, at least one of the sets $Z_{\pi(k)}^{in}$ and $Z_{\pi(k)}^{out}$ is entirely contained in $\bigcup_{d=1}^{k-1} (Z_{\pi(d)}^{in} \cup Z_{\pi(d)}^{out})$ (which is automatically holds when $\pi(k)$ is a minimal couple in $M^{lh} \cup M^{uh}$ since some of $Z_{\pi(k)}^{in}, Z_{\pi(k)}^{out}$ are empty).

Now we argue as follows. If $\pi(k)$ is a minimal couple in M^{lh} and if $\{i\} = \pi(k) \cap C$, then each of $N_{\pi(k)}^1, N_{\pi(k)}^2$ consists of a single edge and, obviously, \tilde{F} contains exactly one of them, namely, the edge incident to s_i . Applying (5.6)(i) to this $\pi(k)$, we obtain that \tilde{F} contains $Z_{\pi(k)}^{out}$ (as well as $Z_{\pi(k)}^{in} = \emptyset$). Similarly, if $\pi(k)$ is a minimal couple in M^{uh} , then \tilde{F} is determined within $L_{\pi(k)}$ and contains $Z_{\pi(k)}^{in}$ (and $Z_{\pi(k)}^{out} = \emptyset$). In a general case, assume by induction that for $d = 1, \dots, k - 1$, \tilde{F} is determined on $L_{\pi(d)}$ and contains $Z_{\pi(d)}^{in} \cup Z_{\pi(d)}^{out}$. Then, due to the above ordering, \tilde{F} contains at least one of $Z_{\pi(k)}^{in}, Z_{\pi(k)}^{out}$. Hence, by (5.6)(ii), $\tilde{F} \cap L_{\pi(k)}$ is $N_{\pi(k)}^\alpha$ for some $\alpha \in \{1, 2\}$. (This remains true when $Z_{\pi(k)}^{in} \cup Z_{\pi(k)}^{out} = \emptyset$, which is possible only if $\pi(k) = ij \in M^{vert}$ and $i = j$.) Moreover, α is determined by considering the end vertices (terminals) of $L_{\pi(k)}$ and checking which of them (or none, or both) belongs to $S_C \cup T_{C'}$ (since such a terminal must be covered by $N_{\pi(k)}^\alpha$). Now (5.6)(i) enables us to conclude with $Z_{\pi(k)}^{in} \cup Z_{\pi(k)}^{out} \subseteq \tilde{F}$, justifying the induction.

So $\tilde{F} = F$. The uniqueness of a $(\bar{C}|\bar{C}')$ -flow is shown similarly. This yields (P1).

(c) To check (P2), consider a proper pair (C, C') for (Y, Y') such that there exist both a $(C|C')$ -flow F and a $(\bar{C}|\bar{C}')$ -flow F' in G . Our goal is to show that $M \in \mathcal{M}(C, C')$, i.e., that the following hold: (c1) $|\pi \cap C| = 1$ for $\pi \in M^{lh}$; (c2) $|\pi \cap C'| = 1$ for $\pi \in M^{uh}$; and (c3) $(i \in C) \Leftrightarrow (j \in C')$ for $\pi = ij \in M^{vert}$. We consider

the above ordering $\pi(1), \dots, \pi(r)$ on M and use induction on k , assuming that the corresponding relation among (c1)–(c3) holds for each $\pi(d)$ with $d < k$.

When $\pi(k)$ is a minimal couple in M^{lh} , $\pi(k) \subseteq C$ would imply that F contains both edges of the 2-edge path $L_{\pi(k)}$, which is impossible (since these edges enter the same vertex). For a similar reason, $\pi(k) \cap C = \emptyset$ would imply the nonexistence of F' . So $\pi(k)$ satisfies (c1). Similarly a minimal couple in M^{uh} satisfies (c2). Also if $\pi(k) = ij \in M^{\text{vert}}$ and $i = j$, then $L_{\pi(k)}$ consists of a single edge, and (c3) is trivial. For a general k , using induction and arguing as in part (b), we may assume that there is a nonempty set among $Z_{\pi(k)}^{\text{in}}, Z_{\pi(k)}^{\text{out}}$ which is contained in both F, F' . Then there are $\alpha, \beta \in \{1, 2\}$ such that $F \cap L_{\pi(k)} = N_{\pi(k)}^{\alpha}$ and $F' \cap L_{\pi(k)} = N_{\pi(k)}^{\beta}$. The matchings $N_{\pi(k)}^{\alpha}, N_{\pi(k)}^{\beta}$ determine the location of the end vertices (terminals) u, v of $L_{\pi(k)}$ w.r.t. C, C' and their complements to Y, Y' , and each of u, v is related to exactly one of $C \cup C'$ and $\bar{C} \cup \bar{C}'$. This implies $\alpha \neq \beta$, and validity of (c1)–(c3) for $\pi(k)$ follows. Finally, (5.6)(i) provides that each of F, F' contains both $Z_{\pi(k)}^{\text{in}}, Z_{\pi(k)}^{\text{out}}$, completing the proof of (P2).

