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On Locally Projective Graphs of Girth 5

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Abstract. Let Γ be a graph and G be a 2-arc transitive automorphism group of Γ. For a vertex $x \in \Gamma$ let $G(x)^{\Gamma(x)}$ denote the permutation group induced by the stabilizer $G(x)$ of x in G on the set $\Gamma(x)$ of vertices adjacent to x in Γ. Then Γ is said to be a locally projective graph of type (n, q) if $G(x)^{\Gamma(x)}$ contains $PSL_n(q)$ as a normal subgroup in its natural doubly transitive action. Suppose that Γ is a locally projective graph of type (n, q), for some $n \geq 3$, whose girth (that is, the length of a shortest cycle) is 5 and suppose that $G(x)$ acts faithfully on $\Gamma(x)$. (The case of unfaithful action was completely settled earlier.) We show that under these conditions either $n = 4$, $q = 2$, Γ has 506 vertices and $G \cong \mathrm{M}_{23}$, or $q = 4$, $PSL_n(4) \leq G(x) \leq PGL_n(4)$, and Γ contains the Wells graph on 32 vertices as a subgraph. In the latter case if, for a given n, at least one graph satisfying the conditions exists then there is a universal graph $W(n)$ of which all other graphs for this n are quotients. The graph $W(3)$ satisfies the conditions and has 2^{20} vertices.

Keywords: locally projective graph, graph of girth 5, 2-arc-transitive graph

1. Introduction

Let Γ be a graph which is assumed to be undirected connected and locally finite (the latter means that every vertex of Γ is adjacent to a finite number of other vertices). The vertex set of Γ will be denoted by the same letter Γ while $E(\Gamma)$ and Aut (Γ) will denote the edge set and the automorphism group of Γ, respectively. For a vertex $x \in \Gamma$ we denote by $\Gamma_i(x)$ the set of vertices at distance i from x with respect to the natural distance on Γ. The set $\Gamma_1(x)$ (which consists of the vertices adjacent to x) will be denoted simply by $\Gamma(x)$. An *s-arc* in Γ is a sequence $x_0, x_1, \ldots, x_s$ of vertices, such that $\{x_i, x_{i+1}\} \in E(\Gamma)$ for $0 \leq i \leq s-1$ and $x_i \neq x_{i+2}$ for $0 \leq i \leq s-2$. Such an arc is a *cycle* of length s if $x_0 = x_s$. The *girth* of Γ is the length of its shortest cycle. For a subset Δ of the vertex set of Γ the *subgraph induced by* Γ on Δ has Δ as vertex set and $\{x, y\}$ is an edge in this subgraph if $x, y \in \Delta$ and $\{x, y\} \in E(\Gamma)$. Let G be a group of automorphisms of Γ, that is a subgroup of Aut (Γ). If $\Delta \subseteq \Gamma$ then $G(\Delta)$ and $G\{\Delta\}$ denote the pointwise and the setwise stabilizers of Δ in G, respectively. We write $G(x, y, \ldots)$ instead of $G(\{x, y, \ldots\})$ and $G\{x, y, \ldots\}$ instead of $G\{\{x, y, \ldots\}\}$. If $H \leq G\{\Delta\}$ then H^Δ denotes the permutation group induced by H on Δ, so that abstractly $H^\Delta \cong H/H(\Delta)$. If G acts transitively on s-arcs in Γ then G is

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said to be *s-arc transitive*. It is easy to see that G is 2-arc transitive if and only if G is vertex-transitive and, for $x \in \Gamma$, the permutation group $G(x)^{\Gamma(x)}$ is doubly transitive. Let $G_1(x) := G(\{x\} \cup \Gamma(x))$ so that $G(x)^{\Gamma(x)} \cong G(x)/G_1(x)$.

Let Γ be a graph and G be a 2-arc transitive automorphism group of Γ. Then Γ is said to be a *locally projective graph of type* (n, q) (with respect to the action of G) if, for $x \in \Gamma$, the permutation group $G(x)^{\Gamma(x)}$ contains, as a normal subgroup, the projective special linear group $PSL_n(q)$ in its natural doubly transitive action. This means that $|\Gamma(x)| = [{n \atop 1}]_q := (q^n - 1)/(q - 1)$ (which is the number of 1-subspaces in an n-dimensional $GF(q)$-space) and $PSL_n(q) \leq G(x)^{\Gamma(x)} \leq P\Gamma L_n(q)$. Examples of locally projective graphs come from actions of finite groups of Lie type on certain incidence graphs of their parabolic geometries and also from certain actions of the sporadic simple groups. In these examples the kernel $G_1(x)$ is large compared with the size of this group for other 2-arc transitive actions, and this is one of the reasons for the attention locally projective graphs have received in the literature, see for example [19, 20].

The present paper contributes to the classification of locally projective graphs of type (n, q), for $n \geq 3$, of small girth. We start with a brief survey of what has already been achieved in this area (cf. [12] for further details).

If Γ is a locally projective graph of type (n, q) and girth 3, then it is a complete graph on $[{n \atop 1}]_q + 1$ vertices and G acts triply transitively on the vertex set of Γ, being a one-point transitive extension of a projective linear group. All such extensions are classified in [9].

Theorem 1.1 *Let Γ be a locally projective graph of type (n, q) with $n \geq 3$, and of girth 3, with respect to a subgroup G of automorphisms of Γ. Then Γ is a complete graph on $[{n \atop 1}]_q + 1$ vertices, and one of the following holds:*
(i) *$q = 2$ and $G \cong AGL_n(2)$;*
(ii) *$q = 4$, $n = 3$ and $M_{22} \leq G \leq Aut\ (M_{22})$.*

Locally projective graphs of girth 4 were considered in [5] where complete classification was achieved in the case $G_1(x) \neq 1$. The case $G_1(x) = 1$ was completed in [6].

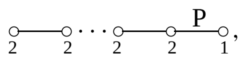
Theorem 1.2 *Let Γ be a locally projective graph of type (n, q) with $n \geq 3$, and of girth 4, with respect to a subgroup G of automorphisms of Γ. Then one of the following holds:*
(i) *Γ is the complete bipartite graph on $2 \cdot [{n \atop 1}]_q$ vertices and $PSL_n(q) \times PSL_n(q) < G \leq P\Gamma L_n(q) \wr 2$;*
(ii) *Γ is the point-hyperplane incidence graph of an $(n + 1)$-dimensional $GF(q)$-space, G contains $PSL_{n+1}(q)$ extended by a contragredient automorphism and is contained in $Aut\ (PSL_{n+1}(q))$;*
(iii) *the vertices of Γ are the maximal totally singular subspaces of a $2n$-dimensional $GF(q)$-space equipped with a non-degenerate orthogonal form of maximal Witt index, two subspaces are adjacent if their intersection has codimension 1 in each; $O^+_{2n}(q) < G \leq Aut\ (O^+_{2n}(q))$;*
(iv) *Γ is the standard doubling $2.K_m$ of the complete graph on $m := [{n \atop 1}]_q + 1$ vertices, i.e., the vertices of Γ are ordered pairs (i, α) where $1 \leq i \leq [{n \atop 1}]_q + 1$, $\alpha \in \{0, 1\}$ with (i, α) and (j, β) adjacent if $i \neq j$ and $\alpha \neq \beta$, moreover either $q = 2$ and $G \cong AGL_n(q) \times 2$ or $q = 4$, $n = 3$ and $M_{22} < G \leq Aut\ (M_{22}) \times 2$;*

(v) *G contains an elementary abelian normal subgroup E which acts regularly on the vertex set of Γ and E is a quotient of the $GF(2)$-permutational module of $G(x)^{\Gamma(x)}$;*
(vi) $(n,q) = (3,4)$, $|\Gamma| = 324$, $U_4(3).2 \le G \le U_4(3).(2^2)_{122}$, $PSL_3(4) \le G(x) \le P\Sigma L_3(4)$;
(vii) $(n,q) = (3,2)$, $|\Gamma| = 72$, $G \cong G_2(2) \cong U_3(3).2$, $G(x) \cong PSL_3(2)$.

Remark 1 In the above theorem $G_1(x) \neq 1$ in the cases (i)–(iii) and $G_1(x) = 1$ in the remaining cases.

Remark 2 In case (v) the graph Γ is a quotient of the $\left[{n \atop 1}\right]_q$-dimensional cube. If q is odd then Γ is either the cube itself or the folded cube (cf. [4]), while if q is even there are more quotients of the permutation module and correspondingly more possibilities for Γ (cf. [14] for some information about these quotients).

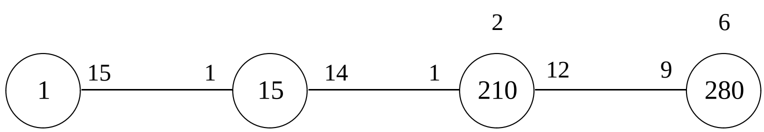
In [11] the classification problem for locally projective graphs of girth 5 in the case $G_1(x) \neq 1$ was reduced to the classification of flag-transitive Petersen type geometries, namely geometries with a diagram of the following type:



where the rightmost edge represents the geometry of edges and vertices of the Petersen graph with the natural incidence relation. The classification of such geometries was recently completed (cf. [15]). All examples come from sporadic simple groups. As a direct consequence of the classification we have the following:

Theorem 1.3 *Let Γ be a locally projective graph of type (n,q) with $n \geq 3$, and of girth 5, with respect to a subgroup G of automorphisms of Γ. Suppose that $G_1(x) \neq 1$. Then $q = 2$, Γ contains the Petersen graph as a subgraph, and there are exactly eight possibilities for the isomorphism type of Γ so that one of the following holds*:
(i) *$n = 3$ and $M_{22} \le G \le Aut\,(M_{22})$ or $3 \cdot M_{22} \le G \le 3 \cdot Aut\,(M_{22})$;*
(ii) *$n = 4$ and $G \cong Co_2$, $3^{23} \cdot Co_2$ or J_4;*
(iii) *$n = 5$ and $G \cong J_4$, F_2 or $3^{4371} \cdot F_2$.*

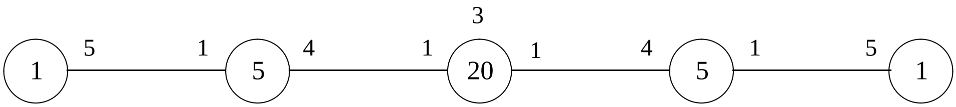
In the present paper we address the classification problem for locally projective graphs of girth 5 in the case $G_1(x) = 1$. One such example, which we denote by $\Gamma(\mathrm{M}_{23})$ comes from the Petersen type geometry associated with the Mathieu group M_{23} and until recently it was the only example known. The vertices of $\Gamma(\mathrm{M}_{23})$ are the blocks of the Steiner system $S(5,8,24)$ which do not contain a given point (there are exactly 506 such blocks); two blocks are adjacent if they are disjoint. The graph is distance-transitive [4] with the following intersection diagram:



$\Gamma(\mathrm{M}_{23})$ is a locally projective graph of type $(4,2)$ with respect to its full automorphism group $G \cong \mathrm{M}_{23}$. If x is a vertex and $\{x,y\}$ is an edge, then $G(x) \cong PSL_4(2) \cong \mathrm{A}_8$,

$G(x, y) \cong 2^3 : PSL_3(2)$ and $G\{x, y\}$ is an extension of $G(x, y)$ by an automorphism which interchanges the two classes of complements to $O_2(G(x, y))$ in $G(x, y)$.

Very recently a new example of a locally projective graph of girth 5 was constructed using computer calculations performed by L.H. Soicher. This graph is locally projective of type (3, 4) and we will denote it by $W(3)$. The automorphism group of $W(3)$ contains a non abelian normal subgroup of order 2^{20} which acts regularly on the vertex set of the graph. Further $W(3)$ contains as a subgraph the Wells graph on 32 vertices which is a distance-transitive graph with intersection diagram



and automorphism group isomorphic to $2^{1+4}_{-}.A_5$. (The Wells graph was constructed by A. L. Wells in [21] and also earlier by C. Armanios [1, 2].)

We summarise the results of this paper in the following theorem. It shows in particular the significance of the Wells graph in our context. For a vertex x of a locally projective graph Γ of type (n, q), where $n \geq 3$ and q is a power of a prime number p, let Π_x denote the projective space structure having $\Gamma(x)$ as point set and preserved by $G(x)$. Choose a line λ of Π_x and let $G(\lambda)$ be the pointwise stabilizer of λ in G. Consider the subgraph in Γ induced by the vertices fixed by $O^p(G(\lambda))$. Let Δ be the connected component of this subgraph containing x. The isomorphism type of Δ is clearly independent of the choices of x and λ.

