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Partitioned Tensor Products and Their Spectra

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Abstract. A pleasant family of graphs defined by Godsil and McKay is shown to have easily computed eigenvalues in many cases.

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Let G and H be directed graphs on the respective vertices U and V, and suppose that the vertex sets have each been partitioned into disjoint subsets $U = U_0 \cup U_1$ and $V = V_0 \cup V_1$. The *partitioned tensor product* $G \underline{\times} H$ of G and H with respect to this partitioning is defined as follows:

a) Each vertex of U_0 is replaced by a copy of $H \mid V_0$, the subgraph of H induced by V_0;
b) Each vertex of U_1 is replaced by a copy of $H \mid V_1$;
c) Each arc of G that runs from U_0 to U_1 is replaced by a copy of the arcs of H that run from V_0 to V_1;
d) Each arc of G that runs from U_1 to U_0 is replaced by a copy of the arcs of H that run from V_1 to V_0.

For example, Figure 1 shows two partitioned tensor products. The example in Figure 1(b) is undirected; this is the special case of a directed graph where each undirected edge corresponds to a pair of arcs in opposite directions. Arcs of G that stay within U_0 or U_1 do not contribute to $G \underline{\times} H$, so we may assume that no such arcs exist (i.e., that G is bipartite).

Figure 2 shows what happens if we interchange the roles of U_0 and U_1 in G but leave everything else intact. (Equivalently, we could interchange the roles of V_0 and V_1.) These graphs, which may be denoted $G^R \underline{\times} H$ to distinguish them from the graphs $G \underline{\times} H$ in Figure 1, might look quite different from their mates, yet it turns out that the characteristic polynomials of $G \underline{\times} H$ and $G^R \underline{\times} H$ are strongly related.

Let E_{ij} be the arcs from U_i to U_j in G, and F_{ij} the arcs from V_i to V_j in H; multiple arcs are allowed, so E_{ij} and F_{ij} are multisets. It follows that $G \underline{\times} H$ has $|U_0|\,|V_0| + |U_1|\,|V_1|$ vertices and $|U_0|\,|F_{00}| + |U_1|\,|F_{11}| + |E_{01}|\,|F_{01}| + |E_{10}|\,|F_{10}|$ arcs. Similarly, $G^R \underline{\times} H$ has $|U_1|\,|V_0| + |U_0|\,|V_1|$ vertices and $|U_1|\,|F_{00}| + |U_0|\,|F_{11}| + |E_{10}|\,|F_{01}| + |E_{01}|\,|F_{10}|$ arcs.

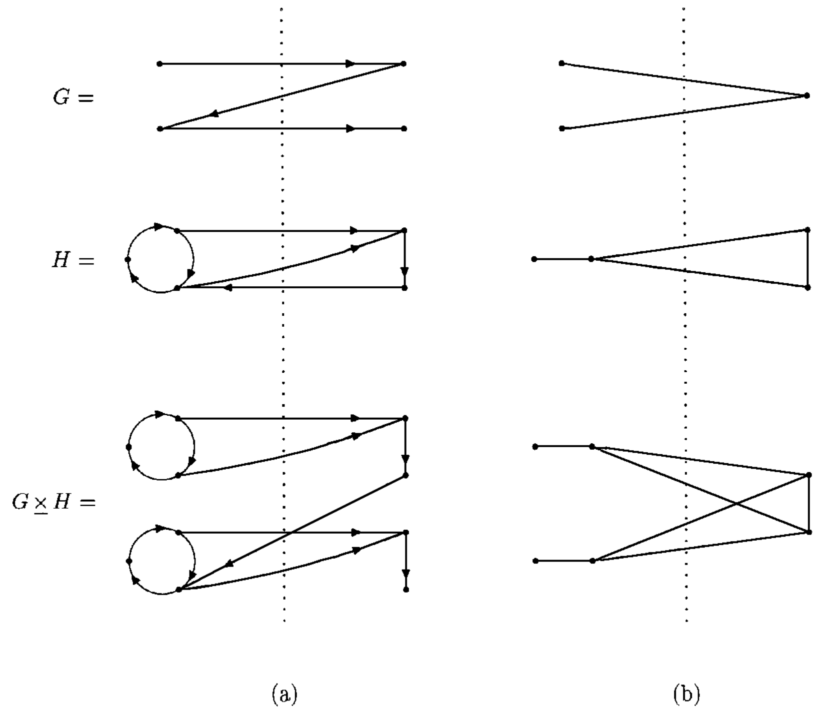


Figure 1. Partitioned tensor products, directed and undirected.

