Journal of Algebraic Combinatorics **8** (1998), 127–138

Distance-Regular Graphs with $c_i = b_{d-i}$ and Antipodal Double Covers

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Received March 24, 1995; Revised May 1, 1997

Abstract. Let Γ be a distance-regular graph of diameter d and valency $k > 2$. Suppose there exists an integer s with $d \leq 2s$ such that $c_i = b_{d-i}$ for all $1 \leq i \leq s$. Then Γ is an antipodal double cover.

Keywords: distance-regular graph, antipodal double cover, box, brox.

1. Introduction

Throughout this paper, we assume Γ is a connected finite undirected graph without loops or multiple edges. We identify Γ with the set of vertices. For vertices u and x in Γ, let $\partial(u, x)$ denote the *distance* between u and x in Γ, i.e., the length of a shortest path connecting u and x. Let $d = d(\Gamma)$ denote the *diameter* of Γ, i.e., the maximal distance between any two vertices in Γ. Let

$$\Gamma_i(u) = \{\, y \in \Gamma \mid \partial(u, y) = i \,\}.$$

For vertices u and x in Γ at distance i, let

$$\begin{aligned} C_i(u,x) &= \Gamma_{i-1}(u) \cap \Gamma_1(x), \\ A_i(u,x) &= \Gamma_i(u) \cap \Gamma_1(x) \qquad \text{and} \\ B_i(u,x) &= \Gamma_{i+1}(u) \cap \Gamma_1(x). \end{aligned}$$

A graph Γ is called a *distance-regular graph* if for any two vertices u and x in Γ at distance i, the numbers

$$c_i = |\, C_i(u,x) \,|, \quad a_i = |\, A_i(u,x) \,| \quad \text{and} \quad b_i = |\, B_i(u,x) \,|$$

depend only on the distance $\partial(u, x) = i$ rather than on individual vertices. When this is the case we call numbers c_i, a_i and b_i the *intersection numbers* of Γ, in particular $k = b_0$ is called *valency* of Γ.

Let h be an integer with $1 \leq h \leq d$, v and x vertices in Γ at distance h. Take any $u \in C_h(x, v)$. The following are well known basic properties which we use implicitly in this paper.

(1) $\Gamma_1(x) = C_h(v,x) \cup A_h(v,x) \cup B_h(v,x)$,
(2) $B_{h-1}(u,x) \supseteq B_h(v,x)$,
(3) $C_{h-1}(u,x) \subseteq C_h(v,x)$,
(4) $C_h(\alpha,\gamma) \subseteq B_{d-h}(\beta,\gamma)$ for any $\gamma \in \Gamma_h(\alpha) \cap \Gamma_{d-h}(\beta)$ with $\partial(\alpha,\beta) = d$,
(5) The numbers $k_i := |\ \Gamma_i(x)\ |$ depend only on i,
(6) The numbers $p^h_{i,j} := |\ \Gamma_i(v) \cap \Gamma_j(x)\ |$ depend only on i, j and $h = \partial(v,x)$.

In particular, we have

(1′) $k = c_i + a_i + b_i$ for $i = 0, \ldots, d$,
(2′) $k = b_0 > b_1 \geq \cdots \geq b_{d-1} \geq 1$,
(3′) $1 = c_1 \leq c_2 \leq \cdots \leq c_d \leq k$,
(4′) $c_h \leq b_{d-h}$ for $1 \leq h \leq d$.

The reader is referred to [3] or [4] for the general theory of distance-regular graphs.

A distance-regular graph Γ of diameter d is called an *antipodal double cover* (of its folded graph), if and only if $c_i = b_{d-i}$, for $i = 1, \ldots, d$.
For more details on antipodal graphs see [5], and § 4.2 of [4].