It remains to consider the situation when some of X, X' or both are nonempty. It reduces to the previous case by replacing each element of $X (X')$ by a couple of elements in Y (resp., Y') and adding such couples to the matching in question. More precisely, an element $i \in X$ is replaced by consecutive elements i', i'' added to Y (which are inserted into the linearly ordered set $X \cup Y$ in place of i). Similarly, an element $j \in X'$ is replaced by consecutive elements j', j'' added to Y' . Then the resulting sets \tilde{Y} and \tilde{Y}' have the same size, equal to $|Y| + 2|X| = |Y'| + 2|X'|$. Accordingly, we extend each planar perfect matching M on $Y \sqcup Y'$ to a planar perfect matching \tilde{M} on $\tilde{Y} \sqcup \tilde{Y}'$ by adding the lower (upper) horizontal couple $\pi^i = i'i''$ for each $i \in X$ (resp., $\pi^j = j'j''$ for each $j \in X'$). Note that the added couples are minimal for the corresponding partial orders, and the pairs $(\tilde{C} \subseteq \tilde{Y}, \tilde{C}' \subseteq \tilde{Y}')$ having \tilde{M} as a feasible matching are exactly those obtained from the pairs $(C \subseteq Y, C' \subseteq Y')$ satisfying $M \in \mathcal{M}(C, C')$ by adding to C one element from $\{i', i''\}$ for each $i \in X$, and adding to C' one element from $\{j', j''\}$ for each $j \in X'$.

Let \tilde{G} be the graph obtained by applying the previous construction to such an \tilde{M} . Then each couple $\pi^i, i \in X$, generates the 2-edge path L_{π^i} connecting the sources $s_{i'}, s_{i''}$, and similarly for X' and sinks. Shrinking each L_{π^i} into one point, regarded as the source s_i when $i \in X$ and as the sink t_i when $i \in X'$, we obtain the desired graph G for X, X', Y, Y' and M . It is straightforward to verify that properties (P1), (P2) for $\tilde{G}, \tilde{Y}, \tilde{Y}', \tilde{M}$ imply those for G, X, X', Y, Y', M .

This completes the proof of Theorem 5.1, and Theorem 3.1 follows. □

Note that any FG-function $f = f_w$ on $\mathcal{E}^{n,n'}$ obtained by the construction in the above proof takes nonnegative integer values (since the weighting w is nonnegative). This together with the fact that the function of minors of a totally nonnegative matrix is an FG-function gives the following:

Corollary 5.2 *2-patterns $\mathcal{A}_0, \mathcal{B}_0 \in \Pi_{m,m'}$ are balanced if and only if the corresponding quadratic relations (concerning any n, n', X, X', Y, Y' as above) hold for any function $f : \mathcal{E}^{n,n'} \rightarrow \mathbb{Z}_{\geq 0}$ which is the function of minors of a totally nonnegative*

$n' \times n$ matrix. Furthermore, when $\mathcal{A}_0, \mathcal{B}_0 \in \Pi_{m,m'}$ are not balanced, for any corresponding n, n', X, X', Y, Y' , there exists, and can be explicitly constructed, a totally nonnegative $n' \times n$ matrix such that the function f of its minors obeys inequality (5.1) (where $\mathcal{A} := \gamma_{Y,Y'}(\mathcal{A}_0)$ and $\mathcal{B} := \gamma_{Y,Y'}(\mathcal{B}_0)$).

6 Applications to Schur functions

It is well known that Schur functions (polynomials) are expressed as minors of a certain matrix, by Jacobi–Trudi’s formula. Therefore, these functions satisfy many quadratic relations, in particular, ones of Plucker type. In [8, 9] and some other works (see a discussion in [8]), one shows how to establish quadratic relations for ordinary and skew Schur functions by use of a lattice paths method based on the Gessel–Viennot interpretation of semistandard Young tableaux [11]. This lattice path method is, in fact, a specialization to a particular planar network of the flow approach that we described in Sects. 2, 3. Below we give a brief discussion on this subject.

