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Imprimitive Q-polynomial Association Schemes*

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Abstract. It is well known that imprimitive P-polynomial association schemes $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ with $k_1 > 2$ are either bipartite or antipodal, i.e., intersection numbers satisfy either $a_i = 0$ for all i, or $b_i = c_{d-i}$ for all $i \ne [d/2]$. In this paper, we show that imprimitive Q-polynomial association schemes $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ with $d > 6$ and $k_1^* > 2$ are either dual bipartite or dual antipodal, i.e., dual intersection numbers satisfy either $a_i^* = 0$ for all i, or $b_i^* = c_{d-i}^*$ for all $i \ne [d/2]$.

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1. Introduction

A *d-class symmetric association scheme* is a pair $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$, where X is a finite set, each R_i is a nonempty subset of $X \times X$ for $i = 0, 1, \ldots, d$ satisfying the following.

(i) $R_0 = \{(x, x) | x \in X\}$.

(ii) $\{R_i\}_{0\le i\le d}$ is a partition of $X \times X$, i.e.,

$$X \times X = R_0 \cup R_1 \cup \cdots \cup R_d,\ R_i \cap R_j = \emptyset \text{ if } i \ne j.$$

(iii) ${}^tR_i = R_i$ for $i = 0, 1, \ldots, d$, where ${}^tR_i = \{(y, x) | (x, y) \in R_i\}$.

(iv) There exist integers $p_{i,j}^h$ such that for all $x, y \in X$ with $(x, y) \in R_h$,

$$p_{i,j}^h = |\{z \in X | (x, z) \in R_i,\ (z, y) \in R_j\}|.$$

We refer to X as the *vertex set* of $\mathcal{X}$, and to the integers $p_{i,j}^h$ as the *intersection numbers* of $\mathcal{X}$.

Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a symmetric association scheme. Let $\mathrm{Mat}_X(\boldsymbol{R})$ denote the algebra of matrices over the reals $\boldsymbol{R}$ with rows and columns indexed by X. The *i-th adjacency matrix* $A_i \in \mathrm{Mat}_X(\boldsymbol{R})$ of $\mathcal{X}$ is defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in X).$$

From $(i) - (iv)$ above, it is easy to see the following.

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$(i)'$ $A_0 = I$.

$(ii)'$ $A_0 + A_1 + \cdots + A_d = J$, where J is the all-1s matrix, and $A_i \circ A_j = \delta_{i,j} A_i$ for $0 \leq i, j \leq d$, where $\circ$ denotes the entry-wise matrix product.

$(iii)'$ ${}^t A_i = A_i$ for $0 \leq i \leq d$.

$(iv)'$ $A_i A_j = \sum_{h=0}^{d} p_{i,j}^h A_h$ for $0 \leq i, j \leq d$.

By the *Bose-Mesner algebra* of $\mathcal{X}$ we mean the subalgebra $\mathcal{M}$ of $\mathrm{Mat}_X(\boldsymbol{R})$ generated by the adjacency matrices $A_0, A_1, \ldots, A_d$. Observe by $(iv)'$ above that the adjacency matrices form a basis for $\mathcal{M}$. Moreover, $\mathcal{M}$ consists of symmetric matrices and it is closed under $\circ$. In particular, $\mathcal{M}$ is commutative in both multiplications.

Since the algebra $\mathcal{M}$ consists of commutative symmetric matrices, there is a second basis $E_0, E_1, \ldots, E_d$ satisfying the following.

$(i)''$ $E_0 = \frac{1}{|X|} J$.

$(ii)''$ $E_0 + E_1 + \cdots + E_d = I$, and $E_i E_j = \delta_{i,j} E_i$ for $0 \leq i, j \leq d$.

$(iii)''$ ${}^t E_i = E_i$ for $0 \leq i \leq d$.

$(iv)''$ $E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^{d} q_{i,j}^h E_h$, $(0 \leq i, j \leq d)$ for some real numbers $q_{i,j}^h$.

$E_0, E_1, \ldots, E_d$ are the primitive idempotents of the Bose-Mesner algebra. The parameters $q_{i,j}^h$ are called *Krein parameters*.

Conventionally, we assume $p_{i,j}^h$ and $q_{i,j}^h$ are zero if one of the indices h, i, j is out of range $\{0, 1, \ldots, d\}$ otherwise mentioned clearly.

A symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ with respect to the ordering $R_0, R_1, \ldots, R_d$ of the relations is called a *P-polynomial association scheme* if the following conditions are satisfied.

$(P1)$ $p_{i,j}^h = 0$ if one of h, i, j is greater than the sum of the other two.

$(P2)$ $p_{i,j}^h \neq 0$ if one of h, i, j is equal to the sum of the other two for $0 \leq h, i, j \leq d$.

In this case we write $c_i = p_{i-1,1}^i$, $a_i = p_{i,1}^i$, $b_i = p_{i+1,1}^i$ and $k_i = p_{i,i}^0$ for $i = 0, 1, \ldots, d$.

A symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ with respect to the ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of the Bose-Mesner algebra is called a *Q-polynomial association scheme* if the following conditions are satisfied.

$(Q1)$ $q_{i,j}^h = 0$ if one of h, i, j is greater than the sum of the other two.

$(Q2)$ $q_{i,j}^h \neq 0$ if one of h, i, j is equal to the sum of the other two for $0 \leq h, i, j \leq d$.

In this case we write $c_i^* = q_{i-1,1}^i$, $a_i^* = q_{i,1}^i$, $b_i^* = q_{i+1,1}^i$ and $k_i^* = q_{i,i}^0$ for $i = 0, 1, \ldots, d$.

If $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ is a P-polynomial association scheme with respect to the ordering $R_0, R_1, \ldots, R_d$, then the graph $\Gamma = (X, R_1)$ with vertex set X, edge set defined by R_1 becomes a distance-regular graph. In this case,

$$R_i = \{(x, y) \in X \times X | \partial(x, y) = i\},$$

where $\partial(x, y)$ denotes the distance between x and y. Conversely, every distance-regular graph is obtained in this way.

Q-polynomial association schemes appear in design theory in connection with tight conditions, but it is not much studied compared with P-polynomial association schemes, though there are extensive studies of P- and Q-polynomial association schemes.

A symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ is said to be *imprimitive* if it satisfies one of the following equivalent conditions.

(A) By a suitable rearrangement of indices $1, 2, \ldots, d$, there exists an index $s\,(0 < s < d)$ such that A_iA_j is a linear combination of $A_0, A_1, \ldots, A_s$ for all $i, j\ (0 \le i, j \le s)$.

(E) By a suitable rearrangement of indices $1, 2, \ldots, d$, there exists an index $t\,(0 < t < d)$ such that $E_i \circ E_j$ is a linear combination of $E_0, E_1, \ldots, E_t$ for all $i, j\ (0 \le i, j \le t)$.

The imprimitivity of association schemes including the equivalence of the above definitions were first studied in [3]. We also refer the readers to sections 2.4, 2.9 and 3.6 in [1] and sections 2.4, 4.1 and 4.2 in [2].