The main result of this paper is the following:

Theorem 1 *Let Γ be a distance-regular graph of diameter d and valency $k > 2$. Suppose there exists an integer s with $d \leq 2s$ such that $c_i = b_{d-i}$ for all $1 \leq i \leq s$. Then Γ is an antipodal double cover.*

In [1], we have already obtained the special case of the main theorem of this paper, i.e., a distance-regular graph of $b_t = 1$, $d \geq 2t$ and valency $k > 2$ is an antipodal double cover, which is one of important facts to prove our theorem.

In general, it is well known that

$$p^i_{d,d-i} \;=\; \frac{b_i\ \cdots\ b_{d-1}}{c_{d-i}\ \cdots\ c_1} \;=\; \frac{b_i}{c_{d-i}} \cdot p^{i+1}_{d,d-i-1} \;\geq\; p^{i+1}_{d,d-i-1}$$

and thus

$$k_d \;=\; p^0_{d,d} \;\geq\; p^1_{d,d-1} \;\geq\; \cdots \;\geq\; p^d_{d,0} = 1.$$

Hence we obtain the following corollary immediately from our theorem.

Corollary 1 *If there exists an integer t with $2t \leq d$ such that $p^t_{d,d-t} = 1$, then Γ is an antipodal double cover.*

By the definition, Γ is an antipodal double cover if and only if $p^i_{d,d-i} = 1$ for all $0 \leq i \leq d$.

We use the following terminology in this paper.

Definition Let u, v, x and y be vertices in Γ.
(1) We write the " *triangle inequalities on* (u, v, x, y)" for the triangle inequalities of (u, v, y) and of (u, x, y).
(2) The quadruple (u, v, x, y) is called an (h, j)-*box* if

$$\partial(u, v) = 1,\ \partial(u, x) = h - 1,\ \partial(x, y) = j,$$
$$\partial(v, x) = h,\ \partial(v, y) = h - j,\ \partial(u, y) = h - j + 1.$$

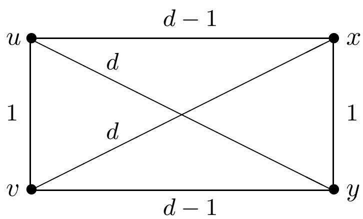
(3) The quadruple (u, v, x, y) is called a j-*brox* if

$$\partial(u, v) = 2,\ \partial(u, x) = d - 1,\ \partial(x, y) = j,$$
$$\partial(v, x) = d,\ \partial(v, y) = d - j,\ \partial(u, y) = d - j + 2.$$

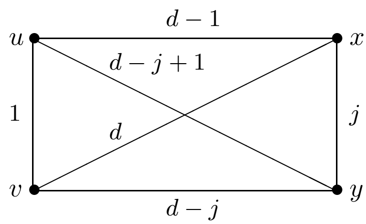
A $(d, 1)$-box is called a *box* that was a key to prove the theorem in [1]. Notice that there are many boxes in an antipodal distance-regular graph Γ of diameter $d \geq 3$; namely, given u, y with $\partial(u, y) = d$, there is a one to one correspondence between $v \in \Gamma_1(u)$ and $x \in \Gamma_1(y)$ such that (u, v, x, y) is a box. Moreover, if Γ has a box, then also Γ has a (d, j)-box, i.e., for $y' \in \Gamma_{d-j}(v) \cap \Gamma_{j-1}(y)$ the quadruple (u, v, x, y') is a (d, j)-box. Whence an antipodal distance-regular graph has a (d, j)-box for any j.

On the other hand, a distance-regular graph which is an antipodal double cover never contains a j-brox (u, v, x, y) by observing the (u, v, x).

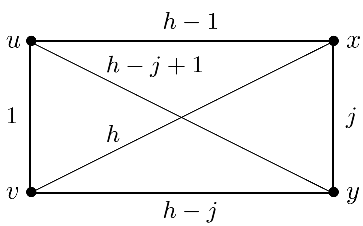
a box:



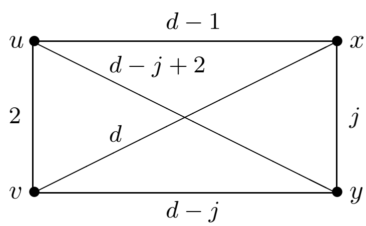
a (d, j)-box:



an (h, j)-box:



a j-brox:



When we characterize graphs, it is important to consider their substructures. One of the characterization of antipodal distance-regular graphs is that they have a box. These configurations are useful tools when we investigate if a graph is antipodal or not as we see § 2, [1] or [2]. Readers who are familiar with distance distribution diagrams may read some of proofs easily, however, we can do without diagrams.

