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# Discriminantal Arrangements, Fiber Polytopes and Formality

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**Abstract.** Manin and Schechtman defined the discriminantal arrangement of a generic hyperplane arrangement as a generalization of the braid arrangement. This paper shows their construction is dual to the fiber zonotope construction of Billera and Sturmfels, and thus makes sense even when the base arrangement is not generic. The hyperplanes, face lattices and intersection lattices of discriminantal arrangements are studied. The discriminantal arrangement over a generic arrangement is shown to be formal (and in some cases 3-formal), though it is in general not free. An example of a free discriminantal arrangement over a generic arrangement is given.

**Keywords:** discriminantal arrangement, hyperplane arrangement, polytope, free

## 1. Introduction

Manin and Schechtman [8] defined discriminantal arrangement as a generalization of the braid arrangement. The discriminantal arrangement $\mathcal{B}(n, k)$ has as its complement the manifold of general position parallel translates of an affine arrangement of $n$ hyperplanes that is in general position in $\mathbf{R}^k$. Manin and Schechtman then studied the intersection lattice of discriminantal arrangements arising from a dense subset of general position affine arrangements. Falk [6] studied the discriminantal arrangement associated with any general position affine arrangement, and described the hyperplanes of the discriminantal arrangement in terms of those of the base arrangement. In particular he showed that the intersection lattice of the discriminantal arrangement does not depend solely on $n$ and $k$. Billera and Sturmfels [2] defined the fiber zonotope as a quotient of a projection map from a cube onto a zonotope. They gave the vectors of the fiber zonotope explicitly.

In this paper we show the following. The discriminantal arrangement as studied by Manin and Schechtman (or more generally Falk) is the arrangement dual to the fiber zonotope of the zonotope dual to the base arrangement. No general position assumption is needed: the discriminantal arrangement can be defined for any essential arrangement. In all cases the complement of the discriminantal arrangement is the manifold of (relatively) general position parallel translates of the base arrangement. The hyperplanes of the discriminantal arrangement correspond to minimal violations of general position conditions that can be achieved by parallel translation of the base arrangement. By Billera and Sturmfels, a face of the fiber zonotope corresponds to a regular zonotopal subdivision of the base zonotope, with vertices corresponding to regular cubical subdivisions. This is matched by a correspondence between the faces of the discriminantal arrangement and the face posets of parallel translates of the base arrangement. In Section 4 we conjecture a description of the intersection lattice of the discriminantal arrangement based on a "very generic" arrangement.

Free discriminantal arrangements can arise from generic base arrangements (see example in Section 5), but not from very generic base arrangements. A necessary condition for the freeness of an arrangement is formality, a condition on the space of linear relations among the hyperplanes. We show that the discriminantal arrangement based on any generic arrangement is formal. If the base arrangement is very generic, the discriminantal arrangement is actually 3-formal.

Useful general references are [9] on hyperplane arrangements and [3] on oriented matroids. Some of the material in this paper is described in the expository paper [1].

## 2. Discriminantal Arrangements

A *hyperplane* in $\mathbf{R}^k$ is a set of the form $H = \{\mathbf{x} \in \mathbf{R}^k : \boldsymbol{\alpha}\cdot\mathbf{x} = b\}$ where $\boldsymbol{\alpha} \in \mathbf{R}^k$ ($\boldsymbol{\alpha} \neq \mathbf{0}$) and $b \in \mathbf{R}$. An *affine arrangement* in $\mathbf{R}^k$ is a finite collection of hyperplanes. It is called a *central arrangement* (or simply an *arrangement*) if each hyperplane passes through the origin ($b = 0$).

Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_n$ be nonzero vectors in $\mathbf{R}^k$ such that no $\boldsymbol{\alpha}_i$ is a multiple of another. Associated with these vectors is the central arrangement of hyperplanes with normals $\boldsymbol{\alpha}_i$ in $\mathbf{R}^k$, $\mathcal{A} = \{H_1^0, H_2^0, \ldots, H_n^0\}$. Associated with the same set of vectors is the zonotope $Z = Z(\mathcal{A})$, which is the Minkowski sum of the line segments $[-\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i]$, $1 \leq i \leq n$. If two vectors are multiples of each other, the same hyperplane occurs twice; the set of hyperplanes is then called a *multiarrangement*. The constructions here work also for multiarrangements, but we avoid parallel vectors to simplify the presentation.

For $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbf{R}^n$, let $\mathcal{A}_\mathbf{b}$ be the affine arrangement of $n$ hyperplanes, $H_i = \{\mathbf{x} \in \mathbf{R}^k : \boldsymbol{\alpha}_i \cdot \mathbf{x} = b_i\}$. Then $\mathcal{A}_\mathbf{0} = \mathcal{A}$ and $\mathcal{A}_\mathbf{b}$ is called a *parallel translate* of $\mathcal{A}$. The affine arrangement $\mathcal{A}_\mathbf{b} = \{H_1, H_2, \ldots, H_n\}$ is in *relatively general position* if for all subsets $S$ of $[n] = \{1, 2, \ldots, n\}$ with $|S| \leq k$, $\dim \bigcap_{i\in S} H_i \leq k - |S|$, and for all subsets $S$ of $[n]$ with $|S| > k$, $\bigcap_{i\in S} H_i = \emptyset$. (It is in *general position* if, moreover, the dimension inequality holds as equality for all $S$ with $|S| \leq k$.) The idea is that the high dimensional intersections, but not the parallelisms, of a nongeneral position arrangement can be eliminated by parallel translation of the hyperplanes. Let $U(\mathcal{A}) = \{\mathbf{b} \in \mathbf{R}^n : \mathcal{A}_\mathbf{b}$ is in relatively general position$\}$. We will show that $U(\mathcal{A})$ is the complement of a central hyperplane arrangement in $\mathbf{R}^n$.

**Example 2.1** Let $\mathcal{A}$ be the 2-arrangement defined by the normals $\boldsymbol{\alpha}_1 = (0,1)$, $\boldsymbol{\alpha}_2 = (-1,1)$, $\boldsymbol{\alpha}_3 = (1,0)$, and $\boldsymbol{\alpha}_4 = (1,1)$. (The corresponding zonotope is an octagon.) Consider the four hyperplanes $D_{[i]}$ in $\mathbf{R}^4$:

$$\begin{aligned}
D_{[1]} &= \{(b_1, b_2, b_3, b_4) : b_2 + 2b_3 - b_4 = 0\}\\
D_{[2]} &= \{(b_1, b_2, b_3, b_4) : b_1 + b_3 - b_4 = 0\}\\
D_{[3]} &= \{(b_1, b_2, b_3, b_4) : 2b_1 - b_2 - b_4 = 0\}
\end{aligned}$$

$$D_{[4]} \;=\; \{(b_1, b_2, b_3, b_4) : b_1 - b_2 - b_3 = 0\}.$$

The hyperplane $D_{[i]}$ consists of those "right-hand sides" $\mathbf{b}$ for which, in the arrangement $\mathcal{A}_{\mathbf{b}}$, the three hyperplanes other than $H_i$ intersect at a point. The complement of the hyperplane arrangement $\{D_{[1]}, D_{[2]}, D_{[3]}, D_{[4]}\}$ thus consists of those $\mathbf{b}$ such that no three of the hyperplanes $H_i$ intersect, that is, the general position parallel translates of $\mathcal{A}$. Note that this arrangement in $\mathbf{R}^4$ is not essentially four dimensional; the intersection of all four hyperplanes is a two-dimensional subspace of $\mathbf{R}^4$. In fact this arrangement is isomorphic to the two-dimensional arrangement $\mathcal{A}$, but this happens only in *very* special cases. □

Assume that the normals to the hyperplanes of $\mathcal{A}$ span $\mathbf{R}^k$. (The arrangement $\mathcal{A}$ is then called *essential*.) Let $C_n$ be the $n$-cube with vertices $\pm\mathbf{e}_i$ ($\mathbf{e}_i$ the standard unit vector); let $\pi$ be the linear map from $C_n$ to $Z$ satisfying $\pi(\mathbf{e}_i) = \boldsymbol{\alpha}_i$. Billera and Sturmfels [2] defined the fiber polytope as $\Sigma(C_n, Z) = \frac{1}{\text{vol}\, Z}\{\int_Z \gamma(\mathbf{x})\, d\mathbf{x}\} \subseteq \mathbf{R}^n$, where $\gamma$ ranges over all measurable right inverses of $\pi$ ($\gamma : Z \to \mathbf{R}^n$). They proved the following facts about the fiber zonotope.

