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On Generators of the Module of Logarithmic 1-Forms with Poles Along an Arrangement

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Abstract. For each element X of codimension two of the intersection lattice of a hyperplane arrangement we define a differential logarithmic 1-forms ω_X with poles along the arrangement. Then we describe the class of arrangements for which forms ω_X generate the whole module of the logarithmic 1-forms with poles along the arrangement. The description is done in terms of linear relations among the functionals defining the hyperplanes. We construct a minimal free resolution of the module generated by ω_X that in particular defines the projective dimension of this module. In order to study relations among ω_X we construct free resolutions of certain ideals of a polynomial ring generated by products of linear forms. We give examples and discuss possible generalizations of the results.

Keywords: hyperplane arrangement, logarithmic form, module, free resolution, ideal

Introduction

Let $\mathcal{A}$ be an arrangement of n (linear) hyperplanes in an l-dimensional linear space V over an arbitrary field F (*an l-arrangement*). For each $H \in \mathcal{A}$ fix a linear form $\alpha_H \in V^*$ such that $\operatorname{Ker}\alpha_H = H$ and put $Q = \prod_{H\in\mathcal{A}} \alpha_H$. Denote by S the symmetric algebra of V^* that is naturally isomorphic to $F[x_1, \dots, x_l]$ for any choice of basis $(x_1, \dots, x_l)$ of V^*. Denote by $\Omega^p[V]$ the S-module of all differential p-forms on V with coefficients from S. Of course $\Omega^p[V]$ is free with the natural basis $(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p})_{1\le i_1<i_2<\cdots i_p\le l}$. Also denote by $\Omega^p(V)$ the S-module of differential p-forms with coefficients from the field (S) of quotients of S, i.e., $\Omega^p(V) = \Omega^p[V] \otimes_S (S)$. Clearly all the modules $\Omega^p(V)$ can be graded by the degrees of coefficients at elements of the natural bases.

The modules of *logarithmic differential forms* with poles along a divisor were defined by Deligne [2] for a divisor with normal crossings and by Saito [9] for an arbitrary divisor. The following specialization of these modules to the union of hyperplanes (and to the category of S-modules) became the subject of extensive studies in arrangement theory. Put $\Omega^p(\mathcal{A}) = \{\omega \in \Omega^p(V) \mid Q\omega \in \Omega^p[V],\ Qd\omega \in \Omega^{p+1}[V]\}$ and call it the S-module of logarithmic p-forms *with poles along* $\mathcal{A}$. Under the condition $\eta = Q\omega \in \Omega^p[V]$ the condition $Qd\omega \in \Omega^{p+1}[V]$ is equivalent to $d\alpha_H \wedge \eta \in \alpha_H\Omega^{p+1}[V]$ for every $H \in \mathcal{A}$.

The structure of modules $\Omega^p(\mathcal{A})$ is known to certain extent for some classes of arrangements. For generic arrangements (i.e., in the case where every l forms α_H are linearly independent) certain free resolutions of these modules were constructed by Rose and Terao in [8]. If $\mathcal{A}$ is just r-generic ($3 \le r \le l-1$), i.e., every r forms α_H are linear independent, then Ziegler [14] announced that every module $\Omega^p(\mathcal{A})$ with $p \le r-2$ was generated by the exterior products of the forms $\frac{d\alpha_H}{\alpha_H}$. He proved this for $r = 3$ and even constructed a resolution of the module $\Omega^1(\mathcal{A})$. For $r > 3$ a complete proof was given by Lee [5]. Another class of arrangements that has been studied is the class of free arrangements, i.e., arrangements $\mathcal{A}$ for which $\Omega^1(\mathcal{A})$ is a free S-module. Then every module $\Omega^p(\mathcal{A})$ is free that is equivalent to the fact that this module is generated by $\binom{l}{p}$ elements. Explicit bases for $\Omega^{l-1}(\mathcal{A})$ have been found in [3] for certain subclasses of free arrangements.

Let us focus our attention now on $\Omega^1(\mathcal{A})$, the module that often defines the structure of all modules $\Omega^p(\mathcal{A})$. The forms $\frac{d\alpha_H}{\alpha_H}$ belong to this module and according to [14] generate the whole module if and only if $\mathcal{A}$ is 3-generic. Introducing the intersection lattice $L = L(\mathcal{A})$, i.e., the set of all intersections of hyperplanes from $\mathcal{A}$ ordered opposite to inclusion, we can view $\frac{d\alpha_H}{\alpha_H}$ as the forms corresponding to the atoms of L. For $\mathcal{A}$ that is not 3-generic it is natural to look for forms in $\Omega^1(\mathcal{A})$ that correspond to the elements X of L of codimension 2, i.e., the intersections of pairs of hyperplanes from $\mathcal{A}$. Then the natural question arises for what arrangements these forms generate the whole module.

In this paper, for each $X \in L$ of codimension 2 we define a unique (up to a constant) form $\omega_X \in \Omega^1(\mathcal{A})$ and describe the class of arrangements for which these forms together with the form $\frac{dx_1}{x_1}$ generate $\Omega^1(\mathcal{A})$. This class is given by the condition that for any subset $B \subset \{\alpha_H \mid H \in \mathcal{A}\}$ of rank 3 all linear relations of length 3 among $\alpha_H \in B$ are linearly independent (see Theorem 3.1). Naturally this class includes all the 3-generic arrangements since they do not have linear relations of length 3 at all. This class can be viewed as the collection of general position central arrangements using a weaker definition of the general position than the usual one. We also find a minimal free resolution of the module $\Omega^1(\mathcal{A})$ for $\mathcal{A}$ from this class. In fact we construct a minimal free resolution of the module generated by the forms ω_X for any arrangement (see Theorem 2.9). In order to find all the relations among these forms we need to resolve certain ideals of S generated by products of linear forms. This is done in Section 1 (see Theorem 1.3). In Section 4, we give examples and suggest possible generalizations of the main results of the paper.

Besides the notation introduce above we will use the following. The lattice L is ranked by codimension of its elements in V. Denote the rank of L by m. Clearly m is the dimension of the subspace W of V^* generated by all α_H whence $m \leq l$. If $m = l$ (equivalently $\bigcap_{H\in\mathcal{A}} H = 0$) then $\mathcal{A}$ is called *essential*. If $\mathcal{A}$ is not essential we will always assume that a basis $(x_1, \dots, x_l)$ in V^* is chosen so that $x_1, \dots, x_m$ are among α_H and thus form a basis in W. The forms α_H define an essential m-arrangement $\mathcal{A}_1$ in W^*. It is easy to compute (e.g., cf. [10]) that $\Omega^1(\mathcal{A}) = (\Omega^1(\mathcal{A}_1)\otimes_F F[x_{m+1}, \dots, x_l])\oplus_S(F[x_1, \dots, x_m]\otimes_F \sum_{i=m+1}^{l} S dx_i)$.

For each i, $0 \leq i \leq m$, we put $L(i) = \{X \in L \mid \operatorname{rank} X = i\}$. Finally, for each $X \in L$, we put $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supset X\}$, $Q_X = \prod_{H\in\mathcal{A}_X} \alpha_H$, and $\pi_X = \frac{Q}{Q_X}$.

1. Polynomial ideals generated by products of linear forms

Let $\mathcal{A}$ be an l-arrangement. With every nonempty $A \subset L$ we associate the (homogeneous) ideal $J(A)$ of S generated by $\{\pi_X \mid X \in A\}$. Clearly $J(A)$ does not change if one substitutes for A the subset of all maximal elements of A. Thus without loss of generality we can assume that all elements of A are pairwise incomparable. The goal of this section is to exhibit a minimal free resolution of the graded S-module $J(A)$.

Denote by $K = (0 \to K_r \to \cdots \to K_i \xrightarrow{d_i} K_{i-1} \to \cdots \to K_0 \to K_{-1} = F \to 0)$ the chain complex over F of the simplex with the set of vertices A. In particular K_i has the basis consisting of all the subsets of A with $i+1$ elements and d_i is given in this basis by a matrix $(d_{\sigma,\tau})$ where $\sigma, \tau \subset A$, $|\sigma| = i$, $|\tau| = i+1$. The entry $d_{\sigma,\tau}$ may be non-zero only if $\sigma \subset \tau$.