2. Proof of the Theorem

Throughout this section, we assume Γ is not an antipodal double cover to derive a contradiction. Then Γ cannot have any boxes, and must have some broxes in Lemma 3 and Lemma 5. The existence and nonexistence of these configurations lead to the inequality in Lemma 6 that causes a contradiction.

For the case $s = 1$, the theorem is trivial and well-known. We may assume $s \geq 2$. Suppose Γ is not an antipodal double cover. Then we have

$$c_j = b_{d-j} \quad \text{for all} \quad 1 \leq j \leq s, \quad c_{s+1} \neq b_{d-(s+1)}$$

for some s with $\frac{d}{2} \leq s < d$ and let $t := d - s$.

If $b_t = 1$, then Γ is an antipodal double cover from [1]. So we may assume $b_t \geq 2$.

Lemma 1 (1) $p^{d-j}_{d,j} = 1$ *for all* $0 \leq j \leq s$ *and* $p^{t-1}_{d,s+1} \geq 2$,
(2) $a_{t-1} < a_t$.

Proof: Using the well known formula of $p^l_{i,j}$

$$p^{d-j}_{d,j} = \frac{b_{d-j} \cdots b_{d-1}}{c_j \ \cdots \ c_1} = \begin{cases} = 1 & \text{if} \quad 0 \leq j \leq s \\ \geq 2 & \text{if} \quad j = s+1 \end{cases}$$

from our assumption. This implies $b_{t-1} = c_{s+1}\, p^{t-1}_{d,s-1} \geq 2c_{s+1}$. Thus we obtain

$$\begin{aligned} a_{t-1} \ \leq \ k - b_{t-1} \ &\leq \ k - 2c_{s+1} \\ &\leq \ k - c_t - c_s \ = \ a_t + b_t - c_s \ = \ a_t. \end{aligned}$$

If the equality holds, then $t = 1$ and $b_t = c_s = c_{s+1} = c_t = 1$. This contradicts $b_t \geq 2$. □

Lemma 2 *Let* u, v, α *and* β *be vertices in* Γ *with* $\partial(u,v) = 1$ *and* $\partial(\alpha,\beta) = d$.
(1) *If* $c_j = c_{j+1}$, *then we have*

$$A_j(v,x) \ \subseteq \ A_{j+1}(u,x) \quad \textit{for any} \quad x \in \Gamma_{j+1}(u) \cap \Gamma_j(v).$$

(2) *For all integer* j *with* $1 \leq j \leq s$. *We have*

$$C_j(\alpha,x) \ = \ B_{d-j}(\beta,x) \quad \textit{for any} \quad x \in \Gamma_j(\alpha) \cap \Gamma_{d-j}(\beta).$$

In particular, if $t \leq j \leq s$, *then* $A_j(\alpha,x) = A_{d-j}(\beta,x)$.
(3) *We have* $\Gamma_{s+1}(\beta) \cap \Gamma_t(\alpha) \neq \phi$ *and*

$$C_{s+1}(\beta,y) \ = \ B_t(\alpha,y) \quad \textit{for any} \quad y \in \Gamma_{s+1}(\beta) \cap \Gamma_t(\alpha).$$

In particular, $c_{s+1} = b_t = c_s$.