1. The fiber polytope is an $(n-k)$-dimensional zonotope.

2. The scaled fiber zonotope $(\text{vol}\, Z)\Sigma(C_n, Z)$ is the Minkowski sum of the line segments $[-\mathbf{E}_S, \mathbf{E}_S]$, where for each $(k+1)$-subset $S$ of $[n]$, $S = \{s_1 < s_2 < \cdots < s_{k+1}\}$,

$$\mathbf{E}_S = \sum_{i=1}^{k+1} (-1)^i \det(\boldsymbol{\alpha}_{s_1}, \ldots, \boldsymbol{\alpha}_{s_{i-1}}, \boldsymbol{\alpha}_{s_{i+1}}, \ldots, \boldsymbol{\alpha}_{s_{k+1}}) \cdot \mathbf{e}_{s_i}.$$

3. The face lattice of the fiber zonotope $\Sigma(C_n, Z)$ is isomorphic to the poset of regular zonotopal subdivisions (to be defined later) of $Z$. The vertices correspond to regular cubical subdivisions.

**Definition 2.2** Let $\mathcal{A}$ be an essential arrangement of $n$ hyperplanes in $\mathbf{R}^k$ with normal vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_n$. The *discriminantal arrangement based on* $\mathcal{A}$ is the arrangement in $\mathbf{R}^n$ of hyperplanes with normal vectors the distinct, nonzero vectors of the form

$$\mathbf{E}_S = \sum_{i=1}^{k+1} (-1)^i \det(\boldsymbol{\alpha}_{s_1}, \ldots, \boldsymbol{\alpha}_{s_{i-1}}, \boldsymbol{\alpha}_{s_{i+1}}, \ldots, \boldsymbol{\alpha}_{s_{k+1}}) \cdot \mathbf{e}_{s_i},$$

as $S = \{s_1 < s_2 < \cdots < s_{k+1}\}$ ranges over the $(k+1)$-subsets of $[n]$.

The discriminantal arrangement is thus the arrangement dual to the fiber zonotope.

**Theorem 2.3** *Let $\mathcal{A}$ be an essential arrangement of $n$ hyperplanes in $\mathbf{R}^k$, and $\mathcal{B}(\mathcal{A})$ the discriminantal arrangement based on $\mathcal{A}$. Then $U(\mathcal{A})$, the set of relatively general position translations, is the complement of $\mathcal{B}(\mathcal{A})$.*

**Proof:** The complement of $\mathcal{B}(\mathcal{A})$ is the set of $\mathbf{b} \in \mathbf{R}^n$ such that for all $(k+1)$-subsets $S$ of $[n]$, $\mathbf{E}_S \neq \mathbf{0}$ implies $\mathbf{E}_S \cdot \mathbf{b} \neq 0$. Recall that $U(\mathcal{A}) = \{\mathbf{b} \in \mathbf{R}^n :$

$\mathcal{A}_{\mathbf{b}}$ is in relatively general position$\}$. Since the normals to the hyperplanes of $\mathcal{A}$ span $\mathbf{R}^k$, a parallel translate $\mathcal{A}_{\mathbf{b}}$ is in relatively general position if and only if for all subsets $T$ of $[n]$ with $|T| > k$, $\bigcap_{i \in T} H_i = \emptyset$, and it is of course equivalent to apply the empty intersection criterion only to sets $T$ of cardinality $k+1$. Thus we need to show the following are equivalent, for the affine hyperplane arrangement $\mathcal{A}_{\mathbf{b}} = \{H_1, H_2, \ldots, H_n\}$:

1. for all $(k+1)$-subsets $T$ of $[n]$, $\bigcap\limits_{i \in T} H_i = \emptyset$
2. for all $(k+1)$-subsets $S$ of $[n]$ for which $\mathbf{E}_S \neq \mathbf{0}$, $\mathbf{E}_S \cdot \mathbf{b} \neq 0$.

We use two facts about linear dependencies. For any set $T$, $\bigcap_{i \in T} H_i \neq \emptyset$ if and only if the set $\{b_i : i \in T\}$ satisfies all linear dependencies satisfied by the set $\{\boldsymbol{\alpha}_i : i \in T\}$. If the rank of the set $\{\boldsymbol{\alpha}_i : i \in S\}$ is $k = |S| - 1$, then every linear dependency on the set $\{\boldsymbol{\alpha}_i : i \in S\}$ is a nonzero multiple of $\mathbf{E}_S$.

Assume (1), and let $S$ be a $(k+1)$-subset of $[n]$ with $\mathbf{E}_S \neq \mathbf{0}$. Thus the rank of $\{\boldsymbol{\alpha}_i : i \in S\}$ is $k$, so $\mathbf{E}_S$ gives the only linear dependency on $\{\boldsymbol{\alpha}_i : i \in S\}$. By (1), $\bigcap_{i \in S} H_i = \emptyset$, so $\{b_i : i \in S\}$ fails to satisfy the linear dependency given by $\mathbf{E}_S$. Thus $\mathbf{E}_S \cdot \mathbf{b} \neq 0$.

Assume (2), and let $T$ be a $(k+1)$-subset of $[n]$. Choose $T_0 \subseteq T$ so that $\{\boldsymbol{\alpha}_i : i \in T_0\}$ is an independent set with $\operatorname{span}\{\boldsymbol{\alpha}_i : i \in T_0\} = \operatorname{span}\{\boldsymbol{\alpha}_i : i \in T\}$; then choose a $(k+1)$-set $S$ containing $T_0$ with $\operatorname{rank}\{\boldsymbol{\alpha}_i : i \in S\} = k$. Thus $\mathbf{E}_S \cdot \mathbf{b} \neq 0$, so $\{b_i : i \in S\}$ fails to satisfy some linear dependency satisfied by $\{\boldsymbol{\alpha}_i : i \in S\}$. So $\bigcap\limits_{i \in S} H_i \neq \emptyset$. Since $\bigcap\limits_{i \in S} H_i \subseteq \bigcap\limits_{i \in T_0} H_i = \bigcap\limits_{i \in T} H_i$, we conclude that $\bigcap\limits_{i \in T} H_i \neq \emptyset$. $\square$

The arrangement $\mathcal{A}$ is called *generic* if $n \geq k$ and every intersection of $k$ hyperplanes has dimension zero. In this case $\mathcal{A}_{\mathbf{b}}$ is in general position for some $\mathbf{b} \in \mathbf{R}^n$.

Manin and Schechtman [8] defined (in the case when $\mathcal{A}$ is generic) the discriminantal arrangement as the complement of $U(\mathcal{A})$. The theorem thus says that the arrangement dual to the fiber zonotope is the same as the Manin-Schechtman arrangement. Since the fiber zonotope is an $(n-k)$-dimensional polytope in $\mathbf{R}^n$, the essential dimension of $\mathcal{B}(\mathcal{A})$ is $n-k$. We will return to the correspondence between the discriminantal arrangement and the fiber zonotope.

The discriminantal arrangement has at most $\binom{n}{k+1}$ hyperplanes. If $\mathcal{A}$ is generic, the vectors $\mathbf{E}_S$ are nonzero and distinct. In that case $\mathcal{B}(\mathcal{A})$ has exactly $\binom{n}{k+1}$ hyperplanes; for each $S \subseteq [n]$ of size $k+1$, the hyperplane with normal $\mathbf{E}_S$ consists of the set of points $\mathbf{b}$ such that in the affine arrangement $\mathcal{A}_{\mathbf{b}}$ the $k+1$ hyperplanes $H_{s_1}, H_{s_2}, \ldots, H_{s_{k+1}}$ have a common intersection. This is the case considered by Falk [6].

To describe the hyperplanes of the discriminantal arrangement in the arbitrary (essential) case, we use the following convention. Let $\mathcal{A}$ be an essential arrangement of $n$ hyperplanes in $\mathbf{R}^k$, with normal vectors $\boldsymbol{\alpha}_i$, $1 \leq i \leq n$. For $S$ a subset of $[n]$ we abbreviate $\operatorname{rank}\{\boldsymbol{\alpha}_i : i \in S\}$ to $\operatorname{rank} S$. A subset $S$ of $[n]$ is called a *dependent set* if $\operatorname{rank} S < |S|$. A set $S$ is dependent if and only if there exists a vector $\mathbf{b} \in \mathbf{R}^n$ such that in the affine arrangement

$\mathcal{A}_{\mathbf{b}}$, $\dim \bigcap_{i\in S} H_i > k - |S|$. A *minimal* dependent set $S$ has rank $|S| - 1$, and for such a set there exists a vector $\mathbf{b} \in \mathbf{R}^n$ such that in $\mathcal{A}_{\mathbf{b}}$, $\dim \bigcap_{i\in S} H_i = k - |S| + 1$.