The following is well known. See the references above.

Theorem 1 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be an imprimitive P-polynomial association scheme with respect to the ordering $R_0, R_1, \ldots, R_d$ of the relations. If $k_1 > 2$, then one of the following holds.*

(i) $a_i = 0$ *for all* $i = 0, 1, \ldots, d$.

(ii) $b_i = c_{d-i}$ *for all* $i = 0, 1, \ldots, d$ *except possibly for* $i = [d/2]$.

If the condition (i) is satisfied, the scheme is called *bipartite*, and if the condition (ii) is satisfied, it is called *antipodal*, by adopting the terminologies of the distance-regular graph associated with the P-polynomial structure.

The following is our main result in this paper.

Theorem 2 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be an imprimitive Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents. If $d > 6$ and $k_1^* > 2$, then one of the following holds.*

(i) $a_i^* = 0$ *for all* $i = 0, 1, \ldots, d$.

(ii) $b_i^* = c_{d-i}^*$ *for all* $i = 0, 1, \ldots, d$ *except possibly for* $i = [d/2]$.

If the condition (i) is satisfied, the scheme is called *dual bipartite*, and if the condition (ii) is satisfied, it is called *dual antipodal*. It is known that if $k_1^* = 2$, then $\mathcal{X}$ is an ordinary polygon.

The proof of Theorem 1 is relatively easy and uses the inequalities based on the combinatorial structure of distance-regular graphs. We substitute that part by matrix identities to prove Theorem 2. These identities were used in Dickie's paper [5], which is a part of [4, Chapter 4].

2. P-polynomial C-algebra

We begin with a definition of P-polynomial C-algebra.

Let d be a positive integer and let c_{i+1}, a_i, b_{i-1} $(i = 0, 1, \ldots, d)$ be real numbers satisfying the following.

(i) $a_0 = b_{-1} = c_{d+1} = 0$ and $c_1 = 1$.

(ii) $c_i + a_i + b_i = b_0 = c_d + a_d$ for $i = 1, \ldots, d-1$.

(iii) $b_i c_{i+1} > 0$ for $i = 0, 1, \ldots, d-1$.

A P-polynomial C-algebra is an algebra over the reals $\boldsymbol{R}$ with basis $x_0, x_1, \ldots, x_d$, which satisfies the following.

$$x_0 x_0 = x_0, \quad x_1 x_i = b_{i-1} x_{i-1} + a_i x_i + c_{i+1} x_{i+1},\ (0 \le i \le d), \tag{1}$$

where x_{-1} and x_{d+1} are indeterminates. Then x_i can be written as a polynomial of x_1 of degree i and $x_0 = 1$, the unit element in this algebra. Define constants $p_{i,j}^h$ by the following.

$$x_i x_j = \sum_{h=0}^{d} p_{i,j}^h x_h,\ 0 \le i, j \le d. \tag{2}$$

Since the algebra becomes commutative, $p_{i,j}^h = p_{j,i}^h$. Let $k_i = p_{i,i}^0$, $n = k_0 + k_1 + \cdots + k_d$, and $ne_0 = x_0 + x_1 + \cdots + x_d$. Then it is easy to check by (i) and (ii) that $k_1 = b_0$ and that $x_1(ne_0) = k_1(ne_1)$.

The algebra $\mathcal{M} =< x_0, x_1, \ldots, x_d >$ defined above becomes a C-algebra in the sense defined in [1, Section 2.5]. See also [1, Section 3.6] and (2) in the following lemma. In particular, $\mathcal{M}$ has another basis $\{e_0, e_1, \ldots, e_d\}$ consisting of primitive idempotents and the dual algebra $\mathcal{M}^*$ defined by $x_i \circ x_j = \delta_{i,j} x_i$ becomes a C-algebra with respect to the basis $ne_0 = x_0 + x_1 + \cdots + x_d, ne_1, \ldots, ne_d$. Let

$$e_i \circ e_j = \frac{1}{n} \sum_{h=0}^{d} q_{i,j}^h e_h.$$

As the intersection numbers and the Krein parameters, by convention we assume the parameters $p_{i,j}^h$ and $q_{i,j}^h$ of C-algebras are zero if one of the indices h, i, j is out of range $\{0, 1, \ldots, d\}$.

Lemma 1 *Let $\mathcal{M} =< x_0, x_1, \ldots, x_d >$ be a P-polynomial C-algebra. Let $k_i = p^0_{i,i}$. Then the following hold.*

(1) $p^h_{i+1,j}c_{i+1} = p^h_{i,j-1}b_{j-1} + p^h_{i,j}(a_j - a_i) + p^h_{i,j+1}c_{j+1} - p^h_{i-1,j}b_{i-1}$.

(2) $p^0_{i,j} = \delta_{i,j}k_i$, $k_h p^h_{i,j} = k_i p^i_{j,h}$ *and* $k_i > 0$ *for* $i = 0, 1, \ldots, d$. *In particular,* $p^h_{i,j} = 0$ *if and only if* $p^i_{h,j} = 0$.

(3) $p^h_{i,j} = 0$ *if one of* h, i, j *is greater than the sum of the other two.*

(4) $p^h_{i,j} \neq 0$ *if one of* h, i, j *is equal to the sum of the other two for* $0 \leq h, i, j \leq d$.

(5) $p^{i+h}_{i,h+1}c_{h+1} = p^{i+h}_{i,h}(a_i + \cdots + a_{i+h} - a_1 - \cdots - a_h)$.

Proof: (1) Compute the coefficient of x_h in the expression of $(x_1x_i)x_j = (x_1x_j)x_i$ by applying (1) and then (2), and we obtain the formula.

(2) First we prove that $c_{i+1}p^0_{i+1,j+1} = \delta_{i,j}b_jp^0_{i,j}$ for $0 \leq i \leq j \leq d-1$ by induction on i. If $i = 0$, then this is obvious. Compute the coefficient of x_0 in the expression of $(x_1x_i)x_j = x_i(x_1x_j)$ in two ways. By induction hypothesis $p^0_{l,m} = 0$ for $l < i+1, m$, we have $c_{i+1}p^0_{i+1,j+1} = b_jp^0_{i,j}$. Since $p^0_{i,j} = \delta_{i,j}p^0_{i,i}$, we have the assertion. Hence we have $p^0_{i,j} = \delta_{i,j}k_i$ and $k_ib_i = k_{i+1}c_{i+1}$. By our assumption $b_ic_{i+1} > 0$, we have $k_i > 0$ as $k_0 = 1$.

Next compute the coefficient of x_0 in the expression of $(x_ix_j)x_h = (x_jx_h)x_i$ in two ways using the formula $p^0_{i,j} = \delta_{i,j}k_i$ just shown above, and we obtain the second formula $k_hp^h_{i,j} = k_ip^i_{j,h}$.

(3) By (2), we may assume that $h > i + j$. Since x_i is expressed as a polynomial of x_1 of degree i, we have the assertion.

(4) By (2), we may assume that $h = i + j$. Then by (1), $p^{i+j}_{i,j}c_i = p^{i+j}_{i-1,j+1}c_{j+1}$. Hence we have the assertion by induction on i.