Theorem 1.4 *Let Γ be a locally projective graph of type (n, q) with $n \geq 3$, and of girth 5, with respect to a subgroup G of automorphisms of Γ. Suppose that $G_1(x) = 1$. Then one of the following holds*:

(i) $n = 4$, $q = 2$, *Δ is the Petersen graph, and $\Gamma \cong \Gamma(M_{23})$, $G \cong M_{23}$*;

(ii) $n \geq 3$, $q = 4$, *Δ is the Wells graph. Graphs with these properties exist if and only if the graph $W(n)$ defined in Proposition* 6.9 *has girth* 5. *Moreover*,

  (a) *if $W(n)$ has girth* 5, *then every graph Γ with these properties is a quotient of $W(n)$*;

  (b) *$W(3)$ has girth* 5, *and has exactly 2^{20} vertices*;

  (c) *for every $n \geq 3$ the automorphism group of $W(n)$ contains a normal subgroup T acting regularly on the vertex set, such that $[T, T, T] = 1$, both $T/[T, T]$ and $[T, T]$ are elementary abelian 2-groups of rank less than $\begin{bmatrix} n \\ 1 \end{bmatrix}_4$ and $\begin{bmatrix} n \\ 2 \end{bmatrix}_4$, respectively (in particular $W(n)$ is finite).*

Our theoretical analysis proves that $W(3)$ has at least 2^{20} vertices (see Proposition 7.7). The fact that equality holds depends on the computer calculations of Leonard Soicher mentioned above. An explicit presentation for a 2-arc transitive automorphism group of $W(3)$ is given in Section 8. We are grateful to Leonard for his assistance. Not only was the new construction a very nice surprise, but also it suggested the line of investigation which resulted in the construction of the graphs $W(n)$ for general $n \geq 3$. We are also very thankful to Sergey Shpectorov for pointing out a few inaccuracies in the first draft of the paper.

2. The stabilizer of an edge

For the remainder of the paper Γ denotes a locally projective graph of type (n, q), where $n \geq 3$, with respect to a 2-arc transitive subgroup G of automorphisms of Γ. We assume that the girth of Γ is 5 and that $G_1(x) = 1$ for $x \in \Gamma$. The latter means that $PSL_n(q) \leq G(x) \leq P\Gamma L_n(q)$, and $G(x)$ acts faithfully as a doubly transitive permutation group on the set $\Gamma(x)$ of size $k := \left[{n \atop 1}\right]_q$. Let p be the characteristic of $GF(q)$, that is, p is a prime and q is a positive power of p.

Let Π_x denote the projective space structure having $\Gamma(x)$ as the point set, which is invariant under the action of $G(x)$. Let L_x denote the set of lines in Π_x considered as a collection of $(q + 1)$-element subsets of $\Gamma(x)$. For $y \in \Gamma(x)$ let $L_x(y)$ denote the set of lines in L_x containing y and for $z \in \Gamma(x) \setminus \{y\}$ let $l_x(y, z)$ denote the unique line in L_x which contains both y and z. We will usually be working with a given vertex $x \in \Gamma$ and a given pair of vertices $y, z \in \Gamma(x)$, and we set $\Gamma(x) = \{y_1 = y, y_2 = z, y_3, \ldots, y_{\left[{n \atop 1}\right]_q}\}$ and $\lambda := l_x(z, y) = \{y_1, y_2, \ldots, y_{q+1}\}$.

We start by recalling some basic properties of the projective linear groups in their natural doubly transitive actions. Let $P_1 := G(x, y)$, $P_2 := G(x) \cap G\{\lambda\}$, $P_{12} := P_1 \cap P_2$, and $R := G(x, y, z)$. This means that P_1 and P_2 are two maximal parabolic subgroups associated with the action of $G(x)$ on Π_x and $R \leq P_{12}$. We can and will identify $G(x)$ and its subgroups with the corresponding subgroups in the automorphism group A of Π_x, where $A \cong P\Gamma L_n(q)$. Let A^0 be the largest subgroup in A which consists of projective linear transformations of Π_x, so that $A^0 \cong PGL_n(q)$. For X being one of the subgroups $G(x)$, P_1, P_2, P_{12}, or R, set $X^0 := X \cap A^0$. Then X^0 is normal in X and X/X^0 is a subgroup of the automorphism group of $GF(q)$ which is independent of the choice of X from the above list.

To describe the action of P_1 on $\Gamma(x) \setminus \{y\}$, we introduce some characteristic subgroups of P_1. First of all $C_1 := O_p(P_1)$ is a characteristic subgroup of P_1. Moreover C_1 is elementary abelian of order q^{n-1}, it stabilizes setwise every line $l \in L_x(y)$ and induces a regular permutation group on $l \setminus \{y\}$. Let H and H^0 denote the permutation groups induced on the set $L_x(y)$ by P_1 and P_1^0, respectively. Then $H^0 \cong PGL_{n-1}(q)$ (notice that this is true even if $G(x)^0$ is a proper subgroup of $PGL_n(q)$) and $PGL_{n-1}(q) \cong H^0 \leq H \leq P\Gamma L_{n-1}(q)$. Moreover, it is easy to see that $H/H^0 \cong G(x)/G(x)^0$. The latter means that the kernel C_2 of the action of P_1 on $L_x(y)$ is contained in P_1^0. Let H^1 be the unique subgroup of H^0 isomorphic to $PSL_{n-1}(q)$. We claim that both H^0 and H^1 are characteristic subgroups of H. Indeed, if $(n-1, q) = (2, 2)$ or $(2, 3)$ then the claim can be checked directly; otherwise H^1 is the unique minimal non abelian normal subgroup of H and hence is characteristic. Let $F = P\Gamma L_{n-1}(q)/H^1$. Then F is a split extension of a cyclic group D of order $q - 1$ (which is the image of H^0) by a cyclic subgroup E of order m, where $q = p^m$. Now if $d \in D$ then the order of $C_F(d)$ is divisible by $q - 1$ and if $e \in F \setminus D$ then the order of $C_F(e)$ is at most $(p^{m/2} - 1)m$. This means that all elements of F with centralizers having order divisible by $q - 1$ are contained in D and hence D is characteristic. Hence P_1^0 and the full preimage P_1^1 of H^1 in P_1^0 are characteristic subgroups of P_1. Let us consider more closely the kernel C_2 of the action of P_1 on $L_x(y)$. In terms of matrix groups, one can see that C_2 is a split extension of C_1 by a cyclic subgroup K_2 whose order divides $q - 1$ and is divisible by $(q - 1)/\gcd(n, q - 1)$. Moreover, every non-trivial element of K_2 acts fixed-point freely on C_1. We claim that C_2 is a characteristic subgroup of P_1. Indeed, if $(n, q) = (3, 2)$ or $(3, 3)$ this is straightforward; otherwise C_2 is the largest solvable normal subgroup in P_1.

Lemma 2.1 *Using the above notation let $t \in G\{x, y\} \setminus G(x, y)$. Then one of the following holds*:

(i) *P_1 contains a unique class of complements to C_1 and t can be chosen to normalize such a complement*;

(ii) *$(n, q) = (4, 2)$ or $(n, q) = (3, 4)$ and $G(x)$ does not contain $PGL_3(4)$ in the latter case*; *there are more than one class of complements to C_1 in P_1 but t can be chosen to normalize such a complement*;

(iii) *$(n, q) = (4, 2)$ and the amalgam $\{G(x), G\{x, y\}\}$ is isomorphic to the amalgam of vertex and edge stabilizers in M_{23} acting on $\Gamma(M_{23})$*;

(iv) *$(n, q) = (3, 4)$; $G(x) \cong PSL_3(4)$ and the amalgam $\{G(x), G\{x, y\}\}$ is isomorphic to the amalgam of vertex and edge stabilizers in the vertex transitive action of Aut (M_{22}) on $2.K_{22}$ as in Theorem* 1.2 (iv);

(v) *$(n, q) = (3, 4)$; $G(x) \cong P\Sigma L_3(4)$ and the amalgam $\{G(x), G\{x, y\}\}$ is isomorphic to the amalgam of vertex and edge stabilizers in the action of Aut (M_{22}) on K_{22} as in Theorem* 1.1 (ii).

Proof: If $(n, q) = (3, 2)$ or $(3, 3)$ then $P_1 \cong S_4$ or $AGL_2(3)$, respectively and obviously there is a unique class of complements to C_1 in P_1; and t can be chosen to normalize one of them. In the remaining cases define C_3 to be the commutator subgroup of P_1^1, so that C_3 is a split extension of C_1 by a subgroup isomorphic to $SL_{n-1}(q)$. Let $(n, q) \neq (4, 2)$ and $(n, q) \neq (3, 2^m)$ for $m \geq 2$. By [3] in this case all complements to C_1 in C_3 are conjugate in C_3. Since every complement to C_1 in P_1 is the normalizer in P_1 of a complement to C_1 in C_3 we are again in case (i). Suppose now that $(n, q) = (3, 2^m)$ where either $m \geq 3$, or $m = 2$ and $G(x)$ contains $PGL_3(4)$. We claim that in both cases K_2 is non-trivial. Indeed, in the former case $q - 1 \neq \gcd(q - 1, n)$ and so $|K_2| \geq (q - 1)/\gcd(q - 1, n) > 1$, and in the latter case it is straightforward to see that $K_2 \cong Z_3$. The subgroup K_2 is a complement to C_1 in C_2 and it is a Hall subgroup of C_1. Hence all the complements to C_1 in C_2 are conjugate. Since K_2 acts fixed-point freely on C_1, every complement to C_1 in P_1 is the normalizer in P_1 of a complement to C_1 in C_2 provided that the latter complement is non-trivial. Hence again we are in case (i).

If $(n, q) = (4, 2)$, then by [3] there are two classes of complements to $C_1 \cong 2^3$ in $P_1 \cong 2^3 : PSL_3(2)$. If these two classes are permuted by the elements from $G\{x, y\}\setminus G(x, y)$ then $G\{x, y\} = \text{Aut}\,(G(x, y))$ which immediately shows that the isomorphism type of the amalgam $\{G(x), G\{x, y\}\}$ is uniquely determined. This amalgam corresponds to the action of M_{23} on $\Gamma(M_{23})$.

Let $(n, q) = (3, 4)$ and $G(x) \cong PSL_3(4)$ or $P\Sigma L_3(4)$. Then $P_1 \cong 2^4 : L$, where $L \cong A_5$ or S_5, respectively. By [3] there are four or two classes of complements to C_1 in P_1, respectively. If $G\{x, y\}$ normalizes one of these classes then of course we are in case (ii). On the other hand if $G\{x, y\}$ is an extension of P_1 by an automorphism acting fixed-point freely on the classes of complements and whose square is an inner automorphism, then its isomorphism type is uniquely determined in each of the two cases. Namely, $G\{x, y\}$ is a split extension by L of the five-dimensional quotient of the six-dimensional permutation module of L. In each of the two cases the automorphism group of $G(x, y)$ is factorized by the normalizers of $G(x, y)$ in $G(x)$ and $G\{x, y\}$. Hence the isomorphism type of the amalgam $\{G(x), G\{x, y\}\}$ is uniquely determined. In both cases the amalgam is contained in Aut (M_{22}). □

We next discuss the relationship between P_1 and P_2. First notice that, even if $G(x)^0$ is a proper subgroup of $PGL_n(q)$, P_2^0 induces $PGL_2(q)$ on the points of λ. Its subgroup $O_p(P_2)$ is elementary abelian of order $q^{2(n-2)}$ and acts fixed-point freely on $\Gamma(x) \setminus \lambda$. Also the subgroup P_{12} is the full preimage in P_1 of the stabilizer in $H = P_1/C_1$ of the line λ; $O_p(P_{12})$ has order $q^{(n-1)+(n-2)}$ and its center is $O_p(P_1) \cap O_p(P_2)$.

Lemma 2.2 *Suppose we are not in case* (iii) *of Lemma* 2.1. *Then there is a bijective mapping φ from $\Gamma_2(x)$ onto the set of ordered pairs of distinct vertices in $\Gamma(x)$, such that φ commutes with the action of $G(x)$ and if $\varphi(u) = (y, z)$ for $u \in \Gamma_2(x)$, then u and y are adjacent.*

Proof: Suppose first that $q = 2$. Then $G(x, y) = P_1 \cong 2^{n-1} : PSL_{n-1}(2)$. Since we are not in case (iii) of Lemma 2.1, we can choose $t \in G\{x, y\} \setminus G(x, y)$ to normalize a complement N to C_1 in P_1. Since t also normalizes C_1 and the latter is the natural module for N, t induces an inner automorphism of N. Hence t can be adjusted to centralize N and C_1 as well. Then t^2 is in the center of P_1 which is trivial. Hence $G\{x, y\} = P_1 \times \langle t\rangle$ where $t^2 = 1$ and t is uniquely determined by $\{x, y\}$. For $u \in \Gamma(y) \setminus \{x\}$ define $\varphi(u)$ to be (y, u^t). It is easy to see that φ is bijective and commutes with the action of $G(x)$.

