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Semimodular Lattices and Semibuildings*

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Abstract. In a ranked lattice, we consider two maximal chains, or "flags" to be i-adjacent if they are equal except possibly on rank i. Thus, a finite rank lattice is a chamber system. If the lattice is semimodular, as noted in [9], there is a "Jordan-Hölder permutation" between any two flags. This permutation has the properties of an S_n-distance function on the chamber system of flags. Using these notions, we define a W-semibuilding as a chamber system with certain additional properties similar to properties Tits used to characterize buildings. We show that finite rank semimodular lattices form an S_n-semibuilding, and develop a flag-based axiomatization of semimodular lattices. We refine these properties to axiomatize geometric, modular and distributive lattices as well, and to reprove Tits' result that S_n-buildings correspond to relatively complemented modular lattices (see [16], Section 6.1.5).

Keywords: semimodular lattice, chamber system, Jordan-Hölder permutation

1. Introduction

The paper [9] studies relationships between maximal chains, or *flags* in finite rank semimodular lattices. We say two flags are i-adjacent if they agree on all ranks except, possibly, rank i. Thus, the flags of the lattice form a chamber system, as used in the study of Coxeter groups and buildings. Furthermore, the Jordan-Hölder function as developed by Stanley in [13] and [14] and by Björner in [4] has many properties in common with an S_n-distance function. In this paper, we develop that analogy. The results here are related to results of Abels in [2]. He developed his own characterizations of the relationships between two flags in a semimodular lattice, and also used the Jordan-Hölder permutation extensively to prove his results. However, his approach is more geometric than the lattice-based viewpoint adopted here.

We define a *semibuilding* over a Coxeter group W as a chamber system with a W-distance function and with some additional properties similar to those used by Tits to define W-buildings in [17]. We define an *upper semibuilding* as an S_n-semibuilding with an additional property that is obeyed by the flags of a semimodular lattice. (We do not define upper W-semibuildings for $W \neq S_n$.)

Upper semibuildings are closely related to upper semimodular lattices. From the results in [9], we show that the chamber system formed by the flags of a semimodular lattice under the relation of i-adjacency is an upper semibuilding. The Jordan-Hölder permutation is the

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required S_n-distance function. Conversely, for an S_n-semibuilding B, we construct a ranked lattice whose flags form a chamber system isomorphic to B. We show that the lattice is semimodular if and only if B is an upper semibuilding. By performing this construction on the upper semibuilding given by the flags of a semimodular lattice, we obtain the original lattice. Thus, we have a flag-based axiomatization of finite rank semimodular lattices: a poset is a rank n semimodular lattice if and only if its maximal chains form an upper semibuilding.

We also show how to add extra constraints to upper semibuildings to determine when they correspond to modular and distributive lattices, and we also give a condition which determines when the lattice for an S_n-semibuilding (not necessarily an upper semibuilding) is relatively complemented. This enables us to prove Tits' result that S_n-buildings correspond to finite rank, relatively complemented modular lattices, and also allows us to characterize finite rank geometric lattices, since a geometric lattice is simply a relatively complemented semimodular lattice (see [12], Proposition 3.3.3).

We review the pertinent definitions and results from the study of buildings and from [9] in Section 2, and in Section 3, we define semibuildings and relate them to semimodular lattices.

2. Preliminaries

We wish to relate the concepts from the paper [9] to the study of buildings. We first recall the definitions concerning buildings, and then present the results from [9].

2.1. Coxeter groups and buildings

To define buildings, we need two sets of preliminary definitions; one set for Coxeter groups, and another for chamber systems.

Definitions for Coxeter groups The group W is a *Coxeter group*, if W is generated by a set of involutions $\{r_i : i \in I\}$ whose only relations are of the form $(r_i r_j)^{m_{ij}} = \mathbf{1}$, the identity in W. The generating involutions are called *simple reflections*. For example, S_n is generated by the adjacent transpositions, $r_i = (i \;\; i+1)$, so these are the simple reflections. A *decomposition* of τ in W is an expression of τ as a product of simple reflections. The decomposition is *reduced* if there is no shorter decomposition of τ. Finally, the *weak Bruhat order on* W is given by $\rho \leq \tau$ if some reduced decomposition of τ begins with a decomposition of ρ.

Definitions for chamber systems A *chamber system* is a collection of elements called *chambers* together with an equivalence relation called i*-adjacency* on the chambers for each i in some indexing set I. We say the chamber system has *finite rank* if the set I is finite. A *gallery of type* $r_{i_1} r_{i_2} \cdots r_{i_m}$ between the chambers X and Y is a sequence of chambers $(X = Z_0, Z_1, \ldots, Z_m = Y)$ such that Z_k and Z_{k+1} are i_k-adjacent for each k.

Remark The more usual terminology for what we call a gallery of type $r_{i_1} r_{i_2} \cdots r_{i_m}$ is "a gallery of type $(i_1, i_2, \ldots, i_m)$." We have adopted this alternate notation for consistency with the notation of Section 7 in [9].

The following definition of a building can be found in [17] and elsewhere.