**Theorem 2.4** *Let $\mathcal{A}$ be an essential arrangement of $n$ hyperplanes in $\mathbf{R}^k$. For $S$ a minimal dependent set, define $D_S$ to be the set of points $\mathbf{b} \in \mathbf{R}^n$ such that in the affine arrangement $\mathcal{A}_{\mathbf{b}}$, $\dim \bigcap_{i\in S} H_i = k - |S| + 1$. Then the discriminantal arrangement based on $\mathcal{A}$ is*

$$\mathcal{B}(\mathcal{A}) = \{D_S : S \text{ is a minimal dependent set}\}.$$

**Proof:** Let $\mathcal{A}$ be a central essential arrangement whose hyperplanes have normals $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_n$. Let $S$ be a minimal dependent set; $\operatorname{rank} S = |S| - 1$. Let $T$ be a $(k+1)$-set containing $S$ with $\operatorname{rank} T = k$. Then $\mathbf{E}_T$ defines a hyperplane of $\mathcal{B}(\mathcal{A})$ consisting of the set of $\mathbf{b} \in \mathbf{R}^n$ which satisfy the linear dependencies of $\{\boldsymbol{\alpha}_i : i \in S\}$. The vector $\mathbf{b}$ satisfies the linear dependencies of $\{\boldsymbol{\alpha}_i : i \in S\}$ if and only if in $\mathcal{A}_{\mathbf{b}}$ $\dim \bigcap_{i\in S} H_i = k - |S| + 1$. So $\mathbf{E}_T$ is the normal to the hyperplane $D_S$. Thus every set of the form $D_S$ is one of the hyperplanes of $\mathcal{B}(\mathcal{A})$.

Conversely, consider $\mathbf{E}_T \neq \mathbf{0}$, where $T$ is a $(k+1)$ subset of $[n]$. Then $\mathbf{E}_T$ is the unique (up to scalar multiple) linear dependence on $\{\boldsymbol{\alpha}_i : i \in T\}$, so $T$ contains a unique minimal dependent set $S$, and as above $\{\mathbf{b} : \mathbf{E}_T \cdot \mathbf{b} = 0\} = D_S$. Thus every hyperplane of $\mathcal{B}(\mathcal{A})$ is of the form $D_S$ for a unique minimal dependent set $S$. □

When $\mathcal{A}$ is a generic arrangement, this gives the correspondence of the hyperplanes of the discriminantal arrangement with $(k+1)$-subsets of $[n]$, as given by Manin and Schechtman [8]. We look now at a nongeneric example.

**Example 2.5** Let $\mathcal{A}$ be the 3-arrangement defined by the normals $\boldsymbol{\alpha}_1 = (1,0,0)$, $\boldsymbol{\alpha}_2 = (0,1,0)$, $\boldsymbol{\alpha}_3 = (1,1,0)$, $\boldsymbol{\alpha}_4 = (0,0,1)$, and $\boldsymbol{\alpha}_5 = (1,0,1)$. The minimal dependent sets are $S_1 = \{1,2,3\}$, $S_2 = \{1,4,5\}$, and $S_3 = \{2,3,4,5\}$. The discriminantal arrangement $\mathcal{B}(\mathcal{A})$ is a 5-arrangement of essential dimension 2, with hyperplanes

$$\begin{aligned} D_{S_1} &= \{(b_1,b_2,b_3,b_4,b_5) : b_1 + b_2 - b_3 = 0\} \\ D_{S_2} &= \{(b_1,b_2,b_3,b_4,b_5) : b_1 + b_4 - b_5 = 0\} \\ D_{S_3} &= \{(b_1,b_2,b_3,b_4,b_5) : b_2 - b_3 - b_4 + b_5 = 0\}. \end{aligned}$$

The zonotope $Z = Z(\mathcal{A})$ is a 3-polytope with four hexagonal faces and eight quadrilateral faces. Figure 1 is a drawing of one side of this polytope, with half of the 2-faces showing. (Since zonotopes are centrally symmetric, this shows enough to determine the whole polytope.) □

## 3. The Face Lattice

In the study of hyperplane arrangements, the term "combinatorial" is applied to properties that depend only on the intersection lattice of the arrangement. By contrast the combinatorial object of interest in zonotopes is the full face lattice: the set of all faces of the zonotope,

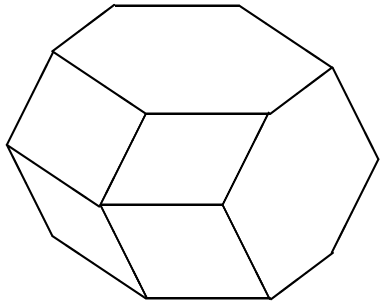

*Figure 1.* 3-dimensional zonotope

partially ordered by inclusion. The intersection lattice can be determined from the face lattice, but not vice versa. We study both lattices of the discriminantal arrangement/fiber zonotope here. To clarify the connection between an arrangement and a zonotope, we turn (finally) to oriented matroids (see [3]).

A *sign vector* on a finite set $E$ is a vector indexed by $E$ with coordinates from the set $\{-, 0, +\}$. We use the notation $\boldsymbol{\sigma}^+ = \{e \in E : \sigma_e = +\}$, and similarly for $\boldsymbol{\sigma}^-$ and $\boldsymbol{\sigma}^0$. The sign vector with all coordinates $0$ is written $\mathbf{0}$, and $-\boldsymbol{\sigma}$ denotes the coordinatewise negation of a sign vector $\boldsymbol{\sigma}$. The *product* $\boldsymbol{\sigma} \cdot \boldsymbol{\tau}$ of sign vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ is given by $(\boldsymbol{\sigma} \cdot \boldsymbol{\tau})_e = \sigma_e$ if $\sigma_e \neq 0$, and $(\boldsymbol{\sigma} \cdot \boldsymbol{\tau})_e = \tau_e$ otherwise. An element $e \in E$ *separates* $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ if $\sigma_e = -\tau_e \neq 0$.

**Definition 3.1** An *oriented matroid* is a pair $M = (E, \mathcal{K})$, where $E$ is a finite set and $\mathcal{K}$ is a set of sign vectors on $E$ satisfying

1. $\mathbf{0} \in \mathcal{K}$;
2. if $\boldsymbol{\sigma} \in \mathcal{K}$ then $-\boldsymbol{\sigma} \in \mathcal{K}$;
3. if $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{K}$ then $\boldsymbol{\sigma} \cdot \boldsymbol{\tau} \in \mathcal{K}$; and
4. if $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{K}$ and $e \in E$ separates $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, then there exists $\boldsymbol{\mu} \in \mathcal{K}$ such that $\boldsymbol{\mu}_e = 0$ and for every $f \in E$ that does not separate $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, $\boldsymbol{\mu}_f = (\boldsymbol{\sigma} \cdot \boldsymbol{\tau})_f = (\boldsymbol{\tau} \cdot \boldsymbol{\sigma})_f$.

The set $\mathcal{K}$ is known as the *signed cocircuit span* of $M$. The signed cocircuit span $\mathcal{K}$ of an oriented matroid $M$ has a natural partial order: $\boldsymbol{\sigma} \preceq \boldsymbol{\tau}$ if and only if $\boldsymbol{\sigma}^+ \subseteq \boldsymbol{\tau}^+$ and $\boldsymbol{\sigma}^- \subseteq \boldsymbol{\tau}^-$. The poset $\mathcal{K}$ is ranked and is generated by its elements of rank $1$, called *cocircuits*. For $I \subseteq E$, the set $\{\boldsymbol{\sigma} \in \mathcal{K}(M) : (\boldsymbol{\sigma}^+ \cup \boldsymbol{\sigma}^-) \cap I = \emptyset\}$ is the cocircuit span of another oriented matroid $M/I$, called the *contraction* of $M$ by $I$.

An oriented matroid is defined from a set of real vectors as follows. Let $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\ldots$, $\boldsymbol{\alpha}_n$ be vectors in $\mathbf{R}^k$, and let $\mathbf{A}$ be the $n \times k$ matrix with the $\boldsymbol{\alpha}_i$ as rows. Associate with each $n$-vector $\mathbf{v}$ in the column space of $\mathbf{A}$, $\mathbf{v} = \mathbf{A}\mathbf{x}$, the sign vector $\boldsymbol{\sigma} \in \{-, 0, +\}^n$ with

$\boldsymbol{\sigma}_i = +$ if $\mathbf{v}_i > 0$, $\boldsymbol{\sigma}_i = -$ if $\mathbf{v}_i < 0$, and $\boldsymbol{\sigma}_i = 0$ if $\mathbf{v}_i = 0$. Let $\mathcal{K}$ be the set of all sign vectors of elements of the column space of $\mathbf{A}$. Then $M = ([n], \mathcal{K})$ is an oriented matroid.