Now suppose that $q \geq 3$. We observed in the paragraph before the statement of the lemma that P_2^0 induces $PGL_2(q)$ on the points in λ. This means in particular that $R = G(x, y, z)$ does not stabilize in $\Gamma(x)$ any vertices other than y and z. On the other hand, if the mapping we are seeking exists, then R stabilizes the vertex $u \in \Gamma(y)$ satisfying $\varphi(u) = (y, z)$. By symmetry $R = G(x, y, u)$ and R stabilizes only the vertex u in $\Gamma(y) \setminus \{x\}$. Conversely, if R stabilizes a (unique) vertex $u \in \Gamma(y) \setminus \{x\}$, then the map φ defined by $\varphi(u) = (y, z)$ has the required properties.

It is clear that R fixes a vertex $u \in \Gamma(y) \setminus \{x\}$ if and only if there is an element $t \in G\{x, y\} \setminus G(x, y)$ which normalizes R. First suppose that we are in case (i) of Lemma 2.1. Then all complements to C_1 in P_1 are conjugated in P_1 and we can choose t to normalize one of them, say N. Since N acts transitively on $L_x(y)$ we can choose t to normalize the stabilizer N_2 of λ in N. Since N is from the unique class of complements, we have $P_{12} = C_1 : N_2$. Moreover $O_p(P_1) \cap O_p(P_2)$ is the centralizer of $O_p(N_2)$ in C_1, and $(O_p(P_1) \cap O_p(P_2)) : N_2$ is the stabilizer in P_1 of a point in $\lambda \setminus \{y\}$, so we can choose z to be this point.

Finally let us assume that we are in case (ii), (iv) or (v) of Lemma 2.1 for $q > 2$. Then $(n, q) = (3, 4)$, $G(x) \cong PSL_3(4)$ or $P\Sigma L_3(4)$, and $P_1 \cong 2^4 : A_5$ or $2^4 : S_5$, respectively. There is a unique class of index 5 subgroups in P_1 and P_{12} is one of them. Hence we can choose t to normalize P_{12}. There are exactly two elementary abelian subgroups of order 2^4 in $O_2(P_{12})$. One of them is C_1 and the other is $O_2(P_2) = O_2(R)$. Since t normalizes

C_1 it normalizes $O_2(R)$. Finally R is the full preimage of the normalizer of a Sylow 3-subgroup in $P_{12}/O_2(R) \cong \mathrm{A}_4$. Hence we can choose t to normalize R and the result follows. $\square$

3. A characterization of $\Gamma(\mathrm{M}_{23})$

Let Γ be a locally projective graph of girth 5 which corresponds to case (iii) of Lemma 2.1 with respect to a 2-arc transitive subgroup G of automorphisms of Γ. We show in this section that under these conditions $\Gamma \cong \Gamma(\mathrm{M}_{23})$ and $G \cong \mathrm{M}_{23}$.

If Γ corresponds to case (iii) of Lemma 2.1, then $G(x) \cong PSL_4(2)$; $G(x, y) \cong 2^3 : PSL_3(2)$; $G\{x, y\}$ is a split extension of an elementary abelian subgroup Q of order 2^4 by $PSL_3(2)$ and Q is an indecomposable $GF(2)$-module for $PSL_3(2)$. Now $O_2(G(x, y))$ stabilizes setwise every line from $L_x(y) \cup L_y(x)$, while $PSL_3(2) \cong G(x, y)/O_2(G(x, y))$ induces on $L_x(y)$ and on $L_y(x)$ two equivalent natural actions of degree 7. Hence there is a unique bijection ψ_{xy} of $L_x(y)$ onto $L_y(x)$ which commutes with the action of $G(x, y)$.

By Lemma 2.1 (iii) the amalgam $\{G(x), G\{x, y\}\}$ is isomorphic to the amalgam $\{M(x), M\{x, y\}\}$ associated with the action of $M \cong \mathrm{M}_{23}$ on $\Gamma(\mathrm{M}_{23})$. Let $\hat{\Gamma}$ be the covering tree of Γ and let $\hat{G}$ be the free amalgamated product of $G(x)$ and $G\{x, y\}$ acting naturally on $\hat{\Gamma}$. Then $\hat{\Gamma}$ is also a covering tree of $\Gamma(\mathrm{M}_{23})$ and $\hat{G}$ is the free amalgamated product of $M(x)$ and $M\{x, y\}$. This means that every local property of the action of G on Γ (that is, a property shared with the action of $\hat{G}$ on $\hat{\Gamma}$) is also shared with the action of M_{23} on $\Gamma(\mathrm{M}_{23})$. This applies in particular to the action of $G(x)$ on $\Gamma_2(x)$ and to the action of $G(x, y)$ on $\Gamma(x) \cup \Gamma(y)$.

As usual, let $\lambda = \{y_1 = y,\ y_2 = z,\ y_3\} \in L_x$ and set

$$\Delta := \{x\} \cup \lambda \bigcup_{i=1}^{3} \psi_{xy_i}(\lambda).$$

The properties of $\Gamma(\mathrm{M}_{23})$ stated in the next lemma follow from standard facts about the action of M_{24} on $S(5, 8, 24)$ and from the Petersen type geometry structure associated with $\Gamma(\mathrm{M}_{23})$, see [13].

Lemma 3.1 *Let $\Gamma \cong \Gamma(M_{23})$ and $G \cong M_{23}$. Then*

(i) *the subgraph induced by Γ on Δ is isomorphic to the Petersen graph*;
(ii) $G\{\Delta\} \cong 2^4 : (3 \times A_5) : 2$, $G\{\Delta\}/G(\Delta) \cong S_5$, *the center of $G(\Delta)$ is trivial and $G(\Delta)$ acts transitively on $\Gamma(x) \setminus \Delta$.*

Since G is 2-arc transitive and Γ is of girth 5, there are $u \in \Gamma(y) \setminus \{x\}$ and $v \in \Gamma(z) \setminus \{x\}$ which are adjacent. Set $\mu := \psi_{yx}(l_y(u, x))$, $\nu := \psi_{zx}(l_z(v, x))$.

Lemma 3.2 *Let Γ be an arbitrary locally projective graph of girth* 5 *with respect to a* 2*-arc transitive subgroup G of automorphisms of Γ, and suppose that Γ corresponds to case* (iii) *in Lemma* 2.1. *Then*

(i) $G(x, y, u)$ *acts transitively on the Cartesian product* $(\mu \setminus \{y\}) \times (\Gamma(x) \setminus \mu)$;
(ii) $\mu = \nu = \lambda$;
(iii) *the subgraph induced by* Γ *on* Δ *is isomorphic to the Petersen graph.*

Proof: Since the action of G on Γ is locally isomorphic to that of M_{23} on $\Gamma(M_{23})$, part (i) follows directly from Lemma 3.1.

To prove part (ii) suppose first that $\mu = \nu$. Then μ contains z and hence must be equal to $l_x(y, z) = \lambda$. So we may assume that $\mu \neq \nu$. Notice that $G(x, y, z, u)$ must fix v since otherwise there would be more than one 2-arc joining u and z which is impossible, since the girth of Γ is 5. Hence $G(x, y, z, u)$ stabilizes $l_z(v, x)$ and also ν. Suppose that $\mu = \lambda$. Then by part (i), $G(x, y, z, u)$ acts transitively on $\Gamma(x) \setminus \mu$ and hence it does not stabilize in Π_x any lines other than $\mu = \lambda$, which is a contradiction. Hence $\mu \neq \lambda$. Because of the obvious symmetry between y and z, also $\nu \neq \lambda$, and so μ, ν and λ are pairwise distinct. Let Ξ be the hyperplane in Π_x which contains λ and μ. By (i), $G(x, y, z, u)$ acts transitively on $\mu \setminus \{y\}$ which means that $G(x, y, z, u)$ does not stabilize lines in Ξ not passing through y. Hence ν does not lie in Ξ. Finally, $G(x) \cap G(\Xi)$ (which is the same as $G(\Xi)$ since there are no 4-cycles in Γ) permutes transitively the eight points in $\Gamma(x) \setminus \Xi$ and $G(x, y, z, u) \cap G(\Xi)$ has index at most 2 in $G(x) \cap G(\Xi)$. Hence $G(x, y, z, u)$ does not stabilize lines outside Ξ, which is a contradiction. Thus part (ii) holds.

Now by (i) and (ii) it is easy to observe that the subgraph induced by Γ on Δ is regular of valency 3, girth 5, with 10 vertices. Hence it is isomorphic to the Petersen graph and (iii) follows. □

Lemma 3.3 *Let* Γ *be a locally projective graph of girth* 5 *which corresponds to case* (iii) *of Lemma* 2.1 *with respect to a subgroup* G *of automorphisms of* Γ. *Then* $\Gamma \cong \Gamma(M_{23})$ *and* $G \cong M_{23}$.

Proof: Let Δ be the Petersen subgraph in Γ as in Lemma 3.2 (iii). By Lemma 3.1 and since the actions of G on Γ and of M_{23} on $\Gamma(M_{23})$ are locally isomorphic, we have $(G(x) \cap G\{\Delta\})^{\Delta} \cong S_3 \times Z_2$. Furthermore, one can see that Δ consists of $\{x, y\}$, λ, $\psi_{xy}(\lambda)$ and the vertices on the shortest paths joining vertices from λ to vertices from $\psi_{xy}(\lambda)$. This observation shows that there exists $t \in G\{x, y\} \setminus G(x, y)$ which stabilizes Δ as a whole. Hence $G\{\Delta\}/G(\Delta) \cong S_5$. We claim that the isomorphism type of the amalgam $\mathcal{G} = \{G(x), G\{x, y\}, G\{\Delta\}\}$ is uniquely determined. Indeed, the isomorphism type of the amalgam $\{G(x), G\{x, y\}\}$ is uniquely determined. Hence $G\{\Delta\}$ is a homomorphic image of the free amalgamated product F of $G(x) \cap G\{\Delta\}$ and $G\{x, y\} \cap G\{\Delta\}$. Let K be the corresponding kernel. Since the center of $G(\Delta)$ is trivial and $G\{\Delta\}/G(\Delta)$ acts faithfully on $G(\Delta)$, we have $K \geq C_F(G(\Delta))$. On the other hand $F/(G(\Delta)C_F(G(\Delta))) \cong S_5$. Hence $K = C_F(G(\Delta))$ and the isomorphism type of $\mathcal{G}$ is uniquely determined. Thus $\mathcal{G}$ is isomorphic to the analogous amalgam $\mathcal{M}$ associated with the action of M_{23} on $\Gamma(M_{23})$. By the main result of [13] the Petersen type geometry associated with $\Gamma(M_{23})$ is 2-simply connected. In accordance with a standard principle (cf. Section 12.4.3 in [17]) this is equivalent to the fact that M_{23} is the universal completion of the amalgam $\mathcal{M}$. Since $\mathcal{M}$ and $\mathcal{G}$ are isomorphic and M_{23} is a non-abelian simple group, M_{23} is the unique completion of the amalgam $\mathcal{G}$ and the result follows. □

4. The geometrical subgraphs

From now on we assume that, for every $x \in \Gamma$, there is a bijection φ of $\Gamma_2(x)$ onto the set of ordered pairs of distinct vertices in $\Gamma(x)$ which commutes with the action of $G(x)$ and, if $\varphi(u) = (y, z)$, then u and y are adjacent (see Lemma 2.2).

Let σ be a subspace in Π_x of projective dimension $m - 1$, where $2 \leq m \leq n$. Consider the subgraph in Γ induced by the fixed vertices of the pointwise stabilizer $G(\sigma)$ of σ in G. Let $\Sigma = \Sigma[\sigma]$ be the connected component of this subgraph containing x. In what follows the subgraph $\Sigma[\sigma]$ will be called the *geometrical subgraph* corresponding to σ. Set $H \cong G\{\Sigma\}^{\Sigma}$ be the action induced by $G\{\Sigma\}$ on Σ.

Lemma 4.1 *With the above notation Σ is a locally projective graph of type (m, q) with respect to H. The set $\Sigma(x)$ of vertices adjacent to x in Σ coincides with σ. The action of $H(x)$ on this set is faithful and if $m \leq n - 1$ then $H(x)^{\Sigma(x)}$ contains $PGL_m(q)$.*

Proof: Since Γ is locally projective with respect to G and every point of Π_x fixed by $G(\sigma)$ is in σ, we see that $\Sigma(x) = \sigma$ and that $H(x)^{\Sigma(x)}$ contains $PGL_m(q)$ provided that $m \leq n - 1$. Hence the elementwise stabilizer of $\Sigma(x)$ in G coincides with $G(\sigma)$ which fixes by the definition every vertex of Σ. Thus all we have to show is that H acts vertex-transitively on Σ. Suppose that $y \in \sigma$. Then a point $u \in \Gamma(y) \setminus \{x\}$ is fixed by $G(\sigma)$ if and only if $\varphi(u) = (y, w)$ with $w \in \sigma$. It is easy to observe that the points in Π_y fixed by $G(\sigma)$ form a subspace σ' of projective dimension m and $G(\sigma) = G(\sigma')$. Now since G is vertex-transitive on Γ and $G(x)$ acts flag-transitively on Π_x, it follows that H acts vertex-transitively on Σ. □

In the rest of the paper we shall use Δ to denote the geometrical subgraph $\Delta = \Sigma[\lambda]$ defined with respect to the line $\lambda = \{y_1 = y,\ y_2 = z, \ldots, y_{q+1}\}$.