Definition A W-*building* is a chamber system Δ over the indexing set I with a function $\delta : \Delta \times \Delta \to W$ (called a W-*distance function*) such that $\delta(X, Y) = r_i$ if and only if X and Y are distinct and i-adjacent, and such that Δ obeys the following conditions.

B0. Every chamber is i-adjacent to at least one other chamber for each i in I.
B1. If $\delta(X, Y) = \tau$, and $\delta(Y, Y') = r_i$, then either $\delta(X, Y') = \tau$ or $\delta(X, Y') = \tau r_i$. Furthermore, if $\tau < \tau r_i$ in the weak Bruhat order, then $\delta(X, Y') = \tau r_i$.
B2. For every reduced decomposition f of $\delta(X, Y)$, there exists a gallery of type f between X and Y. Such a gallery is called a *minimal gallery*.

2.2. *Minimal paths between flags in semimodular lattices*

In [9] finite rank semimodular lattices were studied by considering their maximal chains, or *flags*, and the adjacency relationships between the flags. Two flags are i-*adjacent* if they agree on all ranks except possibly rank i. From this point of view, the flags of a semimodular lattice form a chamber system. A *path* from X to Y is a gallery between X and Y, and a *reduced path* is a minimal gallery from X to Y. Finally, if a minimal gallery has type f, we say the decomposition f *takes* X *to* Y along the path.

Two useful tools for studying these relationships were the Jordan-Hölder permutation and the labeling functions as developed by Stanley in [13] and [14]. We recall the definitions of these concepts.

Definitions If X and Y are two flags in a semimodular lattice, we define $\pi(X, Y)$, the *Jordan-Hölder function of* Y *relative to* X from $[n] = \{1, 2, \ldots, n\}$ to itself by:

$$\pi(X, Y)(j) = \min\{i : y_j \leq x_i \vee y_{j-1}\} = \min\{i : x_i \vee y_{j-1} = x_i \vee y_j\}.$$

The *labeling function with respect to* X is a function from points in the lattice to subsets of $[n]$. It is defined as follows:

$$l_X(z) = \{i \in [n] : x_i \leq x_{i-1} \vee z\} = \{i \in [n] : x_i \vee z = x_{i-1} \vee z\}. \tag{1}$$

We call $l_X(z)$ the X-*label of* z.

The properties in Proposition 2.1 of the Jordan-Hölder permutation and of labels were proved separately in [9].

Proposition 2.1 *If* X, Y *and* Y' *are flags in a semimodular lattice with* $\tau = \pi(X, Y)$ *and* $\tau' = \pi(X, Y')$, *then the following properties hold for the labeling function and the Jordan-Hölder function.*

(i) *The functions* τ *and* τ' *are permutations in* S_n.
(ii) *If* Y *and* Y' *are* j-*adjacent then either* $\tau' = \tau$ *or* $\tau' = \tau r_j$. *Furthermore, if* $Y \neq Y'$ *and* $\tau < \tau r_j$ *in the weak Bruhat order, then* $\tau' = \tau r_j$.

(iii) *The element* i *is in* $l_X(y_j)$ *if and only if* $i = \tau(k)$ *for some* $k \le j$, *i.e.*, $l_X(y_j) = \tau([j]) = \{\tau(1), \ldots, \tau(j)\}$, *so the cardinality of* $l_X(z)$ *equals the rank of* z *for all* z *in the lattice.*

In particular, the statement (ii) says that $\pi(X, Y)$ is an S_n-distance function, and that the flags of a semimodular lattice obey the axiom B1. We can relate the flags to other building axioms using Proposition 2.2 (Proposition 7.1 in [9]).

Proposition 2.2 *Let* S *be the set of reduced decompositions which take* X *to* Y *in some semimodular lattice. Then* S *is nonempty and has the following properties.*
R1. *If* $f r_i r_j h$ *is in* S *and* r_i *and* r_j *commute, then* $f r_j r_i h$ *is in* S.
R2. *If* $f r_i r_{i+1} r_i h$ *is in* S *then* $f r_{i+1} r_i r_{i+1} h$ *is in* S.

To better describe the relation between these properties and the flags of semimodular lattices and other lattices, we develop the notion of semibuildings.

3. Semibuildings

We note that the flags of finite rank semimodular lattices obey axioms similar to those for a building. We therefore make the following definitions.

Definitions A W-*semibuilding* is a chamber system with a W-distance function δ such that:

S1. If $\delta(X, Y) = \tau$, and $\delta(Y, Y') = r_i$, then either $\delta(X, Y') = \tau$ or $\delta(X, Y') = \tau r_i$. Furthermore, if $\tau < \tau r_i$ in the weak Bruhat order, then $\delta(X, Y') = \tau r_i$.
S2. For some reduced decomposition f of $\delta(X, Y)$, there exists a gallery of type f between X and Y.
S3. If r_i and r_j commute and $\delta(X, Y) = r_i r_j$, then there are galleries of type $r_i r_j$ and of type $r_j r_i$ between X and Y.