We will also use other complexes over F defined by A. For every $Z \in L$ put $A_Z = \{X \in A \mid X \geq Z\}$ and fix $Y \in L$, $Y \neq V$. Then for every $Z \leq Y$ define two subcomplexes of K: $K_{\geq Z}$ whose linear spaces $K_{\geq Z,i}$ are generated by subsets of A_Z and $K_{>Z}$ whose linear spaces $K_{>Z,i}$ are generated by subsets σ of A_Z with the extra condition $\bigwedge_\sigma X \wedge Y > Z$. Finally put $K_Z = K_{\geq Z}/K_{>Z}$.

Lemma 1.1 *For every $Z \in L$ $(Z \leq Y)$ the complex K_Z is exact in any dimension i such that $i \geq$ rank Y.*

Proof: First notice that if $A_Z = \emptyset$ then $K_{\geq Z,i} = 0$ for every $i \geq 0$ and the result follows. Thus it suffices to consider the case where $A_Z \neq \emptyset$ whence $K_{\geq Z}$ is the complex of a (non-empty) simplex and thus exact. Denote by $Z_1, Z_2, \ldots, Z_k$ all the successors of Z such that $Z_i \leq Y$, $i = 1, \ldots, k$, and $A_{Z_i} \neq \emptyset$. Clearly $K_{>Z} = \sum_{i=1}^k K_{\geq Z_i}$ (recall that all the complexes under consideration are subcomplexes of K so it is possible to add them). Again it suffices to consider the general case where $k > 0$ since otherwise the result is immediate.

In the general case we need to study $K_{>Z}$. For that we define two posets. The poset P_1 consists of all ordered by inclusion nonempty subsets σ of A such that $\bigwedge_\sigma X \geq Z_i$ for some i. The poset P_2 is the subposet of L^{op} defined by $P_2 = \{U \in \bigcup_i [Y, Z_i] \mid A_U \neq \emptyset\}$. Define the order preserving map $\phi\colon P_1 \to P_2$ via $\phi(\sigma) = \bigwedge_\sigma X \wedge Y$. For every $Z \in P_2$ we have $\phi^{-1}(\{U \in P_2 \mid U \leq Z\}) = \{\sigma \in P_1 \mid \bigwedge_\sigma X \geq Z\}$ that is the poset with the unique maximal element A_Z and thus contractible. Thus by [7] ϕ is a homotopy equivalence. Now consider two cases. If $Y \in P_2$ (i.e., $A_Y \neq \emptyset$) then Y is the greatest element of P_2 whence P_2 is contractible. Then P_1 is contractible also.

Suppose $Y \notin P_2$. Then $H_i(P_1) = 0$ for $i \geq$ rank $Y - 1$ since the length of any linearly ordered set in P_2 is smaller than rank$Y$. In any case since $K_{>Z}$ is homotopy equivalent to the order complex of P_1 we have $H_i(K_{>Z}) = 0$ for $i \geq$ rank$Y - 1$. Applying the homology long exact sequence corresponding to the short exact sequence

$$0 \to K_{>Z} \to K_{\geq Z} \to K_Z \to 0$$

we obtain the result.

Now we define a sequence of S-modules and their homomorphisms. Put $\tilde{K}_i = K_i \otimes_F S$, $i \geq 0$, and define the S-linear map $\tilde{d}_i\colon \tilde{K}_i \to \tilde{K}_{i-1}$ by the following matrix (with respect to the standard bases of K_j)

$$d_{\sigma,\tau}\left(\prod_{H \in \mathcal{A}_\sigma \setminus \mathcal{A}_\tau} \alpha_H\right)$$

where $\mathcal{A}_\nu = \bigcap_{X \in \nu} \mathcal{A}_X$ for every $\nu \subset L$. Also put $\tilde{K}_{-1} = J(A)$ and define the S-linear map $\tilde{d}_0\colon \tilde{K}_0 \to \tilde{K}_{-1}$ by $\tilde{d}_0(X) = \pi_X$ for every $X \in A$. □

The following lemma is straightforward.

Lemma 1.2 *The sequence $\tilde{K} = (\tilde{K}_i, \tilde{d}_i)$ is a complex and $\tilde{d}_0$ is surjective.*

Now we are ready to prove the main result of the section.

Theorem 1.3 *For every nonempty subset A of L the complex $\tilde{K}$ is exact, i.e., it is a free resolution of the S-module $J(A)$.*

Proof: We will prove the result by checking the conditions (a)–(c) of [6, Sect. 6.4, Theorem 15]. For that we use the evaluation of $\tilde{K}$ at points of V (cf. [12, p. 437]). For every $x \in V$ define the complex $K(x)$ of linear spaces putting $K(x)_i = K_i$ for $i \geq -1$ and defining the differential $d_i(x)$ by evaluating at x the matrix of $\tilde{d}_i$. Now we consider two cases.

(1) Let x be in general position with respect to $\mathcal{A}$, i.e., $\alpha_H(x) \neq 0$ for every $H \in \mathcal{A}$. Define for each $\sigma \subset L$ the subarrangement $\mathcal{A}_\sigma = \bigcap_{X \in \sigma} \mathcal{A}_X$ and the polynomial $Q_\sigma = \prod_{H \in \mathcal{A}_\sigma} \alpha_H$. Then the matrix for $d_i(x)$ can be obtained from the matrix for d_i by multiplying the σ-row by $Q_\sigma(x)$ and the τ-column by $\frac{1}{Q_\tau(x)}$. Since all the polynomials Q_σ do not vanish at x the multiplication do not change the rank of the matrices. Thus the exactness of K implies the exactness of $K(x)$. Since besides the evaluation at x does not increase the rank of a matrix we have

$$\operatorname{rank}_F K_r = \operatorname{rank}_S \tilde{K}_r \geq \operatorname{rank}_S \tilde{d}_r \geq \operatorname{rank}_F d_r(x) = \operatorname{rank}_F K_r$$

whence

$$\operatorname{rank}_S \tilde{d}_r = \operatorname{rank}_S \tilde{K}_r.$$

In a similar way, using induction, one can show that

$$\operatorname{rank}_S \tilde{d}_{i+1} + \operatorname{rank}_S \tilde{d}_i = \operatorname{rank}_S \tilde{K}_i$$

for every i, $0 \leq i < r$. These are the conditions (b) and (c) from [6].

(2) Now let $x \in V$ be such that $\mathcal{A}_x = \{H \in \mathcal{A} \mid \alpha_H(x) = 0\} \neq \emptyset$. Then put $Y = \bigcap_{H \in \mathcal{A}_x} H$ and notice that $\mathcal{A}_Y = \mathcal{A}_x$. In particular $Y \neq V$. Now the evaluation at x annihilates some entries of the matrices of $\tilde{d}_i$. More precisely $\tilde{d}_{\sigma,\tau}(x) = 0$ if and only if $\bigwedge_\sigma X \wedge Y \neq \bigwedge_\tau X \wedge Y$. This means that $K(x) = \oplus_Z K_Z(x)$ where the subcomplex $K_Z(x)$ of $K(x)$ is generated by $\{\sigma \subset A \mid \bigwedge_\sigma X \wedge Y = Z\}$. Now we use Lemma 1.1. The matrix of a differential of complex $K_Z(x)$ can be obtained from the matrix of the respective differential of complex K_Z by multiplying its rows and columns by the same factors as in the case (1). Because of the restrictions on the generators σ and τ these factors are again non-zero whence the multiplication preserves the ranks of the differentials. According to Lemma 1.1 the complex K_Z is exact in dimension greater than or equal to $l - \dim Y$. This implies that for $k \geq \operatorname{codim} Y$ at least one minor of size rank d_k of the matrix of d_k is not annihilated at x or, in other words, x does not belong to the variety of the Fitting ideal I_k of d_k. Thus the variety of I_k lies in the union of elements of L of codimension greater than k. Extending F to an algebraically closed field and applying the Hilbert Nullstellensatz one sees that any prime ideal containing I_k contains at least $k + 1$ linearly independent forms α_H $(H \in \mathcal{A})$. Thus depth$I_k \geq k + 1$ which is the condition (a) of [6]. This completes the proof of the theorem. □

Corollary 1.4 *Let $\sum_{X\in L(2)} p_X\pi_X = 0$ for some $p_X \in S$. Then for every X we have $p_X \in \sum_{H\supset X} S\frac{Q_X}{\alpha_H}$. If besides for every X with $p_X \neq 0$ we have $H_1 \supset X$ then $p_X \in S\frac{Q_X}{\alpha_1}$.*

