Journal of Algebraic Combinatorics **7** (1998), 5–15

A Note on Thin P-Polynomial and Dual-Thin Q-Polynomial Symmetric Association Schemes

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Received December 18, 1995

Abstract. Let Y denote a d-class symmetric association scheme, with $d \geq 3$. We show the following: If Y admits a P-polynomial structure with intersection numbers p^h_{ij} and Y is *1-thin* with respect to at least one vertex, then

$$p^1_{11} = 0 \Rightarrow p^i_{1i} = 0 \quad 1 \leq i \leq d-1.$$

If Y admits a Q-polynomial structure with Krein parameters q^h_{ij}, and Y is *dual 1-thin* with respect to at least one vertex, then

$$q^1_{11} = 0 \Rightarrow q^i_{1i} = 0 \quad 1 \leq i \leq d-1.$$

Keywords: Association scheme, distance-regular graph, intersection number, Q-polynomial

1. Introduction

Let Y denote a d-class symmetric association scheme, with $d \geq 3$. It is well-known that if Y admits a P-polynomial structure with intersection numbers p^h_{ij}, then

$$p^1_{11} \neq 0 \;\Rightarrow\; p^i_{1i} \neq 0 \quad 1 \leq i \leq d-1 \tag{1}$$

[1, Theorem 5.5.1]. The first author shows in [3] that if Y admits a Q-polynomial structure with Krein parameters q^h_{ij}, then

$$q^1_{11} \neq 0 \;\Rightarrow\; q^i_{1i} \neq 0 \quad 1 \leq i \leq d-1. \tag{2}$$

In the present paper we show the following: If Y admits a P-polynomial structure with intersection numbers p^h_{ij}, and Y is *1-thin* with respect to at least one vertex, then

$$p^1_{11} = 0 \;\Rightarrow\; p^i_{1i} = 0 \quad 1 \leq i \leq d-1. \tag{3}$$

If Y admits a Q-polynomial structure with Krein parameters q^h_{ij}, and Y is *dual 1-thin* with respect to at least one vertex, then

$$q^1_{11} = 0 \;\Rightarrow\; q^i_{1i} = 0 \quad 1 \leq i \leq d-1. \tag{4}$$

The *1-thin* and *dual 1-thin* conditions are defined in Section 1.4. Our main results are in Theorems 2.1 and 2.2.

In the following sections we introduce notation and recall basic results, following [1, Section 2.1] and [4, Section 3].

1.1. *Symmetric association schemes*

By a d-class *symmetric association scheme* we mean a pair $Y = (X, \{R_i\}_{0 \le i \le d})$, where X is a non-empty finite set, and where

(i) $\{R_i\}_{0 \le i \le d}$ is a partition of $X \times X$;
(ii) $R_0 = \{xx \mid x \in X\}$;
(iii) $R_i = R_i^t$ for $0 \le i \le d$, where $R_i^t = \{yx \mid xy \in R_i\}$;
(iv) there exist integers p_{ij}^h such that for all integers h with $0 \le h \le d$ and all vertices $x, y \in X$ with $xy \in R_h$,

$$p_{ij}^h = |\{z \in X \mid xz \in R_i,\ yz \in R_j\}| \quad 0 \le i,\ j \le d. \tag{5}$$

We refer to X as the *vertex set* of Y, and refer to the integers p_{ij}^h as the *intersection numbers* of Y. Abbreviate $k_i = p_{ii}^0$, and observe k_i is non-zero for $0 \le i \le d$.

1.2. *The Bose-Mesner algebra*

Let $Y = (X, \{R_i\}_{0 \le i \le d})$ denote a symmetric association scheme. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of matrices over $\mathbb{R}$ with rows and columns indexed by X. The *associate matrices* for Y are the matrices $A_0, \ldots, A_d \in \mathrm{Mat}_X(\mathbb{R})$ defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } xy \in R_i, \\ 0 & \text{otherwise} \end{cases} \qquad x, y \in X. \tag{6}$$

From (i)–(iv) above we obtain

$$A_0 + \cdots + A_d = J, \tag{7}$$

$$A_i \circ A_j = \delta_{ij} A_i \quad 0 \le i,\ j \le d, \tag{8}$$

$$A_0 = I, \tag{9}$$

$$A_i = A_i^t \quad 0 \le i \le d, \tag{10}$$

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \quad 0 \le i,\ j \le d, \tag{11}$$

where J is the all-1s matrix and $\circ$ denotes the entry-wise matrix product.