(5) This follows by induction on h using (1). □

By definition, it is easy to see that the Bose-Mesner algebra $\mathcal{M}$ of a P-polynomial association scheme becomes a P-polynomial C-algebra with respect to the basis $A_0, A_1, \ldots, A_d$. Moreover, if we take $\circ$ product, the dual Bose-Mesner algebra $\mathcal{M}^*$ of Q-polynomial association scheme becomes a P-polynomial C-algebra with respect to the basis $|X|E_0, |X|E_1, \ldots, |X|E_d$.

In both of these cases, the structure constants and Krein parameters are nonnegative, i.e., $p^h_{i,j} \geq 0$ and $q^h_{i,j} \geq 0$. The latter inequality is called the Krein condition.

Lemma 2 *Let $\mathcal{M} =< x_0, x_1, \ldots, x_d >$ be a P-polynomial C-algebra. Suppose the structure constants $p^h_{i,j}$ are all nonnegative. Then the following hold.*

(1) *If* $p^h_{i+1,j-1} = p^h_{i+1,j} = p^h_{i+1,j+1} = 0$ *for* $0 \leq i < d$, *then* $p^h_{i,j} = p^h_{i+2,j} = 0$.

(2) *If* $p^h_{l,j-l+i} = p^h_{l,j-l+i+1} = \cdots = p^h_{l,j+l-i} = 0$ *for* $i \leq l$ *and* $0 \leq i < d$, *then* $p^h_{i,j} = p^h_{2l-i,j} = 0$.

(3) *For all* i, j *with* $0 \leq i, h, i+h \leq d$, $a_i = a_{i+1} = \cdots = a_{i+h} = 0$ *implies* $a_1 = \cdots = a_h = 0$.

(4) *For all* h *and* i *with* $0 \leq h, i, i+h \leq d$*, the following hold.*

(i) *If* $p^h_{i,i+h-1} = 0$*, then* $a_i \leq a_{i+h}$*. Moreover if* $a_i = a_{i+h}$*, then* $p^h_{i+1,i+h} = 0$.

(ii) *If* $p^h_{i+1,i+h} = 0$*, then* $a_i \geq a_{i+h}$*. Moreover if* $a_i = a_{i+h}$*, then* $p^h_{i,i+h-1} = 0$.

(iii) *If* $p^h_{i,i+h-1} = p^h_{i+1,i+h} = 0$*, then* $a_i = a_{i+h}$.

(5) *For all* h *and* i *with* $0 \leq i \leq h \leq d$*, the following hold.*

(i) *If* $p^h_{i,h-i+1} = 0$*, then* $a_i \leq a_{h-i}$*. Moreover if* $a_i = a_{h-i}$*, then* $p^h_{i+1,h-i} = 0$.

(ii) *If* $p^h_{i+1,h-i} = 0$*, then* $a_i \geq a_{h-i}$*. Moreover if* $a_i = a_{h-i}$*, then* $p^h_{i,h-i+1} = 0$.

(iii) *If* $p^h_{i,h-i+1} = p^h_{i+1,h-i} = 0$*, then* $a_i = a_{h-i}$.

Proof: (1) Replacing i by $i+1$, by Lemma 1 (1) we have

$$p^h_{i,j} b_i + p^h_{i+2,j} c_{i+2} = p^h_{i+1,j-1} b_{j-1} + p^h_{i+1,j}(a_j - a_{i+1}) + p^h_{i+1,j+1} c_{j+1}.$$

Since $i < d$ by our assumption, $b_i > 0$ and we have the assertion. Note that $b_i = p^i_{1,i+1}$ with $i < d$ is nonzero by the definition of P-polynomial C-algebra and it is nonnegative by our assumption.

(2) We prove the assertion by induction on $m = l - i$. If $l = i$, there is nothing to prove. Suppose the assertion holds for $m = l - i - 1 \geq 0$. Then

$$p^h_{i+1,j-1} = p^h_{i+1,j} = p^h_{i+1,j+1} = p^h_{2l-i-1,j-1} = p^h_{2l-i-1,j} = p^h_{2l-i-1,j+1} = 0.$$

By (1), we have $p^h_{i,j} = p^h_{2l-i,j} = 0$.

(3) This follows from Lemma 1 (4), (5) and the nonnegativity of the a_j's.

(4) Since $p^h_{i-1,i+h} = p^h_{i,i+h+1} = 0$ by Lemma 1 (3), it follows from Lemma 1 (1) by setting $j = i + h$ that

$$p^h_{i+1,i+h} c_{i+1} + p^h_{i,i+h} a_i = p^h_{i,i+h-1} b_{i+h-1} + p^h_{i,i+h} a_{i+h}.$$

Since $p^h_{i,i+h} \neq 0$, we have the assertion.

(5) This is similar to (4). Consider the following.

$$p^h_{i+1,h-i} c_{i+1} + p^h_{i,h-i} a_i = p^h_{i,h-i+1} c_{h-i+1} + p^h_{i,h-i} a_{h-i}. \qquad \square$$

Lemma 3 *Let* $\mathcal{M} =< x_0, x_1, \ldots, x_d >$ *be a* P*-polynomial* C*-algebra such that the structure constants* $p^h_{i,j}$ *are all nonnegative. Suppose for a positive integer* α*,* $p^\alpha_{i,j\alpha} \neq 0$ *only if* $i \equiv 0 \pmod{\alpha}$*. Then* $p^\alpha_{l,m} \neq 0$ *only if* $l \equiv m$ *or* $-m \pmod{\alpha}$.

Proof: It suffices to consider $p^\alpha_{l,m}$ with $0 < m - l < \alpha$ by Lemma 1 (3). We may assume that $(2i-1)\alpha < l + m < 2i\alpha$ or $2i\alpha < l + m < (2i+1)\alpha$. In the first case, there exists $0 \leq \beta \leq [\alpha/2] - 1$ such that $m = i\alpha - \beta$ or $i\alpha + \beta$ as $l < m$. Similarly, in the latter case, there exists $0 \le \beta \le [\alpha/2] - 1$ such that $l = i\alpha - \beta$ or $i\alpha + \beta$. Define γ by the following: $l = (i-1)\alpha + \beta + \gamma$ in the first case and $m = (i+1)\alpha - \beta - \gamma$ in the latter. Since $0 < m - l < \alpha$ and $m + l$ is in the corresponding range, in each case we have $1 \le \gamma \le \alpha - 1$ and that $2\beta + \gamma < \alpha$. Thus there are four cases.

(i) $l = (i-1)\alpha + \beta + \gamma$ and $m = i\alpha - \beta$.

(ii) $l = (i-1)\alpha + \beta + \gamma$ and $m = i\alpha + \beta$.

(iii) $l = i\alpha - \beta$ and $m = (i+1)\alpha - \beta - \gamma$.

(iv) $l = i\alpha + \beta$ and $m = (i+1)\alpha - \beta - \gamma$.