The poset $\mathcal{K}$ has natural interpretations in the hyperplane arrangement $\mathcal{A}$ and in the zonotope $Z$ determined by $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, ..., $\boldsymbol{\alpha}_n$. Every point $\mathbf{x} \in \mathbf{R}^k$ is in the relative interior of a unique face of the arrangement $\mathcal{A}$. Two points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^k$ are in the same (open) face if and only if they are on the same hyperplanes and are on the same sides of the hyperplanes that do not contain them. This is equivalent to equality of the sign vectors of $\mathbf{Ax}$ and $\mathbf{Ay}$. So we can identify elements of $\mathcal{K}$ with faces of $\mathcal{A}$. Furthermore, if $F$ and $G$ are faces of $\mathcal{A}$ with associated sign vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ and $F \subseteq G$, then $\boldsymbol{\sigma} \preceq \boldsymbol{\tau}$. So the poset $\mathcal{K}$ is isomorphic to the face semilattice of $\mathcal{A}$.

It is well known that the face semilattice of the zonotope $Z$ (the face lattice with $\emptyset$ removed) is dual to that of $\mathcal{A}$. Thus there is an order-reversing isomorphism between $\mathcal{K}$ and the face semilattice of $Z$. What is it? To any sign vector $\boldsymbol{\sigma} \in [n]$ (whether it is in $\mathcal{K}$ or not) assign the set $\sum_{\substack{i\in[n]\\ \sigma_i=0}} [-\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i] + \sum_{\substack{i\in[n]\\ \sigma_i\neq 0}} \sigma_i\boldsymbol{\alpha}_i$. This set is a zonotope contained in $Z$. The nonempty faces of $Z$ are exactly the zonotopes of this form associated with $\boldsymbol{\sigma} \in \mathcal{K}$.

Note that the maximal elements of $\mathcal{K}$ are the sign vectors with no zero coordinate. These correspond to the chambers of $\mathcal{A}$ and to the vertices of $Z$.

**Example 3.2** (Example 2.1 continued) For four vectors in $\mathbf{R}^2$ Figure 2 shows the assignments of sign vectors to the face lattices of the zonotope and the hyperplane arrangement. $\square$

The combinatorics of the fiber zonotope and discriminantal arrangement is related to liftings of the oriented matroid. Let $M$ be the oriented matroid of the vectors $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, ..., $\boldsymbol{\alpha}_n$ in $\mathbf{R}^k$. Fix $h_1, h_2, \ldots, h_n \in \mathbf{R}$, and let $M'$ be the oriented matroid of the vectors $(\boldsymbol{\alpha}_1, h_1)$, $(\boldsymbol{\alpha}_2, h_2)$, ..., $(\boldsymbol{\alpha}_n, h_n)$ and $\mathbf{e}_{k+1}$ (the $(k+1)$st standard unit vector) in $\mathbf{R}^{k+1}$. Then $M$ is the contraction of $M'$ by the element $n+1$. Write $\mathcal{K}'$ for the signed cocircuit span of $M'$ and $T(\mathcal{K}') = \{\boldsymbol{\sigma} \in \{-,0,+\}^n : (\boldsymbol{\sigma},+) \in \mathcal{K}'\}$. The set $T(\mathcal{K}')$ contains $\mathcal{K}$. Other elements of $T(\mathcal{K}')$ correspond to subzonotopes of $Z$ that are not faces of $Z$.

A *zonotopal subdivision* $\Delta$ of the zonotope $Z$ is a polyhedral subdivision of $Z$ such that each cell is a zonotope whose edges are translations of edges of $Z$. In [3] it is proved that the zonotopes associated with the elements of $T(\mathcal{K}')$ form a zonotopal subdivision of $Z$. A zonotopal subdivision obtained in this way is called a *regular zonotopal subdivision*. The subdivision is called *cubical* if each cell is combinatorially a cube. The set $T(\mathcal{K}')$ with sign sets ordered as before is isomorphic to the face poset of the subdivision of the zonotope considered as a polyhedral complex. The regular zonotopal subdivision can be described geometrically as follows. Let $Z'$ be the zonotope whose oriented matroid is $M'$. Project the *top* faces of $Z'$ along the vector $\mathbf{e}_{k+1}$ onto $Z$. Their images subdivide $Z$. (These zonotopal subdivisions are also studied in [5]; see that paper also for the appropriate definitions when the generating vectors $\boldsymbol{\alpha}_i$ may occur more than once.)

**Example 3.3** Let $Z$ be the zonotope generated by vectors $\boldsymbol{\alpha}_1 = (1,0)$, $\boldsymbol{\alpha}_2 = (0,1)$, and $\boldsymbol{\alpha}_3 = (1,1)$; $Z$ is a hexagon. Now consider the 3-dimensional zonotope $Z'$ generated by $\boldsymbol{\beta}_1 = (1,0,0)$, $\boldsymbol{\beta}_2 = (0,1,0)$, $\boldsymbol{\beta}_3 = (1,1,1)$, and $\boldsymbol{\beta}_4 = (0,0,1)$. This is the convex hull

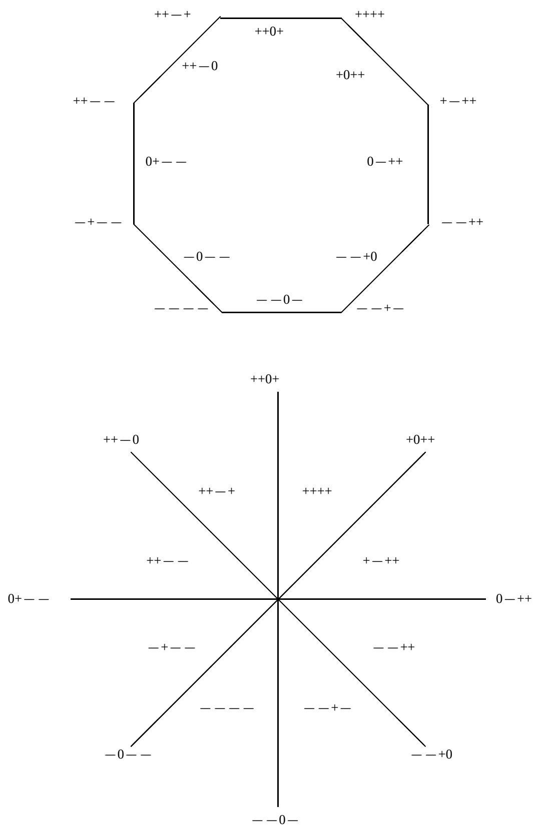


*Figure 2.* Sign patterns on an octagon and arrangement of four lines

of the cubes $C$ and $-C$, where $C$ has vertex set $\{(x_1, x_2, x_3) : x_i \in \{0, 2\}\}$. The vertices of $Z'$ are the 14 nonzero vertices of $C$ and $-C$. The top vertices of $Z'$ are the top nonzero vertices of the two cubes: $(0, 0, 2)$, $(2, 0, 2)$, $(0, 2, 2)$, $(2, 2, 2)$, $(0, -2, 0)$, $(-2, 0, 0)$, and $(-2, -2, 0)$. The top 2-faces are the square with supporting hyperplane $x_3 = 2$ and the two parallelograms with supporting hyperplanes $x_3 - x_1 = 2$ and $x_3 - x_2 = 2$. The projection of these onto $x_3 = 0$ subdivides the hexagon $Z$ into the upper right square and the two lower left parallelograms shown in Figure 3. □

The face lattice of the fiber zonotope $\Sigma(C_n, Z)$ is isomorphic to the poset of regular zonotopal subdivisions of $Z$, with vertices corresponding to regular cubical subdivisions [2]. Thus we have a poset of sign vectors ($T(\mathcal{K}')$) associated with every face of the fiber zonotope. The vertices of the fiber zonotope have associated oriented matroid liftings $M'$ with the heights $h_i$ chosen generically. This transfers to a correspondence between the distinct posets $T(\mathcal{K}')$ of sign vectors (for different liftings $M'$) and the faces of the

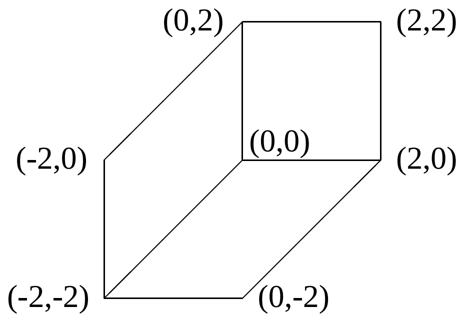


*Figure 3.* Subdivided hexagon

discriminantal arrangement. We want to interpret $T(\mathcal{K}')$ in terms of the base arrangement $\mathcal{A}$.