An *upper* (S_n)-*semibuilding* is an S_n-semibuilding with the additional property:

U4. If $\delta(X, Y) = (k\ \ k+2)$, then there is a gallery between X and Y of type $r_{k+1} r_k r_{k+1}$.

In particular, an S_n-building is an upper S_n-semibuilding, since condition B1 implies S1 and condition B2 implies S2, S3 and U4.

We have chosen to include condition S3 in the definition of a semibuilding because our applications all require this condition. We also focus almost entirely on the case $W = S_n$, so all semibuilding are S_n-semibuildings unless otherwise indicated. We do not define an upper W-semibuilding for $W \neq S_n$.

Proposition 3.1 *The flags of an upper semimodular lattice form the chambers of an upper semibuilding with distance function* $\delta(X, Y) = \pi(X, Y)$, *the Jordan-Hölder permutation.*

Proof: Properties S1, S2, S3 and U4 of upper semibuildings follow, respectively, from Proposition 2.1 (ii), the fact that S is nonempty in Proposition 2.2, and properties R1 and R2 from Proposition 2.2. □

Given an S_n-semibuilding B, we construct a lattice $L(B)$ whose flags are in one-to-one correspondence with the chambers of B and whose paths are in one-to-one correspondence with galleries in B. We show that if B is an upper semibuilding, then $L(B)$ is semimodular, and we relate other constraints on B to properties of $L(B)$. Using this approach, we develop a flag-based axiomatization for semimodular, geometric, modular, and distributive lattices.

To construct $L(B)$ from the semibuilding B, we need some way to take a lattice whose flags form an S_n-semibuilding, and to recover the points of the lattice from the flags. We make an observation: in a semimodular lattice, if the flags Z and Z' both contain the rank k point z_k, then $\pi(Z, Z')([k]) = [k]$. Therefore, a reduced decomposition of $\pi(Z, Z')$ has no r_k's, so all flags in every reduced path from Z to Z' contain z_k. With this motivation, we define the following equivalence relation for S_n-semibuildings.

Definition For every j with $0 \leq j \leq n$, we say the chambers X and Y in a semibuilding are *j-equivalent* and write $X \sim_j Y$ if there is gallery from X to Y in which no consecutive chambers are j-adjacent. In particular, all chambers are 0-equivalent and n-equivalent. For every j, this is an equivalence relation on the chambers of B.

The j-equivalence classes are the rank j points of the lattice we are in the process of constructing.

Proposition 3.2 *For every pair of chambers X and Y in a semibuilding, the following are equivalent*:

(i) $X \sim_j Y$.
(ii) *We have $\delta(X, Y)$ in the "parabolic subgroup" $P_j = \langle r_m : m \neq j\rangle$.*
(iii) *There is a chamber Z such that $X \sim_i Z$ for $i \leq j$ and $Z \sim_k Y$ for $k \geq j$.*

We use Lemma 3.3 to prove this.

Lemma 3.3 *In a semibuilding, if r_i and r_j commute and there is a gallery of type $f r_i r_j g$ between the chambers X and Y, then there is also a gallery of type $f r_j r_i g$ between X and Y. In an upper semibuilding, if there is a gallery of type $f r_k r_{k+1} r_k g$ between X and Y there is a gallery of type $f r_{k+1} r_k r_{k+1} g$ between X and Y.*

Proof: Let X' be the chamber reached after traversing f, and let Y' be the chamber reached after traversing $f r_i r_j$ or $f r_k r_{k+1} r_k$, respectively. Now by applying property S3 or U4, we obtain a new path from X' to Y', and we can follow this new path in our gallery from X to X' to Y' to Y. □

Proof of Proposition 3.2:

(i ⇔ ii). If $X \sim_j Y$, let $(X = Z_0, Z_1, \ldots, Z_m = Y)$ be a gallery from X to Y in which no consecutive chambers are j-adjacent. If $\delta(X, Z_k)$ is in P_j, then $\delta(X, Z_{k+1})$ is in P_j as well, since by property S1, $\delta(X, Z_{k+1})$ equals either $\delta(X, Z_k)$ or $\delta(X, Z_k)r_p$ for $p \neq j$; therefore, by induction, $\delta(X, Y)$ is in P_j. Conversely, if $\delta(X, Y)$ is in P_j, a reduced decomposition of $\delta(X, Y)$ has no r_j's in it. Therefore, by S2, there is a gallery from X to Y in which consecutive chambers are never j-adjacent.

(i $\Leftrightarrow$ iii). Suppose we have a minimal gallery of type f from X to Y. By the equivalence of (i) and (ii), f has no r_j's in it, since f is a reduced decomposition of $\delta(X, Y)$. By Lemma 3.3, if an r_i with $i < j$ precedes an r_k with $k > j$ in f, we may reverse the order. Thus, we may assume that every r_k in f with $k > j$ occurs before every r_i with $i < j$. If Z is the chamber immediately after the last r_k, then $X \sim_i Z$ for all $i \leq j$ and $Z \sim_k Y$ for all $k \geq j$. Conversely, if (iii) holds, we have $X \sim_j Z \sim_j Y$. □

We now define $L(B)$.