Lemma 4.2 *Every cycle of length 5 passing through the 2-arc (y, x, z) is contained in Δ.*

Proof: Since G is 2-arc transitive and the girth of Γ is 5, there is a vertex $u \in \Gamma(y) \setminus \{x\}$ which is adjacent to a vertex $v \in \Gamma(z) \setminus \{x\}$. Let $\varphi(u) = (y, t)$ and $\varphi(v) = (z, s)$. To prove the lemma we have to show that both t and s are contained in λ. Suppose that $t \notin \lambda$ and let μ denote the plane in Π_x which contains λ and t. Observe that $G(x, y, z, u) = G(x, y, z, t)$ must fix v and s, since otherwise there would be more than one 2-path joining u and z which is impossible since the girth of Γ is 5. Hence s is contained in $\Phi \setminus \{z\}$ where Φ is the set of vertices in $\Gamma(x)$ fixed by $G(x, y, z, t)$. It follows from basic properties of projective linear groups that one of the following holds:

(i) $q \geq 3$ and $\Phi = \{y, z, t\}$;
(ii) $q = 2$ and $\Phi = \mu$.

If $s = t$, then $G(x, t)$ acts doubly transitively on λ since $t \notin \lambda$. Hence the vertices $w \in \Gamma_2(x)$ with $\varphi(w) = (r, t)$ for $r \in \lambda$ must be pairwise adjacent, which is impossible since the girth of Γ is 5. If $s = y$ then $G(x, y, z, s) = G(x, y, z)$ and every vertex u' with

$\varphi(u') = (y, t')$, for some $t' \notin \lambda$, must adjacent to both y and v. So in this case we would have 4-cycles in Γ. This rules out case (i).

Now we turn to case (ii). Without loss of generality we may assume that $n = 3$, that is, that the geometrical subgraph $\Sigma[\mu]$ is the whole of Γ. Let Θ denote the graph on $\Gamma_2(x)$ in which u and v are adjacent if they are adjacent in Γ and if they are not contained in any of the images of $\Gamma_2(x) \cap \Delta$ under elements of $G(x)$. Then Θ is a graph on the 42 ordered pairs of points of the projective plane of order 2 which is invariant under the natural action of $PSL_3(2)$. In addition the valency of Θ is at most 6 (since the valency of Γ is 7) and the girth of Θ is at least 5. We claim that such a graph has no edges. Clearly Θ must be a union of orbitals of $G(x) \cong PSL_3(2)$ on $\Gamma_2(x)$. Elementary calculations with characters show that the rank of the action is 15 and that exactly 7 of the orbitals are self-paired. Moreover $G(x, y, z)$ stabilizes exactly 6 vertices in $\Gamma_2(x)$, namely those contained in Δ. By the definition of Θ the corresponding 6 orbitals of valency 1 are not involved in Θ. It is easy to see that exactly 4 of the orbitals of valency 1 are self-paired. This implies that the remaining 9 orbitals all have valency 4 and exactly 3 of them are self-paired. If Θ involved a non-self-paired orbital of valency 4 it would also involve its paired orbital which is impossible since the valency of Θ is less than eight. The self-paired orbitals of valency 4 are the following:

$$R_1 = \{(y, t), (z, s) \mid t = s,\ t \notin l_x(y, z)\};$$

$$R_2 = \{(y, t), (z, s) \mid y = z,\ s \notin l_x(y, t)\};$$

$$R_3 = \{(y, t), (z, s) \mid l_x(y, t) \cap l_x(z, s) \cap \{y, t, z, s\} = \emptyset\}.$$

It is easy to see that the orbital graph associated with each of the above three orbitals has girth 3 and the claim follows. Thus Θ has no edges, and the lemma is proved. □

5. Wells subgraphs

As in the previous section let Δ be the geometrical subgraph in Γ defined with respect to the line $\lambda = \{y_1 = y, y_2 = z, y_3, \ldots, y_{q+1}\}$. By Lemma 4.1, Δ is a locally projective graph of type $(2, q)$ with respect to the action of $H = G\{\Delta\}$ and $H(x)^{\Delta(x)}$ contains $PGL_2(q)$. By Lemma 4.2 the girth of Δ is 5.

Lemma 5.1 *Every* 2*-arc of* Δ *is in exactly* a_2 *cycles of length* 5, *where* a_2 *is independent of the* 2*-arc and equals* 1, $q - 1$ *or* q. *The stabilizer in* H *of a* 5*-cycle induces the dihedral group* D_{10} *on the vertices of the cycle. Moreover*, *if* $a_2 = q$, *then* $q = 2$ *and* Δ *is the Petersen graph.*

Proof: Since H acts transitively on the 2-arcs of Δ, the number of cycles of length 5 containing a given 2-arc is a constant a_2, independent of the 2-arc. Let $u \in \Delta_2(x)$ such that $\varphi(u) = (y, z)$. Then $H(x, u) = H(x, y, z)$ acts transitively on $\lambda \setminus \{y, z\}$. On the other hand, for $2 \le i \le q + 1$, the 3-arc (u, y, x, y_i) is contained in at most one 5-cycle, since the girth of Δ is 5. Thus a_2 equals 1, $q - 1$ or q.

Let C be a 5-cycle containing (x, y, u), and note that there exists an element h in $H(y)$ interchanging x and u. If $a_2 = 1$ or $q - 1$ then $H(x, u)$ acts transitively on the 5-cycles containing (x, y, u), and so we may assume that h fixes C setwise. It follows that $H\{C\}^C = D_{10}$. Finally, if $a_2 = q$ then Δ is a Moore graph of valency $q+1$ and hence (see [4, p. 207]) it is the Petersen graph and we have $H\{C\}^C = D_{10}$ in this case also. □

Already the available properties of Δ are strong enough to restrict dramatically the possibilities for the isomorphism type of Δ, but we shall exploit further the fact that Δ appears as a geometrical subgraph in a larger locally projective graph. We will show that Δ must be isomorphic to the Wells graph of valency 5 on 32 vertices, so that $q = 4$.

Let μ be a plane containing λ and $F = G\{\Sigma[\mu]\}$, where $\Sigma[\mu]$ is the geometric subgraph defined with respect to μ. (The definition of $\Sigma[\mu]$ is given before Lemma 4.1.) We will study the natural homomorphism from $F\{\Delta\}$ into the automorphism group of $O_p(F(\Delta))$. Without loss of generality and to simplify the notation we assume that $n = 3$ which means that $\mu = \Pi_x$ and $F = G$. In this case $G(\Delta)$ is a split extension of $Q := O_p(G(\Delta))$, which is elementary abelian of order q^2, by a cyclic subgroup K such that $(q-1)/\gcd(q-1, 3) \le |K| \le (q-1)$. Let ξ denote the natural homomorphism from $G\{\Delta\}$ into the automorphism group of Q.

Lemma 5.2

(i) *If* $q \neq 4$ *then* $\xi(G\{\Delta\}) = \xi(G(x) \cap G\{\Delta\})$;

(ii) *if* $q = 4$ *with* Γ *and* G *corresponding to case* (iv) *or* (v) *of Lemma* 2.1 *then* $\xi(G\{\Delta\}) \cong S_6$.

Proof: Suppose first that $q \neq 4$. If $q = 2$ then $\xi(G(x) \cap G\{\Delta\}) \cong PSL_2(2) \cong \mathrm{Aut}\,(Q)$ and the claim is obvious. So we may assume that the cyclic subgroup K in $G(\Delta)$ is non-trivial. Observe that in this case, if $q = p^m$ for an integer m, then $|K|$ does not divide $p^a - 1$ for $a < m$. It is clear that $\xi(G\{\Delta\})$ normalizes $\xi(G(\Delta)) \cong K$ and by the above observation we have $\xi(G\{\Delta\}) \le \Gamma L_2(q)$. On the other hand (see Lemma 4.1) $\xi(G(x) \cap G\{\Delta\})$ contains either $GL_2(q)$ or a subgroup of index 3 in $GL_2(q)$. This shows that $\xi(G(x) \cap G\{\Delta\})$ is normal in $\Gamma L_2(q)$ and hence in $\xi(G\{\Delta\})$ as well. Let $C = (x, y, u, v, z)$ be a 5-cycle in Δ. By Lemma 5.1 there are elements t and s in $G\{\Delta\}$ which induce on C the permutations $(x, y)(u, z)(v)$ and $(x)(y, z)(u, v)$, respectively. Note that t generates $G\{\Delta\}$ together with $G(x) \cap G\{\Delta\}$. On the other hand, ts induces a 5-cycle on C and so we may choose t to be a conjugate of s by an element of $\langle ts\rangle \subseteq G\{\Delta\}$. Then, since $s \in G(x) \cap G\{\Delta\}$ and $\xi(G(x) \cap G\{\Delta\})$ is normal in $\xi(G\{\Delta\})$, it follows that $\xi(t) \in \xi(G(x) \cap G\{\Delta\})$, and part (i) follows.

Now suppose that $q = 4$ and we are in case (iv) or (v) of Lemma 2.1. In this case Q is elementary abelian of order 2^4 and hence $A := \mathrm{Aut}\,(Q) \cong PSL_4(2) \cong \mathrm{A}_8$. Let $N_1 = G(x) \cap G\{\Delta\}$ and $N_2 = G\{x, y\} \cap G\{\Delta\}$. Then it is clear that $N_1 = N_{G(x)}(Q)$ and $N_2 = N_{G\{x,y\}}(Q)$. Consider $A_1 = N_1/C_{N_1}(Q)$ and $A_2 = N_2/C_{N_2}(Q)$ as subgroups in A. Since $G\{\Delta\}$ is generated by N_1 and N_2, $\xi(G\{\Delta\})$ is the subgroup in A generated by A_1 and A_2. It is easy to see that the subamalgam $\{A_1, A_2\}$ in A is uniquely determined by the isomorphism type of the amalgam $\{G(x), G\{x, y\}\}$ (i.e., it is independent of particular choice of the completion G of the amalgam). By Lemma 2.1 $\mathrm{Aut}\,(\mathrm{M}_{22})$ is a completion of the amalgam $\{G(x), G\{x, y\}\}$. It is easy to observe that if $G = \mathrm{Aut}\,(\mathrm{M}_{22})$ then $G\{\Delta\} \cong 2^4 : \mathrm{S}_6$ is the stabilizer of a block of the Steiner system $S(3, 6, 22)$ and $Q = O_2(G\{\Delta\})$. This specifies $\{A_1, A_2\}$ uniquely and hence $\xi(G\{\Delta\}) \cong \mathrm{S}_6$ for any completion G. □

Lemma 5.3 *If $q \neq 4$ then there is a normal subgroup N of $G\{\Delta\}/G(\Delta)$ which acts regularly on the vertices of Δ.*

Proof: By Lemma 5.2 it follows that $G\{\Delta\} = C_{G\{\Delta\}}(Q)(G(x) \cap G\{\Delta\})$, and hence that $C_{G\{\Delta\}}(Q)$ is transitive on the vertices of Δ. Moreover, $C_{G\{\Delta\}}(Q) \cap G(x) = Q \subseteq G(\Delta)$, and so $C_{G\{\Delta\}}(Q)$ acts regularly on the vertices of Δ. Thus the subgroup $N := C_{G\{\Delta\}}(Q)G(\Delta)/G(\Delta)$ has the required properties. □

We now come closer to our first objective of showing that Δ is the Wells graph, by showing that either this is true, or Δ is a pentagraph. A connected graph Δ is called a *pentagraph* if it has girth 5, and contains a collection Π of 5-cycles such that every 2-arc of Δ is contained in a unique cycle in Π.

Lemma 5.4 *One of the following holds:*
(i) Δ *is a pentagraph of valency at most* 5;
(ii) Δ *is the Wells graph of valency* 5 *on* 32 *vertices.*

Proof: We consider the possible values of a_2 given in Lemma 5.1. If $a_2 = q$ then by Lemma 5.1, $q = 2$ and Δ is the Petersen graph. Since neither of the 2-arc transitive automorphism groups of the Petersen graph has a normal subgroup acting regularly on the set of vertices, this possibility cannot be realized by Lemma 5.3.

Suppose next that $a_2 = 1$. Then every 2-arc of Δ is in a unique 5-cycle and there are no cycles of length less than 5. Thus, by definition of a pentagraph, Δ is a pentagraph which is a locally projective graph of type $(2, q)$. Since $H(x)^{\Delta(x)} \geq PGL_2(q)$ (by Lemma 4.1), it follows from the main result of [18] that $q \leq 4$. Thus part (i) holds.

