Journal of Algebraic Combinatorics 3 (1994), 177-185

Completely Regular Designs of Strength One

WILLIAM J. MARTIN*
Department of Mathematics & Statistics, University of Vermont, Burlington, VT 05405

Received January 13, 1993; Revised July 16, 1993

Abstract. We study a class of highly regular t-designs. These are the subsets of vertices of the Johnson graph which are completely regular in the sense of Delsarte [2]. In [9], Meyerowitz classified the completely regular designs having strength zero. In this paper, we determine the completely regular designs having strength one and minimum distance at least two. The approach taken here utilizes the incidence matrix of $(t+1)$-sets versus k-sets and is related to the representation theory of distance-regular graphs [1, 5].

Keywords: completely regular subset, equitable partition, Johnson graph, t-design

1. Background

Let G be a graph and let π be a partition of its vertices. We call π *equitable* if, for every cell C and for every vertex x, the number of neighbors of x in C depends only on C and on the cell containing x and not on the choice of x itself. Given a subset C of the vertices of G, the *distance partition* of G with respect to C is the partition whose cells are the nonempty sets of the form

$$C_i = \{x \in V(G) : \text{dist}(x, C) = i\}.$$

Subset C is said to be *completely regular* if the distance partition of G with respect to C is equitable.

This definition of a completely regular subset is due to Neumaier [10] and, in the case where G is distance-regular, it is equivalent to Delsarte's original definition [2]. This states that subset C is completely regular if, for each vertex x of G and for each distance i, the number of vertices of C which lie at distance i from x depends only the distance from x to C and not on the choice of x itself.

Let $\mathcal{V} = \{1, 2, \ldots, v\}$ and let $0 \leq k \leq v/2$. The *Johnson graph* $J(v, k)$ is the graph whose vertices are the k-element subsets of $\mathcal{V}$, two vertices being adjacent precisely when their intersection has cardinality $k-1$. It follows that two vertices x, y are at distance i in $J(v, k)$ precisely when $|x \cap y| = k - i$. The Johnson graph is distance-regular. Let A_i denote the adjacency matrix of the distance-i graph of $J(v, k)$ (i.e., A_i is a 01-matrix with (x, y)-entry equal to one if and only

*Current address: Department of Mathematics and Statistics, University of Winnipeg, 515 Portage Ave., Winnipeg MB R3B 2E9, Canada.

if $|x \cap y| = k - i$). Then A_1 is the adjacency matrix of $J(v,k)$ and the matrices $\{A_0 = I, A_1, \ldots, A_k\}$ form a basis for the *Bose-Mesner algebra* of the *Johnson association scheme* [2]. This association scheme provides an algebraic setting for the study of combinatorial t-designs.

Recall that a collection C of k-subsets of V is a *t-design* if every t-element subset lies in a constant number of elements of C. This number is denoted by $\lambda = \lambda_t$ and, in this case, C is called a $t-(v,k,\lambda)$ *design*. The elements of C are called *blocks*. The *strength* of C is the largest t for which C is a t-design. Any proper subset of the vertices of $J(v,k)$ is a $t-(v,k,\lambda)$ design for some t.

Let $\{E_0, E_1, \ldots, E_k\}$ be the basis of primitive idempotents for the Johnson scheme $J(v,k)$, ordered so that

$$A_1 E_j = \theta_j E_j$$

for $0 \le j \le k$ where $\theta_j = (k-j)(v-k-j) - j$ is the jth largest eigenvalue of $J(v,k)$. Let C be a subset of the vertices of $J(v,k)$ with characteristic vector χ_C. Delsarte proved the following important result:

THEOREM 1.1 (Delsarte [2, Theorem 4.2]). *C is a combinatorial t-design if and only if*

$$E_1\chi_C = E_2\chi_C = \cdots = E_t\chi_C = 0.$$

Our goal is to study completely regular subsets in the Johnson graphs; a *completely regular t-design* is such a subset having strength t. This class of designs includes some very nice designs, for example:

- the Witt designs on 11, 23, and 24 points
- all t-designs with block size $t+1$
- all Steiner systems with block size $t+2$.