Let $\mathcal{A}'$ be the hyperplane arrangement whose oriented matroid is $M'$. Then $\mathcal{A}$ is the induced arrangement on the hyperplane $H_{n+1} = \{\mathbf{x} \in \mathbf{R}^{k+1} : x_{k+1} = 0\}$. The sign vectors $\boldsymbol{\tau}$ of $M'$ with $\tau_{n+1} = +$ correspond to the faces of $\mathcal{A}'$ above (on the positive side of) $H_{n+1}$. Let $\mathcal{A}_*$ be the affine arrangement induced by $\mathcal{A}'$ on the hyperplane $x_{k+1} = 1$, considered as an affine arrangement in $\mathbf{R}^k$. Then $\mathcal{A}_*$ is a parallel translate of the central arrangement $\mathcal{A}$; in fact, $\mathcal{A}_* = \mathcal{A}_{-\mathbf{h}}$. Each face of $\mathcal{A}_*$ is contained in a unique face of $\mathcal{A}'$ above $H_{n+1}$, and hence inherits a sign vector of length $n+1$ with $\sigma_{n+1} = +$. Dropping the last coordinate gives a length $n$ sign vector with the natural interpretation: $\sigma_i = 0$ if the face is contained in the $i$th hyperplane of $\mathcal{A}_*$; $\sigma_i = +$ if the face is on the positive side and $\sigma_i = -$ if the face is on the negative side of the $i$th hyperplane. Thus associated with every face $F$ of the discriminantal arrangement is a poset of sign vectors isomorphic to the face poset of any parallel translate $\mathcal{A}_{\mathbf{b}}$ with $\mathbf{b} \in F$. The “generic” oriented matroid liftings (those associated with cubical subdivisions) give the relatively general position parallel translates. Thus we get the following.

**Theorem 3.4** *Each open face of $\mathcal{B}(\mathcal{A})$ consists of those points $\mathbf{b} \in \mathbf{R}^k$ for which the face poset of $\mathcal{A}_{\mathbf{b}}$ is isomorphic to a fixed poset of sign vectors, $T(\mathcal{K}')$.*

Here we see that a regular zonotopal subdivision of $Z$ corresponds to a family of parallel translates $\mathcal{A}_{\mathbf{b}}$ with the same face poset. What about nonregular zonotopal subdivisions? The face poset of such a subdivision is realized by an affine pseudoarrangement (arrangement of codimension one surfaces) that agrees with the base arrangement outside a bounded region.

**Example 3.5** (Example 2.5 continued) For Example 2.5 we observed that the discriminantal arrangement is of essential dimension two, and has three hyperplanes. So we represent $\mathcal{B}(\mathcal{A})$ by an arrangement of three lines in $\mathbf{R}^2$ (Figure 4). This arrangement has six chambers and six one-dimensional faces, each of which corresponds to a zonotopal subdivision of the original zonotope $Z$. What are these subdivisions? In discussing $Z$ we refer to the hexagonal face with sign vector 000−− as the "front hexagon," and picture it as the hexagon in the lower right of Figure 1; its minus (not pictured) is the "back hexagon." Similarly we talk about the "top" (at the top of the figure) and "bottom" (not pictured) hexagon—they have sign vectors 0++00 and 0−−00, respectively. The quadrilateral faces in the picture have sign vectors (moving clockwise from upper right) −+00−, −0−0−, −0−+0 and −+0+0. The part of $Z$ near these faces is referred to as the "left"; the "right" of the polytope is not pictured. Now we describe regular zonotopal subdivisions of $Z$. Subdivision $\Sigma_1$ corresponds to the affine arrangement $\mathcal{A}_{\mathbf{b}}$, with $\mathbf{b} = (0, 0, 0, 0, 1)$. The new vertices in the subdivision have sign vectors +−−+−, +−++− and ++++−. The maximal cells of $\Sigma_1$ are two hexagonal prisms and two cubes. Subdivision $\Sigma_2$ is the minus of $\Sigma_1$. Subdivisions $\Sigma_3$ and $\Sigma_4$ are combinatorially equivalent to $\Sigma_1$ and $\Sigma_2$; they correspond to $\mathcal{A}_{\mathbf{b}}$, with $\mathbf{b} = \pm(0, 0, 1, 0, 0)$. The new vertices for $\Sigma_3$ have sign vectors ++−−−, ++−−+ and ++−++ (and for $\Sigma_4$ the minuses of these).

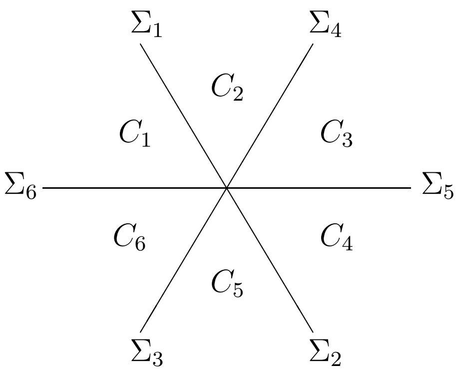


*Figure 4.* Subdivisions of three-dimensional zonotope

These four subdivisions of $Z$ correspond to the 1-faces on two lines of (the 2-dimensional representation of) $\mathcal{B}(\mathcal{A})$. They have cubical subdivisions that are easy to describe. Cubical subdivision $C_1$ has interior vertices with sign vectors +−−+−, +−++−, ++++−, ++−−−, ++−+− and ++−++. It is a refinement of $\Sigma_1$ but of no other $\Sigma_i$ that we have described so far. Cubical subdivision $C_2$ has interior vertices with sign vectors +−−+−, +−++−, ++++−, −−+−−, −−++−, and −−+++. It is a refinement of $\Sigma_1$ and $\Sigma_4$. The cubical subdivision $C_1$ is a refinement of a second subdivision, but it is of a different combinatorial type. Let $\Sigma_5$ be the subdivision corresponding to $\mathcal{A}_{\mathbf{b}}$, with $\mathbf{b} = (1, 0, 0, 0, 0)$. The new vertices of $\Sigma_5$ have sign vectors −++−+, −−+−+ and −−−−+. Four of its maximal cells are cubes. The other is a twelve-sided zonotope, parallel to the zonotope generated by vectors $\boldsymbol{\alpha}_i$, $i = 2, 3, 4, 5$. The final subdivision $\Sigma_6$ is the minus of this. Figure 4 shows how these subdivisions (along with four other cubical subdivisions) are related by refinement. □

## 4. The Intersection Lattice

Let us turn to the intersection lattice of the hyperplane arrangement $\mathcal{A}$. This is the set of all subspaces obtained as the intersection of some of the hyperplanes, ordered (conventionally) by reverse inclusion. The lattice is *ranked*, that is, all chains up to a fixed subspace $K$ are the same length, called the *rank*, $r(K)$, of $K$. The intersection lattice is isomorphic to the lattice of flats of the (unoriented) matroid underlying the oriented matroid. A flat in the underlying matroid is just the zero set of an element in the signed cocircuit span of the oriented matroid; in the hyperplane arrangement this amounts to saying that each face is a subset of a unique subspace (intersection of hyperplanes) of the same dimension, and that all maximal faces on one hyperplane have sign vectors with the same zero set. What in the zonotope corresponds to an intersection of hyperplanes? For $S \subseteq [n]$, let $K = \bigcap_{i \in S} H_i$. Define the $K$-zone of $Z$ to be the set of faces of $Z$ of the form $\sum_{i \in S}[-\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i] + \sum_{i \notin S} \epsilon_i \boldsymbol{\alpha}_i$, where $\epsilon_i \in \{+, -\}$. The $K$-zone is thus the collection of faces of $Z$ whose corresponding faces in $\mathcal{A}$ are contained in and have the same dimension as $K$. (The dimension of a face in the $K$-zone is thus codim $K = \dim\{\boldsymbol{\alpha}_i : i \in S\}$.) Note that for each face $F$ of $Z$ the faces that are translates of $F$ form a zone.

Now we consider the intersection lattice of the discriminantal arrangement and the zones of the fiber zonotope. Recall that a hyperplane of the discriminantal arrangement $\mathcal{B}(\mathcal{A})$ corresponds to a minimal dependent subset of $[n]$ (meaning a minimal dependency of the $\boldsymbol{\alpha}_i$). An element of the intersection lattice of $\mathcal{B}(\mathcal{A})$ is thus a subspace of $\mathbf{R}^n$ of the form $K = \bigcap_{i=1}^{q} D_{S_i}$, where for each $i$, $S_i$ is a minimal dependent subset of $[n]$. The maximal faces of $\mathcal{B}(\mathcal{A})$ contained in $K$ correspond to combinatorial types of parallel translates $\mathcal{A}_{\mathbf{b}}$ such that for each $j$, $1 \leq j \leq q$, $\bigcap_{i \in S_j} H_i = k - |S_j| + 1$.