Finally, suppose that $a_2 = q - 1 > 1$. Then every vertex of $\Delta_2(x)$ is adjacent to exactly one vertex of $\Delta_3(x)$ (by the same reason used inductively we observe that the action of $G\{\Delta\}$ on Δ is distance-transitive). In particular $H(x)$ acts transitively on $\Delta_3(x)$. Let c_3 denote the number of vertices in $\Delta_2(x)$ adjacent to a given vertex of $\Delta_3(x)$. Let $u \in \Delta_2(x)$ with $\varphi(u) = (y, z)$ and $w \in \Delta(x)$. Then the distance from u to w in Δ is 1 if $w = y$, 3 if $w = z$ and 2 otherwise. Let $v \in \Delta_2(x) \cap \Delta(u)$ and $\varphi(v) = (a, b)$. From what we have just observed, $a \neq y, z$, and by symmetry $y \neq a, b$. Also if $z = b$ then, since $q > 2$, we find (by considering the actions of $H(x, y, z)$ and $H(x, a, z)$) that Δ contains a triangle (u, v, w), where $\varphi(w) = (c, z)$ for some $c \in \Delta(x) \setminus \{y, z, a\}$. Since Γ has girth 5, we must therefore have $z \neq b$. Thus the intersection $\{y, z\} \cap \{a, b\}$ must be empty. This shows that each of the $q - 1$ vertices in $\Delta_2(x)$ adjacent to u, and also the vertex y, are in $\Delta_2(z)$. Hence $c_3 \geq q$.

Suppose that $c_3 = q + 1$. Then $\Delta_3(x)$ has size q and $H(x) \geq PGL_2(q)$ acts faithfully and transitively on this set. By [8, Section 263], $q = 5, 7$ or 11, and since $PGL_2(q)$ acts, only $q = 5$ is possible. In this case Δ is a distance-transitive graph with intersection array $\{6, 5, 1; 1, 1, 6\}$, and hence (see [4, p. 223]) Δ is the graph $(6 \cdot K_7)_1$ induced on the set of 42 points at distance 2 from a given vertex in the Hoffman-Singleton graph and $\mathrm{Aut}(\Delta) = \mathrm{S}_7$. Again, since neither of the 2-arc transitive automorphism groups of this graph has a normal subgroup acting regularly on the set of vertices, this possibility contradicts Lemma 5.3.

Thus $c_3 = q$, $|\Delta_3(x)| = q + 1$, and $H(x)/H(\Delta)$ acts naturally and doubly transitively on this set. Now it is easy to conclude that Δ is a distance-transitive graph with intersection array $\{q+1, q, 1, 1; 1, 1, q, q+1\}$. If $q = 4$ then Δ is the Wells graph (cf. [4, p. 223]), so suppose that $q \neq 4$. From the intersection array we see that Δ is antipodal with antipodal classes of size 2. Let $\bar{\Delta}$ be the antipodal quotient of Δ. Then $\bar{\Delta}$ is distance-transitive of diameter 2 with intersection array $\{q + 1, q; 1, 2\}$. By Lemma 5.3, H acting on $\bar{\Delta}$ has a regular normal subgroup $\bar{N}$. For $w \in \Delta$ let $\bar{w}$ denote the image of w in $\bar{\Delta}$. We can identify $\bar{n} \in \bar{N}$ with the vertex $\bar{x}^{\bar{n}}$. The vertices in $\bar{\Delta}_2(\bar{x})$ are the antipodal classes of size 2 contained in $\Delta_2(x)$. By Lemma 2.2 the vertices in $\Delta_2(x)$ are indexed by the ordered pairs of vertices in $\Delta(x)$. It is easy to see that $H(x) \geq PGL_2(q)$ preserves a unique equivalence relation on $\Delta_2(x)$ with classes of size 2. The classes of this relation are indexed by the unordered pairs of vertices in $\Delta(x)$. Hence the vertices in $\bar{\Delta}_2(\bar{x}) = \bar{\Delta} \setminus (\{\bar{x}\} \cup \bar{\Delta}(\bar{x}))$ are indexed by the unordered pairs of vertices in $\Delta(x)$. We claim that the exponent of $\bar{N}$ is 2. In fact by the above description for any $\bar{w} \in \bar{\Delta} \setminus \{\bar{x}\}$ the subgroup $H(\bar{x}) \cap H(\bar{w})$ does not stabilize vertices in $\bar{\Delta}$ other than $\bar{x}$ and $\bar{w}$. On the other hand if $\bar{w}$ (considered as an element of $\bar{N}$) had order greater than 2 then $H(\bar{x}) \cap H(\bar{w})$ would stabilize $\bar{w}^2$, a contradiction. Hence $\bar{N}$ is an elementary abelian 2-group. This means that $\bar{N}$ is a quotient of the $GF(2)$-permutation module for $H(x)^{\Delta(x)}$ and $\bar{\Delta}$ is a quotient of the $(q+1)$-dimensional cube. Since $H(x)^{\Delta(x)}$ is triply transitive, $\bar{\Delta}$ must be the halved cube (cf. [13]). The halved $(q+1)$-dimensional cube has intersection array $\{q + 1, q; 1, 2\}$ if and only if $q = 4$. This contradiction completes the proof. $\square$

Now to obtain the main result of the section it remains to show that Δ does not satisfy case (i) of Lemma 5.4, i.e., that Δ cannot be a pentagraph.

We start by defining a series of pentagraphs coming from a class of Coxeter groups. Let H_k denote the Coxeter group generated by involutions e_i, $i = 1, \ldots, k$, subject to the relations $(e_i e_j)^{m_{ij}} = 1$, where $m_{ij} = 2$ if $|i - j| \geq 2$, $m_{ij} = 3$ if $|i - j| = 1$ and both $i < k$ and $j < k$, and $m_{k-1,k} = m_{k,k-1} = 5$. It is well known that $H_3 \cong \mathrm{A}_5 \times 2$ (the automorphism group of the dodecahedron); $H_4 \cong (SL_2(5) * SL_2(5)).2$ (where $*$ denotes the central product) and H_k is infinite for $k \geq 5$. Let H_k^v, H_k^e and H_k^c be the subgroups in H_k generated by all the generators except for e_k, e_{k-1} and e_{k-2}, respectively. Define Λ_k to be the graph with vertices the right cosets of H_k^v in H_k such that two cosets are adjacent if and only if they intersect non-trivially the same right coset of H_k^e. Then (cf. [10]) H_k acts 2-arc transitively on Λ_k and the stabilizer of a vertex induces the natural action of S_k on the adjacent vertices. Moreover, every 2-arc is in a unique 5-cycle and H_k^c is the setwise stabilizer of one of these 5-cycles. Notice that Λ_3 is just the dodecahedron. Let H_k^+ be the index 2 subgroup in H_k which contains the products of even numbers of generators only. Then H_k^+ acts naturally on Λ_k and the vertex stabilizer induces A_k on the set of adjacent vertices. Clearly H_k contains H_l for $l \leq k$ in the obvious way. We will make use of the following result.

Lemma 5.5 *Every non-trivial homomorphic image of H_5^+ contains an element of order* 15.

Proof: Let K be a normal subgroup of H_5^+. By definition, H_5^+ is the subgroup of H_5 generated by the elements $f_{ij} := e_i e_j$, for $1 \leq i < j \leq 5$. It follows from the presentation given for H_5 that f_{12} has order 3, f_{45} has order 5, and the commutator $[f_{12}, f_{45}] = 1$. If $K \cap \langle f_{12}, f_{45} \rangle = 1$ then H_5^+/K contains an element of order 15, for example $f_{12} f_{45} K$. Suppose then that K intersects $\langle f_{12}, f_{45} \rangle$ non-trivially. Then K contains either f_{12} or f_{45} or both. Set $S_1 := \langle f_{12}, f_{23}, f_{34} \rangle$ and $S_2 := \langle f_{34}, f_{45} \rangle$. Then $S_1 \cong S_2 \cong \mathrm{A}_5$. Since K contains f_{12} or f_{45} at least one of $K \cap S_1$ and $K \cap S_2$ is non-trivial. Hence, since $K \cap S_i$ is normal in S_i and S_i is simple, at least one of S_1, S_2 is contained in K. Then, since $S_1 \cap S_2$ contains f_{34}, both $K \cap S_1$ and $K \cap S_2$ are non-trivial, and so both S_1 and S_2 are contained in K. It is not difficult to see that, for all $1 \leq i < j \leq 5$, f_{ij} belongs to $\langle S_1, S_2 \rangle$ which in turn is contained in K. Hence $K = H_5^+$. □

The following result (cf. [18, 10]) characterizes the locally projective pentagraphs of type $(2, q)$ with $q \leq 4$.

Proposition 5.6 *Let Σ be a pentagraph of valency $q + 1$ with $q \leq 4$, and let F be a 2-arc transitive subgroup of automorphisms of Σ. Suppose that $F(x)^{\Sigma(x)}$ contains $PGL_2(q)$ as a normal subgroup. Then one of the following holds*:

(i) *Σ is a quotient of Λ_{q+1} and F is a factor group of either H_{q+1} or H_{q+1}^+*;

(ii) *$q = 3$ and $PSL_2(11) \leq F \leq (PSL_2(11) \times 3).2$*;

(iii) *$q = 4$ and F is isomorphic to $PSL_2(31)$.*

Lemma 5.7 *The subgraph Δ is isomorphic to the Wells graph.*

Proof: By Lemma 5.2 and its proof there is a homomorphism ξ of $F := G\{\Delta\}/G(\Delta)$ into the automorphism group of an elementary abelian group of order q^2 such that either the image contains $PGL_2(q)$ as a normal subgroup, or $q = 4$ and $\xi(F) = \mathrm{S}_6$. Suppose that Δ is not the Wells graph. Then, by Lemma 5.4, Δ is a pentagraph of valency $q + 1 \leq 5$, and so $F = G\{\Delta\}/G(\Delta)$ satisfies one of (i)–(iii) of Proposition 5.6. It is obvious that there is no such homomorphism ξ in any of these cases except when F is a factor group of H_5 or H_5^+. In these latter cases we have a contradiction by Lemma 5.5. □

Thus we have established the following:

Proposition 5.8 *Let Γ be a locally projective graph of type (n, q), $n \geq 3$ of girth 5 with respect to a 2-arc transitive subgroup G of automorphisms of Γ, such that $G_1(x) = 1$. Then either*

(i) *$\Gamma \cong \Gamma(M_{23})$ and $G \cong M_{23}$; or*

(ii) *$q = 4$ and the geometrical subgraph of valency 5 in Γ is isomorphic to the Wells graph.*

6. The structure of $W(n)$

From now on Γ will be a locally projective graph of type $(n, 4)$ with respect to a 2-arc transitive subgroup G of automorphisms of Γ, such that the geometrical subgraph Δ defined with respect to the line $\lambda = \{y_1 = y, y_2 = z, y_3, y_4, y_5\}$ is isomorphic to the Wells graph.

The Wells graph is the unique distance-transitive graph with intersection array $\{5, 4, 1, 1; 1, 1, 4, 5\}$. An elementary description of this graph may be made as follows. Consider the dodecahedron D as a solid body in three-dimensional space. There are exactly ten 4-subsets T_i, $1 \le i \le 10$, of vertices of D which are the vertex sets of tetrahedrons. Under the action of the rotation group $R \cong A_5$ of D, these tetrahedrons split into two orbits O_1 and O_2, each of length 5. The vertices of the Wells graph may be identified with the vertices of D, the ten tetrahedrons T_i, and the orbits O_1, O_2, with adjacency being the natural adjacency relation in D and set inclusion as appropriate. The full automorphism group is an extension by R of the extraspecial group 2_-^{1+4} of order 32 of minus type. The Wells graph has antipodal classes of size 2 and the folded graph is the folded five-dimensional cube whose complement is known as the Clebsch graph. The following two results are straightforward.