Most of these can be shown to be completely regular using Theorem 1.1 and [2, Theorem 5.13] (or see [4, Lemma 11.7.6]). Any perfect code in the Johnson graph would also be a completely regular design, so the present topic is related to the famous conjecture of Delsarte that there are no nontrivial perfect codes in $J(v,k)$. (See [8] for a discussion of this connection.)

In [9], Meyerowitz proved the following theorem, although the language used there differed from ours.

THEOREM 1.2 (Meyerowitz). *Suppose C is a completely regular design in $J(v,k)$ having strength zero. Then there is a subset $T \subseteq V$ such that either*

$$C = \{w : |w| = k, w \subseteq T\} \quad \text{or} \quad C = \{w : |w| = k, T \subseteq w\}.$$

In this paper, I classify the completely regular designs having strength one and minimum distance at least two.

Background on association schemes and the Johnson graph can be found in [1, 2]. The association scheme approach to t-designs is founded in [2, 3]. Material on equitable partitions can be found in [4]. A general discussion of completely regular designs is presented in [8]. This work is based on a portion of the author's Ph.D. thesis [7], completed at the University of Waterloo.

2. Groupwise complete designs

Next, we introduce the family of 1-designs which are the main focus of this paper.

Consider the Johnson graph $J(v,k)$ where $v = qp$ and $k = sp$ with $p \geq 2$ and $q \geq 2s$. Let

$$\mathcal{X} = \{X_1, X_2, \ldots, X_q\}$$

be a partition of $\mathcal{V}$ into q "groups," each of size p. The blocks of C will be the $\binom{q}{s}$ subsets of the form

$$z = \bigcup_{i \in I} X_i$$

where I is any s-element subset of $\{1, 2, \ldots, q\}$. Since $p \geq 2$ and $q > s$, such a design has strength one. It is also clear that C has minimum distance equal to p. Let us call such a design a *groupwise complete design*. We now determine for which values of p, q, and s such a 1-design is completely regular.

THEOREM 2.1. *Let C be a groupwise complete design in $J(v,k)$. Then C is completely regular if and only if one of the following holds*:

(i) $p = k$ *and* $v = 2k$ *and C is an antipodal class (containing two elements)*;
(ii) $p = 2$;
(iii) $p = 3$ *and* $s = 1$.

Proof. We first deal with the case $q = 2, s = 1$. Here the Johnson graph is $J(2k,k)$. This is an antipodal distance-regular graph and C is an antipodal class. It is known that such a subset is always completely regular [1, p. 349].

Otherwise, let C be a groupwise complete design in $J(v,k)$ with $v = qp$ and $k = sp$ ($p \geq 2, q \geq 2s, q > 2$). Let C be constructed as described above from a partition $\mathcal{X}$ of $\mathcal{V}$ into q groups of size p. Let $z = X_1 \cup X_2 \cup \cdots \cup X_s$ be an element of C. Since $q \geq s+2$, we can find X and X', two distinct groups disjoint from z. Let h and h' be points in X_1, j, k in X and k' in X'. Consider the sets $w := (z \backslash \{h, h'\}) \cup \{j, k\}$ and $w' := (z \backslash \{h, h'\}) \cup \{j, k'\}$, both at distance two from z. Vertex w is at distance $p-2$ from the block $(z \backslash X_1) \cup X$ while w' is at distance at least $p-1$ from every block other than z. If C is completely regular, by Delsarte's definition, w and w' must be in different cells of the distance

Table 1. Association schemes induced on design C and its opposite design C_{opp}.

p	q	s	*design*	*distance*	*induced subgraph*
2	$2s$	s	C_{opp}	1	$H(q,2)$
2	q	s	C	2	$J(q,s)$
p	2	1	C	k	K_2
3	3	1	C_{opp}	1	$H(3,3)$
3	q	1	C	k	K_q

partition of $J(v,k)$ with respect to C. This forces the minimum distance p to be at most three.

Next, we modify the above argument to deal with the case $p = 3$. Suppose $s \geq 2$ and let $z = X_1 \cup X_2 \cup \cdots \cup X_s$ be a block of the design. Let h and h' be elements of z, now in distinct groups. Let X and X' be groups disjoint from z and let j,k be elements of X and k' an element of X'. The sets $w := (z \backslash \{h,h'\}) \cup \{j,k\}$ and $w' := (z \backslash \{h,h'\}) \cup \{j,k'\}$ are both at distance two from C. Yet w is at distance two from three blocks while w' is at distance two from only one. Thus, such a design cannot be completely regular.