We apply Lemma 2 (2). Since $p^{\alpha}_{(i-1)\alpha+\gamma,i\alpha} = \cdots = p^{\alpha}_{(i-1)\alpha+2\beta+\gamma,i\alpha} = 0$, $p^{\alpha}_{l,m} = 0$ in the first two cases. Since $p^{\alpha}_{i\alpha,(i+1)\alpha-2\beta-\gamma} = \cdots = p^{\alpha}_{i\alpha,(i+1)\alpha-\gamma} = 0$, $p^{\alpha}_{l,m} = 0$ in the last two cases. □

The following is Proposition 6.2 in [1] but the description of it involves an error. Hence we restate the corrected version below. Note that we do not know if $b_t = c_{t+1}$ when $\alpha = 2t+1$.

Proposition 1 *Let $\mathcal{M} =< x_i \mid 0 \le i \le d >$ be a P-polynomial C-algebra with respect to the basis $x_0, x_1, \ldots, x_d$. Assume $p^h_{i,j} \ge 0$ and $q^h_{i,j} \ge 0$ for all h, i, j. Let $< x_\beta \mid \beta \in T >$ be a proper C-subalgebra of $\mathcal{M}$. Then*

$$T = \{0, \alpha, 2\alpha, 3\alpha, \ldots\} \text{ for some } \alpha \in \{2, d, \frac{d}{s}, \frac{2d+1}{2s+1}, \frac{2d}{2s+1}\}.$$

Let the following be the array of defining parameters,

$$\left\{ \begin{array}{c} c_i \\ a_i \\ b_i \end{array} \right\} = \left\{ \begin{array}{cccccc} * & 1 & c_2 & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_{d-1} & * \end{array} \right\}.$$

Then $\mathcal{M}$ has a C-subalgebra $< x_\beta \mid \beta \in T >$ with (i) $\alpha = 2$, (ii) $\alpha = d$, (iii) $\alpha \in \{\frac{d}{s}, \frac{2d+1}{2s+1}, \frac{2d}{2s+1}\}$ respectively if and only if the following hold.

(i) $a_2 = a_4 = \cdots = 0$ *and* $a_1 = a_3 = \cdots$.

(ii) $b_i = c_{d-i}$ *for all i except possibly for $i = [d/2]$.*

(iii) *The parameters c_h, a_h, b_h satisfy the following for $0 \le h \le d-1$. $b_i = c_{\alpha-i} = b_{j\alpha+i} = c_{(j+1)\alpha-i}$ for all $1 \le i \le \alpha - 1$ and $1 \le j$ except for $i = [\alpha/2]$, $a_i = a_{\alpha-i} = a_{j\alpha+i} = a_{(j+1)\alpha-i}$ for all $0 \le i \le \alpha$ and $1 \le j$ except for $i = [\alpha/2], [(\alpha+1)/2]$ with odd α. Moreover,*

$$(c_d, a_d) = \begin{cases} (b_0, 0) & \text{if } \alpha = \frac{d}{s} \\ (c_{(\alpha-1)/2}, a_{(\alpha-1)/2} + b_{(\alpha-1)/2}) & \text{if } \alpha = \frac{2d+1}{2s+1} \\ (c_{\alpha/2} + b_{\alpha/2}, a_{\alpha/2}) & \text{if } \alpha = \frac{2d}{2s+1}. \end{cases}$$

Note that (i) and (ii) are special cases of (iii) for $\alpha = 2$ and $\alpha = d$, respectively.

3. Vanishing Conditions of Krein Parameters

Only a few restrictions of the Krein parameters $q_{i,j}^h$ of symmetric association schemes are known except those derived algebraically using Lemma 1. We first list them in the following.

Proposition 2 *Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a symmetric association scheme. Let $E_0, E_1, \ldots, E_d$ be primitive idempotents and let $q_{i,j}^h$ be the Krein parameters. Then the following hold.*

(1) $q_{i,j}^h \ge 0$ *for all* $0 \le h, i, j \le d$.

(2) *For* $0 \le h, i, j \le d$, *we have*

$$q_{i,j}^h = 0 \Leftrightarrow \sum_{u\in X} (E_h)_{ux}(E_i)_{uy}(E_j)_{uz} = 0 \text{ for all } x, y, z \in X.$$

Proposition 2 (1) is known as Krein condition and (2) is in [3]. See also [1, Theorem 2.3.8, Proposition 2.8.3].

Lemma 4 *Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a symmetric association scheme. Let $E_0, E_1, \ldots, E_d$ be primitive idempotents and let $q_{i,j}^h$ be the Krein parameters. Suppose $\{i \mid q_{j,k}^i q_{l,m}^i \ne 0\} \subset \{h\}$. Then for all integers $0 \le h, i, j, k, l, m \le d$ and all vertices a, a', b, b', the following hold.*

$$(1)\quad \sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb} = \frac{q_{l,m}^h}{|X|} \sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_h)_{eb}.$$

$$(2)\quad \sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb'} = \sum_{e,e'\in X} (E_j)_{ea}(E_k)_{ea'}(E_h)_{ee'}(E_l)_{e'b}(E_m)_{e'b'}.$$

Proof: (1) By Proposition 2 (2), we have

$$\begin{aligned}
&\sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb} \\
&= \sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_l \circ E_m)_{eb} \\
&= \frac{1}{|X|} \sum_{i=0}^{d} q_{l,m}^i \sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_i)_{eb} \\
&= \frac{q_{l,m}^h}{|X|} \sum_{e\in X} (E_j)_{ea}(E_k)_{ea'}(E_h)_{eb}.
\end{aligned}$$

(2) Since $I = E_0 + E_1 + \cdots + E_d$, similarly we have

$$\begin{aligned}
&\sum_{e\in X}(E_j)_{ea}(E_k)_{ea'}(E_l)_{eb}(E_m)_{eb'} \\
&= \sum_{e,e'\in X}(E_j)_{ea}(E_k)_{ea'}(I)_{ee'}(E_l)_{e'b}(E_m)_{e'b'} \\
&= \sum_{i=0}^{d}\sum_{e,e'\in X}(E_j)_{ea}(E_k)_{ea'}(E_i)_{ee'}(E_l)_{e'b}(E_m)_{e'b'} \\
&= \sum_{i=0}^{d}\sum_{e'\in X}\left(\sum_{e\in X}(E_j)_{ea}(E_k)_{ea'}(E_i)_{ee'}\right)(E_l)_{e'b}(E_m)_{e'b'} \\
&= \sum_{i=0}^{d}\sum_{e\in X}(E_j)_{ea}(E_k)_{ea'}\left(\sum_{e'\in X}(E_i)_{ee'}(E_l)_{e'b}(E_m)_{e'b'}\right).
\end{aligned}$$

Comparing the last two expressions using Proposition 2 (2) we have the right hand side of (2) by our assumption. □

Proposition 3 *Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents. Suppose that*

$$\{l \mid q^l_{j,h+i}q^l_{i-j,h+j} \neq 0\} \subset \{h+i-j\}.$$

Then for $h \ge 0$, $i \ge j \ge 1$ with $h+i+j \le d$, $q^{h+i}_{i,h+j} = 0$ implies that $q^{h+j}_{j,h+j} = 0$.