Lemma 6.1 *Let Δ be the Wells graph and let H be a 2-arc transitive subgroup of automorphisms of Δ. Let $x \in \Delta$ and $\{x, y\} \in E(\Delta)$. Then*
- (i) *H is the full automorphism group of Δ, it is a semidirect product of a normal subgroup $N \cong 2_-^{1+4}$ and $H(x) \cong A_5$;*
- (ii) *$H(x, y) \cong A_4$ and $H\{x, y\} \cong H(x, y) \times \langle t_{xy} \rangle$ where t_{xy} is an involution from N.*

Lemma 6.2 *Let X be a coset of A_5 in S_5, so that X consists of either all the even permutations or all the odd permutations of a 5-element set. Let M be a group generated by five involutions s_i, $1 \le i \le 5$, subject to the relation $s_{\pi(1)} s_{\pi(2)} \dots s_{\pi(5)} = 1$ for every permutation $\pi \in X$. Then $M \cong 2_-^{1+4}$ and if $\sigma \in S_5 \setminus X$ then $s_{\sigma(1)} s_{\sigma(2)} \dots s_{\sigma(5)}$ is the unique non-trivial element in the center of M.*

Lemma 6.3 *Let Γ and G be as above. Then*
- (i) *$G(x) \cong PSL_n(4)$ or $PGL_n(4)$;*
- (ii) *either $G(x, y) \cong 2^{2(n-1)} : GL_{n-1}(4)$, or $G(x) \cong PSL_n(4)$, n is divisible by 3, and $G(x, y) \cong 2^{2(n-1)} : SL_{n-1}(4)$;*
- (iii) *$G\{x, y\} \cong G(x, y) \times \langle t_{xy} \rangle$, where t_{xy} is an involution which is the unique non-trivial element of the center of $G\{x, y\}$;*
- (iv) *if $v \in \Gamma(y) \setminus \{x\}$ with $\varphi(v) = (y, w)$ then t_{xy} maps w onto v.*

Proof: Since Δ is the Wells graph, by Lemma 6.1, $G(x) \cap G\{\lambda\}$ induces A_5 on the points of λ. Thus part (i) follows and immediately implies part (ii). To prove part (iii) assume first that $n = 3$. By Lemma 6.1, $G\{\Delta\}/G(\Delta) \cong 2_-^{1+4} : A_5$. Let $Q := O_2(G(\Delta)) \cong 2^4$ and let ξ be the natural homomorphism from $G\{\Delta\}$ into Aut $(Q) \cong PSL_4(2)$. Then it is clear that $\xi(G\{\Delta\}/G(\Delta)) \cong A_5$ and by Lemma 5.2 (ii) we are not in case (iv) or (v) of Lemma 2.1. This means that we can choose $t \in G\{x, y\} \setminus G(x, y)$ such that t normalizes a complement K to $O_2(G(x, y))$ in $G(x, y)$. If t can be chosen to centralize K, then it is easy to see that (iii) follows. Otherwise $G\{x, y\} \cong 2^4 : S_5$ or $2^4 : 3 : S_5$. Since $G\{x, y\} \cap G\{\Delta\}$ is of index 5 in $G\{x, y\}$, it must be isomorphic, respectively, to $2^4 : S_4$ or $2^4 : 3 : S_4$, and $|O_2(G\{x, y\})| = 2^6$. On the other hand by Lemma 6.1 (ii), $G\{x, y\} \cap G\{\Delta\} \cong 2^4 : (A_4 \times 2)$ or $2^4 : 3 : (A_4 \times 2)$ and $|O_2(G\{x, y\})| = 2^7$ (notice that $\xi(G\{\Delta\}) \cong A_5$). This is a contradiction, so we have proved part (iii) for $n = 3$. If $n \geq 4$ then we may still choose $t \in G\{x, y\} \setminus G(x, y)$ to normalize a complement $K \cong SL_{n-1}(4)$ to $O_2(G(x, y))$ in the commutator subgroup of $G(x, y)$. Consider the geometrical subgraph Σ in Γ defined with respect to a plane containing y and stabilized by t. Since part (iii) is proved for the case $n = 3$, one can easily see that t cannot induce a non-trivial field automorphism on K and hence t can be chosen to centralize K and part (iii) follows. Finally $G(x, y)$ acts transitively on the set of pairs

$$\mathcal{P} = \left\{ \{y_i, u_i\} \,\middle|\, \varphi(u_i) = (y, y_i),\ 2 \leq i \leq \begin{bmatrix} n \\ 1 \end{bmatrix}_4 \right\}$$

and t_{xy} centralizes this action. Since $q = 4$, $G(x, y, z)$ does not stabilise in $\Gamma(x)$ vertices other than y and z. This implies that different pairs in $\mathcal{P}$ have different stabilizers in $G(x, y)$. Hence the action of t_{xy} on $\mathcal{P}$ is trivial and part (iv) follows. □

In what follows, for $1 \leq i \leq [{n \atop 1}]_4$, we set $t_i := t_{xy_i}$, the involution defined as in Lemma 6.3 (iii) for $G\{x, y_i\}$, and let T be the subgroup of G generated by the elements t_i, $i = 1, \ldots, [{n \atop 1}]_4$.

Lemma 6.4 *Let $N = \langle t_i \mid 1 \leq i \leq 5\rangle$. Then one of the following holds*:
(i) $N \cong 2_-^{1+4}$ *and N acts regularly on the vertex set of* Δ;
(ii) $n = 3$, $G(x) \cong PSL_3(4)$ *and N contains* $G(\Delta)$.

Proof: Since Δ is the Wells graph, by Lemma 6.1 the action induced by N on Δ is isomorphic to 2_-^{1+4}. Let L be the kernel of the action. It is clear that L is contained in $G(\Delta)$ and L is normal in $G\{\Delta\} \cap G(x)$. By Lemma 6.3 (iii) t_i commutes with $G(\Delta)$ for $1 \leq i \leq 5$ and hence L is in the center of $G(\Delta)$. If $n \geq 4$ or $n = 3$ and $G(x) = PGL_3(4)$ then the center of $G(\Delta)$ is trivial and we are in case (i). In the case $n = 3$, $G(x) = PSL_3(4)$ the action of $G\{\Delta\} \cap G(x)$ on $G(\Delta)$ is irreducible and hence if we are not in case (i), L should be equal to the whole $G(\Delta)$. □

Lemma 6.5 *The case* (ii) *in Lemma* 6.4 *is impossible.*

Proof: The result follows from the triviality of groups defined in terms of generators and relations which are implied by the conditions in Lemma 6.4 (ii). Namely $G(x) \cong PSL_3(4)$ and by Lemma 6.3 (iii) $G\{x, y\} \cong G(x, y) \times \langle t_1\rangle$. This specified the amalgam $\mathcal{B} = \{G(x), G\{x, y\}\}$ up to isomorphism. Moreover, for $1 \leq i \leq 5$ the element t_i is equal to $t_1^{a_i}$ where a_i is an element in $G(x)$ which maps $y = y_1$ onto y_i. Notice that the embedding of $G(x)$ into $P\Gamma L_3(4)$ and the fact that $G\{x, y\}$ is a direct product shows that the amalgam $\mathcal{B}$ possesses an automorphism which normalizes the set $\{t_1, t_2, t_3, t_4, t_5\}$ and induces an odd permutation on this set. In view of this symmetry and since Δ is the Wells graph, by Lemma 6.2 we can assume without loss of generality that the product $q := t_1t_2t_3t_4t_5$ is contained in $G(\Delta)$. If the product is the identity element then we are in case (i) of Lemma 6.4.

Otherwise q is one of 15 non-trivial elements in $G(\Delta)$ and we arrive with 15 possible presentations. A coset enumeration with explicit presentations (cf. Section 8) performed by L.H. Soicher has shown that each of the 15 presentations defines the trivial group. $\square$

Let $\hat{\Gamma}$ be the covering tree of Γ, and let $\hat{G}$ be the free amalgamated product of $G(x)$ and $G\{x, y\}$ over $G(x, y)$ acting naturally on $\hat{\Gamma}$. For an edge $\{\hat{u}, \hat{v}\}$ let $\hat{t}_{\hat{u}\hat{v}}$ be the unique non-trivial element (involution) in the center of the stabilizer in $\hat{G}$ of this edge. Let $\hat{x}$ be a preimage of x in $\hat{\Gamma}$, and let $\{\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{[{n \atop 1}]_4}\}$ be the set of vertices adjacent to $\hat{x}$ in $\hat{\Gamma}$. Set $\hat{t}_i := \hat{t}_{\hat{x}\hat{y}_i}$, for $1 \leq i \leq [{n \atop 1}]_4$, and let $\hat{T}$ be the subgroup of $\hat{G}$ generated by these elements $\hat{t}_i$.

Lemma 6.6
(i) *$\hat{T}$ acts transitively on the vertex set of $\hat{\Gamma}$;*
(ii) *$\hat{T}$ is normal in $\hat{G}$ and $\hat{T} \cap \hat{G}(\hat{x}) = 1$;*
(iii) *$\hat{T}$ is freely generated by the involutions $\hat{t}_i$ for $1 \leq i \leq [{n \atop 1}]_4$.*

The proof of the lemma uses the notion of permutation isomorphism of permutation representations. We say that permutation representations $\alpha : G \to \mathrm{Sym}(\Omega)$ and $\alpha' : G' \to \mathrm{Sym}(\Omega')$ (of groups G and G' on sets Ω, Ω' respectively) are *permutationally isomorphic* if there exist a bijection $\varphi : \Omega \to \Omega'$ and an isomorphism $\gamma : (G)\alpha \to (G')\alpha'$ such that, for all $g \in G$ and all $i \in \Omega$, $(i^{(g)\alpha})\varphi = (i\varphi)^{(g)\alpha\gamma}$.

Proof: We prove part (i) by induction. Let Θ be the orbit of $\hat{T}$ containing $\hat{x}$ on the vertex set of $\hat{\Gamma}$. Clearly every vertex adjacent to $\hat{x}$ is in Θ. Suppose that Θ contains all vertices whose distance from $\hat{x}$ is less than or equal to k. Let $\hat{v}$ be at distance $k + 1$ from $\hat{x}$ and let $(\hat{v}, \ldots, \hat{u}, \hat{x})$ be a path of shortest length joining $\hat{v}$ and $\hat{x}$. Then $\hat{u} = \hat{y}_i$ for some i and the image of $\hat{v}$ under $\hat{t}_i$ is at distance k from $\hat{x}$. Thus $\hat{v}^{\hat{t}_i}$, and hence also $\hat{v}$, lie in Θ. Part (i) now follows by induction.

Let $\Omega = \{1, 2, \ldots, [{n \atop 1}]_4\}$. We will construct a homomorphism of $\hat{G}$ into $\mathrm{Sym}(\Omega)$ whose kernel is $\hat{T}$. The bijection $y_i \mapsto i$ is such that the mapping $\alpha : G(x) \to \mathrm{Sym}(\Omega)$ defined by $i^{(g)\alpha} = j$ if and only if $y_i^g = y_j$, for $g \in G(x)$, is a permutation representation of $G(x)$ on Ω which is permutationally isomorphic to the permutation representation of $G(x)$ on $\Gamma(x)$. Also the bijection $\{y_i, y_i^{t_1}\} \mapsto i$ is such that the mapping $\beta : G\{x, y\} \to \mathrm{Sym}(\Omega)$ defined by $i^{(g)\beta} = j$ if and only if $\{y_i^g, y_i^{t_1 g}\} = \{y_j, y_j^{t_1 g}\}$, for $g \in G\{x, y\}$, is a permutation representation of $G\{x, y\}$ on Ω which is permutationally isomorphic to its permutation representation on

$$\left\{ \{y_i, y_i^{t_1}\} \;\middle|\; i = 2, \ldots, \begin{bmatrix} n \\ 1 \end{bmatrix}_4 \right\}.$$

Note that the image $(G\{x, y\})\beta$ is contained in the stabilizer of the point 1 in $\mathrm{Sym}(\Omega)$. Note also that the restrictions of α and β to $G(x, y)$ are identical. Hence α and β define a homomorphism γ of the free amalgamated product $\hat{G}$ of $G(x)$ and $G\{x, y\}$ into $\mathrm{Sym}(\Omega)$. Let K be the kernel of γ. It is clear that K contains $\hat{T}$ and so, by part (i), K is transitive on the vertex set of $\hat{\Gamma}$. On the other hand the restriction of γ to $G(x)$ is an isomorphism. Hence K acts regularly on the vertex set of $\hat{\Gamma}$, so $K = \hat{T}$ and part (ii) follows.

Since $\hat{T}$ acts regularly on a tree, it is easy to see that part (iii) holds. $\square$

It follows from the above lemma that the tree $\hat{\Gamma}$ may be considered as a Cayley graph for $\hat{T}$ with respect to the generating set $\{\hat{t}_i \mid 1 \leq i \leq [{n \atop 1}]_4\}$.

By the definition of $\hat{G}$ there is a group homomorphism $\chi : \hat{G} \to G$, corresponding to the graph covering $\hat{\Gamma} \to \Gamma$ and $\chi(\hat{t}_i) = t_i$ for $1 \leq i \leq [{n \atop 1}]_4$. Let $\hat{N} = \langle \hat{t}_i \mid 1 \leq i \leq 5 \rangle$. By Lemma 6.6 (iii), $\hat{N}$ is freely generated by these five involutions and there is a subgroup of the automorphism group of $\hat{N}$ which is isomorphic to S_5 and which permutes naturally the generators $\hat{t}_1, \dots \hat{t}_5$. Let X_1 and X_2 be the cosets of A_5 in S_5. For $j = 1, 2$, let K_j be the smallest normal subgroup of $\hat{N}$ such that K_j contains the products $\hat{t}_{\pi(1)}\hat{t}_{\pi(2)} \dots \hat{t}_{\pi(5)}$ for all $\pi \in X_j$. By Lemma 6.2 we have the following.

Lemma 6.7 *Let N be as in Lemma* 6.4 *and ν be a homomorphism of $\hat{N}$ onto N which maps $\{\hat{t}_i \mid 1 \leq i \leq 5\}$ onto $\{t_i \mid 1 \leq i \leq 5\}$. Then the kernel of ν is either K_1 or K_2.*

For $j = 1, 2$, let R_j be the smallest normal subgroup of $\hat{G}$ containing K_j. By Lemma 6.7 we have the following.