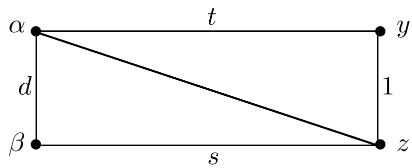
Proof: (1)(2) The assertions follow from basic properties and our assumptions.
(3) Take any $x \in \Gamma_{s+1}(\beta) \cap \Gamma_{t-1}(\alpha)$. Since $c_{s+1} < b_{t-1}$, we have

$$y \ \in \ B_{t-1}(\alpha, x) - C_{s+1}(\beta, x).$$

Since $y \notin C_{s+1}(\beta, x)$ and $y \notin C_{t-1}(\alpha, x) = B_{s+1}(\beta, x)$, we obtain $y \in A_{s+1}(\beta, x)$. This means $y \in \Gamma_{s+1}(\beta) \cap \Gamma_t(\alpha)$, i.e., $\Gamma_{s+1}(\beta) \cap \Gamma_t(\alpha) \neq \phi$.

Next we show that $C_{s+1}(\beta, y) = B_t(\alpha, y)$. Take any $z \in C_{s+1}(\beta, y)$. From the triangle inequalities on (α, β, y, z), we have

$$t \ = \ d - s \ = \ \partial(\beta, \alpha) - \partial(\beta, z) \ \leq \ \partial(\alpha, z) \ \leq \ \partial(\alpha, y) + \partial(y, z) \ = \ t + 1.$$



This implies $\partial(\alpha, z) \in \{ t, \ t+1 \}$. Suppose $\partial(\alpha, z) = t$. Then we have

$$y \ \in \ A_t(\alpha, z) \ = \ A_s(\beta, z)$$

from (2). This contradicts $y \in \Gamma_{s+1}(\beta)$. Hence we obtain $\partial(\alpha, z) = t + 1$ and

$$C_{s+1}(\beta, y) \ \subseteq \ B_t(\alpha, y).$$

The assertion follows from

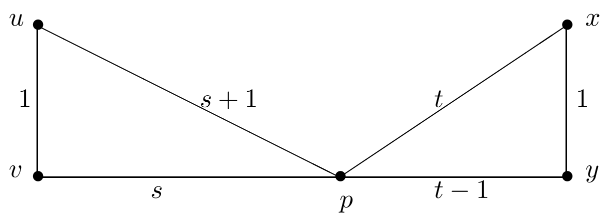
$$c_s \ \leq \ c_{s+1} \ = \ | \ C_{s+1}(\beta, y) \ | \ \leq \ | \ B_t(\alpha, y) \ | \ = \ b_t \ = \ c_s.$$

□

Lemma 3 (1) *There exists no* (d, j)*-box for any* $1 \leq j \leq s$.
(2) *There exists no* $(d - i + 1, 2)$*-box for any* $1 \leq i \leq s - 1$.

Proof: (1) We prove by induction on j.

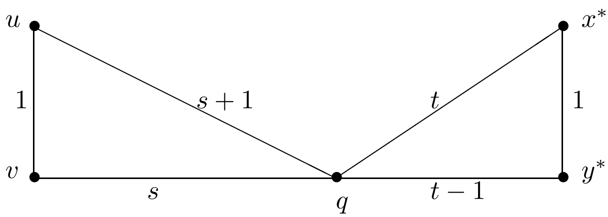
Suppose Γ has a box (u, v, x, y). Take any $p \in \Gamma_s(v) \cap \Gamma_{t-1}(y)$. Then we have $p \in \Gamma_{s+1}(u) \cap \Gamma_t(x)$ by the triangle inequalities on (u, v, y, p) and on (x, v, y, p).



From Lemma 1 (2), there exists $q \in A_t(x, p) - A_{t-1}(y, p)$. Then by Lemma 2

$$q \ \in \ A_t(x, p) \ = \ A_s(v, p) \ \subseteq \ A_{s+1}(u, p).$$

Let $\{ y^* \} = \Gamma_d(u) \cap \Gamma_{t-1}(q)$ as $p^{s+1}_{d,t-1} = 1$. Then we obtain $\partial(v, y^*) = d - 1$ by the triangle inequalities on (v, q, u, y^*). And let $\{ x^* \} = B_{d-1}(v, y^*)$. Also we obtain $\partial(q, x^*) = t$ by the triangle inequalities on (q, v, y^*, x^*).