Recall that a *partition* of length r is an r -tuple λ of weakly decreasing nonnegative integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$. The *Ferrers diagram* of λ is meant to be the array F_λ of cells with r left-aligned rows containing λ_i cells in i th row (the row indices will grow from the bottom to the top). For $N \in \mathbb{N}$, an N -*semistandard Young tableau* of shape λ is a filling T of F_λ with numbers from $[N]$ so that the numbers weakly increase in each row and strictly increase in each column. We associate to T the monomial x^T that is the product of variables x_1, \dots, x_N , each x_k being taken in the degree equal to the number of occurrences of k in T . Then the *Schur function* for λ and N is the polynomial

$$s_\lambda = s_\lambda(x_1, \dots, x_N) := \sum_T x^T,$$

where the sum is over all N -semistandard Young tableaux of shape λ . Besides, one often considers a *skew Schur function* $s_{\lambda/\mu}$, where μ is an r -partition with $\mu_i \leq \lambda_i$; it is defined in a similar way w.r.t. the skew Ferrers diagram $F_{\lambda/\mu}$ obtained by removing from F_λ the cells of F_μ , along with its semistandard fillings. When needed, an “ordinary” diagram F_λ is regarded as the skew one $F_{\lambda/\mu}$, where $\mu = (0, \dots, 0)$, and similarly for tableaux.

There is an important one-to-one correspondence between the r -partitions λ and the r -element subsets A_λ of the set $\mathbb{Z}_{>0}$ of positive integers (or a set $[n]$ for $n \geq \lambda_1 + r$). This is given by

$$\lambda = (\lambda_1 \geq \dots \geq \lambda_r) \iff A_\lambda := \{\lambda_r + 1, \lambda_{r-1} + 2, \dots, \lambda_1 + r\}. \tag{6.1}$$

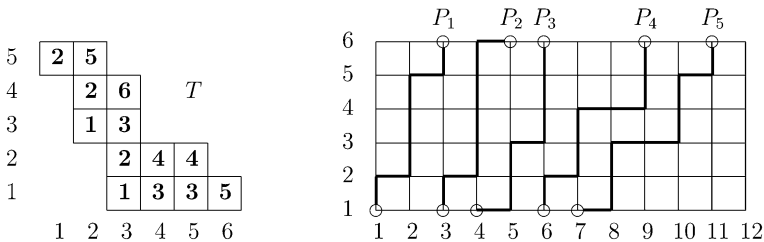
Let us form the directed square grid $\Gamma = \Gamma(N)$ whose vertices are the points (i, j) for $i \in \mathbb{Z}_{>0}$ and $j \in [N]$ and whose edges e are directed upwards or to the right, i.e., $e = ((i, j), (i, j + 1))$ or $((i, j), (i + 1, j))$ (instead, one can take a finite truncation of this grid). The vertices $s_i := (i, 1)$ and $t_i := (i, N)$ are regarded as the sources and sinks in Γ , respectively, and we assign to each horizontal edge e at level h the weight to be the indeterminate x_h :

$$w(e) := x_h \quad \text{for } e = ((i, h), (i + 1, h)), \quad i \in \mathbb{Z}_{>0}, \quad h = 1, \dots, N. \tag{6.2}$$

Now using the Gessel–Viennot model [11] (in a slightly different form), one can associate to an N -semistandard skew Young tableau T with shape λ/μ the system $\mathcal{P}_T = (P_1, \dots, P_r)$ of directed paths in Γ , where for $k = 1, \dots, r$:

- (6.3) P_k corresponds to $(r + 1 - k)$ th row of T ; it goes from the source $s_{k+\mu_{r+1-k}}$ to the sink $t_{k+\lambda_{r+1-k}}$; and for $h = 1, \dots, N$, the number of horizontal edges of P_k at level h is equal to the number of occurrences of h in the k th row of T .