Proof: Let $A = L(2)$ and consider the resolution $\tilde{K}$ of $J(A)$. Since $(p_X)_X \in \operatorname{Ker}\tilde{d}_0 = \operatorname{Im}\tilde{d}_1$ we have $\sum_{X\in A} p_X X = \sum_{\{X,Y\}\subset A} \frac{c_{XY}}{\alpha_{XY}}(Q_X X - Q_Y Y)$ for some $c_{XY} \in S$ where $\alpha_{XY} = \alpha_H$ if $\mathcal{A}_X \cap \mathcal{A}_Y = \{H\}$ and $\alpha_{XY} = 1$ if $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$. Comparing the coefficients of X we obtain the result. □

Remark 1.5 Since ideal $J = J(A)$ is homogeneous one can consider its Hilbert series $P(J,t)$ or the polynomial $p(J,t)$ where $P(J,t) = \frac{p(J,t)}{(1-t)^l}$. Then Theorem 1.3 gives

$$p(J,t) = \sum_{k=0}^{n-1} \chi_k t^{n-k}$$

where $\chi_k = \sum_{i\geq 0}(-1)^i d_i(k)$ with $d_i(k) = |\{\sigma \subset A \mid |\sigma| = i+1, |\mathcal{A}_\sigma| = k\}|$.

2. A free resolution of a module of logarithmic forms

In this section we define certain canonical logarithmic 1-forms with poles along $\mathcal{A}$ and construct a minimal free resolution of the module generated by these forms.

To make the notation simplier let us agree that any time when we use a lower or an upper index for $H \in \mathcal{A}$ we use the same index for α_H. For instance, we linearly order $\mathcal{A}$ and use α_i for α_{H_i}. We will always assume that $\alpha_i = x_i$ for $i = 1, \dots, m$. For every $X \in L(2)$ we denote by H_1^X, H_2^X the first two elements from $\mathcal{A}_X$ in this ordering.

Recall that $\Omega^1 = \Omega^1(\mathcal{A})$ is the S-module of all logarithmic 1-forms with poles along $\mathcal{A}$. For each $X \in L(2)$ we define the form ω_X by

$$\omega_X = \frac{1}{Q_X}\left(\alpha_1^X d\alpha_2^X - \alpha_2^X d\alpha_1^X\right).$$

One checks easily that $\omega_X \in \Omega^1$. Also if one changes the ordering, i.e., uses other α_1', α_2' from $\mathcal{A}_X$ in the definition of ω_X, then ω_X is multiplied by the determinant of the transition matrix from the basis (α_1^X, α_2^X) to the basis (α_1', α_2') of $\operatorname{Ann}X \subset V^*$. In particular the S-module $\Omega^1\langle\mathcal{A}\rangle$ generated by all the ω_X ($X \in L(2)$) does not depend on the ordering of $\mathcal{A}$. Since each ω_X is homogeneous in the natural grading of $\Omega^1(V)$ the module $\Omega^1\langle\mathcal{A}\rangle$ has the structure of a graded S-module.

In the rest of the section the elements X of L with $X \subset H_1$ will play a special part. Every $H \in \mathcal{A}$, $H \neq H_1$ defines $X_H = H_1 \cap H \in L(2)$ (of course it is possible that $X_{H_i} = X_{H_j}$ for $i \neq j$). Then we have $\alpha_1^{X_H} = x_1$ and $\alpha_H = t_H x_1 + s_H \alpha_2^{X_H}$. To simplify computations we will normalize every α_H ($H \neq H_1$) by the condition $s_H = 1$ and by virtue of this assume from now on that $\alpha_H = t_H x_1 + \alpha_2^{X_H}$. Putting $Q_H = Q_{X_H}$ we have $\omega_{X_H} = \frac{1}{Q_H}(x_1 d\alpha_{H'} - \alpha_{H'} dx_1)$ for any $H' \in \mathcal{A}_{X_H}$, $H' \neq H_1$.

Now denote by E_0 the free S-module with basis $L(2)$ and by δ_0 the S-linear surjective map $E_0 \to \Omega^1\langle\mathcal{A}\rangle$ sending X to ω_X. First we study the kernel of δ_0.

Denote by $C = (0 \to C_k \to C_{k-1} \to \cdots \to C_0 \to 0)$ the chain complex over F of the atomic complex of the subposet $L(1) \cup L(2)$ of L. In particular C_i is the linear space spanned by all $i+1$-element subsets of $\mathcal{A}$, each lying in $\mathcal{A}_X$ for some $X \in L(2)$. Let $z = \sum_{1 \le i < j \le n} c_{ij}\{H_i, H_j\}$ be a 1-cycle in C. Every 2-element set $\{H_i, H_j\}$ defines $X_{ij} = H_i \cap H_j \in L(2)$. We define $r_1(z) \in E_0$ by

$$r_1(z) = \sum_{i,j} c_{ij} \Delta_{ij} \frac{Q_{ij}}{\alpha_i \alpha_j} X_{ij}$$

where $Q_{ij} = Q_{X_{ij}}$ and Δ_{ij} is the determinant of the transition matrix from (α_1^X, α_2^X) to (α_i, α_j) for $X = X_{ij}$.

Lemma 2.1

(i) *The map $z \mapsto r_1(z)$ is F-linear.*
(ii) *The element $r_1(z)$ depends only on the homology class of z.*
(iii) *For every 1-cycle z we have $r_1(z) \in \operatorname{Ker}\delta_0$.*

Proof:

(i) is clear.

(ii) Due to (i) it suffices to show that $r_1(z) = 0$ for z being the boundary of a basic element u of C_2. Suppose $u = \{H_i, H_j, H_k\} \subset X$ $(X \in L(2))$ where $i < j < k$ and $\alpha_k = a\alpha_i + b\alpha_j$ with $a, b \in F$. Then $r_1(z) = (\frac{Q_X}{\alpha_i\alpha_j} - a\frac{Q_X}{\alpha_i\alpha_k} - b\frac{Q_X}{\alpha_i\alpha_k})X = \frac{Q_X}{\alpha_i\alpha_j\alpha_k}(\alpha_k - a\alpha_i - b\alpha_j)X = 0$.

(iii) For arbitrary 1-cycle $z = \sum_{i<j} c_{ij}\{H_i, H_j\}$ we have

$$\begin{aligned}\delta_0 r_1(z) &= \sum_{i,j} c_{ij}\Delta_{ij}\frac{Q_{ij}}{\alpha_i\alpha_j}\omega_{X_{i,j}} = \sum_{i,j} c_{ij}\left(\frac{d\alpha_j}{\alpha_j} - \frac{d\alpha_i}{\alpha_i}\right) \\ &= \sum_i \left(\sum_{j<i} c_{ij} - \sum_{j>i} c_{ji}\right)\frac{d\alpha_i}{\alpha_i} = 0\end{aligned}$$

because z is a cycle.

From now on for any $z \in H_1(C)$ we put $r_1(z) = r_1(\bar{z})$ where $\bar{z}$ is an arbitrary cycle from the class z.

Now let $\lambda\colon C_0 \to V^*$ be the linear map sending $H \in \mathcal{A}$ to α_H and $R = R(\mathcal{A}) = \operatorname{Ker}\lambda$, i.e., R is the space of all F-linear relations among α_H. Also let R_0 be the subspace of R generated by all the relations of length 3 that include x_1. According to the convention above each of these relations is a scalar multiple of

$$x_1 + c\alpha_i - c\alpha_j = 0 \tag{2.1}$$

for some non-zero $c \in F$.

For every $p = \sum_{H \in \mathcal{A}} a_H H \in R$ we put

$$r_2(p) = \sum_{H \ne H_1} a_H \frac{Q_H}{x_1} X_H \in E_0$$

(recall that $X_H = H \cap H_1$ and $Q_H = Q_{X_H}$). □

Lemma 2.2

(i) *The map* $r_2: R \to E_0$ *is* F*-linear.*
(ii) $r_2(R) \subset \operatorname{Ker} \delta_0$.
(iii) $\operatorname{Ker} r_2 = R_0$.