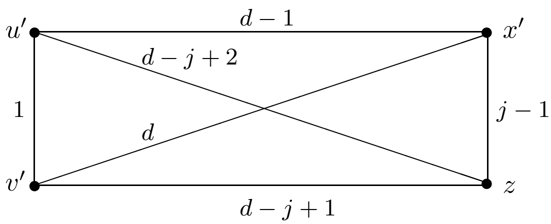
This implies $x = x^*$ as $\{ x,\, x^* \} \subseteq \Gamma_d(v) \cap \Gamma_t(q)$ and $p^s_{d,t} = 1$.

Then $\{ y,\, y^* \} \subseteq B_{d-1}(u, x)$ and $y \neq y^*$ as $\partial(y^*, q) = t-1 \neq \partial(y, q)$. This contradicts $b_{d-1} = 1$. Hence Γ does not have a box.

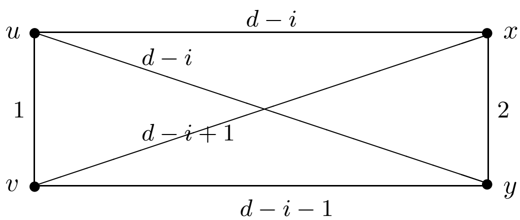
Now we assume $2 \leq j \leq s$ and there exists a (d, j)-box (u', v', x', y') in Γ. Take $z \in B_{d-j+1}(u', y')$ as $2 \leq j$. Then we have

$$z \in B_{d-j+1}(u', y') \subseteq B_{d-j}(v', y') = C_j(x', y')$$

by Lemma 2 (2).



This implies (u', v', x', z) is a $(d, j-1)$-box, contradicting our inductive assumption.
(2) Suppose that there exists a $(d-i+1, 2)$-box (u, v, x, y) for some $1 \leq i \leq s-1$.



Let $\{ v^* \} = \Gamma_d(v) \cap \Gamma_{i-1}(x)$ as $p^{d-i+1}_{d,\,i-1} = 1$. Then we have $\partial(u, v^*) = d - 1$ and $\partial(y, v^*) = i + 1$ from the triangle inequalities on (u, v, x, v^*) and on (y, v, x, v^*). This implies (u, v, v^*, y) is a $(d, i+1)$-box, contradicting (1). $\square$

Lemma 4 *Let α and β be vertices in Γ with $\partial(\alpha,\beta) = d$ and $x \in \Gamma_t(\alpha) \cap \Gamma_s(\beta)$.*
(1)

$$A_d(\alpha,\beta) \;=\; B_s(x,\beta) \quad \textit{and} \quad A_d(\beta,\alpha) \;=\; B_t(x,\alpha).$$

In particular, we have $b_s = a_d = b_t \geq 2$.
(2) $a_1 = 0$,
(3) $b_{s+1} = b_s = c_t$.

Proof: (1) Suppose there exists $z \in A_d(\alpha,\beta) - B_s(x,\beta)$. Then we have $\partial(x,z) = s$, by the triangle inequalities on (x,α,β,z) and $z \notin B_s(x,\beta)$. This means that $\{\,\beta,\, z\,\} \in \Gamma_d(\alpha)\cap\Gamma_s(x)$. However, this contradicts $p^t_{d,s} = 1$. Thus we obtain $A_d(\alpha,\beta) \subseteq B_s(x,\beta)$.
On the other hand, if there exists $y \in B_s(x,\beta) - A_d(\alpha,\beta)$, then (y,β,α,x) is a (d,t)-box, contradicting Lemma 3 (1). Hence we have $A_d(\alpha,\beta) \;=\; B_s(x,\beta)$.
In the same way, we obtain $A_d(\beta,\alpha) = B_t(x,\alpha)$.
(2) Suppose $a_1 > 0$. Take any $\gamma \in A_d(\alpha,\beta)$ and $\delta \in A_1(\beta,\gamma)$. Then we have $\delta \in A_d(\alpha,\beta)$ as $b_{d-1} = 1$. This means $B_s(x,\beta)$ contains an edge $\{\,\gamma,\,\delta\,\}$ from (1). Let