Lemma 6.8 *The kernel of the homomorphism $\chi : \hat{G} \to G$ contains R_j for $j = 1$ or* 2.

Thus, for $j = 1$ or 2, K_j is contained in $\hat{T}$ and the latter is normal in $\hat{G}$, and hence R_j is also contained in $\hat{T}$. We may consider $\hat{G}$ as a semidirect product of $\hat{T}$ by $G(x) \cong \hat{G}(\hat{x})$. The latter acts as a permutation group on the set $\hat{D} = \{\hat{t}_i \mid 1 \leq i \leq [{n \atop 1}]_4\}$ of generators of $\hat{T}$ preserving on $\hat{D}$ a projective space structure isomorphic to Π_x. Let B be the full automorphism group of Π_x, that is $B \cong P\Gamma L_n(4)$. Then the semidirect product $\hat{B}$ of $\hat{T}$ by B contains $\hat{G}$ as a normal subgroup. Since the setwise stabilizer of λ in B induces S_5 on the points of λ, the normal subgroups R_1 and R_2 are conjugate in $\hat{B}$. In particular the groups $\hat{T}/R_1$ and $\hat{T}/R_2$ are isomorphic.

Next we shall determine the structure of $\hat{T}/R$, where $R := R_1R_2$. First of all if $K := K_1K_2$ then, by Lemma 6.2, $L := \hat{N}/K$ is an elementary abelian group of order 2^4. Since L is abelian, it is a quotient of $\bar{N} := \hat{N}/\hat{N}' \cong 2^5$. In turn $\bar{N}$ may be considered as the $GF(2)$-vector space of all subsets of λ (considered as a set of size five) with addition defined by the symmetric difference operation. Then L is the quotient of $\bar{N}$ over the one-dimensional subspace consisting of the improper subsets of λ (that is, the empty set and the set λ itself). Since any two points of Π_x are collinear, every pair of the generators in $\hat{D}$ are contained in a conjugate of $\hat{N}$ and hence their images in $\hat{T}/R$ commute. This means that $\hat{T}/R$ is abelian and so it is a quotient of $\bar{T} := \hat{T}/\hat{T}^{(1)}$ where $\hat{T}^{(1)} = [\hat{T}, \hat{T}]$. We may identify $\bar{T}$ with the $GF(2)$-vector space of all subsets of the point set of Π_x. The image of K in $\bar{T}$ is one-dimensional and contains λ (considered as a subset of points of Π_x). Thus $\hat{T}/R$ is the quotient $\bar{T}/M$, where M is the subspace generated by all lines of Π_x. It is well known and easy to see (cf. [14]) that this quotient is always non trivial. Let $\bar{\Gamma}$ be the Cayley graph of $\hat{T}/R$ with respect to the image $\hat{D}R/R$ of $\hat{D}$. Then $\bar{\Gamma}$ is a quotient of the $[{n \atop 1}]_4$-dimensional cube. Let $W(n)$ be the Cayley graph of $\hat{T}/R_1$ with respect to the image $\hat{D}R_1/R_1$ of the generating set $\hat{D}$. Then obviously there is a graph covering $\varrho : W(n) \to \bar{\Gamma}$. In particular $W(n)$ is locally projective of type $(n, 4)$ with respect to the semidirect product of $\hat{T}/R_1$ by $G(x)$. Moreover, ϱ is a proper covering unless $R_1 = R_2 = R$. If ϱ is proper then

$\hat{N} \cap R_1 = K_1$, K_1 is of index 2 in K and the images in $\hat{T}/R_1$ of the $\hat{t}_i$, for $1 \leq i \leq 5$, generate the extraspecial group 2_-^{1+4}. This means that the geometrical subgraph in $W(n)$ defined with respect to λ is isomorphic to the Wells graph. It is easy to see that in this case $W(n)$ does not have cycles of length less than 5, that is, the girth of the graph is 5. We summarise the above discussion in the following proposition which also implies Theorem 1.4 (ii) (a).

Proposition 6.9 *For $n \geq 3$, let $\hat{T}$ be the group freely generated by the involutions from the set $\hat{D} = \{\hat{t}_i \mid 1 \leq i \leq [{n \atop 1}]_4\}$. Suppose that a structure Π of an $(n-1)$-dimensional projective $GF(4)$-space is defined on $\hat{D}$, so that $\lambda = \{\hat{t}_1, \ldots, \hat{t}_5\}$ is a line. Let A be a subgroup of the automorphism group of Π isomorphic to $PGL_n(4)$ and let $\hat{G}$ be the semidirect product of $\hat{T}$ by A with respect to the natural action of A. Let R_1 be the normal closure in $\hat{G}$ of the element $\hat{t}_1\hat{t}_2\hat{t}_3\hat{t}_4\hat{t}_5$. Let $W(n)$ be the Cayley graph of $\hat{T}/R_1$ with respect to the image $\hat{D}R_1/R_1$ of the generating set $\hat{D}$. Then $W(n)$ is a locally projective graph of type $(n, 4)$, and its girth is 5 if and only if $\hat{T}/R_1$ is not abelian. If the girth of $W(n)$ is 5 then any graph satisfying Proposition 5.8 (ii) for this value of n is a quotient of $W(n)$.*

Notice that if in the above proposition we consider $PSL_n(4)$ instead of $PGL_n(4)$ then R_1 and hence the resulting graph $W(n)$ will be the same.

7. The girth of $W(3)$ is 5

In this section we show that the graph $W(3)$ has girth 5 and that it has at least 2^{20} vertices.

Let $\hat{T}$ be a group freely generated by 21 involutions from the set $\hat{D} = \{\hat{t}_i \mid 1 \leq i \leq 21\}$. Suppose that a structure Π of projective plane of order 4 is defined on $\hat{D}$ so that $\lambda = \{\hat{t}_1, \ldots, \hat{t}_5\}$ is a line, and set $\hat{N} := \langle \hat{t}_i \mid 1 \leq i \leq 5\rangle$. We will use the same letter Π to denote the point set of Π. Let $B \cong P\Gamma L_3(4)$ be the full automorphism group of Π and let A be a normal subgroup of B isomorphic to $PGL_3(4)$. Then A and B are permutation groups on $\hat{D}$ and hence they may be identified with the corresponding subgroups in the automorphism group of $\hat{T}$. Let $\hat{G}$ and $\hat{B}$ be the semidirect products of $\hat{T}$ by A and B respectively, so that $\hat{B}$ contains $\hat{G}$ as a normal subgroup. Let R_1 and R_2 be the normal closures in $\hat{G}$ of the elements $\hat{t}_e := \hat{t}_1\hat{t}_2\hat{t}_3\hat{t}_4\hat{t}_5$ and $\hat{t}_o := \hat{t}_1\hat{t}_2\hat{t}_3\hat{t}_5\hat{t}_4$ respectively, and notice that both R_1 and R_2 are contained in $\hat{T}$. Since the setwise stabilizer of λ in B induces S_5 on the points of λ, it follows that R_1 and R_2 are conjugate in $\hat{B}$. Let $W(3)$ be the Cayley graph of $\hat{T}/R_1$ defined with respect to the image $\hat{D}R_1/R_1$ of the generating set $\hat{D}$. Then $W(3)$ is a locally projective graph with respect to the action of $\hat{G}/R_1$ and by Proposition 6.9 the girth of $W(3)$ is 5 if and only if $\hat{T}/R_1$ is non abelian, or equivalently, if and only if $R_1 \neq R_2$.

As above let $\hat{T}^{(1)} = [\hat{T}, \hat{T}]$, $\bar{T} = \hat{T}/\hat{T}^{(1)}$ and put $\hat{T}^{(2)} = [\hat{T}, \hat{T}, \hat{T}]$. For an element $\hat{h} \in \hat{T}$ let $\bar{h}$ denote its image in $\bar{T}$. Then $\bar{T}$ is an elementary abelian 2-group of rank 21 with basis $\{\bar{t}_i \mid 1 \leq i \leq 21\}$ and the elements of this basis are indexed by the points of Π. This enables us to identify $\bar{T}$ with the $GF(2)$-vector space of all subsets of Π with addition defined by the symmetric difference operation. We will use the following well-known result (cf. [14]) on the structure of $\bar{T}$ as a module for B.

Lemma 7.1 *With the above notation the following assertions hold:*
(i) $\bar{T} = \bar{T}^e \oplus \bar{T}_1$, *where* $\bar{T}^e$ *consists of the even subsets of* Π *and* $\bar{T}_1$ *consists of the empty set and the whole set* Π;
(ii) $\bar{T}^e$ *is uniserial*: $0 < \bar{T}_9^e < \bar{T}_{11}^e < \bar{T}^e$, *where* $\bar{T}_i^e$ *has dimension* i *for* $i = 9$ *and* 11;
(iii) $\bar{T}_9^e$ *is generated by the complements of lines in* Π *and it is dual to* $\bar{T}^e/\bar{T}_{11}^e$;
(iv) B *induces* S_3 *on* $\bar{T}_{11}^e/\bar{T}_9^e$, *so that* $C_B(\bar{T}_{11}^e/\bar{T}_9^e) \cong PSL_3(4)$;
(v) $\bar{T}_{11}^e$ *and* $\bar{T}^e/\bar{T}_9^e$ *are indecomposable, even as modules for* $PSL_3(4)$;
(vi) B *has three orbits on the non-zero elements of* $\bar{T}_9^e$, *with lengths* 21, 210 *and* 280, *which consist, respectively, of the complements of lines, the symmetric differences of pairs of lines, and the complements of the symmetric differences of triples of lines in general position.*

Let $\bar{S}$ be the $GF(2)$-permutation module of A acting on the line set of Π. Then $\bar{S}$ is the image of $\bar{T}$ under the contragredient automorphism of A so the structure of $\bar{S}$ can be deduced from Lemma 7.1. Namely $\bar{S} = \bar{S}^e \oplus \bar{S}_1$ where $\bar{S}^e$ is uniserial $0 < \bar{S}_9^e < \bar{S}_{11}^e < \bar{S}^e$. Moreover $\bar{T}^e/\bar{T}_{11}^e \cong \bar{S}_9^e$ and $\bar{T}_9^e \cong \bar{S}^e/\bar{S}_{11}^e$ as A-modules.

The element $\bar{t}_e$ (which is equal to $\bar{t}_o$) can be identified with λ (considered as a 5-element subset of Π). By Lemma 7.1 this means the following:

Lemma 7.2 *The image* $\bar{R}_1$ *of* R_1 *in* $\bar{T}$ *coincides with* $\bar{T}_9^e \oplus \bar{T}_1$, *in particular it is* 10-*dimensional and hence* $\hat{T}/R_1\hat{T}^{(1)}$ *is* 11-*dimensional.*

Next we study the quotient $\tilde{T} := \hat{T}^{(1)}/(R_1 \cap \hat{T}^{(1)})\hat{T}^{(2)}$. Let $\tilde{s}_{ij}$ be the image in $\tilde{T}$ of the element $[\hat{t}_i, \hat{t}_j]$ from $\hat{T}^{(1)}$ for $1 \leq i, j \leq 21, i \neq j$. Since the generators $\hat{t}_i$ of $\hat{T}$ are involutions and the image in $\tilde{T}$ of $[\hat{t}_i, \hat{t}_j, \hat{t}_k]$ is trivial, it is easy to see that the $\tilde{s}_{ij}$ are pairwise commuting involutions which generate $\tilde{T}$. In particular $\tilde{T}$ is an elementary abelian 2-group. Furthermore, if $1 \leq i, j, k, l \leq 5$ with $i \neq j, k \neq l$ then both $[\hat{t}_i, \hat{t}_j]$ and $[\hat{t}_k, \hat{t}_l]$ are equal to the unique non-trivial element in $[\hat{N}, \hat{N}]$. Hence the image of $[\hat{t}_i, \hat{t}_j]$ in $\tilde{T}$ depends only on the line of Π containing i and j. Thus we have the following:

Lemma 7.3 *The group* $\tilde{T}$ *is generated by* 21 *pairwise commuting elements* m_l *indexed by the lines of* Π *such that* $m_l^2 = 1$. *If* l *is the line containing the points* i *and* j *then* $m_l = \tilde{s}_{ij}$ *is the image in* $\tilde{T}$ *of the element* $[\hat{t}_i, \hat{t}_j]$. *Thus* $\tilde{T}$ *is isomorphic to a quotient of the* $GF(2)$-*permutation module* $\bar{S}$ *of* A *acting on the line set of* Π.

For $\hat{t} \in \hat{T}$ let $S(\hat{t})$ be the subset of $\{1, 2, \ldots, n\}$ such that if $\hat{u}$ is the product (in some order) of the $\hat{t}_i$, for $i \in S(\hat{t})$, then $\bar{u} = \bar{t}$. Note that $S(\hat{t})$ is well-defined since $\bar{T}$ is abelian.