Proof:

(i) is clear.
(ii) For every $p = \sum_{H \in \mathcal{A}} a_H H \in R$ we have $\sum_{H \neq H_1} a_H \alpha_H = -a_{H_1} x_1$ whence

$$\begin{aligned} \delta_0 r_2(p) &= \sum_{H \neq H_1} a_H \frac{Q_H}{x_1} \omega_{X_H} \\ &= -\sum a_H \alpha_H \frac{dx_1}{x_1} + \sum a_H d\alpha_H \\ &= a_1 x_1 \frac{dx_1}{x_1} - d(a_1 x_1) = 0. \end{aligned}$$

(iii) One computes easily that $r_2(p) = 0$ for every p of the form (2.1) whence $R_0 \subset \operatorname{Ker} r_2$. Conversely let $p = \sum_{H \in \mathcal{A}} a_H H \in \operatorname{Ker} r_2$, i.e, $\sum_{H \neq H_1} a_H \frac{Q_H}{x_1} X_H = 0$. Thus for every $X \in L(2)$ such that $X \subset H_1$ we have

$$\sum_{H \supset X, H \neq H_1} a_H = 0. \tag{2.2}$$

On the other hand

$$\sum_{H \supset X} a_H \alpha_H = 0 \tag{2.3}$$

since $p \in R$. Put $H_X = H_2^X$ and recall that $\alpha_H = t_H x_1 + \alpha_X$ for every $X \subset H_1$ and $H \supset X$, $H \neq H_1$. Thus projecting (2.3) to x_1 we obtain

$$\sum_{H \supset X} a_H t_H = 0. \tag{2.4}$$

Now we can put $p_H = H - t_H H_1 - H_{X_H} \in R_{X_H}$ for every $H \neq H_1, H_{X_H}$ and compute

$$\begin{aligned} p - \sum_{H \neq H_1, H_{X_H}} a_H p_H &= a_{H_1} H_1 + \sum_{X \subset H_1} a_{H_X} H_X + \sum_{H \neq H_1, H_{X_H}} a_H t_H H_1 \\ &\quad + \sum_{X \subset H_1} \left(\sum_{H \neq H_1, H_{X_H}} a_H \right) H_X = 0 \end{aligned}$$

by (2.2) and (2.4). Thus $p = \sum a_H p_H \in R_0$ which completes the proof. □

Lemmas 2.1 and 2.2 give two F-linear subspaces of $\operatorname{Ker} \delta_0$, namely $K_1 = r_1(H_1(C))$ and $K_2 = r_2(R)$. Their importance can be shown as follows.

Theorem 2.3 *The sets* K_1 *and* K_2 *generate the whole* S*-module* $\operatorname{Ker} \delta_0$.

Proof: Let $q = \sum_{X \in L(2)} c_X X \in \operatorname{Ker}\delta_0$ for some $c_X \in S$. This means that $\sum_X c_X \omega_X = 0$. Multiplying by Q we have $\sum_X c_X \pi_X \beta_i(X) = 0$ for every $i = 1, \dots, l$ where $\beta_i(X) = a_{2i}^X \alpha_1^X - a_{1i}^X \alpha_2^X$ with $\alpha_j^X = \sum_{i=1}^l a_{ji}^X x_i$ for every $j = 1, 2$. Due to Corollary 1.4 we have $c_X \beta_i(X) \in \sum_{H \in \mathcal{A}_X} S \frac{Q_X}{\alpha_H}$. Since two of $\beta_1(X), \dots, \beta_l(X)$ are linearly independent and thus α_1^X can be expressed as their linear combination we have

$$c_X \alpha_1^X = \sum_{H \in \mathcal{A}_X} q_H \frac{Q_X}{\alpha_H} \tag{2.5}$$

for some $q_H \in S$. Clearly (2.5) implies that α_1^X divides $q_{H_1^X}$ whence upon canceling α_1^X from (2.5) we have

$$c_X \in \sum S \frac{Q_X}{\alpha_{H_i} \alpha_{H_j}} \tag{2.6}$$

where summation is taken over all 2-subsets $\{H_i, H_j\}$ of $\mathcal{A}_X$. Now for every $X \in L(2)$ such that $X \not\subset H_1$ and for every $H_i, H_j \supset X$ denote by z_{ij} the 1-cycle $\{H_1, H_i\} - \{H_1, H_j\} - \{H_i, H_j\}$ of C. Due to (2.6) there exists a linear combination $q_1 = \sum s_{ij} r_1(z_{ij}) \in K_1$ ($s_{ij} \in S$) such that in the representation $q - q_1 = \sum_X d_X X$ we have $d_X \neq 0$ only if $X \subset H_1$. Using that $q - q_1 \in \operatorname{Ker}\delta_0$ we have similarly to the above $\sum_X d_X \pi_X x_1 = 0$ and upon canceling x_1 we obtain $\sum_X d_X \pi_X = 0$. Applying now the second part of Corollary 1.4 we have

$$d_X \in S \frac{Q_X}{x_1} \tag{2.7}$$

for every X. Recall that $(x_1, x_2, \dots, x_m)$ is a maximal linear independent system in $\mathcal{A}$ and $x_i = \alpha_i$ ($i = 1, \dots, m$). For each X put $p_X = H_2^X - \sum_{i=1}^m a_i(X) H_i \in R$ for some $a_i(X) \in F$. Then due to (2.7) there exists a linear combination $q_2 = \sum_X s_X r_2(p_X) \in K_2$ ($s_X \in S$) such that in the representation $q - q_1 - q_2 = \sum_X e_X X$ we have $e_X \neq 0$ only if $x_i \in \mathcal{A}_X$ for some i, $2 \le i \le m$. For such an X we have $\omega_X = f_X(\frac{dx_i}{x_i} - \frac{dx_1}{x_1})$ for some rational function f_X. Since dx_i are linearly independent over S we have $q - q_1 - q_2 = 0$ whence $q \in K_1 + K_2$. This completes the proof. □

Corollary 2.4 *Put $E_1 = (H_1(C) \oplus R) \otimes_F S$ and define the S-linear map $\delta_1: E_1 \to E_0$ via $\delta_1(z) = r_1(z)$ for every $z \in H_1(C)$ and $\delta_1(p) = r_2(p)$ for every $p \in R$. Then*

$$E_1 \xrightarrow{\delta_1} E_0 \xrightarrow{\delta_0} \Omega^1\langle \mathcal{A} \rangle \to 0$$

is exact.

The result follows directly from Theorem 2.3.

Our next goal is to study $\operatorname{Ker}\delta_1$. First of all it is convenient to choose a specific basis of $H_1(C)$. For every $X \in L(2)$ such that $X \not\subset H_1$ and for every $H \supset X$, $H \neq H_1^X$, put $\bar{z}(X, H) = \{H_1, H_1^X\} - \{H_1, H\} - \{H_1^X, H\}$. Clearly $\bar{z}(X, H)$ is a 1-cycle of C. Denote by $z(X, H)$ the respective homology class.

Lemma 2.5 *The classes $z(X, H)$ form a basis of $H_1(C)$.*

Proof: First of all let us compute dim $H_1(C)$. One of many ways to do this is to use the Euler characteristic of the graph $L(1) \cup L(2)$ that coincide with the Euler characteristic of C. We have $\dim H_1(C) = \sum_{X \in L(2)} (m_X - 1) - (n-1) = \sum_{X \in L(2), X \not\subset H_1} (m_X - 1)$ where $m_X = |\mathcal{A}_X|$. Since on the other hand this is the number of classes $z(X, H)$ it suffices to prove that they generate $H_1(C)$. But this follows easily from the fact that each 2-subset of $\mathcal{A}$ belongs to C_1.

Now we put $R_X = R(\mathcal{A}_X)$ for every $X \in L(2)$ and notice that $R_X \subset R$. If for every $H \in \mathcal{A}_X$ we write

$$\alpha_H = a_{XH}\alpha_1^X + b_{XH}\alpha_2^X$$

with $a_{XH}, b_{XH} \in F$ then the elements $H - a_{XH}H_1^X - b_{XH}H_2^X$ with $H \in \mathcal{A}_X \setminus \{H_1^X, H_2^X\}$ form a basis in R_X.