These faces correspond to regular zonotopal subdivisions of $Z$. What do the zonotopal subdivisions corresponding to maximal faces of the same subspace $K$ have in common? Given any two such zonotopal subdivisions, $\Delta_1$ and $\Delta_2$, there is a bijection $\phi$ from the faces of $\Delta_1$ to the faces of $\Delta_2$ so that for every face $F$ of $\Delta_1$, $\phi(F)$ is a translate of $F$. Conversely, any two zonotopal subdivisions related in this way correspond to two maximal faces of the same subspace in the intersection lattice of the discriminantal arrangement. Note that for any regular zonotopal subdivision $S$, $-S$ is also a regular zonotopal subdivision, whose cells are translates of the cells of $S$. In the discriminantal arrangement every face has an opposite face, which of course spans the same subspace.

**Example 4.1** (Example 2.5 continued) The only proper subspaces in the intersection lattice of $\mathcal{B}(\mathcal{A})$ are the hyperplanes themselves. These have only two maximal faces each, corresponding to opposite zonotopal subdivisions of $Z$. The chambers of the arrangement all span the top element of the intersection lattice. They correspond to cubical regular subdivisions of $Z$. All cubical subdivisions of $Z$ have the same collection of cubes; in different subdivisions they are in different positions. Each cubical subdivision has exactly one translate of the cube $\sum_{i \in S}[-\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i]$ for each independent set $S$ of size $\dim Z$. □

We turn now to the sizes of the lattices of the discriminantal arrangements. Recall (note after Theorem 2.3) that for $\mathcal{A}$ a $k$-arrangement of $n$ hyperplanes, $\mathcal{B}(\mathcal{A})$ has at most $\binom{n}{k+1}$ hyperplanes, with equality if $\mathcal{A}$ is generic. Actually it is clear that $\mathcal{B}(\mathcal{A})$ has exactly $\binom{n}{k+1}$

hyperplanes if and only if $\mathcal{A}$ is generic. Falk observed that the intersection lattices of the discriminantal arrangements of different generic arrangements may differ in the number of rank 2 elements.

Suppose first that $\mathcal{A}$ is a generic arrangement. Then the intersection lattice $L(\mathcal{B}(\mathcal{A}))$ has as a sublattice a truncated Boolean lattice $L_{n,k}$, the lattice of all subsets of $[n]$ with at least $k+1$ elements (plus the empty set). To see this, consider a set $S \subseteq [n]$ with $|S| \geq k+1$. Then there is a parallel translate $\mathcal{A}_{\mathbf{b}(S)}$ of $\mathcal{A}$ for which $\bigcap_{i\in S} H_i = 0$ and all other hyperplanes are in general position. The minimal subspaces of $\mathcal{B}(\mathcal{A})$ containing the $\mathbf{b}(S)$ for the different sets $S$ are all distinct. (Those for which $|S| = k+1$ are, of course, the hyperplanes of $\mathcal{B}(\mathcal{A})$.) In this sublattice $L_{n,k}$ of $L(\mathcal{B}(\mathcal{A}))$ every subspace of rank $j$ (dimension $n-k-j$ when $\mathcal{B}(\mathcal{A})$ is considered as an $(n-k)$-arrangement) is contained in $\binom{k+j}{k+1}$ hyperplanes.

Now consider the rank two elements of $L(\mathcal{B}(\mathcal{A}))$. In any central arrangement every two hyperplanes intersect in a rank two subspace, and every rank two subspace is contained in at least two hyperplanes. Each rank two element in the truncated Boolean sublattice is contained in $k+2$ hyperplanes. The largest number of rank two elements would occur if every rank two element not in the truncated Boolean sublattice is contained in exactly two hyperplanes. According to Manin and Schechtman [8], this occurs for arrangements of $k+3$ hyperplanes in $\mathbf{R}^k$ that form an open Zariski dense subset of all arrangements of that size. Falk [6] gives an example of a generic arrangement $\mathcal{A}$ of six planes in $\mathbf{R}^3$ for which $\mathcal{B}(\mathcal{A})$ has fewer rank two elements. He refers to Manin and Schechtman's arrangements as "sufficiently general."

**Definition 4.2** An arrangement $\mathcal{A}$ of $n$ hyperplanes in $\mathbf{R}^k$ is *very generic* if for all $r$, $L(\mathcal{B}(\mathcal{A}))$ achieves the maximum number of rank $r$ elements possible for a discriminantal arrangement based on a $k$-arrangement with $n$ hyperplanes.

We conjecture the following description of the intersection lattice $L(\mathcal{B}(\mathcal{A}))$ for a discriminantal arrangement $\mathcal{B}(\mathcal{A})$ of a very generic arrangement $\mathcal{A}$. For $n \geq k+1 \geq 2$ let $P(n,k)$ be the following poset. The elements are sets $\{S_1, S_2, \ldots, S_m\}$ of subsets of $\{1, 2, \ldots, n\}$ satisfying

1. for each $i$, $|S_i| \geq k+1$
2. for each $I \subseteq \{1, 2, \ldots, m\}$ with $|I| \geq 2$, $|\bigcup_{i\in I} S_i| > k + \sum_{i\in I}(|S_i| - k)$.

The ordering is given by $\{S_1, S_2, \ldots, S_m\} \preceq \{T_1, T_2, \ldots, T_p\}$ if and only if for each $i$ there exists $j$ such that $S_i \subseteq T_j$. This is a ranked poset, with

$$\text{rank}\{S_1, S_2, \ldots, S_m\} = \sum_{i=1}^{m}(|S_i| - k).$$

**Conjecture 4.3** *Let $n \geq k+1 \geq 2$.*

1. *There exist arrangements $\mathcal{A}$ of $n$ hyperplanes in $\mathbf{R}^k$ that are very generic, that is, all rank sets of the intersection lattice $L(\mathcal{B}(\mathcal{A}))$ are of maximum size.*

2. *The intersection lattice $L(\mathcal{B}(\mathcal{A}))$ of the discriminantal arrangement based on a very generic arrangement of $n$ hyperplanes in $\mathbf{R}^k$ is isomorphic to $P(n,k)$.*

The conjecture holds for $k+1 \leq n \leq k+3$ ([8, Proposition 4]). Falk's arrangement fails to be very generic because of "second-order" dependencies: while the normals of the base hyperplanes are in general position, the minimal linear dependencies have extra linear relations beyond the Grassmann-Plücker relations (see [3]). Our guess is that very generic arrangements can be constructed by choosing the coordinates of the normal vectors from an algebraically independent set. Another candidate for a very generic arrangement is the *cyclic arrangement* [11]. This is obtained by taking normal vectors $\boldsymbol{\alpha}_i = (1, t_i, t_i^2, \ldots, t_i^{k-1})$ for $n$ arbitrary, distinct real numbers $t_i$.

## 5. Freeness and Formality

Of special interest in the study of hyperplane arrangements is the question of whether a given arrangement is free [9]. This is often difficult to determine. The notions of formality [10] and $i$-formality [4] are useful as tools for deciding freeness. If an arrangement is free, then it is $i$-formal for all $i$, but not conversely. We see in this section that the discriminantal arrangement based on a generic arrangement is formal, but not necessarily free. We give one example where the discriminantal arrangement is free. This cannot happen if the base arrangement is very generic, but in that case we can prove higher formality for the discriminantal arrangement. Edelman and Reiner [5] classified all multiarrangements $\mathcal{A}$ in $\mathbf{R}^2$ for which the discriminantal arrangement $\mathcal{B}(\mathcal{A})$ is free.

**Example 5.1 (A free discriminantal arrangement)** Let $\mathcal{A}$ be the central arrangement of six planes in $\mathbf{R}^3$ defined by the forms in the product

$$Q_{\mathcal{A}} = xyz(2x+2y+z)(6x+3y+z)(4x+y+z).$$

By the adjoint construction in [6], the discriminantal arrangement can be written as an arrangement of fifteen planes in $\mathbf{R}^3$, with

$$\begin{aligned} Q_{\mathcal{B}(\mathcal{A})} = {} & x(x-y)(x+y)(x-z)(x+2z)(5x+3y-8z)(x+3y-4z) \times \\ & (x-3y+2z)(x-3y+8z)(7x-3y+8z)(11x+3y-8z) \times \\ & (2x+3y-2z)(4x-3y+2z)(5x+3y-2z)(x-3y-4z). \end{aligned}$$

A supersolvable arrangement can be obtained by adding ten hyperplanes of the form $cx - 3y - 4z = 0$, in such a way that the addition-deletion theorem ([9, Theorem 4.51]) proves $\mathcal{B}(\mathcal{A})$ free.

Discriminantal arrangements are not in general free, however. When $\mathcal{A}$ is a very generic arrangement of $n = k+3$ hyperplanes in $\mathbf{R}^k$, the characteristic polynomial of the lattice $L(\mathcal{B}(\mathcal{A}))$ is known not to factor. Thus in this case the discriminantal arrangement is not free ([9, Proposition 5.120]). Reiner (private communication) points out that since a localization of a free arrangement is free ([9, Theorem 4.37]), the following more general statement holds.