The only cases which remain are $p = 3, s = 1$, and $p = 2$ (any permissible value of s). It is not difficult to verify in both these cases that C is completely regular. □

The above class of 1-designs may seem somewhat trivial, but we will show that any completely regular 1-design having minimum distance at least two is of this type.

There are certainly other completely regular 1-designs. We refer to the collection of k-sets at maximal distance from a completely regular design as the *opposite design* and denote this by C_{opp}. The opposite design of a completely regular t-design is again completely regular of strength t [8]. For a completely regular groupwise complete design, C_{opp} has the form

$$\{w : |w| = k, |X_i \cap w| \leq \left\lceil \frac{k}{q} \right\rceil \text{ for each } i = 1,2,\ldots,q\}$$

except when $p = 2$ and $q < k$, in which case C_{opp} is described by

$$\{w : |w| = k, X_i \subseteq w \text{ for at most } k-q \text{ values of } i\}.$$

Remark. In some cases, the subset C in Theorem 2.1 induces a distance-regular subgraph of some (distance) graph of the Johnson scheme. These are described in Table 1.

Godsil and Praeger [6] discovered several other families of completely regular 1-designs. Let $v = qp$ and $\mathcal{X}$ a partition of $\mathcal{V}$ into q groups of size p as above. Then the following designs (and their opposites) are completely regular of strength one:

- For $p = 2$ and $k = 2s + 1$, we can set

  $$C := \{x : x \text{ contains } s \text{ groups of } \mathcal{X} \text{ and 1 point from one other group}\};$$

- For $q = 2$ and $p \geq k$ and for $k = 3$, $p, q \geq 3$, we can set $C := \{x : x \subseteq X_i$ for some $i\}$.

Also, by Theorem 1.1 and [2, Theorem 5.13], any 1-design with block size two is completely regular. These are the same as regular graphs and such 1-designs abound. Finally, we point out that, when $k = 1$, the entire vertex set of $J(v, 1)$ is a trivial 1-design. Henceforth, we assume $k \geq 2$ and $v \geq 2k$.

3. A representation approach

Let C be a completely regular design of strength $t = 1$ in $J(v, k)$. To study C, we represent C in a way in which its strength and complete regularity are evident. Since the number of elements of C containing a 2-set varies, we represent C as a function on pairs. Let W be the incidence matrix of 2-subsets versus k-subsets of $\mathcal{V}$. For each vertex x of $J(v, k)$ we map x to the xth column of W and we map C to the sum of the columns indexed by its elements. In this way, C becomes a nonconstant function on sets of size two. Moreover, as we will see, the inner product of the xth column of W and the image of C depends only on the distance from x to C.

A similar approach was used by Meyerowitz [9] to prove Theorem 1.2.

For a vertex x of $J(v, k)$, we define $u(x) := W\mathbf{e}_x$, which is the xth column of W. For a subset C, we denote by χ_C the characteristic vector of C and we map C to $u(C) := W\chi_C$. It is not difficult to see that, for vertices x and y of $J(v, k)$ we have inner product

$$\langle u(x), u(y)\rangle = \binom{|x \cap y|}{2};$$

(see [2, Equation 4.22]).

LEMMA 3.1. *Given a completely regular 1-design C in $J(v, k)$ and the function u as defined above, the inner product $\langle u(x), u(C)\rangle$ is a constant ω_h which only depends on the distance h from x to C. Moreover, $\omega_0 > \omega_1 > \cdots > \omega_\rho$ where*

$$\rho = \max_{x \in J(v,k)} dist(x, C).$$

Proof. Let $\pi = \{C_0, C_1, \ldots, C_\rho\}$ be the distance partition of $J(v,k)$ with respect to C. Then, for $x \in C_h$,

$$\langle u(x), u(C)\rangle = \sum_{i=0}^{k} R_{hi}\binom{k-i}{2}$$

where R_{hi} is the number of elements of C at distance i from x; this depends only on h and i. So the inner product is constant for $x \in C_h$. Denote this constant by ω_h.