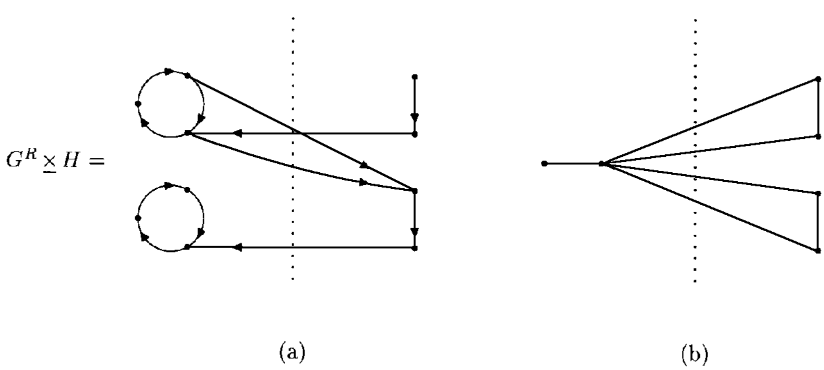


Figure 2. Dual products after right-left reflection of G.

The definition of partitioned tensor product is due to Godsil and McKay [3], who proved the remarkable fact that

$$p(G \underline{\times} H)\ p(H \mid V_0)^{|U_1|-|U_0|} = p(G^T \underline{\times} H)\ p(H \mid V_1)^{|U_1|-|U_0|},$$

where p denotes the characteristic polynomial of a graph. They also observed [4] that Figures 1(b) and 2(b) represent the smallest pair of connected undirected graphs having the same spectrum (the same p). The purpose of the present note is to refine their results by showing how to calculate $p(G \underline{\times} H)$ explicitly in terms of G and H.

We can use the symbols G and H to stand for the adjacency matrices as well as for the graphs themselves. Thus we have

$$G = \begin{pmatrix} G_{00} & G_{01} \\ G_{10} & G_{11} \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{pmatrix}$$

in partitioned form, where G_{ij} and H_{ij} denote the respective adjacency matrices corresponding to the arcs E_{ij} and F_{ij}. (These submatrices are not necessarily square; G_{ij} has size $|U_i| \times |U_j|$ and H_{ij} has size $|V_i| \times |V_j|$.) It follows by definition that

$$G \underline{\times} H = \begin{pmatrix} I_{|U_0|} \otimes H_{00} & G_{01} \otimes H_{01} \\ G_{10} \otimes H_{10} & I_{|U_1|} \otimes H_{11} \end{pmatrix}$$

where $\otimes$ denotes the Kronecker product or tensor product [7, page 8] and I_k denotes an identity matrix of size $k \times k$.

Let $H \uparrow \sigma$ denote the graph obtained from H by σ-fold repetition of each arc that joins V_0 to V_1. In matrix form

$$H \uparrow \sigma = \begin{pmatrix} H_{00} & \sigma H_{01} \\ \sigma H_{10} & H_{11} \end{pmatrix}.$$

This definition applies to the adjacency matrix when σ is any complex number, but of course $H \uparrow \sigma$ is difficult to "draw" unless σ is a nonnegative integer. We will show that the characteristic polynomial of $G \underline{\times} H$ factors into characteristic polynomials of graphs $H \uparrow \sigma$, times a power of the characteristic polynomials of H_{00} or H_{11}. The proof is simplest when G is undirected.