So the sources used in \mathcal{P}_T are the s_i for $i \in A_\mu$, and the sinks are the t_j for $j \in A_\lambda$. Observe that the semistandardness of T implies that these paths are pairwise disjoint, i.e., \mathcal{P}_T is an $(A_\mu|A_\lambda)$ -flow in Γ . One can see the converse as well: if \mathcal{P} is an $(A_\mu|A_\lambda)$ -flow in Γ , then the filling T of $F_{\lambda/\mu}$ determined, in a due way, by the horizontal edges of paths in \mathcal{P} is just a semistandard skew Young tableau, and one has $\mathcal{P}_T = \mathcal{P}$. This gives a nice bijection between corresponding flows and tableaux. The next picture illustrates an example of a semistandard Young tableau T with $N = 6$, $r = 5$, $\lambda = (6, 5, 3, 3, 2)$ and $\mu = (2, 2, 1, 1, 0)$, and its corresponding flow $\mathcal{P}_T = (P_1, \dots, P_5)$.



Note that when T is “ordinary” (i.e., $\mu = \mathbf{0}$), the sources used in \mathcal{P}_T are s_1, s_2, \dots, s_r . We say that this \mathcal{P}_T is a *co-flag flow* (it becomes a flag flow if we reverse the edges of Γ and swap the sources and sinks).

The above bijection between the N -semistandard skew Young tableaux with shape λ/μ and the $(A_\mu|A_\lambda)$ -flows in $\Gamma(N)$ implies that (ordinary of skew) Schur functions are “values” of the flow-generated function f_w for Γ and the weighting w as in (6.2). (It leads to no confusion that the weights are given on the horizontal edges of Γ and belong to a polynomial ring.) This gives rise to establishing quadratic relations on Schur functions, by properly translating SQ-relations on FG-functions. Below we give two examples (the reader may try to extend the list of examples by using SQ-relations from Sect. 4).

1. A particular relation on ordinary Schur functions with $r = 2$ can be derived from PSQ-relations on quadruples. This reads as

$$s_{(k,i)}s_{(\ell,j)} = s_{(\ell,i)}s_{(k,j)} + s_{(j-1,i)}s_{(\ell,k+1)}, \tag{6.4}$$

where $i < j \leq k < \ell$. Letting $(i', j', k', \ell') := (i + 1, j + 1, k + 2, \ell + 2)$ and $f := f_w$, one can see that (6.4) turns into

$$f([2] | i'k') f([2] | j'\ell') = f([2] | i'\ell') f([2] | j'k') + f([2] | i'j') f([2] | k'\ell'),$$

which, in view of $i' < j' < k' < \ell'$, is nothing else than the co-flag counterpart of the AP4-relation (1.2) in case $X = \emptyset$. (Note that relation (6.4) can be generalized by adding to each 2-component partition a fixed partition $(\lambda_1, \dots, \lambda_{r'})$ such that either $\lambda_{r'} \geq \ell$ or $i \geq \lambda_1$.)

2. The next example is shown in [9] by use of Dodgson’s condensation formula for matrix minors. It says that a partition $(\lambda_1, \dots, \lambda_r)$ with $\lambda_r > 0$ yields

$$S(\lambda_1, \dots, \lambda_{r-1}) S(\lambda_2, \dots, \lambda_r) = S(\lambda_2, \dots, \lambda_{r-1}) S(\lambda_1, \dots, \lambda_r) + S(\lambda_2 - 1, \dots, \lambda_r - 1) S(\lambda_1 + 1, \dots, \lambda_{r-1} + 1). \tag{6.5}$$

For each of the six partitions $\lambda^{(i)}$ in this relation, $i = 1, \dots, 6$ (from left to right), we take the set $A_{\lambda^{(i)}}$ as in (6.1) and form the corresponding subsets $S^{(i)}, T^{(i)}$ of sources and sinks in Γ , respectively. In addition, for $i = 1, 3, 5$, we shift each of the sets $S^{(i)}, T^{(i)}$ by one position to the right (which leads to equivalent sets of flows, as well as their weights, in Γ). Then we obtain the following six source-sink index pairs (from left to right, as before), denoting $X := \{2, \dots, r - 1\}$ and $X' := \{\lambda_2 + r - 1, \lambda_3 + r - 2, \dots, \lambda_{r-1} + 2\}$:

$$(Xr|X'(\lambda_1 + r)), (1X|(\lambda_r + 1)X'), (X|X'), (1Xr|(\lambda_r + 1)X'(\lambda_1 + r)), \\ (Xr|(\lambda_r + 1)X'), (1X|X'(\lambda_1 + r)).$$

Now define $i := 1, k := r, i' := \lambda_r + 1, k' := \lambda_1 + r$. Then $i < k, i' < k', X \cap \{i, k\} = \emptyset, X' \cap \{i', k'\} = \emptyset$, and (6.5) turns into the following relation for $f = f_w$:

$$f(Xk|X'k') f(iX|i'X') = f(X|X') f(iXk|i'X'k') + f(Xk|i'X') f(iX|X'k'),$$

which is just Dodgson’s condensation formula (cf. (1.8)).