Proof: Since $q^{h+i}_{i,h+j} = 0$, by Proposition 2,

$$\begin{aligned}
0 &= \frac{q^i_{j,i-j}}{|X|}\sum_{u\in X}(E_{h+i})_{ux}(E_i)_{uy}(E_{h+j})_{uz} \\
&= \sum_{u\in X}\left(\frac{q^i_{j,i-j}}{|X|}(E_i)_{uy}\right)(E_{h+i})_{ux}(E_{h+j})_{uz} \\
&= \sum_{u\in X}((E_j\circ E_{i-j})E_i)_{uy}(E_{h+i})_{ux}(E_{h+j})_{uz} \\
&= \sum_{u\in X}\sum_{v\in X}(E_j)_{uv}(E_{i-j})_{uv}(E_i)_{vy}(E_{h+i})_{ux}(E_{h+j})_{uz} \\
&= \sum_{v\in X}(E_i)_{vy}\left(\sum_{u\in X}(E_j)_{uv}(E_{h+i})_{ux}(E_{i-j})_{uv}(E_{h+j})_{uz}\right).
\end{aligned}$$

Since $\{l \mid q^l_{j,h+i}q^l_{i-j,h+j} \neq 0\} \subset \{h+i-j\}$, by Lemma 4 (2),

$$= \sum_{u\in X}\sum_{v\in X}\sum_{w\in X}(E_i)_{vy}(E_j)_{uv}(E_{h+i})_{ux}(E_{h+i-j})_{uw}(E_{i-j})_{wv}(E_{h+j})_{wz}.$$

Since this holds for arbitrary x, y, z, we have

$$
\begin{aligned}
0 &= \sum_{x,y,z\in X}(E_{h+i+j})_{xy}(E_{h+j})_{yz}(E_j)_{xz}\times \\
&\qquad \sum_{u,v,w\in X}(E_i)_{vy}(E_j)_{uv}(E_{h+i})_{ux}(E_{h+i-j})_{uw}(E_{i-j})_{wv}(E_{h+j})_{wz} \\
&= \sum_{y,z,v,w\in X}(E_{h+j})_{yz}(E_i)_{vy}(E_{i-j})_{wv}(E_{h+j})_{wz}\times \\
&\qquad \sum_{x,u\in X}(E_{h+i+j})_{xy}(E_j)_{xz}(E_{h+i})_{ux}(E_j)_{uv}(E_{h+i-j})_{uw}.
\end{aligned}
$$

Since $\{l \mid q^l_{h+i+j,j}q^l_{j,h+i-j}\neq 0\}\subset\{h+i\},$ by Lemma 4 (2) we have

$$
\begin{aligned}
&= \sum_{y,z,v,w\in X}(E_{h+j})_{yz}(E_i)_{vy}(E_{i-j})_{wv}(E_{h+j})_{wz}\times \\
&\qquad \sum_{x\in X}(E_{h+i+j})_{xy}(E_j)_{xz}(E_j)_{xv}(E_{h+i-j})_{xw} \\
&= \sum_{x,z,w\in X}(E_{h+j})_{wz}(E_j)_{xz}(E_{h+i-j})_{xw}\times \\
&\qquad \sum_{y,v\in X}(E_{h+i+j})_{yx}(E_{h+j})_{yz}(E_i)_{yv}(E_j)_{xv}(E_{i-j})_{wv}.
\end{aligned}
$$

Since $\{l \mid q^l_{h+i+j,h+j}q^l_{j,i-j}\neq 0\}\subset\{i\},$ by Lemma 4 (2) we have

$$
\begin{aligned}
&= \sum_{x,z,w\in X}(E_{h+j})_{wz}(E_j)_{xz}(E_{h+i-j})_{xw}\times \\
&\qquad \sum_{y\in X}(E_{h+i+j})_{yx}(E_{h+j})_{yz}(E_j)_{xy}(E_{i-j})_{wy} \\
&= \sum_{x,z,w\in X}(E_{h+j})_{wz}(E_j)_{xz}(E_{h+i-j})_{xw}\times \\
&\qquad \sum_{y\in X}(E_{h+j})_{yz}(E_{i-j})_{yw}(E_j)_{yx}(E_{h+i+j})_{yx}.
\end{aligned}
$$

Since $\{l \mid q^l_{h+j,i-j}q^l_{j,h+i+j}\neq 0\}\subset\{h+i\},$ by Lemma 4 (1) we have

$$
\begin{aligned}
&= \frac{q^{h+i}_{j,h+i+j}}{|X|}\sum_{x,z,w\in X}(E_{h+j})_{wz}(E_j)_{xz}(E_{h+i-j})_{xw}\times \\
&\qquad \sum_{y\in X}(E_{h+j})_{yz}(E_{i-j})_{yw}(E_{h+i})_{yx} \\
&= \frac{q^{h+i}_{j,h+i+j}}{|X|}\sum_{y,z\in X}(E_{h+j})_{yz}\times \\
&\qquad \sum_{x,w\in X}(E_{h+i})_{xy}(E_j)_{xz}(E_{h+i-j})_{xw}(E_{i-j})_{wy}(E_{h+j})_{wz}.
\end{aligned}
$$

Since $\{l \mid q^l_{h+i,j}q^l_{i-j,h+j} \neq 0\} \subset \{h+i-j\},$ by Lemma 4 (2), we have

$$\begin{aligned}
&= \frac{q^{h+i}_{j,h+i+j}}{|X|}\sum_{y,z\in X}(E_{h+j})_{yz}\sum_{x\in X}(E_{h+i})_{xy}(E_j)_{xz}(E_{i-j})_{xy}(E_{h+j})_{xz}\\
&= \frac{q^{h+i}_{j,h+i+j}}{|X|}\sum_{x,z\in X}(E_j)_{xz}(E_{h+j})_{xz}\sum_{y\in X}(E_{i-j})_{xy}(E_{h+i})_{xy}(E_{h+j})_{yz}\\
&= \frac{q^{h+i}_{j,h+i+j}}{|X|}\sum_{x,z\in X}(E_j)_{xz}(E_{h+j})_{xz}\sum_{y\in X}((E_{i-j})\circ(E_{h+i}))_{xy}(E_{h+j})_{yz}\\
&= \frac{q^{h+i}_{j,h+i+j}q^{h+j}_{i-j,h+i}}{|X|^2}\sum_{x,z\in X}(E_j)_{xz}(E_{h+j})_{xz}(E_{h+j})_{xz}\\
&= \frac{q^{h+i}_{j,h+i+j}q^{h+j}_{i-j,h+i}}{|X|^2}\sum_{x\in X}\left(\sum_{z\in X}((E_j)\circ(E_{h+j}))_{xz}(E_{h+j})_{zx}\right)\\
&= \frac{q^{h+i}_{j,h+i+j}q^{h+j}_{i-j,h+i}q^{h+j}_{j,h+j}}{|X|^3}\sum_{x\in X}(E_{h+j})_{xx}.
\end{aligned}$$