Theorem 1 *Let G be an undirected graph, and let $(\sigma_1, \ldots, \sigma_l)$ be the singular values of $G_{01} = G_{10}^T$, where $l = \min(|U_0|, |U_1|)$. Then*

$$p(G \underline{\times} H) = \begin{cases} \left(\prod_{j=1}^{l} p(H \uparrow \sigma_j)\right) p(H_{00})^{|U_0|-|U_1|}, & \text{if } |U_0| \ge |U_1|; \\ \left(\prod_{j=1}^{l} p(H \uparrow \sigma_j)\right) p(H_{11})^{|U_1|-|U_0|}, & \text{if } |U_1| \ge |U_0|. \end{cases}$$

Proof: Any real $m \times n$ matrix A has a singular value decomposition

$$A = QSR^T$$

where Q is an $m \times m$ orthogonal matrix, R is an $n \times n$ orthogonal matrix, and S is an $m \times n$ matrix with $S_{jj} = \sigma_j \geq 0$ for $1 \leq j \leq \min(m, n)$ and $S_{ij} = 0$ for $i \neq j$ [6, page 16]. The numbers $\sigma_1, \ldots, \sigma_{\min(m,n)}$ are called the singular values of A.

Let $m = |U_0|$ and $n = |U_1|$, and suppose that QSR^T is the singular value decomposition of G_{01}. Then $(\sigma_1, \ldots, \sigma_l)$ are the nonnegative eigenvalues of the bipartite graph G, and we have

$$\begin{pmatrix} Q^T \otimes I_{|V_0|} & O \\ O & R^T \otimes I_{|V_1|} \end{pmatrix} G \underline{\times} H \begin{pmatrix} Q \otimes I_{|V_0|} & O \\ O & R \otimes I_{|V_1|} \end{pmatrix} = \begin{pmatrix} I_{|U_0|} \otimes H_{00} & S \otimes H_{01} \\ S^T \otimes H_{10} & I_{|U_1|} \otimes H_{11} \end{pmatrix}$$

because $G_{10} = RS^T Q^T$. Row and column permutations of this matrix transform it into the block diagonal form

$$\begin{pmatrix} H \uparrow \sigma_1 & & & \\ & \ddots & & \\ & & H \uparrow \sigma_l & \\ & & & D \end{pmatrix},$$

where D consists of $m - n$ copies of H_{00} if $m \geq n$, or $n - m$ copies of H_{11} if $n \geq m$. $\square$

A similar result holds when G is directed, but we cannot use the singular value decomposition because the eigenvalues of G might not be real and the elementary divisors of $\lambda I - G$ might not be linear. The following lemma can be used in place of the singular value decomposition in such cases.

Lemma *Let A and B be arbitrary matrices of complex numbers, where A is $m \times n$ and B is $n \times m$. Then we can write*

$$A = QSR^{-1}, \qquad B = RTQ^{-1},$$

where Q is a nonsingular $m \times m$ matrix, R is a nonsingular $n \times n$ matrix, S is an $m \times n$ matrix, T is an $n \times m$ matrix, and the matrices (S, T) are triangular with consistent diagonals:

$$S_{ij} = T_{ij} = 0 \qquad \text{for } i > j;$$
$$S_{jj} = T_{jj} \quad \text{or} \quad S_{jj}T_{jj} = 0 \qquad \text{for } 1 \leq j \leq \min(m, n).$$

Proof: We may assume that $m \leq n$. If AB has a nonzero eigenvalue λ, let σ be any square root of λ and let x be a nonzero m-vector such that $ABx = \sigma^2 x$. Then the n-vector $y = Bx/\sigma$ is nonzero, and we have

$$Ay = \sigma x, \qquad Bx = \sigma y.$$

On the other hand, if all eigenvalues of AB are zero, let x be a nonzero vector such that $ABx = 0$. Then if $Bx \neq 0$, let $y = Bx$. If $Bx = 0$, let y be any nonzero vector such that

$Ay = 0$; this is possible unless all n columns of A are linearly independent, in which case we must have $m = n$ and we can find y such that $Ay = x$. In all cases we have therefore demonstrated the existence of nonzero vectors x and y such that

$$Ay = \sigma x\,, \qquad Bx = \tau y\,, \qquad \sigma = \tau \quad \text{or} \quad \sigma\tau = 0\,.$$