Definition For a semibuilding B, let $L(B)$ consist of the j-equivalence classes for $0 \leq j \leq n$ with the order relation: if w_i and z_j are i- and j-equivalence classes, then $w_i \leq z_j$ if $w_i \cap z_j \neq \emptyset$ and $i \leq j$.

Proposition 3.4 is a consequence of this definition.

Proposition 3.4 *Let L be a semimodular lattice, and let B be the upper semibuilding whose chambers are the flags of L and whose distance function is the Jordan-Hölder permutation. Then $L(B) \cong L$.*

Proof: Let X and Y be flags in L (or chambers in B). Now a path from X to Y in which consecutive flags are never j-adjacent exists if and only if we can go from X to Y without changing the rank j point. Hence, we have $X \sim_j Y$ in B if and only if X and Y contain the same rank j point, and the j-equivalence classes in $L(B)$ correspond to the rank j points in L. Furthermore, if $x_i \leq x_j$ in L, let X be some flag that goes through both these points. Then in B, the chamber X is in the intersection of the equivalence classes that correspond to x_i and x_j. Hence, the equivalence classes are comparable in $L(B)$. Conversely, if y_i and y_j are comparable equivalence classes in $L(B)$, then some flag Y is in $y_i \cap y_j$, and the rank i and j points of Y are comparable in L. □

We know from Proposition 3.1 that an upper semimodular lattice gives rise to an upper semibuilding. Proposition 3.4 implies that if B is a semibuilding that is constructed from a semimodular lattice, then $(L(B), \leq)$ is a poset isomorphic to the original lattice. We show that for every S_n-semibuilding B $(L(B), \leq)$ is a ranked lattice, and that the chamber system of $L(B)$ is isomorphic to B for every semibuilding B. We begin by showing $L(B)$ is a poset with a $\hat{0}$ and $\hat{1}$. We then show $L(B)$ is ranked, and that its flags form a chamber system isomorphic to B. After that, we define a labeling function on semibuildings and use it to show that $L(B)$ is a lattice. Finally, we relate various conditions on B to lattice properties of $L(B)$, including a proof that $L(B)$ is semimodular if and only if B is an upper semibuilding.

Proposition 3.5 *If B is a semibuilding, then $(L(B), \leq)$ is a poset. The* 0- *and* n-*equivalence classes are* $\hat{0}$ *and* $\hat{1}$ *in the poset.*

Proof: Reflexivity and antisymmetry are trivial, and the 0- and n-equivalence classes are obviously $\hat{0}$ and $\hat{1}$ in the poset if $L(B)$ is in fact a poset. For transitivity, suppose $x_i \leq z_j$ and $z_j \leq y_k$. Let X be a chamber in $x_i \cap z_j$ and let Y be a chamber in $z_j \cap y_k$. Now $X \sim_j Y$, so there is some chamber Z in the j-equivalence class z_j such that $X \sim_i Z$ and $Z \sim_k Y$, by Proposition 3.2. Therefore, Z is in $x_i \cap y_k$, so $x_i \leq y_k$. □

Proposition 3.6 *A collection of points $\mathcal{F}$ in $L(B)$ is a flag in $L(B)$ if and only if $\mathcal{F}$ consists of all equivalence classes of some chamber in B. Hence, there is a one-to-one correspondence between flags in $L(B)$ and chambers in B. Furthermore, $L(B)$ is ranked, since a j-equivalence class is a rank j point in $L(B)$. Two flags in $L(B)$ are i-adjacent (i.e., they agree except, possibly, on rank i), if and only if the corresponding chambers in B are i-adjacent in the chamber system. Thus, the flags in $L(B)$ form a chamber system which is isomorphic to B.*

To prove this, we use Lemma 3.7. This lemma is a particular instance of a more general result on parabolic subgroups of Coxeter groups (see [10], Corollary 5.10(c), for example).

Lemma 3.7 *Let S be a subset of $[n-1]$, and let P_S be the intersection*

$$P_S = \bigcap_{j \in S} P_j.$$

Then P_S is given by

$$P_S = \langle r_m : m \notin S \rangle.$$

Proof of Proposition 3.6: To show the correspondence between flags and chambers, let $\{z_1 < z_2 < \cdots < z_m\}$ be a chain in $L(B)$, and suppose by induction that the intersection $z_1 \cap z_2 \cap \cdots \cap z_p$ is nonempty. Let X be a chamber in this intersection and let Y be a chamber in $z_p \cap z_{p+1}$. If z_p is a j-equivalence class, then $X \sim_j Y$, so by Proposition 3.2, there is a chamber Z such that $X \sim_i Z$ for $i \leq j$ and $Y \sim_k Z$ for $k \geq j$. Thus, Z is in $z_1 \cap \cdots \cap z_p \cap z_{p+1}$, and by induction, the intersection $z_1 \cap z_2 \cap \cdots \cap z_m$ is nonempty. Hence, a maximal chain in $L(B)$ consists of all the equivalence classes of some chamber. In particular, a maximal chain in $L(B)$ consists of $n+1$ equivalence classes, and j is the rank of every j-equivalence class.