By the *Bose-Mesner* algebra of Y we mean the subalgebra M of $\mathrm{Mat}_X(\mathbb{R})$ generated by the associate matrices $A_0, \ldots, A_d$. Observe by (8) and (11) that the associate matrices form a basis for M. In particular, M is symmetric and closed under $\circ$.

The algebra M has a second basis $E_0, \ldots, E_d$ such that

$$E_0 + \cdots + E_d = I, \tag{12}$$
$$E_i E_j = \delta_{ij} E_i \quad 0 \le i, \ j \le d, \tag{13}$$
$$E_0 = \frac{1}{|X|} J, \tag{14}$$
$$E_i = E_i^t \quad 0 \le i \le d, \tag{15}$$

[1, Theorem 2.6.1]. We refer to $E_0, \ldots, E_d$ as the *primitive idempotents* of Y. Since M is closed under $\circ$, there exist real numbers q_{ij}^h satisfying

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^{d} q_{ij}^h E_h \quad 0 \le i, \ j \le d. \tag{16}$$

The numbers q_{ij}^h are the *Krein parameters* for Y. Abbreviate $k_i^* = q_{ii}^0$ for $0 \le i \le d$.

By (8), (9), and the fact that $A_0, \ldots, A_d$ is a basis for M, the primitive idempotents have constant diagonal; in fact

$$(E_i)_{xx} = \frac{k_i^*}{|X|} \quad 0 \le i \le d, \ x \in X \tag{17}$$

and $k_i^* \neq 0$ [1, p. 45]. We apply (17) in the proof of Lemma 4.1.

1.3. The dual Bose-Mesner algebra

Let Y denote a d-class symmetric association scheme with vertex set X, associate matrices $A_0, \ldots, A_d$, primitive idempotents $E_0, \ldots, E_d$, and Bose-Mesner algebra M. Fix a vertex $x \in X$.

For each integer i with $0 \le i \le d$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ defined by

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad y \in X. \tag{18}$$

We refer to $A_0^*, \ldots, A_d^*$ as the *dual associate matrices* for Y with respect to x. Let $M^* = M^*(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ generated by the dual associate matrices. We refer to M^* as the *dual Bose-Mesner algebra* for Y with respect to x. From (16) we obtain

$$A_i^* A_j^* = \sum_{h=0}^{d} q_{ij}^h A_h^* \quad 0 \le i, \ j \le d. \tag{19}$$

In particular, the dual associate matrices form a basis for M^*.

For each integer i with $0 \le i \le d$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ defined by

$$(E_i^*)_{yy} = (A_i)_{xy} \quad y \in X. \tag{20}$$

From (7), (8) we obtain

$$E_0^* + \cdots + E_d^* = I, \tag{21}$$
$$E_i^* E_j^* = \delta_{ij} E_i^* \quad 0 \le i, \ j \le d. \tag{22}$$

We refer to $E_0^*, \ldots, E_d^*$ as the *dual idempotents* for Y with respect to x. Note that the dual idempotents form a second basis for M^*.

1.4. *The thin and dual-thin conditions*

Let Y denote a d-class symmetric association scheme with vertex set X. Fix a vertex $x \in X$, and write $M^* = M^*(x)$.

Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ generated by M and M^*. We refer to T as the *subconstituent algebra* for Y with respect to x. By a *T-module* we mean a subspace of the standard module $V = \mathbb{R}^X$ which is closed under multiplication by T. A T-module is said to be *irreducible* if it properly contains no T-modules other than 0. Recall that T is semi-simple, so that V may be decomposed as a direct sum of irreducible T-modules [4, Lemma 3.4].

An irreducible T-module W is said to be *thin* if

$$\dim E_i^* W \le 1 \quad 0 \le i \le d, \tag{23}$$

and *dual thin* if

$$\dim E_i W \le 1 \quad 0 \le i \le d. \tag{24}$$

We say Y is *i-thin* with respect to x if every irreducible T-module W with $E_i^* W \ne 0$ is thin. We say Y is *dual i-thin* with respect to x if every irreducible T-module W with $E_i W \ne 0$ is dual thin.