Let X be a nonsingular $m \times m$ matrix whose first column is x, and let Y be a nonsingular $n \times n$ matrix whose first column is y. Then

$$X^{-1}AY = \begin{pmatrix} \sigma & a \\ 0 & A_1 \end{pmatrix}, \qquad Y^{-1}BX = \begin{pmatrix} \tau & b \\ 0 & B_1 \end{pmatrix}$$

where A_1 is $(m-1) \times (n-1)$ and B_1 is $(n-1) \times (m-1)$. If $m = 1$, let $Q = X$, $R = Y$, $S = (\sigma a)$, and $T = \binom{\tau}{0}$. Otherwise we have $A_1 = Q_1 S_1 R_1^{-1}$ and $B_1 = R_1 T_1 Q_1^{-1}$ by induction, and we can let

$$Q = X\begin{pmatrix} 1 & 0 \\ 0 & Q_1 \end{pmatrix}, \quad R = Y\begin{pmatrix} 1 & 0 \\ 0 & R_1 \end{pmatrix}, \quad S = \begin{pmatrix} \sigma & aR_1 \\ 0 & S_1 \end{pmatrix}, \quad T = \begin{pmatrix} \tau & BQ_1 \\ 0 & T_1 \end{pmatrix}.$$

All conditions are now fulfilled. □

Theorem 2 *Let G be an arbitrary graph, and let $(\sigma_1, \ldots, \sigma_l)$ be such that $\sigma_j = S_{jj} = T_{jj}$ or $\sigma_j = 0 = S_{jj}T_{jj}$ when $G_{01} = QSR^{-1}$ and $G_{10} = RTQ^{-1}$ as in the lemma, where $l = \min(|U_0|\,, |U_1|)$. Then $p(G \underline{\times} H)$ satisfies the identities of Theorem 1.*

Proof: Proceeding as in the proof of Theorem 1, we have

$$\begin{pmatrix} Q^{-1} \otimes I_{|V_0|} & O \\ O & R^{-1} \otimes I_{|V_1|} \end{pmatrix} G \underline{\times} H \begin{pmatrix} Q \otimes I_{|V_0|} & O \\ O & R \otimes I_{|V_1|} \end{pmatrix} = \begin{pmatrix} I_{|U_0|} \otimes H_{00} & S \otimes H_{01} \\ T \otimes H_{10} & I_{|U_1|} \otimes H_{11} \end{pmatrix}.$$

This time a row and column permutation converts the right-hand matrix to a block *triangular* form, with zeroes below the diagonal blocks. Each block on the diagonal is either $H \uparrow \sigma_j$ or H_{00} or H_{11}, or of the form

$$\begin{pmatrix} H_{00} & \sigma H_{01} \\ \tau H_{10} & H_{11} \end{pmatrix}, \qquad \sigma\tau = 0.$$

In the latter case the characteristic polynomial is clearly $p(H_{00})p(H_{11}) = p(H \uparrow 0)$, so the remainder of the proof of Theorem 1 carries over in general. □

The proof of the lemma shows that the numbers $\sigma_1^2, \ldots, \sigma_p^2$ are the characteristic roots of $G_{01}G_{10}$, when $|U_0| \le |U_1|$, otherwise they are the characteristic roots of $G_{10}G_{01}$. Either square root of σ_j^2 can be chosen, since the matrix $H \uparrow \sigma$ is similar to $H \uparrow (-\sigma)$.

We have now reduced the problem of computing $p(G \underline{\times} H)$ to the problem of computing the characteristic polynomial of the graphs $H \uparrow \sigma$. The latter is easy when $\sigma = 0$, and some graphs G have only a few nonzero singular values. For example, if G is the complete bipartite graph having parts U_0 and U_1 of sizes m and n, all singular values vanish except for $\sigma = \sqrt{mn}$.