To show that the chamber corresponding to a maximal chain is unique, let X and Y be two chambers which correspond to the same maximal chain. Then $X \sim_j Y$ for all j. Now by Lemma 3.7, $\delta(X, Y) = 1$ and so $X = Y$ by S2. Conversely, given a chamber Z in B, if we let z_i be the i-equivalence class of Z in B, then $\{z_0, z_1, \ldots, z_n\}$ is a maximal chain in $L(B)$. The intersection $z_0 \cap z_1 \cap \cdots \cap z_n$ is nonempty since it contains Z. Finally, suppose two flags in $L(B)$ agree on all ranks except rank i, and let X and Y be the chambers in B which correspond to these flags. Now by Lemma 3.7, either $\delta(X, Y) = 1$ or $\delta(X, Y) = r_i$;

hence X and Y are i-adjacent. Conversely, if X and Y are i-adjacent in B, they will be j-equivalent for all $j \neq i$, so the corresponding flags in $L(B)$ will agree on all ranks except i. □

We digress briefly to consider other Coxeter groups. The proof of Proposition 3.5 that $L(B)$ is a poset only uses Proposition 3.2 and Lemma 3.3. But all we require of $W = S_n$ for these results is that r_i and r_j commute if $|j - i| = 1$. The Proof of Proposition 3.6 that $L(B)$ is ranked and its chamber system is isomorphic to the original semibuilding requires the additional Lemma 3.7, but this lemma can be generalized to all Coxeter groups. Thus, if the each connected component of the Coxeter graph of W is a line, we can order the generating reflections of W so that $L(B)$ is a ranked poset for any W-semibuilding B. Furthermore, the flags in $L(B)$ form a W-chamber system isomorphic to B. These results and a converse was shown for buildings by Björner and Wachs. It appears as Proposition 4.18 in [5], and we repeat the statement here.

Proposition 3.8 (Björner and Wachs) *Let Δ be a Coxeter complex or building of finite rank. Then $\Delta \cong \Delta(P)$, the simplicial complex of all finite chains of some poset P if and only if the corresponding Coxeter diagram is linear.*

To show that $L(B)$ is a lattice if $W = S_n$, we define a labeling function on its points, the j-equivalence classes, with respect to a chamber. Motivated by Proposition 2.1(iii), we make the following definition, which agrees with the definition of labels for semimodular and modular lattices in Eq. (1).

Definition Let X be a chamber in an S_n-semibuilding B. For every j-equivalence class z_j, choose some representative Z. Then the *labeling function with respect to X* is defined by

$$l_X(z_j) = \delta(X, Z)([j]).$$

Proposition 3.9 *The labeling function as defined on semibuildings has the following properties.*

(i) *The label $l_X(z_j)$ is independent of the equivalence class representative chosen, so the function is well-defined.*
(ii) *If $z_j \leq z_k$, then $l_X(z_j) \subseteq l_X(z_k)$.*
(iii) *We have $[i] \subseteq l_X(z_j)$ if and only if $x_i \leq z_j$.*

Proof: For (i), let Z and Z' be two representatives of z_j. Since $Z \sim_j Z'$, there is some gallery $(Z = Z_0, Z_1, \ldots, Z_m = Z')$ in which no two consecutive chambers are j-adjacent. Since either $\delta(X, Z_{p+1}) = \delta(X, Z_p)$ or $\delta(X, Z_{p+1}) = \delta(X, Z_p)r_k$ for some $k \neq j$, and $\delta(X, Z_{p+1})([j]) = \delta(X, Z_p)([j])$ in either case, we find $\delta(X, Z')([j]) = \delta(X, Z)([j])$ by induction. The statement (ii) follows by choosing the same representative Z for both z_j and z_k, since their intersection is nonempty. Then $l_X(z_j) = \delta(X, Z)([j]) \subseteq \delta(X, Z)([k]) = l_X(z_k)$.

From (ii), we see that $x_i \leq z_j$ implies $[i] \subseteq l_X(z_j)$. To prove the converse, choose a representative Z in z_j, and use induction on the length of $\rho = \delta(X, Z)$. Take a minimal gallery from X to Z and let Z' be the last chamber in the gallery before Z. Thus, Z and Z' are k-adjacent for some k. If $k \neq j$ then $Z' \sim_j Z$, so by induction we have $x_i \leq z'_j = z_j$. If $k = j$ we let $\rho' = \delta(X, Z') = \rho r_j < \rho$ since we started with a minimal gallery from X to Z. But if $[i] \subseteq \rho([j])$ and $\rho r_j < \rho$, then $\rho(j) > \rho(j+1) > i$. Thus, $[i] \subseteq \rho([j-1]) = \rho'([j-1])$. Now by induction, (iii) applies to Z', so $x_i \leq z'_{j-1} < z_j$ as desired. □

We need one more lemma to prove $L(B)$ is a lattice.