1.5. *P- and Q-polynomial structures*

Let Y denote a d-class symmetric association scheme, with vertex set X, intersection numbers p_{ij}^h, and Krein parameters q_{ij}^h. We say that an ordering $A_0, \ldots, A_d$ of the associate matrices is a *P-polynomial structure* for Y whenever

$$p_{ij}^h = 0 \quad \text{if one of } h, i, j \text{ is greater than the sum of the other two,} \tag{25}$$
$$p_{ij}^h \ne 0 \quad \text{if one of } h, i, j \text{ is equal to the sum of the other two} \tag{26}$$

for $0 \leq h, i, j \leq d$. Recall that if $A_0, \ldots, A_d$ is a P-polynomial structure for Y, then A_1 generates M [4, Lemma 3.8].

We say that an ordering $E_0, \ldots, E_d$ of the primitive idempotents is a *Q-polynomial structure* for Y whenever

$$q^h_{ij} = 0 \quad \text{if one of } h, i, j \text{ is greater than the sum of the other two,} \tag{27}$$

$$q^h_{ij} \neq 0 \quad \text{if one of } h, i, j \text{ is equal to the sum of the other two} \tag{28}$$

for $0 \leq h, i, j \leq d$. Recall that if $E_0, \ldots, E_d$ is a Q-polynomial structure for Y, then for each $x \in X$ the dual associate matrix $A^*_1(x)$ generates $M^*(x)$ [4, Lemma 3.11].

2. Results

Our main results are the following:

Theorem 2.1 *Let Y denote a d-class symmetric association scheme, with $d \geq 3$. Suppose $A_0, \ldots, A_d$ is a P-polynomial structure for Y with intersection numbers p^h_{ij}, and suppose Y is 1-thin with respect to at least one vertex. Then*

$$p^1_{11} = 0 \Rightarrow p^i_{1i} = 0 \quad 1 \leq i \leq d-1. \tag{29}$$

We prove Theorem 2.1 in Section 3.

Theorem 2.2 *Let Y denote a d-class symmetric association scheme, with $d \geq 3$. Suppose $E_0, \ldots, E_d$ is a Q-polynomial structure for Y with Krein parameters q^h_{ij}, and suppose Y is dual 1-thin with respect to at least one vertex. Then*

$$q^1_{11} = 0 \Rightarrow q^i_{1i} = 0 \quad 1 \leq i \leq d-1. \tag{30}$$

We prove Theorem 2.2 in Section 4.

3. Proof of Theorem 2.1

Define a symmetric bilinear form on $\mathrm{Mat}_X(\mathbb{R})$ (where X is any set) by

$$\langle B, C\rangle = \mathrm{tr}(B^t C) \quad B, C \in \mathrm{Mat}_X(\mathbb{R}). \tag{31}$$

Observe that $\langle B, C\rangle$ is just the sum of the entries of $B \circ C$. In particular, the form is positive definite.

Lemma 3.1 (Terwilliger [4]) *Let $Y = (X, \{R_i\}_{0 \leq i \leq d})$ denote a symmetric association scheme with associate matrices $A_0, \ldots, A_d$ and intersection numbers p^h_{ij}. Fix a vertex $x \in X$, and write $E^*_i = E^*_i(x)$ for $0 \leq i \leq d$. Then*:

(i) *for* $0 \le h, h', i, i', j, j' \le d$,

$$\langle E_i^* A_h E_j^*, E_{i'}^* A_{h'} E_{j'}^* \rangle = \delta_{hh'} \delta_{ii'} \delta_{jj'} k_h p_{ij}^h; \tag{32}$$

(ii) *for* $0 \le h, i, j \le d$,

$$E_h^* A_i E_j^* = 0 \Leftrightarrow p_{ij}^h = 0. \tag{33}$$

Proof of (i): Observe

$$(E_i^* A_h E_j^*)_{yz} = (E_i^*)_{yy} (A_h)_{yz} (E_j^*)_{zz} \tag{34}$$
$$= (A_i)_{xy} (A_h)_{yz} (A_j)_{xz}, \tag{35}$$

so that $(E_i^* A_h E_j^*)_{yz} \neq 0$ if and only if $xy \in R_i$, $yz \in R_h$, and $xz \in R_j$. Since the relations $R_0, \dots, R_d$ are disjoint, the matrices $E_i^* A_h E_j^*$ and $E_{i'}^* A_{h'} E_{j'}^*$ have no non-zero entries in common unless $h = h', i = i', j = j'$. In this case there are precisely $k_h p_{ij}^h$ non-zero entries, each equal to 1. The result follows.