7 FG-functions over a semiring with division

In this section, we assume that \mathfrak{S} is a commutative semiring with division, i.e., \mathfrak{S} contains $\underline{1}$ and the operation \odot is invertible (i.e., (\mathfrak{S}, \odot) is an abelian group). Two important special cases mentioned in the Introduction are: the set $\mathbb{R}_{>0}$ of positive reals, and the tropicalization $\mathfrak{L}^{\text{trop}}$ of a totally ordered abelian group \mathfrak{L} , in particular, the set \mathbb{R}_{\max} of reals with the operations $\oplus = \max$ and $\odot = +$. It turns out that for such a \mathfrak{S} , the set $\mathbf{FG} = \mathbf{FG}_n(\mathfrak{S})$ of flag-flow-generated functions on $2^{[n]}$ possesses the following nice properties:

- (7.1) (i) All these functions f can be generated by flows in one planar network, namely, in the half-grid Γ_n^Δ (see Fig. 1);
- (ii) \mathbf{FG} coincides with the set of functions $f : 2^{[n]} \rightarrow \mathfrak{S}$ satisfying P3-relation (4.1) for all i, j, k, X (so (4.1) provides the other PSQ-relations);
- (iii) \mathbf{FG} has as a basis the set \mathcal{I}_n of intervals in $[n]$ (including the “empty interval” \emptyset), called the *standard* basis for \mathbf{FG} ;
- (iv) The values of f are expressed as (algebraic or tropical) Laurent polynomials in its values on \mathcal{I}_n .

Here a *basis* for this **FG** is meant to be a collection $\mathcal{I}' \subset 2^{[n]}$ such that the restriction map $f \mapsto f|_{\mathcal{I}'}$ gives a bijection between **FG** and $\mathfrak{S}^{\mathcal{I}'}$; in other words, any function in **FG** is determined by its values on \mathcal{I}' , and the latter values can be chosen arbitrarily in \mathfrak{S} .

The facts exhibited in (7.1) are discussed in [4] (mostly for $\mathfrak{S} := \mathbb{R}_{\max}$) and in [2, 7] (concerning (iv)). The arguments given there can be directly extended to an arbitrary \mathfrak{S} as above, and below we give a brief outline (which is sufficient to restore the details with help of [4]). As before, an interval $\{p, p + 1, \dots, q\}$ in $[n]$ is denoted by $[p..q]$.

A. An important feature of $\Gamma_n^\Delta = (V, E)$ is that for any nonempty interval $I = [q..r]$ in $[n]$, there exists exactly one feasible flow ϕ_I from S_I to the sinks $t_1, \dots, t_{|I|}$; namely, ϕ_I goes through the vertices (i, j) occurring in the rectangle $[r] \times [r - q + 1]$ (more precisely, satisfying $i \leq r, j \leq r - q + 1$ and $i \geq j$). Therefore, given a weighting $w : V \rightarrow \mathfrak{S}$, the values of $f = f_w$ on the nonempty intervals $[q..r]$ are viewed as

$$f([q..r]) = \bigodot_{j \leq i \leq r, 1 \leq j \leq r - q + 1} w(i, j). \tag{7.2}$$

Note that the number $\frac{n(n+1)}{2}$ of vertices of Γ_n^Δ is equal to the number of nonempty intervals in $[n]$ and system (7.2) is non-degenerate. So, using the division in \mathfrak{S} , denoted as $/$, we can in turn express the weights of vertices via the values of f on the intervals. This is computed as

$$w(i, j) = \begin{cases} (f(I_{i,j}) \odot f(I_{i-1,j-1})) / (f(I_{i-1,j}) \odot f(I_{i,j-1})) & \text{for } i > j, \\ f(I_{i,j}) / f(I_{i,j-1}) & \text{for } i = j, \end{cases} \tag{7.3}$$

denoting by $I_{i',j'}$ the interval $[(i' - j' + 1)..i']$ and letting $f(I_{i',0}) := \underline{1}$.

Thus, the correspondence $w \mapsto f_w$ gives a bijection between the set of weightings $w : V \rightarrow \mathfrak{S}$ and the set $\mathfrak{S}^{\mathcal{I}_n^+}$, where \mathcal{I}_n^+ denotes the set of nonempty intervals in $[n]$.