If H is small, and if only a few nonzero σ need to be considered, the computation of $p(H \uparrow \sigma)$ can be carried out directly. For example, it turns out that

$$\begin{pmatrix} \lambda & -1 & -\sigma & 0 & 0 \\ -1 & \lambda & 0 & 0 & -\sigma \\ -\sigma & 0 & \lambda & -1 & 0 \\ 0 & 0 & -1 & \lambda & -1 \\ 0 & -\sigma & 0 & -1 & \lambda \end{pmatrix} = (\lambda^2 + \lambda - \sigma^2)\big(\lambda^3 - \lambda^2 - (2+\sigma^2)\lambda + 2\big)\,;$$

so we can compute the spectrum of $G \underline{\times} H$ by solving a few quadratic and cubic equations, when H is this particular 5-vertex graph (a partitioned 5-cycle). But it is interesting to look for large families of graphs for which simple formulas yield $p(H \uparrow \sigma)$ as a function of σ.

One such family consists of graphs that have only one edge crossing the partition. Let H_{00} and H_{11} be graphs on V_0 and V_1, and form the graph $H = H_{00} \bullet\!\!-\!\!\bullet H_{11}$ by adding a single edge between designated vertices $x_0 \in V_0$ and $x_1 \in V_1$. Then a glance at the adjacency matrix of H shows that

$$p(H \uparrow \sigma) = p(H_{00})p(H_{11}) - \sigma^2 p(H_{00} \mid V_0 \backslash x_0) p(H_{11} \mid V_1 \backslash x_1)\,.$$

(The special case $\sigma = 1$ of this formula is Theorem 4.2(ii) of [5].)

Another case where $p(H \uparrow \sigma)$ has a simple form arises when the matrices

$$H_0 = \begin{pmatrix} H_{00} & 0 \\ 0 & H_{11} \end{pmatrix} \quad \text{and} \quad H_1 = \begin{pmatrix} 0 & H_{01} \\ H_{10} & 0 \end{pmatrix}$$

commute with each other. Then it is well known [2] that the eigenvalues of $H_0 + \sigma H_1$ are $\lambda_j + \sigma\mu_j$, for some ordering of the eigenvalues λ_j of H_0 and μ_j of H_1. Let us say that (V_0, V_1) is a *compatible partition* of H if $H_0 H_1 = H_1 H_0$, i.e., if

$$H_{00}H_{01} = H_{01}H_{11} \quad \text{and} \quad H_{11}H_{10} = H_{10}H_{00}\,.$$

When H is undirected, so that $H_{00} = H_{00}^T$ and $H_{11} = H_{11}^T$ and $H_{10} = H_{01}^T$, the compatibility condition boils down to the single relation

$$H_{00}H_{01} = H_{01}H_{11}. \qquad (*)$$

Let $m = |V_0|$ and $n = |V_1|$, so that H_{00} is $m \times m$, H_{01} is $m \times n$, and H_{11} is $n \times n$. One obvious way to satisfy $(*)$ is to let H_{00} and H_{11} both be zero, so that H is bipartite as well as G. Then $H \uparrow \sigma$ is simply σH, the σ-fold repetition of the arcs of H, and its eigenvalues are just those of H multiplied by σ. For example, if G is the M-cube P_2^M and H is a path P_N on N points, and if U_0 consists of the vertices of even parity in G while V_0 is one

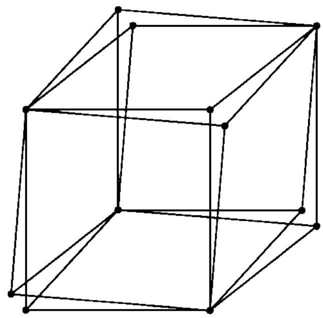
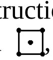


Figure 3. $P_2^3 \underline{\times} P_3$.

of H's bipartite parts, the characteristic polynomial of $G \underline{\times} H$ is

$$\prod_{\substack{1\le j\le M\\ 1\le k\le N}} \left(\lambda - (2N-4j)\cos\frac{k\pi}{N+1}\right)^{\binom{M}{j}/2},$$

because of the well-known eigenvalues of G and H [1]. Figure 3 illustrates this construction in the special case $M = N = 3$. The smallest pair of cospectral graphs,  and , is obtained in a similar way by considering the eigenvalues of $P_3 \underline{\times} P_3$ and $P_3^T \underline{\times} P_3$ [4].