$$m = \max\{\; j = \partial(u,v) \mid B_j(u,v) \text{ contains an edge } \}.$$

By our observation,

$$s \;\leq\; m \;<\; d-1.$$

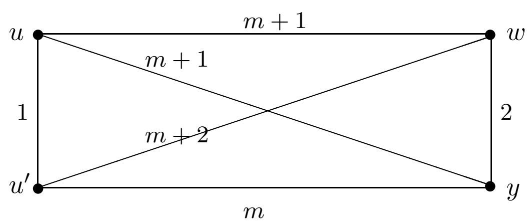
Let u and v be vertices in Γ with $\partial(u,v) = m$ and $B_m(u,v)$ contains an edge $\{\,w,\,z\,\}$. We can take

$$u' \;\in\; B_{m+1}(w,u) \;\subseteq\; B_m(v,u).$$

From the triangle inequalities on (u',v,w,z) and the maximality of m, we have $\partial(u',z) = m+1$. Since

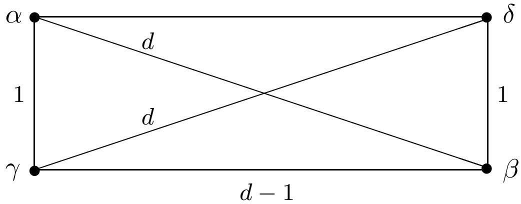
$$1 \;=\; |\,\{\,v\,\}\,| \;\leq\; |\,C_{m+1}(u,z) - C_{m+1}(u',z)\,| \;=\; |\,C_{m+1}(u',z) - C_{m+1}(u,z)\,|,$$

there exists $y \in C_{m+1}(u',z) - C_{m+1}(u,z)$. Then we have $\partial(y,u) = m+1$ and $\partial(w,y) = 2$ by the triangle inequalities on (y,u',z,u) and observing (u',y,w). Thus (u,u',w,y) is an $(m+2,2)$-box.



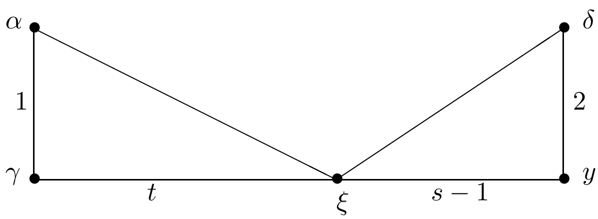
Since $t \;\leq\; s \;\leq\; m$, this contradicts Lemma 3 (2). Therefore a_1 must be zero.

(3) Fix $\gamma \in C_d(\beta, \alpha)$ and $\delta \in B_{d-1}(\gamma, \beta)$. Then $\partial(\alpha, \delta) = d$ as otherwise $(\alpha, \gamma, \delta, \beta)$ is a box.



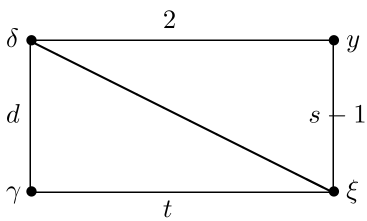
Since $a_d > 1 = b_{d-1}$, we have $y \in A_d(\alpha, \beta) - B_{d-1}(\gamma, \beta)$. By the triangle inequality of (γ, α, y), we obtain $y \in A_{d-1}(\gamma, \beta)$ as $y \notin B_{d-1}(\gamma, \beta)$. Since $a_1 = 0$ and considering (γ, δ, y), we have $\partial(\delta, y) = 2$.