B. We know that for a weighting w , the value of $f = f_w$ on any nonempty subset $A \subseteq [n]$ is represented by a “polynomial” in variables $w(v), v \in V$, namely, by an \oplus -sum of products $\odot(w(v) : v \in V')$ for some subsets $V' \subseteq V$. Substituting into this polynomial the corresponding terms from (7.3), we obtain an expression of the form

$$f(A) = \oplus(\mathcal{P}_k : k = 1, \dots, N),$$

where each \mathcal{P}_k is a “monomial” $\odot(f(I)^{\odot \sigma_k(I)} : I \in \mathcal{I}_n^+)$ with integer (possibly negative) degrees $\sigma_k(I)$. This means that $f(A)$ is a Laurent polynomial (w.r.t. the addition \oplus and multiplication \odot) in variables $f(I), I \in \mathcal{I}_n^+$.

(Analyzing possible flows in Γ_n^Δ , one can show that the degrees $\sigma_k(I)$ are bounded and, moreover, belong to $\{-1, 0, 1, 2\}$. This is proved in [4] for the tropical case and can be directly extended to an arbitrary commutative semiring \mathfrak{S} with division.)

C. A simple fact (cf. [4]) is that any function $f : 2^{[n]} \rightarrow \mathbb{R}$ obeying TP3-relation (1.6) is determined by its values on \mathcal{I}_n . The proof of this fact is directly extended to \mathfrak{S} in question, as follows (a sketch). If $S \subseteq [n]$ is not an interval, define $i := \min(S)$, $k := \max(S)$, $X := S - \{i, k\}$, and choose an element j in $[i..k] - S$. Then for a function f on $2^{[n]}$ obeying P3-relation (4.1), the value $f(S)$ is expressed via the values $f(S')$ on five sets $S' = Xi, Xj, Xk, Xij, Xjk$. Since $\max(S') - \min(S') < \max(S) - \min(S)$, we can apply induction on $\max(S) - \min(S)$.

Using this fact and reasonings above, we obtain that \mathcal{I}_n is indeed a basis for the functions in $\mathbf{FG}_n(\mathfrak{S})$ and that all these functions are generated by flows in Γ_n^Δ (so they are bijective to weightings $w : V \rightarrow \mathfrak{S}$, up to their values on \emptyset , and possess the Laurentness property as above).

Next we explain how to extend the properties exhibited in (7.1) to the set $\mathbf{FG}_{n,n'}(\mathfrak{S})$ of flow-generated functions on $\mathcal{E}^{n,n'}$ taking values in a commutative semiring \mathfrak{S} with division. Instead of P3-relation (4.1) which has shown its importance in the flag flow case, a central role will now will be played by three special SQ-relations. The first one is

$$f(Xik|X'k') \odot f(Xj|X') = (f(Xij|X'k') \odot f(Xk|X')) \oplus (f(Xjk|X'k') \odot f(Xi|X')), \tag{7.4}$$

where $i < j < k$ and X are as before, $k' \in [n']$ and $X' \subseteq [n'] - \{k'\}$. We refer to (7.4) as the *generalized P3-relation*. (The pair of 2-patterns for it is equivalent to the pair of 1-patterns for (4.1). In fact, for our purposes it suffices to assume that $k' > \max(X')$.) The second one is the SQ-relation symmetric to (7.4):

$$f(Xk|X'i'k') \odot f(X|X'j') = (f(Xk|X'i'j') \odot f(X|X'k')) \oplus (f(Xk|X'j'k') \odot f(X|X'i')). \tag{7.5}$$

And the third one is Dodgson’s type relation (4.7) for i, k, X, i', k', X' such that (cf. (1.8)):

$$k - i = k' - i', \quad X = [i + 1..k - 1], \quad \text{and} \quad X' = [i' + 1..k' - 1]. \tag{7.6}$$

Let $\mathbf{K}_{n,n'}(\mathfrak{S})$ be the set of functions $f : \mathcal{E}^{n,n'} \rightarrow \mathfrak{S}$ satisfying (7.4), (7.5), and (4.7) with (7.6). Besides, define $\mathcal{I}_{n,n'}$ to be the set of pairs $(I \subseteq [n], I' \subseteq [n'])$ such that both I and I' are intervals and $|I| = |I'|$; we refer to (I, I') as a *(consistent) double interval*. Two subsets of double intervals are distinguished: let $\mathcal{D}_{n,n'}^1$ consist of those (I, I') that the first interval I is initial (i.e., contains 1), and $\mathcal{D}_{n,n'}^2$ of those (I, I') that the second interval I' is initial; we say that such an (I, I') is a *pressed double interval*.