Proof of (ii): Immediate from (i). □

Let Y denote a d-class symmetric association scheme, with vertex set X. Suppose $A_0, \dots, A_d$ is a P-polynomial structure for Y, with intersection numbers p_{ij}^h. Fix a vertex $x \in X$, and write $T = T(x)$, $M^* = M^*(x)$, and $E_i^* = E_i^*(x)$ for $0 \le i \le d$.

There are three matrices in T which are of particular interest to us (their duals will be used in Section 4). These are the *lowering* matrix $L = L(x)$, the *flat* matrix $F = F(x)$, and the *raising* matrix $R = R(x)$, defined by

$$L = \sum_{i=1}^{d} E_{i-1}^* A_1 E_i^*, \tag{36}$$

$$F = \sum_{i=0}^{d} E_i^* A_1 E_i^*, \tag{37}$$

$$R = \sum_{i=0}^{d-1} E_{i+1}^* A_1 E_i^*. \tag{38}$$

It is easily shown using (25), (21), and (33) that

$$A_1 = L + F + R. \tag{39}$$

Recall that A_1 generates the Bose-Mesner algebra M, so that A_1 and $E_0^*, \dots, E_d^*$ generate T. In particular, L, F, R, and $E_0^*, \dots, E_d^*$ generate T by (39).

Lemma 3.2 *Let Y denote a d-class symmetric association scheme, with vertex set X. Suppose $A_0, \dots, A_d$ is a P-polynomial structure for Y, with intersection numbers p_{ij}^h. Fix* *a vertex* $x \in X$, *and write* $T = T(x)$, $L = L(x)$, *and* $E_i^* = E_i^*(x)$ *for* $0 \le i \le d$. *If* Y *is 1-thin with respect to* x, *then*:

(i) *for any irreducible* T*-module* W *with* $E_1^* W \ne 0$,

$$L E_i^* W = 0 \;\Rightarrow\; E_i^* W = 0 \quad 2 \le i \le d; \tag{40}$$

(ii) *for* $w \in T E_1^* V$,

$$L E_i^* w = 0 \;\Rightarrow\; E_i^* w = 0 \quad 2 \le i \le d; \tag{41}$$

(iii) *for* $B \in T E_1^*$,

$$L E_i^* B = 0 \;\Rightarrow\; E_i^* B = 0 \quad 2 \le i \le d. \tag{42}$$

Proof of (i): Let W be given. Fix an integer i with $2 \le i \le d$, and suppose $L E_i^* W = 0$. Let W' denote the subspace of W defined by

$$W' = E_i^* W + \cdots + E_d^* W. \tag{43}$$

Observe by (36)–(38) and (13) that W' is closed under multiplication by L, F, R, and $E_0^*, \ldots, E_d^*$. Since T is generated by these matrices, W' is a T-module. Since $E_1^* W' = 0$ and $E_1^* W \ne 0$, W' is a proper submodule of W. Since W is irreducible, we now have $W' = 0$, and $E_i^* W \subseteq W'$ is zero as desired.

Proof of (ii): Since V may be decomposed into a direct sum of irreducible T-modules, it suffices to show that the result holds for $w \in T E_1^* W$ where W is an irreducible T-module. Fix an integer i with $2 \le i \le d$ and an irreducible T-module W, and suppose $w \in T E_1^* W$ has $L E_i^* w = 0$.

Suppose $E_i^* w \ne 0$. Observe $E_1^* W \ne 0$, since $0 \ne E_i^* w \in E_i^* T E_1^* W$. Since Y is 1-thin with respect to x, W is thin and $\dim E_i^* W \le 1$. In particular, $E_i^* w \in E_i^* W$ spans $E_i^* W$, and $L E_i^* W = 0$. By (i) we have $E_i^* W = 0$, and $E_i^* w = 0$ for a contradiction. Thus $E_i^* w = 0$ as desired.

Proof of (iii): Immediate from (ii). □

Lemma 3.3 *Let* Y *denote a* d*-class symmetric association scheme, with vertex set* X. *Suppose* $A_0, \ldots, A_d$ *is a P-polynomial structure for* Y, *with intersection numbers* p_{ij}^h. *Fix a vertex* $x \in X$, *and write* $L = L(x)$ *and* $E_i^* = E_i^*(x)$ *for* $0 \le i \le d$. *Then*:

(i) *for* $1 \le i \le d-1$,

$$L E_i^* A_{i+1} E_1^* = p_{1,i+1}^i E_{i-1}^* A_i E_1^*; \tag{44}$$

(ii) *for* $1 \le i \le d$, *if* $p_{1,i-1}^{i-1} = 0$ *then*

$$L E_i^* A_i E_1^* = p_{1i}^i E_{i-1}^* A_i E_1^*. \tag{45}$$

Proof of (i): Let i be given. Observe by (22), (25), (33), (21), and (11) that

$$LE_i^* A_{i+1} E_1^* = E_{i-1}^* A E_i^* A_{i+1} E_1^* \tag{46}$$

$$= E_{i-1}^* A \left(\sum_{h=0}^{d} E_h^* \right) A_{i+1} E_1^* \tag{47}$$

$$= E_{i-1}^* A A_{i+1} E_1^* \tag{48}$$

$$= E_{i-1}^* \left(\sum_{h=0}^{d} p_{1,i+1}^h A_h \right) E_1^* \tag{49}$$

$$= p_{1,i+1}^i E_{i-1}^* A_i E_1^*, \tag{50}$$

as desired.

Proof of (ii): Let i be given, with $p_{1,i-1}^{i-1} = 0$. Observe as in (i) that

$$LE_i^* A_i E_1^* = E_{i-1}^* A E_i^* A_i E_1^* \tag{51}$$

$$= E_{i-1}^* A \left(\sum_{h=0}^{d} E_h^* \right) A_i E_1^* \tag{52}$$

$$= E_{i-1}^* A A_i E_1^* \tag{53}$$

$$= E_{i-1}^* \left(\sum_{h=0}^{d} p_{1i}^h A_h \right) E_1^* \tag{54}$$

$$= p_{1i}^i E_{i-1}^* A_i E_1^*, \tag{55}$$

as desired. □

Proof of Theorem 2.1: Suppose Y is *1-thin* with respect to x, and write $L = L(x)$ and $E_i^* = E_i^*(x)$ for $0 \leq i \leq d$. Suppose $p_{11}^1 = 0$, and suppose for a contradiction that $p_{1i}^i \neq 0$ for some i with $2 \leq i \leq d-1$. Fix $i \geq 2$ minimal with $p_{1i}^i \neq 0$. Then by Lemma 3.3,

$$0 = L\left(p_{1i}^i E_i^* A_{i+1} E_1^* - p_{1,i+1}^i E_i^* A_i E_1^*\right), \tag{56}$$

and by Lemma 3.2(iii),

$$0 = p_{1i}^i E_i^* A_{i+1} E_1^* - p_{1,i+1}^i E_i^* A_i E_1^*. \tag{57}$$

The summands in (57) are nonzero by (33) and orthogonal by (32), for a contradiction. Thus $p_{1i}^i = 0$ for $2 \leq i \leq d-1$, as desired. □

4. Proof of Theorem 2.2

Our proof is based upon the following result:

Lemma 4.1 (Cameron, Goethals, Seidel [2]) *Let Y denote a d-class symmetric association scheme, with vertex set X, primitive idempotents $E_0, \ldots, E_d$, and Krein parameters q_{ij}^h. Fix a vertex $x \in X$, and write $A_i^* = A_i^*(x)$ for $0 \le i \le d$. Then*:
(i) *for* $0 \le h, h', i, i', j, j' \le d$,

$$\langle E_i A_h^* E_j, E_{i'} A_{h'}^* E_{j'} \rangle = \delta_{hh'} \delta_{ii'} \delta_{jj'} k_h^* q_{ij}^h; \tag{58}$$

(ii) *for* $0 \le h, i, j \le d$,

$$E_h A_i^* E_j = 0 \ \Leftrightarrow \ q_{ij}^h = 0. \tag{59}$$

Proof of (i): Recall $\mathrm{tr}(BC) = \mathrm{tr}(CB)$, and observe by (15), (13), (18), (16), and (17) that