Now ω_h is the xth entry in the vector $W^T W \chi_C$ for each $x \in C_h$. Delsarte shows that

$$W^T W = \sum_{j=0}^{2}\binom{k-j}{2-j}\binom{v-2-j}{k-2}E_j$$

(see [2, p. 47]). This gives

$$W^T W \chi_C = \binom{k}{2}\binom{v-2}{k-2}E_0\chi_C + \binom{v-4}{k-2}E_2\chi_C$$

using Theorem 1.1. Hence

$$\omega_h = \binom{k}{2}\binom{v-2}{k-2}\sum_{y\in C}(E_0)_{xy} + \binom{v-4}{k-2}\sum_{y\in C}(E_2)_{xy}.$$

Now E_0 is a multiple of the all-ones matrix. So it is sufficient to prove that the sequence $\zeta_0, \zeta_1, \ldots, \zeta_\rho$ given by

$$\zeta_h = \sum_{y\in C}(E_2)_{xy}$$

for $x \in C_h$, is strictly decreasing. Let Π be the $\binom{v}{k} \times (\rho + 1)$ matrix whose ith column is the characteristic vector of C_i. It is well known that, since π is equitable, there is a $(\rho+1)\times(\rho+1)$ matrix B satisfying $A_1\Pi = \Pi B$. Now it is not difficult to see that the vector $[\zeta_0, \zeta_1 \cdots \zeta_\rho]$ is a right eigenvector for B with eigenvalue θ_2 (cf. [4, Lemma 2.2]). Note that θ_2 is the second largest eigenvalue of B, a tridiagonal matrix. Now the fact that the sequence is strictly decreasing follows from known facts about Sturm sequences (cf. [4, 5] or [1, p. 129–130]). Accordingly, the sequence $\omega_0, \omega_1, \ldots, \omega_\rho$ is strictly decreasing as well. □

Now we can prove our main theorem.

THEOREM 3.1. *Suppose C is a completely regular 1-design in $J(v,k)$ having minimum distance at least two. Then C is a groupwise complete design.*

Proof. As above, we analyze the vector $u(C)$ which we view as an integral function on the 2-element subsets of $\mathcal{V}$. From the lemma, the distance partition of $J(v,k)$ with respect to C can be described by the equivalence relation

$$x \approx y \Leftrightarrow \langle u(x), u(C)\rangle = \langle u(y), u(C)\rangle.$$

For a 2-element subset $\{i,j\}$ of $\mathcal{V}$, define $\xi_{ij} = \xi_{ji}$ to be the entry of $u(C)$ corresponding to $\{i,j\}$. Note that ξ_{ij} is simply the number of blocks of C containing both i and j and so $\sum_{j\neq i} \xi_{ij} = (k-1)\lambda_1$. For convenience, define $\xi_{ii} = 0$ for all $i \in \mathcal{V}$. Then we define, for all k-subsets x,

$$\xi(x) := \sum_{i\in x}\sum_{j\in x} \xi_{ij} = 2\langle u(x), u(C)\rangle$$

and we can express partition π by the equivalence relation

$$x \approx y \Leftrightarrow \xi(x) = \xi(y).$$

If x is adjacent to y, say $y = (x\backslash\{h\}) \cup \{j\}$, then

$$\xi(y) = \xi(x) - 2\sum_{i\in x\backslash h} \xi_{hi} + 2\sum_{i\in x\backslash h} \xi_{ij}.$$

Let x be a vertex in $C = C_0$. Since C has minimum distance at least two, every neighbor of x is in cell C_1 of π. Thus, for $h \in x$ and $j \notin x$,

$$\sum_{i\in x\backslash h} (\xi_{ij} - \xi_{ih}) = \omega_1 - \omega_0 \tag{1}$$

in terms of the notation of the above lemma, and so the left-hand side is constant over all choices of $h \in x$ and all choices of $j \notin x$.

The proof is given in six steps.

Let x be a fixed block of the design C. For $v \in \mathcal{V}$, define $c_v := \sum_{i\in x} \xi_{iv}$.