Since E_{h+j} is a nonzero idempotent,

$$\sum_{x\in X}(E_{h+j})_{xx} = \mathrm{trace}(E_{h+j}) = \mathrm{rank}(E_{h+j}) \neq 0.$$

Moreover, $q^{h+i}_{j,h+i+j} \neq 0$ and $q^{h+j}_{i-j,h+i} \neq 0$ by $(Q2)$. Hence $q^{h+j}_{j,h+j} = 0$. □

Corollary 1 *Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of primitive idempotents.*

(1) *For $h \geq 0$, $i \geq 1$ with $h+i+1 \leq d$,*

$$q^{h+i}_{i,h+1} = q^{h+i}_{1,h+i} = 0 \ \textit{ implies that } \ q^{h+1}_{1,h+1} = 0.$$

(2) *For $h \geq 0$, $i \geq 2$ with $h+i+2 \leq d$,*

$$q^{h+i}_{i,h+2} = q^{h+i}_{2,h+i} = q^{h+i}_{2,h+i-1} = 0 \ \textit{ implies that } \ q^{h+2}_{2,h+2} = 0.$$

Proof: (1) Since $q^{h+i}_{1,h+i} = 0$, by $(Q2)$ we have the following.

$$\{l \mid q^l_{1,h+i}q^l_{i-1,h+1} \neq 0\} \subset \{h+i-1\}.$$

Hence we have the assertion from Proposition 3 by setting $j = 1$.

(2) Since $q^{h+i}_{2,h+i} = q^{h+i}_{2,h+i-1}$, by $(Q2)$ we have the following.

$$\{l \mid q^l_{2,h+i}q^l_{i-2,h+2} \neq 0\} \subset \{h+i-2\}.$$

Hence we have the assertion from Proposition 3 by setting $j = 2$. □

By setting $h = 0$ in Corollary 1, we obtain the result of G. A. Dickie in [4, 5]. Hence the proposition is a generalization of it. The following result for P-polynomial association schemes is not used elsewhere in this paper but it is the dual of the result above, which can be proved very similarly. The proof suggests a possible way to find vanishing conditions and its proof in Q-polynomial association schemes.

Proposition 4 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a P-polynomial association scheme with respect to the ordering $R_0, R_1, \ldots, R_d$ of relations. Suppose that*

$$\{l \mid p^l_{j,h+i} p^l_{i-j,h+j} \neq 0\} \subset \{h+i-j\}.$$

Then for $h \ge 0$, $i \ge j \ge 1$, and $h+i+j \le d$, $p^{h+i}_{i,h+j} = 0$ implies that $p^{h+j}_{j,h+j} = 0$.

Proof: Suppose $p^{h+j}_{j,h+j} \neq 0$. Then there are vertices $\alpha, \beta, \gamma \in X$ such that

$$(\alpha, \beta),\ (\alpha, \gamma) \in R_{h+j} \text{ and } (\beta, \gamma) \in R_j.$$

Since $p^{h+j}_{i-j,h+i} \neq 0$ by $(P2)$, there exists a vertex $\delta \in X$ such that $(\alpha, \delta) \in R_{i-j}$ and $(\delta, \beta) \in R_{h+i}$. Consider two triples (δ, β, γ) and (δ, α, γ). Since $\{l \mid p^l_{h+i,j} p^l_{i-j,h+j} \neq 0\} \subset \{h+i-j\}$, $(\delta, \gamma) \in R_{h+i-j}$. Since $(\beta, \delta) \in R_{h+i}$ and $p^{h+i}_{h+i+j,j} \neq 0$ by $(P2)$, there exists a vertex $\epsilon \in X$ such that $(\beta, \epsilon) \in R_{h+i+j}$ and $(\epsilon, \delta) \in R_j$. Consider two triples $(\epsilon, \beta, \alpha)$ and $(\epsilon, \delta, \alpha)$. Since $\{l \mid p^l_{h+i+j,h+j} p^l_{j,i-j} \neq 0\} \subset \{i\}$, we have $(\epsilon, \alpha) \in R_i$.

Next consider two triples $(\epsilon, \beta, \gamma)$ and $(\epsilon, \delta, \gamma)$. Since $\{l \mid p^l_{h+i+j,j} p^l_{j,h+i-j} \neq 0\} \subset \{h+i\}$, we have $(\epsilon, \gamma) \in R_{h+i}$. Finally consider a triple $(\epsilon, \alpha, \gamma)$. Since

$$(\epsilon, \gamma) \in R_{h+i},\ (\epsilon, \alpha) \in R_i, \text{ and } (\alpha, \gamma) \in R_{h+j},$$

$p^{h+i}_{i,h+j} \neq 0$, which is a contradiction. □

4. Proof of Main Theorem

In this section, we prove the following result. It is obvious that Theorem 2 is a direct consequence of it.

Theorem 3 *Let $\mathcal{X} = (X, \{R_i\}_{0 \le i \le d})$ be a Q-polynomial association scheme with respect to the ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents. Suppose $\mathcal{X}$ is imprimitive. Or more precisely, suppose the linear span of $\{E_i \mid i \in T\}$ is closed under $\circ$ product for some proper subset T of $\{0, 1, \ldots, d\}$ with $T \neq \{0\}$. In addition, assume that $k^*_1 > 2$. Then $T = \{0, \alpha, 2\alpha, 3\alpha, \ldots\}$ for some $\alpha \ge 2$, and one of the following holds.*

(i) *$\alpha = 2$ and $a^*_i = 0$ for all i.*

(ii) *$\alpha = d$ and $b^*_i = c^*_{d-i}$ for all $i = 0, 1, \ldots, d$ except possibly for $i = [d/2]$.*

(iii) $d = 4$, $\alpha = 3$, *and the parameters satisfy the following conditions.*

$$\left\{ \begin{array}{c} c_i^* \\ a_i^* \\ b_i^* \end{array} \right\} = \left\{ \begin{array}{ccccc} * & 1 & c_2^* & c_3^* & 1 \\ 0 & 0 & a_2^* & 0 & k^*-1 \\ k^* & k^*-1 & 1 & b_3^* & * \end{array} \right\}.$$

(iv) $d = 6$, $\alpha = 3$, *and the parameters satisfy the following conditions.*

$$\left\{ \begin{array}{c} c_i^* \\ a_i^* \\ b_i^* \end{array} \right\} = \left\{ \begin{array}{ccccccc} * & 1 & c_2^* & c_3^* & 1 & c_5^* & k^* \\ 0 & 0 & a_4^*+a_5^* & 0 & a_4^* & a_5^* & 0 \\ k^* & k^*-1 & 1 & b_3^* & b_4^* & 1 & * \end{array} \right\}.$$

It is not difficult to see from Proposition 1 that if one of the conditions $(i) - (iv)$ of the theorem above holds, then the linear span of $\{E_i \mid i \in T\}$ is closed under $\circ$ product, where $T = \{0, \alpha, 2\alpha, 3\alpha, \ldots\}$. In particular, the association scheme $\mathcal{X}$ is imprimitive.