Lemma 3.10 *Suppose the rank k points x_k and y_k are both upper bounds of x_i and y_j in $L(B)$. Then either $x_k = y_k$ or there are rank $(k-1)$ points $x_{k-1} < x_k$ and $y_{k-1} < y_k$ which are also upper bounds of x_i and y_j.*

Proof: We find y_{k-1}; to find x_{k-1}, reverse the roles of X and Y. If $x_k \neq y_k$ let X and Y be chambers in $x_i \cap x_k$ and $y_j \cap y_k$, respectively, and consider a minimal gallery from X to Y. Let Y' be the last chamber in the gallery which is not in the equivalence class y_k and let Y'' be the chamber immediately following Y' in the gallery (so Y'' is in y_k). Also, let $\rho' = \delta(X, Y')$ and $\rho'' = \delta(X, Y'')$. From (iii), we have $[i] \subseteq l_X(y_k) = \rho''([k]) = \rho' r_k([k])$, since $x_i \leq y_k$. But Y' precedes Y'' in a minimal gallery, so $\rho' < \rho''$, and so $[i] \subseteq \rho'([k])$. Therefore, $[i] \subseteq \rho'([k-1]) = \rho''([k-1])$. Hence, letting y_{k-1} be the $(k-1)$-equivalence class of Y' and Y'', we have $x_i \leq y_{k-1} < y_k$, though we still must show $y_j \leq y_{k-1}$. Proceeding by induction, we find that x_i is less than the k-equivalence class of every chamber in the minimal gallery, and therefore, less than or equal to the $(k-1)$-equivalence classes of the chambers in the gallery. Similarly, we can use the Y-labels to show that y_j is less than or equal to all the $(k-1)$-equivalence classes in the gallery. Thus, $y_j \leq y_{k-1}$. □

Theorem 3.11 *$(L(B), \leq)$ is a lattice.*

Proof: Since $L(B)$ has a $\hat{1}$, every pair of points has an upper bound. To show each pair has a least upper bound, suppose z_k and w_m are upper bounds of x_i and y_j, with $k \leq m$. Lemma 3.10 shows that if there are distinct upper bounds of the same rank, then neither one is minimal. Thus, if we choose some rank m point $z_m \geq z_k$, we find the only way w_m can be minimal is if $w_m = z_m = z_k$. Therefore, z_k and w_m cannot be distinct minimal upper bounds, and a least upper bound exists. Since $L(B)$ is a finite rank poset with least upper bounds and a $\hat{0}$, it must be a lattice. □

We now show that $L(B)$ is semimodular if B is an upper semibuilding.

Theorem 3.12 *B is an upper semibuilding if and only if $L(B)$ is an upper semimodular lattice. Thus, by virtue of Propositions* 3.4 *and* 3.6, *upper semibuildings are in one-to-one correspondence with finite rank upper semimodular lattices, and the axioms* S1, S2, S3, *and* U4 *give us a flag-based axiom system of rank n semimodular lattices.*

Proof: Since the chamber system formed by the flags in $L(B)$ is isomorphic to B, Proposition 3.1 says that B is an upper semibuilding if $L(B)$ is semimodular. Conversely, suppose

B is an upper semibuilding, and suppose x_j and y_j both cover x_{j-1} in $L(B)$. Let X be a flag containing x_{j-1} and x_j and let Y be a flag that contains x_{j-1} and y_j. We construct a minimal gallery from X to Y with exactly one r_j. Then letting X' and Y' be the flags immediately before and after the r_j in this minimal gallery, we have $x'_i = x_i$ and $y'_i = y_i$ for $i \leq j$, and $x'_{j+1} = y'_{j+1} = x_j \vee y_j$. Therefore, the join covers x_j and y_j, and $L(B)$ is semimodular.

To construct the desired minimal gallery, start with any minimal gallery, and consider the first occurrence of $r_j r_{j+1} \ldots r_k$ in the decomposition of $\delta(X, Y)$. If this is not at the end of the decomposition, let r_p be the first simple reflection after this string. If $p = k$, the decomposition is not reduced. If $p = k+1$, we can lengthen the string. If $p < j$ or $p > k+1$, we can choose a different gallery to replace $r_j r_{j+1} \ldots r_k r_p$ by $r_p r_j r_{j+1} \ldots r_k$ via repeated application of S3. If $j \leq p < k$, we replace $r_j r_{j+1} \ldots r_k r_p$ by $r_j r_{j+1} \ldots r_{p-1}(r_p r_{p+1} r_p) r_{p+2} \ldots r_k$ using S3. Then, we replace this string with the string $r_j r_{j+1} \ldots r_{p-1}(r_{p+1} r_p r_{p+1}) r_{p+2} \ldots r_k$ using U4, and finally we replace this by $r_{p+1} r_j r_{j+1} \ldots r_k$, again using S3. When we reach the end of the string, there is only one r_j in the type of the gallery. □

We now extend this characterization to modular and distributive lattices. To obtain an upper semimodular lattice from a semibuilding, we needed condition U4, which requires a gallery of type $r_{k+1} r_k r_{k+1}$ between X and Y whenever $\delta(X, Y) = (k \;\; k+2)$. By duality, we would get lower semimodular lattices by requiring a gallery of type $r_k r_{k+1} r_k$ between X and Y. Hence, we obtain all modular lattices by requiring conditions S1, S2, S3, and replacing U4 with the following condition M4.