We are going to construct an F-linear map $\rho: \oplus_X R_X \to \operatorname{Ker}\delta_1$. Fix $X \in L(2)$ and $t_X = \sum_{H \supset X} t_{XH} H \in R_X \subset R$ ($t_{XH} \in F$). If $X \subset H_1$ put $\rho(t_X) = t_X$. If $X \not\subset H_1$ put

$$\rho(t_X) = -\sum_{H \supset X, H \neq H_1^X} t_{XH}\alpha_H z(X, H) + t_X.$$

In any case $\rho(t_X)$ can be viewed as an element of E_1. □

Lemma 2.6

(i) *The map $\rho: \oplus_X R_X \to E_1$ given by $(t_X)_X \mapsto \sum_X \rho(t_X)$ is F-linear.*
(ii) $\rho(\oplus_X R_X) \subset \operatorname{Ker}\delta_1$.

Proof: (i) is obvious. (ii) Due to (i) it suffices to consider the case where $t = H - a_{XH}H_1^X - b_{XH}H_2^X$ for some X and $H \supset X, H \neq H_i^X, i = 1, 2$. If $X \subset H_1$ then $\delta_1\rho(t) = r_2(t) = 0$ by Lemma 2.2. (iii) Suppose $X \not\subset H_1$. In this case

$$\begin{aligned}\delta_1\rho(t) = &-\alpha_H\left(\frac{Q_{X'}}{x_1\alpha_1^X}X' - \frac{Q_{X_H}}{x_1\alpha_H}X_H - b_{XH}\frac{Q_X}{\alpha_1^X\alpha_H}X\right)\\ &+ b_{XH}\alpha_2^X\left(\frac{Q_{X'}}{x_1\alpha_1^X}X' - \frac{Q_{X''}}{x_1\alpha_2^X}X'' - \frac{Q_X}{\alpha_1^X\alpha_2^X}X\right)\\ &+\left(\frac{Q_{X_H}}{x_1}X_H - a_{XH}\frac{Q_{X'}}{x_1}X' - b_{XH}\frac{Q_{X''}}{x_1}X''\right)\end{aligned} \tag{2.8}$$

where $X' = H_1 \cap H_1^X$, $X'' = H_1 \cap H_2^X$, and $X_H = H_1 \cap H$. One can easily check that all the terms in (2.8) cancel out which proves the result. □

Theorem 2.7 *The S-module* $\operatorname{Ker}\delta_1$ *is generated by* $\rho(\oplus_X R_X)$.

Proof: Fix a basis $(p_1, \ldots, p_k)$ of R and fix $s \in E_1$, i.e.,

$$s = \sum_{X \not\subset H_1, H \supset X, H \neq H_1^X} s(X, H)z(X, H) + \sum_{i=1}^{k} s_i p_i$$

for some $s(X, H), s_i \in S$. Assume that $s \in \operatorname{Ker}\delta_1$. Applying δ_1 to s and considering the coefficients of X we obtain for each $X \not\subset H_1$

$$\sum_{H\supset X,\ H\neq H_1^X} s(X, H)\frac{Q_X}{\alpha_1^X\alpha_H}b_{XH} = 0. \tag{2.9}$$

The equality (2.9) implies that $s(X, H) = s'(X, H)\alpha_H$ for some $s'(X, H) \in S$ such that

$$\sum_{H\supset X} b_{XH}s'(X, H) = 0. \tag{2.10}$$

Now for each $X \not\subset H_1$ we put $s'(X, H_1^X) = -\sum_{H\supset X, H\neq H_1^X} s'(X, H)a_{XH}$ to achieve

$$\sum_{H\supset X} a_{XH}s'(X, H) = 0. \tag{2.11}$$

If one fixes an F-basis B in S (e.g., consisting of monomials) and represent $s'(X, H) = \sum_{\beta\in B} s'_\beta(X, H)\beta$ then (2.10) and (2.11) imply

$$\sum_{H\supset X} s'_\beta(X, H)\alpha_H = \left(\sum s'_\beta(X, H)a_{XH}\right)\alpha_1^X + \left(\sum s'_\beta(X, H)b_{XH}\right)\alpha_2^X = 0 \tag{2.12}$$

for each $\beta \in B$. The equality (2.12) means that $s'_\beta(X) = \sum_{H\supset X} s'_\beta(X, H)H$ belongs to R_X.

Now consider $s_1 = s + \sum_{X\not\subset H_1}(\sum_{\beta\in B}\beta\rho(s'_\beta(X)))$. Clearly

$$s_1 = \sum_{i=1}^{k} s_i p_i + \sum_X\left(\sum_\beta \beta s'_\beta(X)\right) \in R\otimes S.$$

Since besides $s_1 \in \operatorname{Ker}\delta_1$ and $\delta_1 = r_2 \otimes 1_S$ on $R\otimes S$ we have by Lemma 2.2(iii) that $s_1 \in R_0 \otimes S$. Thus s_1 can be represented as a linear combination (over S) of $\rho(t_X)$ where $X \subset H_1$ and $t_X \in R_X$ which completes the proof.

Theorem 2.7 justifies the following construction. Put $E_2 = \oplus_{X\in L(2)} R_X \otimes S$ and define an S-linear map $\delta_2\colon E_2 \to E_1$ via $\delta_2(t) = \rho(t)$ for every $t \in \oplus_X R_X$. Now we are able to prove the main result of this section. □

Theorem 2.8 *The sequence*

$$0 \to E_2 \xrightarrow{\delta_2} E_1 \xrightarrow{\delta_1} E_0 \xrightarrow{\delta_0} \Omega^1\langle\mathcal{A}\rangle \to 0 \tag{2.13}$$

of S-modules and their homomorphisms is a free resolution of $\Omega^1\langle\mathcal{A}\rangle$.

Proof: Modulo Corollary 2.4 and Theorem 2.7 we need to prove only that δ_2 is injective. For that let us compute the ranks (over S) of the modules in (2.13). Put $N = |L(2)|$ and recall that $m_X = |\mathcal{A}_X|$ for every $X \in L(2)$. Then it is easy to see that

$$\operatorname{rank}(\Omega^1\langle\mathcal{A}\rangle) = m - 1,$$
$$\operatorname{rank}(E_0) = N,$$

$$\begin{aligned}\operatorname{rank}(E_1) &= \sum_{X\in L(2)} (m_X - 1) - (n-1) + (n-m)\\ &= \sum_{X\in L(2)} (m_X - 1) - m + 1\end{aligned}$$

(cf. proof of Lemma 2.5), and

$$\operatorname{rank}(E_2) = \sum_{X\in L(2)} (m_X - 2).$$

This implies that $\sum_{i=0}^{2}(-1)^{i-1}\operatorname{rank}(E_i) = \operatorname{rank}(\Omega^1\langle\mathcal{A}\rangle)$ whence $\operatorname{rank}(\operatorname{Ker}\delta_2) = 0$. Since $\operatorname{Ker}\delta_2$ is a submodule of a free module it vanishes which concludes the proof.

In general the resolution (2.13) is not minimal. Our next goal is to modify it in oder to get a minimal resolution. First put $\mathcal{X} = \{X \in L(2) \mid m(X) \geq 3 \text{ or } X \subset H_1\}$ and $\tilde{E}_0 = \sum_{X\in\mathcal{X}} SX \subset E_0$. Then denote by $\tilde{\delta}_0$ the restriction of δ_0 to $\tilde{E}_0$. Also denote by $\tilde{C}$ the subcomplex of C generated by all the subsets of $\mathcal{A}$ each lying in one of sets $\mathcal{A}_X$ with $X \in \mathcal{X}$. Consider the F-linear map $\zeta\colon \oplus_{X\in L(2)} R_X \to R$ generated by the embeddings $R_X \subset R$ and put $T = T(\mathcal{A}) = \operatorname{Ker}\zeta$ and $U = U(\mathcal{A}) = \operatorname{Coker}\zeta$. Finally put $\tilde{E}_1 = (H_1(\tilde{C}) \oplus U) \otimes S$ and $\tilde{E}_2 = T \otimes S$. □

Theorem 2.9

(i) *δ_1 induces an S linear map $\tilde{\delta}_1\colon \tilde{E}_1 \to \tilde{E}_0$ and δ_2 induces an S-linear map $\tilde{\delta}_2\colon \tilde{E}_2 \to \tilde{E}_1$ such that the sequence*

$$0 \to \tilde{E}_2 \xrightarrow{\tilde{\delta}_2} \tilde{E}_1 \xrightarrow{\tilde{\delta}_1} \tilde{E}_0 \xrightarrow{\tilde{\delta}_0} \Omega^1\langle\mathcal{A}\rangle \to 0 \tag{2.14}$$

is exact.