$$\begin{aligned}
\langle E_i A_h^* E_j, E_{i'} A_{h'}^* E_{j'} \rangle &= \mathrm{tr}(E_j A_h^* E_i E_{i'} A_{h'}^* E_{j'}) && (60)\\
&= \mathrm{tr}(E_{j'} E_j A_h^* E_i E_{i'} A_{h'}^*) && (61)\\
&= \delta_{ii'} \delta_{jj'} \mathrm{tr}(E_j A_h^* E_i A_{h'}^*) && (62)\\
&= \delta_{ii'} \delta_{jj'} \sum_{y,z \in X} (E_j)_{yz} (A_h^*)_{zz} (E_i)_{zy} (A_{h'}^*)_{yy} && (63)\\
&= \delta_{ii'} \delta_{jj'} |X|^2 \sum_{y,z \in X} (E_j)_{yz} (E_h)_{xz} (E_i)_{zy} (E_{h'})_{xy} && (64)\\
&= \delta_{ii'} \delta_{jj'} |X|^2 \sum_{y \in X} ((E_i \circ E_j) E_h)_{yx} (E_{h'})_{xy} && (65)\\
&= \delta_{ii'} \delta_{jj'} |X| q_{ij}^h \sum_{y \in X} (E_h)_{yx} (E_{h'})_{xy} && (66)\\
&= \delta_{ii'} \delta_{jj'} |X| q_{ij}^h (E_{h'} E_h)_{xx} && (67)\\
&= \delta_{hh'} \delta_{ii'} \delta_{jj'} |X| q_{ij}^h (E_h)_{xx} && (68)\\
&= \delta_{hh'} \delta_{ii'} \delta_{jj'} k_h^* q_{ij}^h, && (69)
\end{aligned}$$

as desired.

Proof of (ii): Immediate from (i). □

Let Y denote a d-class symmetric association scheme, with vertex set X. Suppose $E_0, \ldots, E_d$ is a Q-polynomial structure for Y, with Krein parameters q_{ij}^h. Fix a vertex $x \in X$, and write $T = T(x)$, $M^* = M^*(x)$, and $A_i^* = A_i^*(x)$ for $0 \le i \le d$.

The *dual lowering* matrix $L^* = L^*(x)$, the *dual flat* matrix $F^* = F^*(x)$, and the *dual raising* matrix $R^* = R^*(x)$ are defined by

$$L^* = \sum_{i=1}^{d} E_{i-1} A_1^* E_i, \tag{70}$$

$$F^* = \sum_{i=0}^{d} E_i A_1^* E_i, \tag{71}$$

$$R^* = \sum_{i=0}^{d-1} E_{i+1} A_1^* E_i. \tag{72}$$

It is easily shown using (27), (12), and (59) that

$$A_1^* = L^* + F^* + R^*. \tag{73}$$

Recall that A_1^* generates the dual Bose-Mesner algebra M^*, so that A_1^* and $E_0, \ldots, E_d$ generate T. In particular, L^*, F^*, R^*, and $E_0^*, \ldots, E_d^*$ generate T by (73).

Lemma 4.2 *Let Y denote a d-class symmetric association scheme, with vertex set X. Suppose $E_0, \ldots, E_d$ is a Q-polynomial structure for Y, with Krein parameters q_{ij}^h. Fix a vertex $x \in X$, and write $T = T(x)$, $L^* = L^*(x)$, and $A_i^* = A_i^*(x)$ for $0 \le i \le d$. If Y is dual 1-thin with respect to x, then*:

(i) *for any irreducible T-module W with $E_1 W \neq 0$,*

$$L^* E_i W = 0 \;\Rightarrow\; E_i W = 0 \quad 2 \le i \le d; \tag{74}$$

(ii) *for $w \in TE_1V$,*

$$L^* E_i w = 0 \;\Rightarrow\; E_i w = 0 \quad 2 \le i \le d; \tag{75}$$

(iii) *for $B \in TE_1$,*

$$L^* E_i B = 0 \;\Rightarrow\; E_i B = 0 \quad 2 \le i \le d. \tag{76}$$

Proof: Similar to the proof of Lemma 3.2. □

Lemma 4.3 *Let Y denote a d-class symmetric association scheme, with vertex set X. Suppose $E_0, \ldots, E_d$ is a Q-polynomial structure for Y, with Krein parameters q_{ij}^h. Fix a vertex $x \in X$, and write $L^* = L^*(x)$ and $A_i^* = A_i^*(x)$ for $0 \le i \le d$. Then*:

(i) *for $1 \le i \le d$,*

$$L^* E_i A_{i+1}^* E_1 = q_{1,i+1}^i E_{i-1} A_i^* E_1; \tag{77}$$

(ii) *for $1 \le i \le d$, if $q_{1,i-1}^{i-1} = 0$ then*

$$L^* E_i A_i^* E_1 = q_{1i}^i E_{i-1} A_i^* E_1. \tag{78}$$

Proof: Similar to the proof of Lemma 3.3. □

Proof of Theorem 2.2: Similar to the proof of Theorem 2.1. □

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