**Theorem 5.2** *If $\mathcal{A} \subset \mathbf{R}^k$ contains a very generic subarrangement of $k+3$ hyperplanes, then $\mathcal{B}(\mathcal{A})$ is not free.*

Other known examples (such as Falk's) of generic (but not very generic) arrangements of $k+3$ hyperplanes also give rise to nonfree discriminantal arrangements.

The *rank* (or essential dimension) $r(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$ is defined to be the rank of the top element of the intersection lattice $L(\mathcal{A})$. An arrangement $\mathcal{A}$ is formal if the space of linear relations among the normals to the hyperplanes of $\mathcal{A}$ is generated by the relations associated to the rank 2 subspaces in $L(\mathcal{A})$. Let $F(\mathcal{A})$ and $I(\mathcal{A})$ be the kernel and image, respectively, of the map $\bigoplus_{H\in\mathcal{A}} \mathbf{R}\mathbf{e}_H \longrightarrow \mathbf{R}^k$ induced by $\mathbf{e}_H \to \boldsymbol{\alpha}_H$. The elements of $F(\mathcal{A})$ are called relations. Note that $I(\mathcal{A})$ has dimension equal to the rank $r(\mathcal{A})$ of $\mathcal{A}$, so $\dim F(\mathcal{A}) = |\mathcal{A}| - r(\mathcal{A})$. For a subspace $X$ in $L(\mathcal{A})$, $\mathcal{A}_X$ denotes the subarrangement of $\mathcal{A}$ consisting of the hyperplanes containing $X$. There is a natural inclusion map, $F(\mathcal{A}_X) \hookrightarrow F(\mathcal{A})$.

**Definition 5.3** The arrangement $\mathcal{A}$ is *formal* if the inclusions, $F(\mathcal{A}_X) \hookrightarrow F(\mathcal{A})$, induce a surjection

$$\pi_2 : \bigoplus_{\substack{X\in L(\mathcal{A})\\ r(X)=2}} F(\mathcal{A}_X) \longrightarrow F(\mathcal{A}).$$

A more general property known as $i$-formality ($2 \le i < r(\mathcal{A})$) involves certain relations among relations corresponding to the codimension $i$ subspaces in $L(\mathcal{A})$. When $i = 2$ this reduces to formality. We define 3-formality as follows. Suppose $\mathcal{A}$ is formal, and let $R(\mathcal{A})$ be the kernel of the map $\pi_2$. View $R(\mathcal{A})$ as a space of relations among the relations corresponding to the rank 2 elements of $L(\mathcal{A})$. For each $Y \in L(\mathcal{A})$ there is an inclusion map $R(\mathcal{A}_Y) \hookrightarrow R(\mathcal{A})$.

**Definition 5.4** The arrangement $\mathcal{A}$ is 3-*formal* if $\mathcal{A}$ is formal and the inclusions, $R(\mathcal{A}_Y) \hookrightarrow R(\mathcal{A})$, induce a surjection

$$\pi_3 : \bigoplus_{\substack{Y\in L(\mathcal{A})\\ r(Y)=3}} R(\mathcal{A}_Y) \longrightarrow R(\mathcal{A}).$$

For examples and the general definition of $i$-formality, see [4].

We use the following notation. Let $P$ be a finite set of integers. Let $C(P,j)$ denote the set of subsets of $P$ having $j$ elements. For $i \ge j$, let $C(i,j) = C([i],j)$. The sets $C(P,j)$ are ordered lexicographically, so we may speak of the maximal element of $C(P,j)$.

We now return to discriminantal arrangements. Here $\mathcal{A}_\mathbf{0}$ refers to a base arrangement, which is assumed to be a generic arrangement of $n$ hyperplanes in $\mathbf{R}^k$, and we write $\mathcal{B} = \mathcal{B}(\mathcal{A}_\mathbf{0})$. Since $\mathcal{A}_\mathbf{0}$ is generic, the truncated Boolean lattice $L_{n,k}$ can be viewed as a sublattice of $L(\mathcal{B})$. Also, there is a bijection between the atoms of $L(\mathcal{B})$ (the hyperplanes of $\mathcal{B}$) and those of $L_{n,k}$. Thus $\mathcal{B}$ is identified with $C(n,k+1)$ and ordered accordingly, and all elements of $C(n,j)$ for $k+1 \le j \le n$ can be considered as elements of $L(\mathcal{B})$. Since $|\mathcal{B}| = \binom{n}{k+1}$ and $r(\mathcal{B}) = n-k$, it follows that $\dim F(\mathcal{B}) = \binom{n}{k+1} - n + k$.

Let $X \in C(n, k+2)$. Thus, as an element of $L(\mathcal{B})$, $X$ is the intersection of $k+2$ hyperplanes and its rank is $r(X) = 2$. So $\dim F(\mathcal{B}_X) = (k+2) - 2 = k$. Any three distinct hyperplanes in $C(X, k+1)$ determine a relation in $F(\mathcal{B}_X)$, since their normals are linearly dependent. We use the following notation to describe such relations.

**Definition 5.5** For $X \in C(n, k+2)$ and $Q \in C(X, k-1)$, let $f_Q(X) \in F(\mathcal{B}_X)$ denote the relation determined by the three hyperplanes in $C(X, k+1)$ that contain $Q$.

For each $Y \in C(n, j)$ with $k+2 \leq j \leq n$, we define a map that assigns a relation in $F(\mathcal{B}_Y)$ to each hyperplane that is "large enough" in the lexicographical order on $\mathcal{B}_Y$. Let $Y[k]$ denote the first $k$ elements of $Y$ and let $\mathcal{B}_{Y[k]}$ be the set of hyperplanes of $\mathcal{B}_Y$ which contain $Y[k]$. The map $g_Y : \mathcal{B}_Y - \mathcal{B}_{Y[k]} \longrightarrow F(\mathcal{B}_Y)$ is defined as follows.

**Definition 5.6** Let $Y \in C(n, j)$ with $k+2 \leq j \leq n$, and $H = \{a_1, \dots, a_{k+1}\} \in \mathcal{B}_Y - \mathcal{B}_{Y[k]}$ (with $a_1 < a_2 < \cdots < a_{k+1}$). Let $i$ be the smallest element of $Y[k] - H$ and let $Q = \{a_1, \dots, a_{k-1}\}$. Then

$$g_Y(H) = f_Q(\{i\} \cup H).$$

Note that $H$ is the maximal hyperplane occurring in the relation $g_Y(H)$ with nonzero coefficient; call $H$ the *last hyperplane* in $g_Y(H)$. Also the set $X = \{i\} \cup H$ is an element of $C(Y, k+2)$, so $g_Y(H) = g_X(H) \in F(\mathcal{B}_X)$. Thus each $g_Y(H)$ is a relation associated to a rank 2 element in $L(\mathcal{B})$.

**Theorem 5.7** *Let $\mathcal{B} = \mathcal{B}(\mathcal{A}_\mathbf{0})$ where $\mathcal{A}_\mathbf{0}$ is generic. For each $Y \in C(n, j)$ with $k+2 \leq j \leq n$, the arrangement $\mathcal{B}_Y$ is formal. In particular, $\mathcal{B}$ is formal.*

**Proof:** We know that $\dim F(\mathcal{B}_Y) = \binom{j}{k+1} - j + k = |\mathcal{B}_Y - \mathcal{B}_{Y[k]}|$. Since each $g_Y(H) \in \mathrm{Im}(g_Y)$ has distinct last hyperplane, it follows that $\mathrm{Im}(g_Y)$ is a basis for $F(\mathcal{B}_Y)$. We have already seen that each $g_Y(H) \in F(\mathcal{B}_X)(= F((\mathcal{B}_Y)_X))$ for some $X \in C(Y, k+2)$. □

Next we show that if $\mathcal{A}_\mathbf{0}$ is very generic, then $\mathcal{B} = \mathcal{B}(\mathcal{A}_\mathbf{0})$ is actually 3-formal. In fact we do not use the entire strength of the definition of "very generic," but only the condition on rank two elements of $L(\mathcal{B})$. This is equivalent to the condition that every rank two element not in $C(n, k+2)$ is contained in exactly two hyperplanes.

Thus for $\mathcal{A}_\mathbf{0}$ very generic and a rank two element $X \in L(\mathcal{B})$ not in $C(n, k+2)$, we have $F(\mathcal{B}_X) = 0$. Recall that $\dim F(\mathcal{B}) = \binom{n}{k+1} - n + k$, and for each $X \in C(n, k+2)$, $\dim F(\mathcal{B}_X) = k$. By Theorem 5.7 there is an exact sequence

$$0 \longrightarrow R(\mathcal{B}) \longrightarrow \bigoplus_{X \in C(n,k+2)} F(\mathcal{B}_X) \longrightarrow F(\mathcal{B}) \longrightarrow 0,$$

and thus

$$\dim R(\mathcal{B}) = k\binom{n}{k+2} - \binom{n}{k+1} + n - k.$$

Next we define an important subset of $F(\mathcal{B})$.