Another simple way to satisfy the compatibility condition $(*)$ with symmetric matrices H_{00} and H_{11} is to let H_{01} consist entirely of 1s, and to let H_{00} and H_{11} both be regular graphs of the same degree d. Then the eigenvalues of H_0 are $(\lambda_1, \ldots, \lambda_m, \lambda_1', \ldots, \lambda_n')$, where $(\lambda_1, \ldots, \lambda_m)$ belong to H_{00} and $(\lambda_2', \ldots, \lambda_n')$ belong to H_{11} and $\lambda_1 = \lambda_1' = d$. The eigenvalues of H_1 are $(\sqrt{mn}, -\sqrt{mn}, 0, \ldots, 0)$. We can match the eigenvalues of H_0 properly with those of H_1 by looking at the common eigenvectors $(1, \ldots, 1)^T$ and $(1, \ldots, 1, -1, \ldots, -1)^T$ that correspond to d in H_0 and $\pm\sqrt{mn}$ in H_1; the eigenvalues of $H \uparrow \sigma$ are therefore

$$(d + \sigma\sqrt{mn}, \lambda_2, \ldots, \lambda_m, d - \sigma\sqrt{mn}, \lambda_2', \ldots, \lambda_n').$$

Yet another easy way to satisfy $(*)$ is to assume that $m = n$ and to let $H_{00} = H_{11}$ commute with H_{01}. One general construction of this kind arises when the vertices of V_0 and V_1 are the elements of a group, and when $H_{00} = H_{11}$ is a Cayley graph on that group. In other words, two elements α and β are adjacent in H_{00} iff $\alpha\beta^{-1} \in X$, where X is an arbitrary set of group elements closed under inverses. And we can let $\alpha \in V_0$ be adjacent to $\beta \in V_1$ iff $\alpha\beta^{-1} \in Y$, where Y is any normal subgroup. Then H_{00} commutes with H_{01}. The effect is to make the cosets of Y fully interconnected between V_0 and V_1, while retaining a more interesting Cayley graph structure inside V_0 and V_1. If Y is the trivial subgroup, so that H_{01} is simply the identity matrix, our partitioned tensor product $G \underline{\times} H$ becomes simply the ordinary Cartesian product $G \oplus H = I_{|U|} \otimes H + G \otimes I_{|V|}$. But in many other cases this construction gives something more general.

A fourth family of compatible partitions is illustrated by the following graph H in which $m = 6$ and $n = 12$:

$$
\begin{pmatrix}
0&0&1&1&1&0&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&1&1&0&1&0&0&0&0&0&0&1&0&0&0\\
1&0&0&0&1&1&0&0&1&0&0&0&0&0&0&1&0&0\\
1&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0\\
1&1&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&1&1&1&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
1&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0\\
0&1&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&1\\
0&0&1&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0\\
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0&1&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1\\
0&0&0&1&0&0&1&0&0&0&0&0&1&0&1&0&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0&0&1&0&1&0&0
\end{pmatrix}
$$

In general, let C_{2k} be the matrix of a cyclic permutation on $2k$ elements, and let $m = 2k$, $n = 4k$. Then we obtain a compatible partition if

$$
H_{00} = \left(C_{2k}^{j} + C_{2k}^{k} + C_{2k}^{-j}\right), \quad H_{01} = (I_{2k} \;\; C_{2k}), \quad H_{11} = \begin{pmatrix} C_{2k}^{j} + C_{2k}^{-j} & C_{2k}^{k+1} \\ C_{2k}^{k-1} & C_{2k}^{j} + C_{2k}^{-j} \end{pmatrix}.
$$

The 18×18 example matrix is the special case $j = 2$, $k = 3$. The eigenvalues of $H \uparrow \sigma$ in general are

$$
\omega^{jl} + \omega^{-jl} + 1, \qquad \omega^{jl} + \omega^{-jl} - 1 + \sqrt{2}\,\sigma, \qquad \omega^{jl} + \omega^{-jl} - 1 - \sqrt{2}\,\sigma
$$

for $0 \le l < 2k$, where $\omega = e^{\pi i/k}$.

Compatible partitionings of digraphs are not difficult to construct. But it would be interesting to find further examples of undirected graphs, without multiple edges, that have a compatible partition.

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