Throughout this section assume the following:

$\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ *is a Q-polynomial association scheme with respect to the ordering* $E_0, E_1, \ldots, E_d$ *of the primitive idempotents such that the linear span of* $\{E_i \mid i \in T\}$ *is closed under* $\circ$ *product for some proper subset* T *of* $\{0, 1, \ldots, d\}$ *with* $T \neq \{0\}$.

Under the assumption above, $\mathcal{M}^* =< |X|E_0, |X|E_1, \ldots, |X|E_d >$ with $\circ$ product is a P-polynomial C-algebra with nonnegative $p_{i,j}^h$ and $q_{i,j}^h$. Hence we can apply Proposition 1. In particular, we have the following two lemmas as direct consequences.

Lemma 5 (1) $T = \{0, \alpha, 2\alpha, 3\alpha, \ldots\}$ *for some* $\alpha \geq 2$.

(2) *Let* $\beta = [\alpha/2]$. *Then* $d \equiv 0$ *or* $\beta \pmod{\alpha}$.

Lemma 6 *Let* $\beta = [\alpha/2]$. *Then the following hold.*

(1) $q_{l,m}^{\alpha} \neq 0$ *only if* $l \equiv m$ *or* $-m \pmod{\alpha}$.

(2) $q_{\alpha-h+1,h}^{\alpha} = 0$, *unless* $\alpha = 2\beta + 1$ *with* $h = \beta + 1$.

(3) $q_{\alpha+h-1,h}^{\alpha} = 0$, *unless* $\alpha = 2\beta + 1$ *with* $h \equiv \beta + 1 \pmod{\alpha}$.

(4) *Suppose* $\alpha > 2$ *and* $2 \leq h \leq \alpha$. *Then* $q_{\alpha-h+2,h}^{\alpha} \neq 0$, *unless* $\alpha = 2\beta$ *with* $h = \beta + 1$.

Proof: (1) By Proposition 2, $q_{i,j}^h \geq 0$ for all h, i and j. Hence this is a direct consequence of Lemma 3.

(2) By (1) we have that $2h - 1 \equiv 0 \pmod{\alpha}$, if the value is not zero. Hence α is odd and $h = \beta + 1$.

(3) This is similar to (2).

(4) By (1) we have that $2h - 2 \equiv 0 \pmod{\alpha}$, if the value is not zero. Since $2 \leq h \leq \alpha$, α is even and $h = \beta + 1$. $\square$

Lemma 7 *If $\alpha < d$, then $a_i^* = 0$ for all $i = 0, 1, \ldots, \alpha$ except for $i = \beta + 1$ with $\alpha = 2\beta + 1$.*

Proof: By Lemma 6 (2), $q_{\alpha-i+1,i}^{\alpha} = 0$, unless $\alpha = 2\beta + 1$ with $i = \beta + 1$. Since $q_{1,\alpha}^{\alpha} = 0$ by our assumption, we have $a_i^* = q_{1,i}^{i} = 0$ by Corollary 1 (1) as desired. □

Lemma 8 *The following hold.*

(1) *Suppose $\alpha = 2\beta$. Then for each $0 \le h \le \alpha$ and $i \ge 0$ with $0 \le h + i\alpha \le d$, $a_h^* = a_{h+i\alpha}^*$.*

(2) *Suppose $\alpha = 2\beta + 1$. Then for each $0 \le h \le \alpha$ and $i \ge 0$ with $0 \le h + i\alpha \le d$, $a_h^* = a_{h+i\alpha}^*$ unless $h = \beta, \beta + 1$. Moreover, $a_{(i-1)\alpha+\beta}^* \le a_{i\alpha+\beta}^*$, and $a_{(i-1)\alpha+\beta+1}^* \ge a_{i\alpha+\beta+1}^*$.*

Proof: By Lemma 6 (3), we have that $q_{\alpha+h-1,h}^{\alpha} = 0$, unless $\alpha = 2\beta + 1$ with $h \equiv \beta + 1 \pmod{\alpha}$.

(1) Suppose $\alpha = 2\beta$. Then $q_{\alpha+h-1,h}^{\alpha} = 0$ for every h. Hence by Lemma 2 $(4)(iii)$, we have that $a_h^* = a_{h+i\alpha}^*$, for each $0 \le h \le \alpha$ and $i \ge 0$ with $0 \le h + i\alpha \le d$.

(2) Suppose $\alpha = 2\beta + 1$. Then $q_{\alpha+h-1,h}^{\alpha} = 0$, unless $h \equiv \beta + 1 \pmod{\alpha}$. Hence by Lemma 2 (4), $a_h^* = a_{h+i\alpha}^*$ unless $h = \beta, \beta + 1$, for each $0 \le h \le \alpha$ and $i \ge 0$ with $0 \le h + i\alpha \le d$. Moreover, $a_{(i-1)\alpha+\beta}^* \le a_{i\alpha+\beta}^*$, and $a_{(i-1)\alpha+\beta+1}^* \ge a_{i\alpha+\beta+1}^*$. □

Lemma 9 *If $\alpha = 2\beta$ with $\alpha < d$, then $a_i^* = 0$ for all i.*

Proof: By Lemma 7, $a_i^* = 0$ for all $i = 0, 1, \ldots, \alpha$. Hence we have the assertion by Lemma 8 (1). □

Lemma 10 *Suppose $\alpha = 2\beta + 1$. If $a_h^* = 0$ for all $h = 0, 1, \ldots, \alpha$. Then $a_j^* \ne 0$ only when $j = d$ and $d \equiv \beta \pmod{\alpha}$.*

Proof: Choose an integer i so that $i\alpha + 1 \le j \le (i+1)\alpha$. We prove by induction on i. There is nothing to prove when $i = 0$.

By induction hypothesis and Lemma 8, we may assume that $j = i\alpha + \beta$ or $j = i\alpha + \beta + 1$. Since $a_{(i-1)\alpha+\beta+1}^* \ge a_{i\alpha+\beta+1}^*$, $a_{i\alpha+\beta+1}^* = 0$ by induction hypothesis.

Suppose $i\alpha + \beta < d$. Then $q_{(i-1)\alpha+\beta+2,i\alpha+\beta+1}^{\alpha} = 0$ and $a_{(i-1)\alpha+\beta+1}^* = a_{i\alpha+\beta+1}^*$ implies $q_{(i-1)\alpha+\beta+1,i\alpha+\beta}^{\alpha} = 0$ by Lemma 2 $(4)(ii)$. Since $q_{(i-1)\alpha+\beta,i\alpha+\beta-1}^{\alpha} = 0$, we have $0 = a_{(i-1)\alpha+\beta}^* = a_{i\alpha+\beta}^*$ by Lemma 2 $(4)(iii)$ as desired. □

Lemma 11 *Suppose $\alpha = 2\beta + 1 \ge 5$ with $\alpha < d$. Then $a_i^* = 0$ for all $i < d$. Moreover, $a_d^* \ne 0$ only if $d \equiv \beta \pmod{\alpha}$.*

Proof: By Lemma 10 and Lemma 7, it suffices to show that $a_{\beta+1}^* = 0$.