M4. If $\delta(X, Y) = (k \;\; k+2)$, then there are galleries between X and Y of type $r_{k+1} r_k r_{k+1}$ and of type $r_k r_{k+1} r_k$.

However, conditions S2, S3, and M4 are equivalent to condition B2, since we get all reduced decompositions or $\delta(X, Y)$ by virtue of Lemma 3.3. Therefore, we characterize semibuildings corresponding to finite rank modular lattices in Theorem 3.13.

Theorem 3.13 *If B is an S_n-semibuilding, $L(B)$ is modular if and only if B obeys condition* B2. *In this case, we call B a* modular (S_n)-semibuilding, *or simply a* modular semibuilding.

Theorem 3.13 describes $L(B)$ for S_n-semibuildings which obey B2. Theorem 3.14 describes the effects of B0.

Theorem 3.14 *If B is an S_n-semibuilding, $L(B)$ is relatively complemented if and only if B obeys condition* B0.

Proof: If $L(B)$ is relatively complemented, then every interval of length 2 is relatively complemented; hence, to find a flag X' that is i-adjacent to X, choose x'_i to be a complement of x_i in the interval $[x_{i-1}, x_{i+1}]$. Thus, B satisfies B0.

Conversely, suppose B is a semibuilding which obeys condition B0, and suppose $x_i < x_j < x_k$ in $L(B)$. We must show that x_j has a complement in the interval $[x_i, x_k]$. Toward this end, let X be a flag through x_i, x_j, and x_k, and let τ be the permutation

$$\tau = (1\ 2\ \ldots\ i \quad k\ k-1\ \ldots\ i+1 \quad k+1\ k+2\ \ldots\ n)$$

in one line notation. We show there is a flag Y through x_i and x_k such that $\pi(X, Y) = \tau$. Once we find Y, the complement of x_j is the rank $(i + k - j)$ point of Y, since the X-label of this point is $[i] \cup ([k] \setminus [j])$, the complement of $[j]$ in the interval $\{z : [i] \subseteq z \subseteq [k]\}$. To find Y, note that if Z is a flag which contains all x_m with $m \leq i$ and $m \geq k$, then either $\pi(X, Z) = \tau$ so we can use $Y = Z$, or there is some p with $i < p < k$ such that $\pi(X, Z)r_p > \pi(X, Z)$. By condition B0, we may choose a new flag Z' that is p-adjacent to Z, and by B1, $\pi(X, Z') = \pi(X, Z)r_p$. We repeat this process until we find Y. □

As one corollary of this result, we obtain Tits' result ([16], Section 6.1.5, Proposition 6, or in [2], Corollary 3.8). We also obtain an axiomatization of finite rank geometric lattice, since a finite rank lattice is geometric if and only if it is relatively complemented and semimodular (see [12], Proposition 3.3.3).

Corollary 3.15 (Tits) *B is an S_n-building if and only if $L(B)$ is a relatively complemented modular lattice.*

Corollary 3.16 *$L(B)$ is geometric if and only if B is an upper semibuilding which obeys condition* B0.

We now turn to distributive lattices. A modular lattice is distributive if and only if it does not contain a sublattice which is isomorphic to M_3 in Figure 1 ([3], Section II.8, Theorem 13).

This condition lets us extend our work to distributive lattices; we show that all distributive lattices can be obtained as $L(B)$ for a modular semibuilding B which obeys condition D0.

D0. Every chamber is i-adjacent to at most one other chamber for each i in the indexing set for the chamber system.

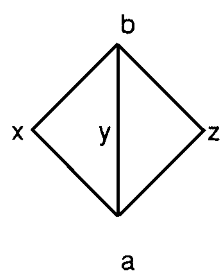

Figure 1. M_3: the unique five element modular nondistributive lattice.

As part of Theorem 3.17 we show that for a modular semibuilding, the condition D0 is equivalent to the either of the conditions D1 or D1′. Theorem 3.17 is similar to Abels' Theorem 3.9 in [2]. He gives several flag-based conditions which describe when a finite rank semimodular lattice can be embedded as a join sublattice into a distributive lattice of the same rank.