**Definition 5.8** Let $G = \bigcup_{X \in C(n,k+2)} \mathrm{Im}(g_X)$.

Recall that for any $Y \in C(n,j)$ with $k+2 \leq j \leq n$ and any $H \in \mathcal{B}_Y - \mathcal{B}_{Y[k]}$, the relation $g_Y(H)$ is an element of $F(\mathcal{B}_X)$ for some $X \in C(n,k+2)$. Thus for each $Y$, we have $\mathrm{Im}(g_Y) \subset G$. Since $G$ has $k\binom{n}{k+2}$ elements and each $\mathrm{Im}(g_X)$ is a basis for $F(\mathcal{B}_X)$, there is an isomorphism

$$\bigoplus_{X \in C(n,k+2)} F(\mathcal{B}_X) \simeq \bigoplus_{f \in G} \mathbf{R} f.$$

Hence $R(\mathcal{B})$ can be viewed as the kernel of the map $\bigoplus_{f \in G} \mathbf{R} f \longrightarrow F(\mathcal{B})$.

Order $G$ by last hyperplanes in the relations as follows. For $g = g_X(H)$ and $g' = g_{X'}(H)$, say $g < g'$ if $H < H'$ or if $H = H'$ and $X < X'$. Then for any $Y \in C(n,j)$ with $k+2 \leq j \leq n$ and for any $H \in \mathcal{B}_Y - \mathcal{B}_{Y[k]}$, the relation $g_Y(H) \in G$ is minimal among the relations in $G \cap F(\mathcal{B}_Y)$ having last hyperplane $H$. In particular when $Y = [n]$, $g_{[n]}(H)$ is minimal among all relations in $G$ having last hyperplane $H$.

The proof of the next theorem is much like the proof of Theorem 5.7. To show that $\mathcal{B}$ is 3-formal, we demonstrate sufficiently many linearly independent elements of $R(\mathcal{B})$, where each one is actually an element of $R(\mathcal{B}_Y)$ for some rank 3 subspace $Y \in L(\mathcal{B})$.

**Theorem 5.9** *If $\mathcal{A}_\mathbf{0}$ is very generic, then $\mathcal{B} = \mathcal{B}(\mathcal{A}_\mathbf{0})$ is 3-formal.*

**Proof:** First note that

$$|G - \mathrm{Im}(g_{[n]})| = k\binom{n}{k+2} - \binom{n}{k+1} + n - k.$$

Choose some $f \in G - \mathrm{Im}(g_{[n]})$. Let $H$ be the last hyperplane in $f$, and let $g = g_{[n]}(H)$. Then there are sets $X$ and $X'$ in $C(n,k+2)$, with $g = g_X(H)$, $f = g_{X'}(H)$, $X < X'$ and $X \cap X' = H$. Thus both $f$ and $g$ are elements of $F(\mathcal{B}_Y)$, where $Y = X \cup X' \in C(n,k+3)$.

We know that $g \in \mathrm{Im}(g_Y)$. Since $f$ and $g$ have the same last hyperplane, $f \notin \mathrm{Im}(g_Y)$. The set $\mathrm{Im}(g_Y)$ is a basis for $F(\mathcal{B}_Y)$, so there is a relation $w(f) \in R(\mathcal{B}_Y)$ having nonzero coefficient on $f$ and zero coefficients outside the set $\mathrm{Im}(g_Y) \cup \{f\}$. By the ordering on $G$, the coefficient on $f$ is the last nonzero coefficient of $w(f)$. Hence the set $\{w(f) \mid f \in G - \mathrm{Im}(g_{[n]})\}$ is a basis for $R(\mathcal{B})$, and $\mathcal{B}$ is 3-formal. □

We remark that our techniques might be used to show that $\mathcal{B}$ is $i$-formal ($i \geq 4$) in the very generic case, but we do not have enough information about the lattice $L(\mathcal{B})$ (see Conjecture 4.3). We know of no example of an arrangement $\mathcal{A}_\mathbf{0}$, generic or otherwise, for which $\mathcal{B}(\mathcal{A}_\mathbf{0})$ fails to be $i$-formal. In the following example, $\mathcal{A}_\mathbf{0}$ is generic, but not very generic, yet $\mathcal{B}$ is 3-formal.

**Example 5.10** Let $\mathcal{A}_\mathbf{0}$ have normals $\boldsymbol{\alpha}_1 = (2,2,1)$, $\boldsymbol{\alpha}_2 = (2,3,2)$, $\boldsymbol{\alpha}_3 = (1,2,2)$, $\boldsymbol{\alpha}_4 = (0,0,1)$, $\boldsymbol{\alpha}_5 = (0,1,0)$, $\boldsymbol{\alpha}_6 = (1,0,0)$, $\boldsymbol{\alpha}_7 = (3,-1,1)$. Then $\mathcal{B} = \mathcal{B}(\mathcal{A}_\mathbf{0})$ is an arrangement of 35 planes in $\mathbf{R}^7$ with rank 4, so $\dim F(\mathcal{B}) = \binom{7}{4} - 7 + 3 = 31$. In ([6, Example 3.2]) it was shown that the arrangement $\mathcal{A}_{\mathbf{0}} - \boldsymbol{\alpha}_7$ is not very generic. Thus $\mathcal{A}_{\mathbf{0}}$ is not very generic. In particular, there are four rank 2 elements $X$ of $L(\mathcal{B}) - C(7,5)$ that are the intersection of three or more hyperplanes of $\mathcal{B}$. They are:

$$\begin{aligned} X_1 &= 1234 \cap 1456 \cap 2356 \\ X_2 &= 1236 \cap 1245 \cap 3456 \\ X_3 &= 1245 \cap 1346 \cap 2356 \\ X_4 &= 1256 \cap 1346 \cap 2345. \end{aligned}$$

Each $F(\mathcal{B}_{X_i})$ has dimension 1, so by the earlier exact sequence, $\dim R(\mathcal{B}) = 3\binom{7}{5} + 4 - 31 = 36$. For $i = 1, \ldots, 4$ let $f_i$ be a nonzero relation in $F(\mathcal{B}_{X_i})$. Let

$$G = \left( \bigcup_{X \in C(7,5)} \mathrm{Im}(g_X) \right) \cup \{f_1, f_2, f_3, f_4\},$$

ordered as before, with the additional relations as the largest four elements. As in the proof of Theorem 5.9, $R(\mathcal{B})$ contains 32 linearly independent elements from the $R(\mathcal{B}_Y)$, for rank 3 elements $Y$ of $L(\mathcal{B})$.

We know that the set $\mathrm{Im}(g_{[n]})$ is a basis for $F(\mathcal{B})$ so it is clear that for each $i$, there is a relation $w(f_i) \in R(\mathcal{B})$ having nonzero coefficient on $f_i$. What is needed is such a relation in $R(\mathcal{B}_Y)$ for some rank 3 element of $L(\mathcal{B})$. It turns out that each $X_i$ contains the rank 3 element $Y$ that is the intersection of the following hyperplanes: 1234, 1235, 1236, 1245, 1246, 1256, 1345, 1346, 1356, 1456, 2345, 2346, 2356, 2456, 3456. Observe that $|\mathcal{B}_Y| = 15$, so $\dim F(\mathcal{B}_Y) = 12$. It is not difficult to check that for each of the twelve $H \in \mathcal{B}_Y - \{1234, 1235, 1236\}$, the relation $g_{[n]}(H)$ is in $F(\mathcal{B}_Y)$, so $\mathrm{Im}(g_{[n]})$ contains a basis for $F(\mathcal{B}_Y)$. Since each $f_i \in F(\mathcal{B}_Y)$, we can choose each $w(f_i) \in R(\mathcal{B}_Y)$. Thus $\mathcal{B}$ is 3-formal. □

Discriminantal arrangements can be defined over the complex numbers **C**. In this case Theorem 5.7 and Theorem 5.9 still hold. An arrangement $\mathcal{A}$ in $\mathbf{C}^n$ is called a $K(\pi,1)$ arrangement if the complement $\mathbf{C}^n - \bigcup_{H \in \mathcal{A}} H$ is a $K(\pi,1)$ space. All $K(\pi,1)$ arrangements are formal [7]. It would be interesting to know which (if any) discriminantal arrangements are $K(\pi,1)$. Edelman and Reiner ([5, Section 4]) give an example of a free discriminantal arrangement (based on a nongeneric arrangement) which is not $K(\pi,1)$.

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