Theorem 7.1 *For $\mathbf{F} := \mathbf{FG}_{n,n'}(\mathfrak{S})$, $\mathbf{K} := \mathbf{K}_{n,n'}(\mathfrak{S})$, and $\mathcal{D} := \mathcal{D}_{n,n'}^1 \cup \mathcal{D}_{n,n'}^2$, the following properties hold:*

- (i) \mathbf{K} coincides with \mathbf{F} .
- (ii) \mathcal{D} is a basis for the functions in \mathbf{F} .

(iii) For each $f \in \mathbf{F}$, the values of f are Laurent polynomials (over \mathfrak{S}) in the values $f(I|I')$, $(I, I') \in \mathcal{D}$.

Proof Instead of the half-grid Γ_n^Δ used in the flag flow case, we now work with the grid $\Gamma = \Gamma_{n,n'} = (V, E)$ (see Fig. 1). Let us associate to each vertex (k, k') of Γ the integer rectangle $R(k, k') := [k] \times [k']$ and the pressed double interval $D(k, k') := (I, I') \in \mathcal{D}$, where $I = [i..k]$ and $I' = [i'..k']$ (then $\min\{i, i'\} = 1$). ($D(k, k')$ is well-defined due to $|I| = |I'|$.) Observe that Γ has a unique $(I|I')$ -flow ϕ : its vertices are exactly those in $R(k, k')$. Therefore, for a weighting $w : V \rightarrow \mathfrak{S}$, the FG-function $f = f_w$ satisfies

$$f(D(k, k')) = \odot(w(p, q) : (p, q) \in R(k, k')).$$

Then w is expressed via the values of f on \mathcal{D} as

$$w(k, k') = (f(D(k, k')) \odot f(D(k - 1, k' - 1))) / (f(D(k - 1, k')) \odot f(D(k, k' - 1))),$$

letting $f(D(p, q)) := \underline{1}$ if $p = 0$ or $q = 0$. Thus, $w \mapsto f_w$ gives a bijection between the set of weightings $w : V \rightarrow \mathfrak{S}$ and $\mathfrak{S}^{\mathcal{D}^+}$, where $\mathcal{D}^+ := \mathcal{D} - \{(\emptyset, \emptyset)\}$.

Next, let $f \in \mathbf{K}$ and consider a pair $(S, S') \in \mathcal{E}^{n,n'}$. Let $i := \min(S)$, $k := \max(S)$, $i' := \min(S')$, $k' := \max(S')$. We show that $f(S|S')$ is determined by the values of f on \mathcal{D} , by considering three cases.

- (a) Suppose that S is not an interval. Letting $X := S - \{i, k\}$ and $X' := S' - \{k'\}$, choosing an element $j \in [i..k] - S$, and using (7.4), we express $f(S|S')$ via the values of f on the other five pairs occurring there. Since for each of those five pairs $(\tilde{S}|\tilde{S}')$, the number $\max(\tilde{S}) + \max(\tilde{S}') - \min(\tilde{S}) - \min(\tilde{S}')$ is strictly less than $k + k' - i - i'$, we can apply induction and conclude that $f(S|S')$ is determined by the values of f on those pairs in $\mathcal{E}^{n,n'}$ where the first term is an interval.
- (b) Suppose that S is an interval but S' is not. Acting symmetrically to the previous case and using (7.5), we conclude that $f(S|S')$ is determined by the values of f on double intervals.
- (c) Suppose that (S, S') is a double interval but not a pressed one. Set $\tilde{i} := i - 1$ and $\tilde{i}' := i' - 1$; then $\tilde{i} \geq 1$ and $\tilde{i}' \geq 1$. Let $X := S - \{k\}$ and $X' := S' - \{k'\}$ and apply (4.7) to $\tilde{i}, k, X, \tilde{i}', k', X'$. Then $f(S|S')$ is expressed via the values of f on five double intervals $(I|I')$ such that $\max(I) + \max(I') + \min(I) + \min(I')$ is strictly less than $k + k' + i + i'$. So we can apply induction.