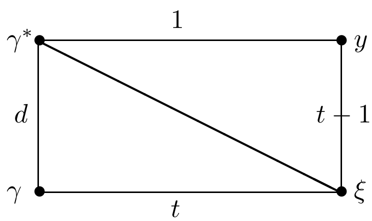
Let $\xi \in \Gamma_t(\gamma) \cap \Gamma_{s-1}(y)$. We claim that $\partial(\alpha, \xi) = t + 1$ and $\partial(\delta, \xi) = s + 1$.



We have $\partial(\alpha, \xi) = t + 1$ by the triangle inequalities on (α, γ, y, ξ).

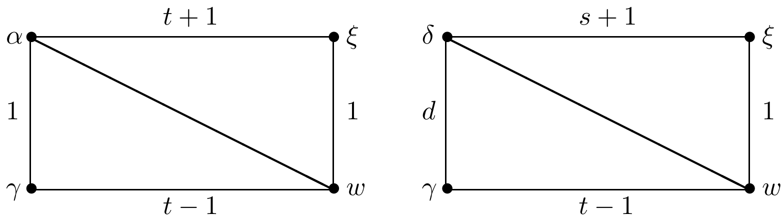
From the triangle inequalities on (δ, γ, y, ξ), we have $\partial(\delta, \xi) \in \{ s,\ s+1 \}$. Let $\{ \gamma^* \} = B_{d-1}(\gamma, y)$. Then $\partial(\xi, \gamma^*) = s$ and $\delta \neq \gamma^*$ by the triangle inequalities on $(\gamma^*, \gamma, y, \xi)$ and considering (y, δ, γ^*).





If $\partial(\delta, \xi) = s$, then $\{ \delta,\ \gamma^* \} \subseteq \Gamma_d(\gamma) \cap \Gamma_s(\xi)$ with $\partial(\gamma, \xi) = t$. This contradicts $p^t_{d,s} = 1$. Hence we obtain $\partial(\delta, \xi) = s + 1$ as claimed.

Next we show that $C_t(\gamma, \xi) = B_{s+1}(\delta, \xi)$. Take any $w \in C_t(\gamma, \xi)$. Then we obtain $\partial(\alpha, w) = t$ and $\partial(\delta, w) \in \{ s+1, s+2 \}$ from the triangle inequalities on (α, γ, ξ, w) and on (δ, γ, ξ, w).



If $\partial(\delta, w) = s+1$, then $w \in \Gamma_t(\alpha) \cap \Gamma_{s+1}(\delta)$ with $\partial(\alpha, \delta) = d$. Hence we have

$$\xi \in B_t(\alpha, w) = C_{s+1}(\delta, w)$$

from Lemma 2 (3). This contradicts $\partial(\delta, \xi) = s+1$. Thus we have $\partial(\delta, w) = s+2$, i.e., $C_t(\gamma, \xi) \subseteq B_{s+1}(\delta, \xi)$. The assertion follows from

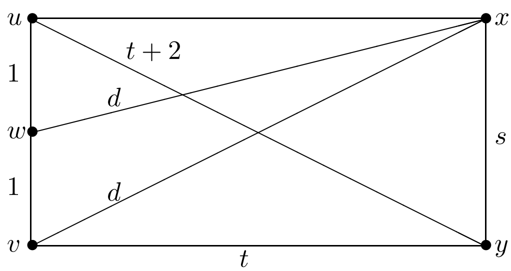
$$c_t = |\, C_t(\gamma, \xi) \,| \leq |\, B_{s+1}(\delta, \xi) \,| = b_{s+1} \leq b_s = c_t.$$

□

Lemma 5 *There exists an i-brox for all $2 \leq i \leq s$.*

Proof: We prove by induction on $s - i$.

Suppose there exists no s-brox in Γ. Let x and v be vertices in Γ with $\partial(x, v) = d$. Take $y \in \Gamma_s(x) \cap \Gamma_t(v)$ and $w \in A_d(x, v)$. Then from Lemma 4 (1), we have $\partial(y, w) = t+1$. Take any $u \in B_{t+1}(y, w)$.