D1. If $\delta(X, Y) = \tau$, and $\delta(Y, Y') = r_i$, then $\delta(X, Y') = \tau r_i$.
D1′. If $\delta(X, Y) = \tau$, and $\delta(Y, Z) = \rho$, then $\delta(X, Z) = \tau\rho$,

Theorem 3.17 *If B is a modular semibuilding the following are equivalent*:

(i) $L(B)$ *is distributive.*
(ii) $L(B)$ *does not contain a sublattice isomorphic to* M_3*.*
(iii) $L(B)$ *does not have distinct points* x, y, *and* z *which all cover* $x \wedge y \wedge z$ *and are covered by* $x \vee y \vee z$*.*
(iv) *B does not contain three distinct mutually adjacent chambers*, *i.e.*, *B obeys condition* D0*.*
(v) *If* $\delta(X, Y) = \tau$, *and* $\delta(Y, Y') = r_i$, *then* $\delta(X, Y') = \tau r_i$, *i.e.*, D1 *holds.*
(vi) *If* $\delta(X, Y) = \tau$, *and* $\delta(Y, Z) = \rho$, *then* $\delta(X, Z) = \tau\rho$, *i.e.*, D1′ *holds.*

We call an S_n-semibuilding which obeys these conditions a *distributive semibuilding.*

Proof:

(i $\Leftrightarrow$ ii). This is well known as previously cited.
(ii $\Rightarrow$ iii). This is clear.
(iii $\Rightarrow$ iv). If X, Y, and Z are distinct and j-adjacent, then $x_j \wedge y_j \wedge z_j = x_{j-1}$ and $x_j \vee y_j \vee z_j = x_{j+1}$, contrary to (iii).
(iv $\Rightarrow$ v). Suppose $\tau = \delta(X, Y)$ and $r_j = \delta(Y, Y')$. If $\tau r_j < \tau$, then there is a reduced decomposition $f r_j$ of τ. Thus, by B2, there is a gallery of type $f r_j$ from X to Y. The last chamber before Y in this gallery must be strictly j-adjacent to Y, but Y' is the only such chamber since no other chamber can be j-adjacent to both Y and Y' by (iv). Hence, there is a gallery of type f from X to Y'. Since f is a reduced expression, $\delta(X, Y') = \tau r_j$. If $\tau r_j > \tau$ and f is a reduced decomposition of τ, there is a gallery of type f from X to Y, and appending a step from Y to Y' gives a gallery from Y to Y' of type $f r_j$. But $f r_j$ is a reduced decomposition, so $\delta(X, Y') = \tau r_j$. In either case, (v) holds.
(v $\Rightarrow$ vi). Let $\tau = \delta(X, Y)$ and $\rho = \delta(Y, Z)$, and let $\rho = s_1 s_2 \cdots s_m$ be a reduced decomposition of ρ. By condition B2, there is a minimal gallery $(Y = Y_0, Y_1, \ldots, Y_m = Z)$ of type $s_1 s_2 \cdots s_m$, and by induction, (v) implies that $\delta(X, Y_k) = \tau s_1 \cdots s_k$, so $\delta(X, Z) = \tau\rho = \delta(X, Y)\delta(Y, Z)$.
(vi $\Rightarrow$ ii). Suppose the points a, x, y, z, and b in $L(B)$ form a sublattice isomorphic to M_3. We may assume $a = \hat{0}$ and $b = \hat{1}$ by restricting our attention to the interval $[a, b]$. We first note that if the lattice has rank n, then $\mathrm{rank}(x) = \mathrm{rank}(y) = \mathrm{rank}(z) = \frac{n}{2}$, for which we use the symbol r. This is so because if x and y are complements in a modular lattice, then $\mathrm{rank}(x) + \mathrm{rank}(y) = \mathrm{rank}(\hat{0}) + \mathrm{rank}(\hat{1}) = n$. Similarly, we have

$\text{rank}(x) + \text{rank}(y) = \text{rank}(x) + \text{rank}(z) = \text{rank}(x) + \text{rank}(z) = n$, which forces the rank of each point to be r.

Let X be any flag containing $x = x_r$, and let Y and Z be the flags $Y = \{\hat{0} = x_r \wedge y < x_{r+1} \wedge y < \cdots < x_n \wedge y = y = \hat{0} \vee y < x_1 \vee y < \cdots < x_r \vee y = \hat{1}\}$ and $Z = \{\hat{0} = x_r \wedge z < x_{r+1} \wedge z < \cdots < x_n \wedge z = z = \hat{0} \vee z < x_1 \vee z < \cdots < x_r \vee z = \hat{1}\}$. The inequalities are all strict since in the interval $[\hat{0}, y]$ there are at most r distinct points, and the rank difference between consecutive points in these sets is at most 1 by modularity, but the total difference in rank between $\hat{0}$ and y is r. A similar argument applies to the inequalities in the intervals $[y, \hat{1}]$, $[\hat{0}, z]$, and $[z, \hat{1}]$. Now $\delta(X, Y) = (r+1 \ \ r+2 \ldots n \ \ 1 \ \ 2 \ldots r)$, since for $j \leq r$ we have $y_j \leq x_{r+j} \vee y_{j-1} = x_{r+j}$, but $y_j \not\leq x_{r+j-1} \vee y_{j-1} = x_{r+j-1}$, and $x_i \leq y_{r+i}$ for $i \leq r$, so $[i] \subseteq l_X(y_{r+i})$. Similarly, $\delta(X, Z) = (r+1 \ \ r+2 \ldots n \ \ 1 \ \ 2 \ldots r)$, but since $Y \neq Z$ and $\delta(Y, Z) \neq 1$, this contradicts 3.17.6.

□

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