As a result, we obtain that any function $f \in \mathbf{K}$ is determined by its values on \mathcal{D} . On the other hand, we have seen that any choice of $f_0 : \mathcal{D}^+ \rightarrow \mathfrak{S}$ determines a (unique) weighting w in Γ , which in turn determines a function $f \in \mathbf{F}$ with $f|_{\mathcal{D}^+} = f_0$. These observations imply that $\mathbf{K} = \mathbf{F}$ and that \mathcal{D} is a basis for \mathbf{F} . The Laurentness concerning \mathbf{F} and \mathcal{D} is clear. □

In light of this theorem, when we deal with a commutative semiring with division, any SQ-relation is a consequence of the SQ-relations of three types: the generalized

P3-relation, its symmetric one, and Dodgson’s type relation. In particular, this is so when we deal with SQ-relations on minors of totally positive matrices.

One more property involving the basis $\mathcal{D} = \mathcal{D}_{n,n'}^1 \cup \mathcal{D}_{n,n'}^2$ extends the corresponding fact for totally positive matrices.

Proposition 7.2 *Let $A = (a_{ij})$ be a real $n \times n'$ matrix (regarding i as the column index) and let $f^A(I|I')$, $(I, I') \in \mathcal{E}^{n,n'}$, be the function of minors of A . Suppose $f^A(I|I') \neq 0$ for all pressed double intervals $(I, I') \in \mathcal{D}$. Then f^A is a flow-generated function.*

Proof Let w be the weighting on vertices of the grid $\Gamma_{n,n'}$ determined by the values of f^A on \mathcal{D} , as in the proof of Theorem 7.1 (w is well-defined since f^A is nowhere zero within \mathcal{D}). Let A' be the $n \times n'$ matrix obtained from w by Lindström’s method. Then the minors of A and A' related to \mathcal{D} coincide; so it suffices to show that A is determined by these minors. To see the latter, observe that the entries a_{ij} with $i = 1$ or $j = 1$ are such minors. Suppose we have already “restored” the entries a_{ij} for a set Q of index pairs such that $ij \in Q$, $i' \leq i$, and $j' \leq j$ imply $i'j' \in Q$; we assume that Q contains all ij with $\min\{i, j\} = 1$ as well. Then, unless $Q = [n] \times [n']$, there exists a pair $ij \notin Q$ such that Q contains each pair $i'j' \neq ij$ with $i' \leq i$ and $j' \leq j$. Let (I, J) and (I', J') be the pressed double intervals with $\max(I) = i$, $\max(J) = j$, $\max(I') = i - 1$ and $\max(J') = j - 1$. Then the product $a_{ij} \cdot f^A(I'|J')$ is expressed as a polynomial in the value $f^A(I|J)$ and entries a_{pq} with $pq \in Q$. Now $f^A(I'|J') \neq 0$ enables us to compute a_{ij} (and the new set $Q \cup \{a_{ij}\}$ of “restored” entries satisfies the above condition). □

We conclude this paper with extending the SQ-relations to minors of matrices.

Proposition 7.3 *Let A be an $n \times n'$ matrix over a commutative ring \mathcal{R} . Then the function f^A of minors of A obeys all SQ-relations for n, n' .*

Proof Assuming $\mathcal{R} = \mathbb{R}$, consider the parameterized matrix $P_{(t)} = A + tB$, where B is an arbitrary totally positive $n' \times n$ matrix and $t \in \mathbb{R}$. (By the way, such a B can be generated by use of the grid $\Gamma_{n,n'}$ with a weighting in $\mathbb{R}_{>0}$.) When t is large enough, $P_{(t)}$ becomes totally positive, and therefore the function $f_{(t)} := f^{P_{(t)}}$ of its minors becomes an FG-function, implying that such an f obeys all SQ-relations S . Substituting f into S gives a polynomial Q in t . Since Q turns into zero when t is large, Q is the zero polynomial. Hence $f_{(0)} = f^A$ obeys S as well (when $\mathcal{R} = \mathbb{R}$). Moreover, Q at 0 is a polynomial, with integer coefficients, in the entries a_{ij} of the matrix A (each being regarded as indeterminate). So it is the zero polynomial in a_{ij} , and we can take an arbitrary commutative ring for \mathcal{R} , obtaining the result. □

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Note Added in Proof As is shown in the arxiv version of this paper (see [arXiv:1102.2578v2](https://arxiv.org/abs/1102.2578v2) [math.CO], the Appendix), the function of minors of any matrix over a commutative ring is an FG-function.

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