(ii) *The sequence* (2.14) *is a minimal resolution of* $\Omega^1\langle\mathcal{A}\rangle$.

Proof:

(i) First put $E_0' = \sum_{X\notin\mathcal{X}} SX$. Then fix some splittings $\epsilon\colon U \to R$ and $\psi\colon \oplus R_X \to T$ of the projection $R \to U$ and the embedding $T \subset \oplus R_X$ respectively. Thus we have $R = U' \oplus \epsilon(U)$ and $\oplus R_X = T \oplus T'$ where $U' = \operatorname{Im}\zeta$ and $T' = \operatorname{Ker}\psi$. Also denote by H' the subspace of $H_1(C)$ generated by the homology classes of the cycles $\{H_1, H_i\} - \{H_1, H_j\} - \{H_i, H_j\}$ with $m(H_i \cap H_j) = 2$. Then $H_1(C) = H_1(\tilde{C}) \oplus H'$. This implies that

$$E_0 = \tilde{E}_0 \oplus E_0', \tag{2.15}$$

$$E_1 = \tilde{\epsilon}(\tilde{E}_1) \oplus (H' \otimes S) \oplus (U' \otimes S), \tag{2.16}$$

and

$$E_2 = \tilde{E}_2 \oplus (T' \otimes S) \tag{2.17}$$

where $\tilde{\epsilon} = (1_{H_1(\tilde{C})} \oplus \epsilon) \otimes 1_S$. Denote by π_i $(i = 0, 1, 2)$ the projections of E_i to the first summands in (2.15)–(2.17) and by $\bar{\delta}_i$ the restrictions of δ_i to the first summands. Now we can put $\tilde{\delta}_0 = \bar{\delta}_0$, $\tilde{\delta}_1 = \pi_0\bar{\delta}_1\tilde{\epsilon}$, and $\tilde{\delta}_2 = \tilde{\epsilon}^{-1}\pi_1\bar{\delta}_2$.

The surjectivity of $\tilde{\delta}_0$ follows from the fact that for every $X \in L(2)$ with $\mathcal{A}_X = \{\alpha_i, \alpha_j\}$ $(i < j)$ we have $\omega_X = \frac{d\alpha_i}{\alpha_i} - \frac{d\alpha_j}{\alpha_j}$ whence ω_X is a linear combination over S

of forms ω_Y with $Y \subset H_1$. The exactness of (2.14) in the other terms follows from the fact that the projection of $\delta_1(H' \otimes S)$ to E_0' in the decomposition (2.15) and the projection of $\delta_2(T' \otimes S)$ to $U' \otimes S$ in (2.16) are isomorphisms.

(ii) To prove the minimality of the sequence (2.14) let us notice that this sequence can be made into a sequence of graded S-modules and homogeneous homomorphisms (with the natural grading on $\Omega^1\langle\mathcal{A}\rangle$). One way to do this is to put $\deg X = n - m_X$, $\deg z = 0$, $\deg u = 1$, and $\deg t = 0$ for every $X \in L(2)$, $z \in H_1(\tilde{C})$, $u \in U$, and $t \in T$. Thus not only all the maps $\tilde{\delta}_i$ become homogeneous but also all entries of their matrices in the natural bases have positive degrees. Then the minimality of (2.14) follows from a well-known criterion (e.g., see [11, p. 54. Lemma 4.4]). □

Corollary 2.10

(i) $\mathrm{pd}_S(\Omega^1\langle\mathcal{A}\rangle) \le 2$.
(ii) $\mathrm{pd}_S(\Omega^1\langle\mathcal{A}\rangle) \le 1$ *if and only if* $T = 0$.
(iii) *The S-module $\Omega^1\langle\mathcal{A}\rangle$ is free if and only if $\mathcal{A}$ is formal (i.e., $U = 0$) and $H_1(\tilde{C}) = 0$.*

3. Generating $\Omega^1\langle\mathcal{A}\rangle$

In this section we are concerned with generators of the S-module $\Omega^1 = \Omega^1\langle\mathcal{A}\rangle$ for a (non-empty) arrangement $\mathcal{A}$. To get rid of a trivial summand consider the S-linear map $\phi: \Omega^1 \to S$ defined by $dx_i \mapsto x_i$ for every $i = 1, 2, \ldots, l$. Put $\Omega_0^1 = \Omega_0^1\langle\mathcal{A}\rangle = \mathrm{Ker}\,\phi$. Since the map $f \mapsto f\frac{dx_1}{x_1}$ $(f \in S)$ splits ϕ we have $\Omega^1 = \Omega_0^1 \oplus_S S(\frac{dx_1}{x_1})$ whence it suffices to find generators of Ω_0^1. Clearly all the forms ω_X $(X \in L(2))$ belong to $\Omega_0^1\langle\mathcal{A}\rangle$. If $\mathcal{A}$ is not essential there are also forms $\eta_i = dx_i - x_i\frac{dx_1}{x_1}$ $(i = m+1, \ldots, l)$ in $\Omega_0^1\langle\mathcal{A}\rangle$. Denote the module generated by all these forms by $\bar{\Omega}^1\langle\mathcal{A}\rangle$. Clearly

$$\bar{\Omega}^1\langle\mathcal{A}\rangle = \Omega^1\langle\mathcal{A}\rangle \oplus_S \Omega_0^1\langle\mathcal{A}\rangle$$

where $\Omega_0^1\langle\mathcal{A}\rangle$ is the free module generated by all η_i.

The main goal of this section is to prove the following result.

Theorem 3.1 *The equality*

$$\Omega_0^1\langle\mathcal{A}\rangle = \bar{\Omega}^1\langle\mathcal{A}\rangle \tag{3.1}$$

holds if and only if $T(\mathcal{A}_Y) = 0$ for every $Y \in L(3)$.

Notice that for an essential arrangement (3.1) means that all the forms ω_X together with $\frac{dx_1}{x_1}$ generate $\Omega^1\langle\mathcal{A}\rangle$.

Also notice that if $m < 3$ then $L(3) = \emptyset$. On the other hand $\Omega_0^1\langle\mathcal{A}\rangle = \sum_{i=2}^{l} S\eta_i$ for $m = 1$ and $\Omega_0^1\langle\mathcal{A}\rangle = S\omega_Z \oplus \sum_{i=3}^{l} S\eta_i$ for $m = 2$ where $Z = \bigcap_{\mathcal{A}} H$. In any case (3.1) holds and this can be used as the base of induction on n.

We will need the following lemma whose proof appeared first in [1, Lemma 3.3.7].

Lemma 3.2 $\mathrm{pd}_S\Omega_0^1 = \mathrm{pd}_S\Omega^1 \le l - 2$.

Proof: First observe that the submodule $Q\Omega^1$ of $\Omega^1[V]$ is the kernel of the S-linear map $\gamma\colon \Omega^1[V] \to M = \Omega^2[V]/Q\Omega^2[V]$ defined by $\omega \mapsto dQ \wedge \omega$. Since $\Omega^2[V]$ is free $\mathrm{pd}_S M \leq 1$. Thus due to [4, p. 199, Prop. 1.8] $\mathrm{pd}_S \mathrm{Im}\,\gamma \leq l-1$. Now applying the same proposition to the exact sequence

$$0 \to \Omega^1 \xrightarrow{Q} \Omega^1[V] \xrightarrow{\gamma} \mathrm{Im}\,\gamma \to 0$$

we obtain the result.

The following result shows that the property (3.1) is hereditary. ☐

Theorem 3.3 *Let* (3.1) *hold for $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{A}$, $\mathcal{B} \neq \emptyset$. Then* (3.1) *holds for $\mathcal{B}$.*

Proof: It suffices to consider the case where $n \geq 2$ and $\mathcal{B} = \mathcal{A}\backslash\{\alpha_H\}$ for some $H \in \mathcal{A}$. We can assume that $H = H_m$, i.e., $\alpha_H = x_m$. Fix $\omega \in \Omega^1_0(\mathcal{B}) \subset \Omega^1_0(\mathcal{A})$. We need to prove that $\omega \in \bar{\Omega}^1\langle\mathcal{B}\rangle$.