Assume that $\alpha \ge 7$. Then we have

$$a_{\beta+2}^* = \cdots = a_{\alpha}^* = a_{\alpha+1}^* = a_{\alpha+2}^*$$

by Lemma 8 (2) as $a_1^* = a_2^* = 0$. Note that $d \geq \alpha + 2$ by Lemma 5 (2). Since $d \geq \alpha + 2$, we have by Lemma 2 (3) that $a_{\beta+1}^* = 0$ as $(\alpha + 2) - (\beta + 2) = \beta + 1$.

Suppose $\alpha = 5$. Then we have

$$a_1^* = a_2^* = a_4^* = a_5^* = a_6^* = 0.$$

Now $q_{2,4}^5 = 0$ by Lemma 6. Since $q_{2,5}^4 = 0$ and $a_2^* = a_6^*$, by Lemma 2 $(4)(i)$, we have $q_{3,6}^4 = 0$. Hence $q_{4,3}^6 = q_{1,6}^6 = 0$, and by Corollary 1 (1), $a_3^* = 0$ as $d \geq 7$ by Lemma 5 (2). □

Lemma 12 *Suppose $\alpha = 3 < d$. Then one of the following holds.*

(1) $a_i^* = 0$ *for all* $i < d$*, and* $a_d^* \neq 0$ *only if* $d \equiv \beta \pmod 3$.

(2) $d = 6$, $a_1^* = a_3^* = a_6^* = 0$*, and* $a_2^* = a_4^* + a_5^* \neq 0$.

(3) $d = 4$, $a_1^* = a_3^* = 0$*, and* $a_2^* \neq 0$.

Proof: It is easy to see that $a_1^* = a_{3i}^* = 0$ for every i. Suppose $d \geq 7$. Since $q_{3,4}^6 = 0$, $a_4^* = 0$ by Corollary 1 (1). Since $q_{1,3}^3 = 0$ and $a_1^* = a_4^* = 0$, $q_{2,4}^3 = 0$ and $a_2^* = a_5^*$ by Lemma 2 (4) $(i), (iii)$. Moreover, $q_{3,2}^4 = 0$ with $a_4^* = 0$ implies $a_2^* = 0$ by Corollary 1 (1). Therefore we have $a_1^* = a_2^* = a_3^* = 0$. Hence we have (1) in this case.

If $d \leq 6$, then $d = 4$ or 6 by Lemma 5 (2). If $d = 4$ or 6, then we have $a_1^* = a_{3i}^* = 0$ for every i. Moreover, since $q_{3,3}^5 = 0$, $a_4^* + a_5^* = a_2^*$. Clearly if $a_2^* = 0$, we have case (1) by Lemma 10. Hence we have one of the three cases above. □

Lemma 13 *Suppose $d \geq \alpha + 2$ with $\alpha > 2$ and $a_i^* = 0$ for all $i = 1, 2, \ldots, d-1$. Then $k_1^* = 2$.*

Proof: Suppose $k_1^* > 2$. Observe by Lemma 6 that $q_{\alpha-h+2,h}^{\alpha} = 0$, unless $\alpha = 2\beta$ and $h = \beta + 1$ for $2 \leq h \leq \alpha$. Moreover $q_{2,\alpha}^{\alpha} = q_{2,\alpha-1}^{\alpha} = 0$ by our assumption. Hence by Corollary 1 (2), $q_{2,h}^{h} = 0$ when $q_{\alpha-h+2,h}^{\alpha} = 0$.

Suppose $\alpha = 2\beta + 1$. Then $q_{2,h}^{h} = 0$ for $2 \leq h \leq \alpha$. Hence

$$c_h^* b_{h-1}^* + b_h^* c_{h+1}^* - k_1^* = 0.$$

Now by induction we show that $c_h^* \leq 1$ when h is odd. The assertion is trivial if $h = 1$. Suppose $c_{h-1}^* \leq 1$. Since $q_{2,h}^h = 0$, we have

$$\begin{aligned}
0 &= c_h^* b_{h-1}^* + b_h^* c_{h+1}^* - k_1^* \\
&= c_h^* (k_1^* - c_{h-1}^*) + b_h^* c_{h+1}^* - k_1^* \\
&\geq c_h^* (k_1^* - 1) + b_h^* + b_h^* (c_{h+1}^* - 1) - k_1^* \\
&\geq c_h^* + b_h^* - k_1^* + b_h^* (c_{h+1}^* - 1) \\
&\geq b_h^* (c_{h+1}^* - 1)
\end{aligned}$$

Thus $c^*_{h+1} - 1 \le 0$. Since α is odd, $c^*_{\alpha-2} \le 1$. Now by Proposition 1, $b^*_2 = c^*_{\alpha-2} \le 1$. Note that this holds for $\alpha = 5$ as well because $a^*_2 = 0$ in our case. Since $q^2_{2,2} = 0$, we have

$$0 = c^*_2 b^*_1 + b^*_2 c^*_3 - k^*_1 > c^*_2 b^*_1 - k^*_1 \ge (k^*_1 - 1)^2 - k^*_1.$$

This is impossible. Hence we have the assertion when α is odd.

Suppose $\alpha = 2\beta$. Then the argument above shows that $c^*_h \le 1$ when h is odd and $h \le \beta + 1$. Note that $q^\beta_{2,\beta} = 0$. Suppose h is odd and $h \le \beta$. Then

$$0 = c^*_h b^*_{h-1} + b^*_h c^*_{h+1} - k^*_1 > (k^*_1 - 1)c^*_{h+1} - k^*_1.$$

Since $k^*_1 \ge 3$ as k^*_1 is an integer, $c^*_{h+1} < 3/2$. Therefore, we have the following.

$c^*_h \le 1$ if h is odd and $h \le \beta + 1$.

$c^*_h < 3/2$ if h is even and $h \le \beta + 1$.

Suppose β is odd. Then $b^*_{\beta-1} = c^*_{\beta+1} < 3/2$ by Proposition 1. Since $c^*_{\beta-1} < 3/2$, $3/2 > b^*_{\beta-1} = k^*_1 - c^*_{\beta-1} > k^*_1 - 3/2$. Thus $k^*_1 < 3$. This contradicts our assumption.

Suppose β is even. Then $c^*_{\beta-1} \le 1$ and $b^*_{\beta-1} = c^*_{\beta+1} \le 1$. Thus we obtain that $k^*_1 \le 2$.

This proves the assertion. □

Proof of Theorem 3: Suppose $\alpha = 2$. Then by Lemma 9, we have (i). Suppose $\alpha = d$. Then by Proposition 1, we have (ii). Suppose $2 < \alpha < d$. If α is even, then $a^*_i = 0$ for every i by Lemma 9. If α is odd, then by Lemma 11, Lemma 12 and Proposition 1, we have $a^*_i = 0$ for all $i = 1, 2, \ldots, d-1$ unless $\alpha = 3$ and (iii) or (iv) holds. Now by Lemma 13, we cannot have other cases.

This completes the proof of Theorem 3. □

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