(a) *For any fixed block x of the design, c_i is constant for $i \in x$ and c_j is constant for $j \notin x$.*

Let $j \notin x$ be fixed. Summing (1) over all $h \in x$, we get

$$(k-1)c_j - \xi(x) = k(\omega_1 - \omega_0),$$

which is independent of j. Then (1) can be written

$$c_j - c_h - \xi_{hj} = \omega_1 - \omega_0 \tag{2}$$

and this is used to complete the proof of the claim.

(b) *For any fixed block x of the design, there is a constant η such that $\xi_{ij} = \eta$ whenever $i \in x$ and $j \notin x$.*

This follows immediately from part (a) and (2).

(c) *The value of η is independent of the choice of x.*

For each block $x \in C$, denote by η_x the common value of ξ on pairs ij with $i \in x$ and $j \notin x$. For $h \in x$ fixed, we have

$$(k-1)\lambda_1 = \sum_{i \in \mathcal{V}} \xi_{hi} = c_h + (v-k)\eta_x.$$

and c_h is independent of $h \in x$. But,

$$\sum_{h \in x} c_h = \xi(x) = 2\omega_0$$

is independent of x. Consequently, η_x is also independent of the choice of $x \in C_0$. Denote this common value by η.

Let us define a partition $\mathcal{X}$ of $\mathcal{V}$ which is the coarsest partition simultaneously refining all the partitions $\{x, \mathcal{V} \backslash x\}$ for $x \in C$. Concretely, two points are in the same cell of $\mathcal{X}$ if and only if every block which contains one of them contains both.

(d) *The blocks of C are precisely those k-sets which can be expressed as a union of cells of partition $\mathcal{X}$.*
Clearly, every block of the design can be expressed as a union of cells of $\mathcal{X}$. The earlier steps of the proof confirm that $\xi_{ij} = \eta$ whenever i and j lie in different cells of $\mathcal{X}$. Moreover, from Lemma 3.1, C consists of precisely the k-sets z having $\xi(z) = 2\omega_0$. So, since the sum $\sum_{j \in \mathcal{V}} \xi_{ij}$ of the weights on the pairs containing a point i is constant, we see that every k-element subset which can be expressed as a union of cells of $\mathcal{X}$ is a block.

(e) *The function ξ is constant on pairs (i, j), where $i \neq j$ and i and j lie in the same cell of $\mathcal{X}$.*
Since every block that contains i also contains j, we have $\xi_{ij} = \lambda_1$ for all such pairs.

(f) *The cells of $\mathcal{X}$ have constant size.*
Suppose X is a cell of $\mathcal{X}$. We can sum the weights on the edges incident to a given point $h \in X$ to obtain

$$(|X|-1)\lambda_1 + (v - |X|)\eta = (k-1)\lambda_1.$$

This shows that the cell size $|X|$ must be constant over $\mathcal{X}$. The edge weights are equal to λ_1 within a cell and equal to η between cells. Hence C is a groupwise complete design. □

Finally, I remark that a more general form of Lemma 3.1 holds for completely regular designs of any strength, where we represent a t-design using the incidence matrix of $(t+1)$-sets versus k-sets. In fact, in a slightly different form, such a result can be proved for completely regular subsets in arbitrary distance-regular graphs.

Acknowledgments

I wish to thank A. Meyerowitz for discussions regarding his work. I am also grateful to the anonymous referees for their careful reading of the manuscript and for suggesting several improvements.

References

1. A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.
2. P. Delsarte, "An algebraic approach to the association schemes of coding theory," *Philips Res. Reports Suppl.* **10** (1973).
3. P. Delsarte, "Hahn polynomials, discrete harmonics, and *t*-designs," *SIAM J. Appl. Math.* **34** (1978), 157–166.
4. C.D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, London, 1993.
5. C.D. Godsil, "Bounding the diameter of distance regular graphs," *Combinatorica* **8** (1988), 333–343.
6. C.D. Godsil and C.E. Praeger, "On completely transitive designs," preprint.
7. W.J. Martin, "Completely regular subsets," Ph.D. thesis, University of Waterloo, 1992.
8. W.J. Martin, "Completely regular designs," In preparation.
9. A.D. Meyerowitz, "Cycle-balanced partitions in distance-regular graphs," preprint.
10. A. Neumaier, "Completely regular codes," *Discrete Math.* **106/107** (1992), 353–360.