Lemma 7.4 *If* $\hat{t} \in \hat{T}$ *then* $\hat{t}^2 \in \hat{T}^{(1)}$. *The image* $\tilde{t}^2$ *of* $\hat{t}^2$ *in* $\tilde{T}$ *is equal to the product of the elements* $\tilde{s}_{ij}$ *taken for all ordered pairs* (i, j) *with* $i, j \in S(\hat{t})$ *and* $i < j$.

Proof: Since $\bar{T} = \hat{T}/\hat{T}^{(1)}$ is of exponent 2, it is clear that $\hat{t}^2 \in \hat{T}^{(1)}$. Since $\hat{T}^{(1)}/\hat{T}^{(2)}$ is in the center of $\hat{T}/\hat{T}^{(2)}$ and $\tilde{T}$ is of exponent 2, the image $\tilde{t}^2$ of $\hat{t}^2$ in $\tilde{T}$ depends only on the image of $\hat{t}$ in $\bar{T}$, that is on $S(\hat{t})$. To see that the assertion made in the second sentence is true, we argue as follows. Without loss of generality we may assume that $S(\hat{t}) = \{1, 2, \dots, r\}$ for some $r \leq n$. By the definition of $S(\hat{t})$, $\hat{t} = \hat{t}_1 \dots \hat{t}_r x$ for some $x \in \hat{T}^{(1)}$. Then since $\hat{T}^{(1)}/\hat{T}^{(2)}$ is central in $\hat{T}/\hat{T}^{(2)}$, we have $\hat{t}^2 = (\hat{t}_1 \dots \hat{t}_r)^2$ modulo $\hat{T}^{(2)}$. We "collect" $\hat{t}^2$ as follows. Working modulo $\hat{T}^{(2)}$, for each $i = 1, \dots, r$, and each $j = r, r-1, \dots, i+1$, in turn replace $\hat{t}_j \hat{t}_i$ by $\hat{t}_i \hat{t}_j [\hat{t}_j, \hat{t}_i]$ and move $[\hat{t}_j, \hat{t}_i]$ to the right hand end of the expression for $\hat{t}^2$. (Note that we may do this since $[\hat{t}_j, \hat{t}_i]$ is central modulo $\hat{T}^{(2)}$). Then, since each $\hat{t}_i^2 = 1$, we have modulo $\hat{T}^{(2)}$, $\hat{t}^2 = \prod_{1 \leq i < j \leq r} [\hat{t}_j, \hat{t}_i]$, and the result follows. □

For $\hat{t} \in \hat{T}$ Lemma 7.4 gives us a method for expressing the image $\tilde{t}^2$ of $\hat{t}^2$ in $\tilde{T}$ in terms of the generators m_l as in Lemma 7.3. Namely the generator m_l is involved in the decomposition of $\tilde{t}^2$ if and only if $S(\hat{t}) \cap l$ contains an odd number of pairs of distinct points, that is, if $|S(\hat{t}) \cap l| = 2$ or 3. Now if $\hat{t} \in R_1$, then clearly $\hat{t}^2 \in R_1$ and hence $\tilde{t}^2$ is the identity. In this way we will obtain further relations on the generators m_l. By Lemma 7.2, $\hat{t} \in R_1$ if and only if $\bar{t} \in \bar{T}_9^e \oplus \bar{T}_1$. By Lemma 7.1 (vi) there are 6 types of non empty subsets of $\bar{T}_9^e \oplus \bar{T}_1$ and the collection of these subsets is closed under taking complements. Now if a line l intersects a subset S in 2 or 3 points then it intersects the complement of S in 3 or 2 points, respectively. Hence it is sufficient to consider just one subset from each complementary pair. We do this below.

(1) Let S be a line. Then there are no lines intersecting S in 2 or 3 points.
(2) Let S be the symmetric difference of two lines l_1 and l_2 and set $p := l_1 \cap l_2$. Then a line l intersects S in 2 or 3 points if and only if l does not contain p (clearly in this case l intersects S in 2 points).
(3) Let S be the symmetric difference of three lines l_1, l_2, l_3 in general position, that is, the $p_{ij} := l_i \cap l_j$ are pairwise different for $1 \leq i < j \leq 3$. Then a line l intersects S in 3 points if and only if it intersects $\{p_{12}, p_{13}, p_{23}\}$ in an even number of elements (that is, in zero or 2 elements). This means that l intersects S in 3 points if and only if l is in the symmetric difference of the sets $M(p)$ for $p = p_{12}$, p_{13} and p_{23}, where $M(p)$ denotes the set of lines missing p.

One can see that the relations implied by (3) are consequences of the relations implied by (2). This can be summarised as follows.

Lemma 7.5 *Let p be a point of Π and $M(p)$ be the set of lines missing p. Then in terms of Lemma* 7.3 *the product of the m_l for all $l \in M(p)$ is the identity in $\tilde{T}$. In other words $\tilde{T}$ is a quotient of the module $(\bar{S}^e/\bar{S}_9^e)$ and the only faithful B-section involved in $\tilde{T}$ is isomorphic to $\bar{S}^e/\bar{S}_{11}^e \cong \bar{T}_9^e$ (in terms introduced in and after Lemma* 7.1).

The group $\hat{T}^{(1)}/\hat{T}^{(2)}$ is generated by 210 linearly independent pairwise commuting involutions $\hat{s}_{ij}$ which are images of the commutators $[\hat{t}_i, \hat{t}_j]$ for $1 \leq i, j \leq 21$ and $i \neq j$. Let ν be the homomorphism of $\hat{T}^{(1)}/\hat{T}^{(2)}$ onto $\bar{S}^e/\bar{S}_9^e$ which commutes with the natural action of A. This means that ν maps the generator $\hat{s}_{ij}$ onto the involution m_l where l is the line containing i and j and the involutions m_l satisfy the relations described in Lemma 7.5. Let

U be the full preimage in $\hat{T}^{(1)}$ of the kernel of ν, so that $\hat{T}^{(1)}/U \cong \bar{S}^e/\bar{S}^e_9$. Let W be the full preimage of $\bar{R}_1 = \bar{T}^e_9 \oplus \bar{T}_1$ with respect to the homomorphism $\hat{T} \to \bar{T} = \hat{T}/\hat{T}^{(1)}$ and set $V := W/U$.

Lemma 7.6 *With the above notation V is an elementary abelian 2-group of rank 22 and as a module for A it involves exactly two non-trivial irreducible sections, both isomorphic to $\bar{T}^e_9$.*

Proof: By the arguments given before Lemma 7.5, if $w \in W$, then $w^2 \in U$. Hence the image of w in V is of order at most 2. Thus all non trivial elements in V are involutions and V is elementary abelian. The proof that the order of V is 2^{22} is straightforward. Finally the non trivial irreducible sections of $V/(\hat{T}^{(1)}/U) \cong W/\hat{T}^{(1)} \cong \bar{T}^e_9 \oplus \bar{T}_1$, and $\hat{T}^{(1)}/U$ are both isomorphic to $\bar{T}^e_9$. □

By the above lemma we may consider V as a $GF(2)$-module for $A \cong PGL_3(4)$. Let v_e and v_o be the images in V of the elements $\hat{t}_e$ and $\hat{t}_o$, respectively. Notice that if $\hat{t}$ is the product of the five elements $\hat{t}_i$ for $1 \leq i \leq 5$, in any order, then the image of $\hat{t}$ in V coincides either with v_e or with v_o and that $v_e v_o = m_\lambda$. Since the stabilizer of λ in A induces A_5 on the points of λ, it stabilizes v_e. So there are 21 images of v_e under A indexed by the lines of Π. Since V is abelian, these 21 images generate R_1U/U. This means that R_1U/U is a quotient of the $GF(2)$-permutation module $\bar{S}$ of A acting on the line set of Π. By the remark after Lemma 7.1, $\bar{S}$ involves only one section isomorphic to $\bar{T}^e_9$. By Lemma 7.6, V involves two such copies of $\bar{T}^e_9$. We know that $R_1\hat{T}^{(1)}/\hat{T}^{(1)}$ involves a copy of $\bar{T}^e_9$, and hence the second copy is involved in $(W \cap \hat{T}^{(1)})/(R_1 \cap \hat{T}^{(1)})$. In particular the latter has order at least 2^9. Thus we have proved the following:

Proposition 7.7 *The graph $W(3)$ has girth 5 and the order of $\hat{T}/R_1\hat{T}^{(2)}$ is at least 2^{20}. Moreover,*

(i) $\hat{T}/R_1\hat{T}^{(1)} \cong \bar{T}/(\bar{T}^e_9 \oplus \bar{T}_1)$ *is of order* 2^{11};

(ii) $\hat{T}^{(1)}/(R_1 \cap \hat{T}^{(1)})\hat{T}^{(2)}$ *involves a section isomorphic to $\bar{T}^e_9 \cong \bar{S}^e/\bar{S}_{11}$, in particular it has order at least 2^9.*

L. H. Soicher has shown, by running a coset enumeration on a presentation exploiting the description from Proposition 6.9, that the index of AR_1/R_1 in $\hat{G}/R_1$ is 2^{20} (cf. Section 8 for an explicit presentation). This means that $|\hat{T}/R_1| = 2^{20}$, and hence the bound in Proposition 7.7 is exact, and $W(3)$ has exactly 2^{20} vertices. This, together with Proposition 7.7 imply Theorem 1.4 (ii) (b) and also the following result:

Lemma 7.8 *Let $n = 3$ and $\hat{T}$ be as in Proposition 6.9. Then R_1 contains $\hat{T}^{(2)}$.*

Now suppose that $n \geq 4$. Then for every 21-point subplane Ω in Π the image in $\hat{T}/R_1$ of the subgroup generated by the $\hat{t}_i$, for $i \in \Omega$, is a quotient of the group $\hat{T}/R_1$ corresponding to the case $n = 3$. Since any triple i, j, k of points in Π are in a subplane, $[\hat{t}_i, \hat{t}_j, \hat{t}_k] \in R_1$ by Lemma 7.8 and hence $\hat{T}^{(2)}$ is contained in R_1 for all $n \geq 3$. This and the obvious analogue of Lemma 7.3 imply Theorem 1.4 (ii) (c). Thus the proof of Theorem 1.4 is complete.

8. Generators and relations

In this final section we present generators and relations for a 2-arc transitive subgroup G of automorphisms of the graph $W(3)$. It is included for the convenience of readers who may wish to construct the graph $W(3)$ by computer.

We start with a presentation for $G(x) \cong PSL_3(4)$ which is similar to the Steinberg presentation. Suppose that the field $GF(4) = \{0, 1, \alpha, \beta\}$. We treat the elements of $G(x)$ as 3×3 matrices over $GF(4)$. Let e_{ij} denote the elementary matrix all of whose entries are equal to zero except for the ij-entry which is equal to 1, and let I denote the identity matrix. Then $G(x) = \langle a, b, c, d, e, f, g, h, i\rangle$, where

$$a = I + e_{12},\ b = I + \alpha e_{12},\ c = I + e_{31},\ d = I + \alpha e_{31},\ e = I + e_{32},$$
$$f = I + \alpha e_{32}, g = e_{11} + \alpha e_{22} + \beta e_{33},\ h = e_{11} + e_{23} + e_{32},\ i = e_{12} + e_{21} + e_{33}.$$

The relations defining $G(x)$ are the following:

$$a^2,\ b^2,\ c^2,\ d^2,\ e^2,\ f^2,\ [a, b],\ [a, c],\ [a, d],\ [a, e]c,\ [a, f]d,\ [b, c],\ [b, d],$$
$$[b, e]d,\ [b, f]cd,\ [c, d],\ [c, e],\ [c, f],\ [d, e],\ [d, f], [e, f],\ g^3,\ a^t b,\ b^t ba,$$
$$c^t cd,\ d^t c,\ e^t f,\ f^t ef,\ h^2,\ i^2,\ t^h t,\ t^i t,\ (eh)^3,\ (ai)^3,\ a^h c,\ b^h d,\ e^i c,\ f^i d,\ (ih)^3.$$

We assume that $G(x, y) = \langle a, b, c, d, e, f, g, h\rangle$ and $G(x) \cap G\{\lambda\} = \langle a, b, c, d, e, f, g, i\rangle$. Then $G\{x, y\} = G(x, y) \times \langle t\rangle$, where $t^2 = 1$ and $[t, x] = 1$ for $x \in \{a, b, c, d, e, f, g, h\}$. Let Δ be the geometrical subgraph corresponding to λ. Then $G(\Delta) = \langle c, d, e, f\rangle$.

The additional relation which guarantees that Δ is the Wells graph is of the form

$$t\, t^i\, t^{ia}\, t^{ib}\, t^{iab} = p,$$

where p is an element from $G(\Delta) = \langle c, d, e, f\rangle$. Let G be the group with generators $a, b, c, d, e, f, g, h, i, t$ subject to the above relations. If p is one of the 15 non trivial elements of $G(\Delta)$ then it turns out that G is the trivial group. However if p is the identity element then the group G has order 2^{20}. A verification of these assertions was done by a coset enumeration performed by L. H. Soicher.

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