By the condition of the theorem

$$\omega = \sum_X s_X\omega_X + \sum_{i=m+1}^{l} s_i\eta_i \tag{3.2}$$

for some $s_X, s_i \in S$. Put $L' = L(\mathcal{B})$ and notice that $L' \subset L$. If $X \in L'(2)$ then denote by ω'_X the respective form from $\Omega^1\langle\mathcal{B}\rangle$. If $X \in L(2)$ and $X \not\subset H_m$ then $X \in L'(2)$ and $\omega'_X = \omega_X$. Thus without loss of generality we can assume that if in (3.2) $s_X \neq 0$ then $X \subset H_m$. Also since $\eta_i \in \bar{\Omega}^1\langle\mathcal{B}\rangle$ we can assume that each $s_i = 0$ $(i = m+1, \ldots, l)$.

Since $\omega \in \Omega^1_0(\mathcal{B})$ we know that x_m divides $Q\omega$. For every $X \subset H_m$ we can write $\omega_X = c_X\frac{1}{Q_X}(\alpha_X dx_m - x_m d\alpha_X)$ where c_X is a non-zero scalar and $\alpha_X = \alpha_1^X$ or $\alpha_X = \alpha_2^X$ if $\alpha_1^X = x_m$. Then the divisibility condition amounts to

$$\sum_{X\in L(2), X\subset H_m} c_X\bar{s}_X\bar{\pi}_X\bar{\alpha}_X = 0$$

where the bar above a polynomial means its evaluation at $x_m = 0$. Since α_X divides π_Y for every $Y \neq X$ every $\bar{\alpha}_X$ cancels out. Now by similar reason if $H \supset X$ and $\alpha_H \neq x_m, \alpha_X$ then $\bar{\alpha}_H$ divides $\bar{s}_X$. This implies that

$$c_X s_X = q_X\frac{Q_X}{x_m\alpha_X} + x_m r_X \tag{3.3}$$

for some $q_X, r_X \in S$ such that $\sum_X q_X = 0$. Now notice that if $X \in L(2)$ and $X \subset H_m$ then either $X \in L'(2)$ and $\omega'_X = x_m\omega_X$ or $\omega_X = \pm(\frac{dx_m}{x_m} - \frac{d\alpha_X}{\alpha_X})$. In any case $x_m\omega_X \in \bar{\Omega}^1\langle\mathcal{B}\rangle$. Thus we can ignore the summands $x_m r_X$ in (3.3) and assume that $\omega = \sum_X q_X\frac{Q_X}{x_m\alpha_X}\frac{\omega_X}{c_X}$ where $\sum_X q_X = 0$. This assumption leads to

$$\omega \in \sum_{X,Y\in L(2), X,Y\subset H_m} S\left(\frac{Q_X}{x_m\alpha_X}\frac{\omega_X}{c_X} - \frac{Q_Y}{x_m\alpha_Y}\frac{\omega_Y}{c_Y}\right).$$

Then since $\frac{Q_X}{x_m\alpha_X}\frac{\omega_X}{c_X} - \frac{Q_Y}{x_m\alpha_Y}\frac{\omega_Y}{c_Y} = \frac{d\alpha_Y}{\alpha_Y} - \frac{d\alpha_X}{\alpha_X} \in \Omega^1\langle\mathcal{B}\rangle$ we have $\omega \in \bar{\Omega}^1\langle\mathcal{B}\rangle$ which completes the proof. ☐

Now using the results of Section 2 we will prove a partial converse of Theorem 3.3.

Theorem 3.4 *Suppose that $m \geq 3$ and for every subarrangement $\mathcal{B}$ of $\mathcal{A}$* (3.1) *holds. Then* (3.1) *holds for $\mathcal{A}$ also if either $m > 3$ or $m = 3$ and $T(\mathcal{A}) = 0$.*

Proof: First notice that the statement can be reduced to essential arrangements. Indeed if $\mathcal{A}$ is not essential then recall that there exists an essential m-arrangement $\mathcal{A}_1$ such that $\Omega^1(\mathcal{A}) = (\Omega^1(\mathcal{A}_1) \otimes F[x_{m+1}, \dots, x_l]) \oplus (F[x_1, \dots, x_m] \otimes \sum_{i=m+1}^{l} S dx_i)$. Thus (3.1) holds for $\mathcal{A}$ if and only if it holds for $\mathcal{A}_1$. Also $T(\mathcal{A}') = T(\mathcal{A})$. Besides there exists a natural bijection between the sets of all subarrangements of $\mathcal{A}$ and those of $\mathcal{A}_1$ preserving (3.1).

From now on we assume that $\mathcal{A}$ is essential, i.e., $m = l$. If we fix $H \in \mathcal{A}$ and put $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ then we have

$$\alpha_H \Omega_0^1(\mathcal{A}) \subset \Omega_0^1(\mathcal{A}') = \bar{\Omega}^1\langle\mathcal{A}'\rangle \subset \Omega^1\langle\mathcal{A}\rangle.$$

Since this holds for every $H \in \mathcal{A}$ and $\mathcal{A}$ is essential (i.e., $\sum_{H \in \mathcal{A}} S\alpha_H = S_+$ where S_+ is the irrelevant ideal of S) we have $S_+\Omega_0^1(\mathcal{A}) \subset \Omega^1\langle\mathcal{A}\rangle$ whence the S-module $M = \Omega_0^1(\mathcal{A})/\Omega^1\langle\mathcal{A}\rangle$ is either 0 or has Krull dimension 0 and $\mathrm{pd}_S M = l$. Suppose that $M \neq 0$. Then applying Lemma 3.2 and [4, p. 199, Prop. 1.8] to the exact sequence

$$0 \to \Omega^1\langle\mathcal{A}\rangle \to \Omega_0^1(\mathcal{A}) \to M \to 0$$

and using that $l \geq 3$ we have

$$\mathrm{pd}_S \Omega^1\langle\mathcal{A}\rangle = \mathrm{pd}_S M - 1 = l - 1. \tag{3.4}$$

If $l > 3$ then (3.4) contradicts Corollary 2.10(i). Thus in this case $M = 0$ always. If $l = 3$ then according to Corollary 2.10(ii) the equality (3.4) is possible only if $T(\mathcal{A}) \neq 0$. Thus if $T(\mathcal{A}) = 0$ again $M = 0$. This completes the proof. □

Corollary 3.5 *Suppose $l = 3$. Then* (3.1) *holds if and only if $T(\mathcal{A}) = 0$.*

Proof: If $T(\mathcal{A}) \neq 0$ then by Corollary 2.10(ii) and Lemma 3.2 $\Omega_0^1(\mathcal{A}) \neq \bar{\Omega}^1\langle\mathcal{A}\rangle$ since their projective dimensions are different. Suppose $T(\mathcal{A}) = 0$. Then (3.1) can be easily proved by induction on n using Theorem 3.4. □

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. Suppose that (3.1) holds for $\mathcal{A}$ and $Y \in L(3)$. By Theorem 3.3 the equality (3.1) holds for $\mathcal{A}_Y$ whence by Corollary 3.5 we have $T(\mathcal{A}_Y) = 0$.

Conversely suppose that for every $Y \in L(3)$ we have $T(\mathcal{A}_Y) = 0$. Then by Corollary 3.5 the equality (3.1) holds for $\mathcal{A}_Y$. Now the fact that (3.1) holds for $\mathcal{A}$ follows by induction on n using Theorem 3.4.

4. Examples and possible generalizations

In order to make the condition of Theorem 3.1 more understandable we consider several examples.

The first one is the simplest example of a 3-arrangement $\mathcal{A}$ with $T(\mathcal{A}) \neq 0$.

Example 4.1 Let $\mathcal{A}$ be given by the functionals

$$x, y, z, x-y, x-z, y-z$$

(essentially, $\mathcal{A}$ is the braid 4-arrangement or the reflection arrangement of type A_3). Recall that R is the space of all linear relations among the functionals, i.e., the kernel of the map $C_0 \to V^*$ sending H to α_H. Here the space C_0 is 6-dimensional (according to the number of hyperplanes) and the space V^* is 3-dimensional. Since besides the functionals α_H generate the whole V^* we have $\dim R = 3$. On the other hand, there are 4 elements X_i $(i = 1, \dots, 4)$ of $L(2)$ that can be described by the respective arrangements $\mathcal{A}_{X_i}$ as

$$\{x, y, x-y\}, \{x, z, x-z\}, \{y, z, y-z\}, \{x-y, x-z, y-z\}.$$

For each of those X_i there is a unique (up to a constant) linear relation among the functionals, i.e., $\dim R_{X_i} = 1$ for every i (every other $X \in L(2)$ has $R_X = 0$). Since R is generated by relations of length 3, the map $\zeta: \oplus R_X \to R$ is surjective, and thus $\dim T = 1$. More explicitly, the following elements generate the images of R_{X_i} in R

$$r_1 = H_1 - H_2 - H_4, r_2 = H_1 - H_3 - H_5, r_3 = H_2 - H_3 - H_6, r_4 = H_4 - H_5 + H_6$$

(the functionals are enumerated in the order they are introduced). These elements are subject to the relation

$$r_1 - r_2 + r_3 + r_4 = 0$$

which corresponds to a generator of T.

In any way, since $T \neq 0$ the forms ω_X do not generate $\Omega_0^1(\mathcal{A})$.

Example 4.2 Consider the 4-arrangement $\mathcal{A}$ given by

$$x, y, z, w, x+z, x+w, x+y+z, x+y+w, x+y+z+w.$$

There are 6 linear relations of length 3 among these functionals. As in Example 4.1, this shows that $\dim T = 1$. However using similar computation for $\mathcal{A}_Y$ for every $Y \in L(3)$ one can show that $T(\mathcal{A}_Y) = 0$. This means that the only (up to a constant) relation among the 3-relations involves the set of functionals of rank 4 (in fact, the set of all of them). Thus for this $\mathcal{A}$ the condition of Theorem 3.1 holds and the forms ω_X do gnerate $\Omega_0^1(\mathcal{A})$.

The class of arrangements for which the condition of Theorem 3.1 holds is not combinatorial, that is the lattice $L(\mathcal{A})$ does not define in general whether $\mathcal{A}$ belongs to the class. To show this we can use an example from [13].

Example 4.3 Suppose that $\mathrm{char}(F) = 0$ or is sufficiently large. Define two 3-arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ by the seven common functionals $\alpha_1 = x$, $\alpha_2 = y$, $\alpha_3 = z$, $\alpha_4 = x+y+z$, $\alpha_5 = 2x+y+z$, $\alpha_6 = 2x+3y+z$, $\alpha_7 = 2x+3y+4z$ and by the two more $\alpha_8 = 3x+5z$, $\alpha_9 = 3x+4y+5z$ for $\mathcal{A}_1$ and $\alpha_8 = x+3z$, $\alpha_9 = x+2y+3z$ for $\mathcal{A}_2$.

For each $\mathcal{A}_i$ there are 6 relevant $X \in L(2)$, for each of those X we have $\dim R_X = 1$, and the images of R_X in R are generated by

$$\left.\begin{array}{r} H_1 + H_4 - H_5 \\ 2H_2 + H_5 - H_6 \\ 3H_3 + H_6 - H_7 \\ 3H_1 + 5H_3 - H_8 \\ 4H_2 + H_8 - H_9 \\ H_4 + H_7 - H_9 \end{array}\right\} \tag{4.1}$$

for $\mathcal{A}_1$ and

$$\left.\begin{array}{r} H_1 + H_4 - H_5 \\ 2H_2 + H_5 - H_6 \\ 3H_3 + H_6 - H_7 \\ H_1 + 3H_3 - H_8 \\ 2H_2 + H_8 - H_9 \\ H_4 - H_7 + H_9 \end{array}\right\} \tag{4.2}$$

for $\mathcal{A}_2$. One can easily see that while the elements (4.1) are linearly independent, there is a unique (up to a constant) relation among the elements (4.2). In other words $T(\mathcal{A}_1) = 0$ while $\dim T(\mathcal{A}_2) = 1$. On the other hand, the one-to-one correspondence between $\mathcal{A}_1$ and $\mathcal{A}_2$ given by the enumeration of the functionals generates an isomorphism between their intersection lattices.

Notice that there is another principle difference between $\mathcal{A}_1$ and $\mathcal{A}_2$: while $\mathcal{A}_1$ is a formal arrangement, i.e., the map $\zeta: \oplus R_X \to R$ is surjective, $\mathcal{A}_2$ is not formal. If we restrict our consideration to the class of formal arrangements then the condition of Theorem 3.1 becomes combinatorial. More precisely the following proposition follows easily from Theorem 3.1.

Proposition 4.4 *Let $\mathcal{A}$ be an arrangement such that for every $Y \in L(3)$ the arrangement $\mathcal{A}_Y$ is formal. Then $\Omega_0^1(\mathcal{A})$ is generated by ω_X if and only if*

$$\sum_{X \in L(2), X < Y} (|\mathcal{A}_X| - 2) = |\mathcal{A}_Y| - 3 \tag{4.3}$$

for every $Y \in L(3)$.

Notice that among all the arrangements with a given intersection lattice the formal ones form a Zariski open set. One can deduce from this that if a geometric lattice L satisfies (4.3) then a sufficiently general arrangement having L as the intersection lattice satisfies the condition of Theorem 3.1. For a concrete example of such an L one can take the intersection lattice of the arrangements from Example 4.3.

There are at least two directions in which it would be natural to try to generalize the results of this paper.

First, one can study generators of $\Omega^p(\mathcal{A})$ for $p > 1$. More precisely, the S-linear map ϕ generalizes to $\phi_p: \Omega^p(\mathcal{A}) \to \Omega^{p-1}(\mathcal{A})$ for every p ($0 < p \le l$) via $dx_{i_1} \wedge \cdots \wedge dx_{i_p} \mapsto \sum_{j=1}^{p} (-1)^{j-1} x_{i_j} dx_{i_1} \wedge \cdots \wedge dx_{i_{j-1}} \wedge dx_{i_{j+1}} \wedge \cdots \wedge dx_{i_p}$ (or equivalently via $\phi_p(\omega) = [\omega, \theta_E]$ where θ_E is the Euler derivation $\theta_E = \sum_{i=1}^{l} x_i \frac{\partial}{\partial x_i}$ and $[\cdot, \cdot]$ is the interior product of forms and derivations). The maps ϕ_p form a chain complex that is homotopy equivalent to 0. Indeed as a homotopy between identity map and 0 map of this complex one can take the collection of maps $\frac{dx_1}{x_1}\wedge\colon \Omega^{p-1}(\mathcal{A}) \to \Omega^p(\mathcal{A})$. Since the homotopy maps form a cochain complex too there is a splitting $\Omega^p(\mathcal{A}) = \Omega_0^p(\mathcal{A}) \oplus_S (\frac{dx_1}{x_1} \wedge \Omega_0^{p-1}(\mathcal{A}))$ where $\Omega_0^i(\mathcal{A}) = \operatorname{Ker}\phi_i$ for every i. Thus to find generators of $\Omega^p(\mathcal{A})$ it suffices to find generators of $\Omega_0^p(\mathcal{A})$ and $\Omega_0^{p-1}(\mathcal{A})$. On the other hand, for every $X \in L(p+1)$ one can define $\omega_X \in \Omega_0^p(\mathcal{A})$ via $\omega_X = \phi_{p+1}(\frac{d\alpha_1^X \cdots d\alpha_{p+1}^X}{Q_X})$ where $(\alpha_1^X, \ldots, \alpha_{p+1}^X)$ is a maximal linearly independent system from $\mathcal{A}_X$. Clearly a change of the linearly independent system changes ω_X by a non-zero multiplicative constant. It would be interesting to find conditions on $\mathcal{A}$ under which $\Omega_0^p(\mathcal{A})$ is generated by ω_X ($X \in L(p+1)$).

Second, it is possible to give an algorithm that starts with an element $t \in T(\mathcal{A})$ and produces a form $\omega_t \in \Omega_0^1(\mathcal{A})$. For instance, for the arrangement of Example 4.1 this form is (up to a constant)

$$\omega_t = \frac{dx}{x(x-y)(x-z)} - \frac{dy}{y(x-y)(y-z)} + \frac{dz}{z(x-z)(y-z)}$$

and it generates $\Omega_0^1(\mathcal{A})$ together with ω_X. Perhaps this process can be continued to obtain an increasing sequence of classes of arrangements with canonical generators of $\Omega^1(\mathcal{A})$ constructed from some kind of "higher syzygies" of the space of relations among the functionals.

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