Then $\partial(x, u) = d$ as otherwise (u, v, x, y) is an s-brox. Thus we obtain

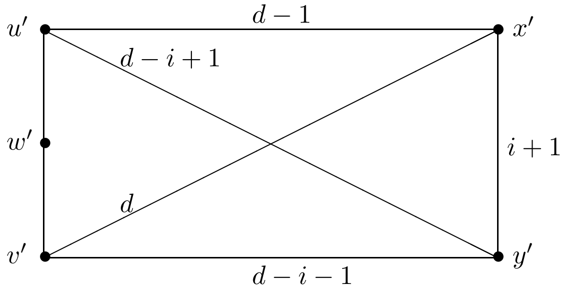
$$B_{t+1}(y, w) \subseteq A_d(x, w) - \{ v \}, \quad i.e., \quad b_{t+1} \leq a_d - 1.$$

On the other hand, we have

$$b_{t+1} \geq b_{s+1} = b_s = a_d$$

from Lemma 4 (1)(3). This is a contradiction. Hence there exist s-broxes.

For the case $s = 2$, the lemma is already proved. We may assume $s \geq 3$. Suppose $2 \leq i \leq s-1$ and there exists no i-brox to derive a contradiction. From the inductive assumption, we have an $(i+1)$-brox (u', v', x', y'). Fix any $w' \in C_2(u', v')$.

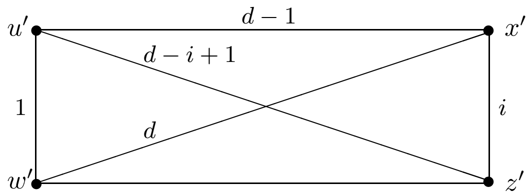


It is clear that $\partial(w', y') = d - i$ by the triangle inequalities on (w', u', v', y') and that $\partial(x', w') = d$ as otherwise (w', v', x', y') is a $(d, i+1)$-box.
Claim: $C_{i+1}(x', y') \subseteq B_{d-i}(w', y') - B_{d-i+1}(u', y')$.

Take any $z' \in C_{i+1}(x', y')$. From Lemma 2 (2), we obtain

$$z' \in C_{i+1}(x', y') = B_{d-(i+1)}(v', y'), \quad i.e., \quad \partial(v', z') = d - i.$$

It is clear that $\partial(u', z') \in \{ d-i, d-i+1, d-i+2 \}$ by the triangle inequality of (u', y', z'). If $\partial(u', z') = d - i$, then (z', y', u', v') is a $(d-i+1, 2)$-box. This contradicts Lemma 3 (2). If $\partial(u', z') = d + 2 - i$, then (u', v', x', z') is an i-brox. This contradicts our assumption. Thus $\partial(u', z') = d - i + 1$, i.e., $z' \notin B_{d-i+1}(u', y')$.



We have $\partial(w', z') \in \{ d-i, d-i+1 \}$ by the triangle inequalities on (w', u', v', z'). If $\partial(w', z') = d - i$, then (u', w', x', z') is a (d, i)-box, contradicting Lemma 3 (1). Hence we obtain $\partial(w', z') = d - i + 1$, i.e., $z' \in B_{d-i}(w', y')$. Whence the claim is proved.

This implies

$$c_{i+1} \leq b_{d-i} - b_{d-i+1}.$$

However we have

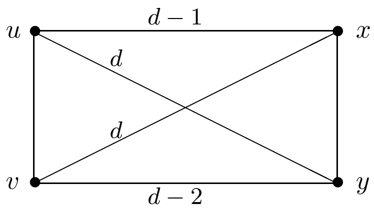
$$c_{i+1} \geq c_i = b_{d-i}.$$

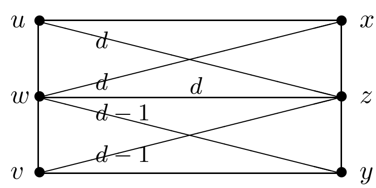
This is a contradiction as $i \geq 2$. $\square$

Lemma 6 *We have*

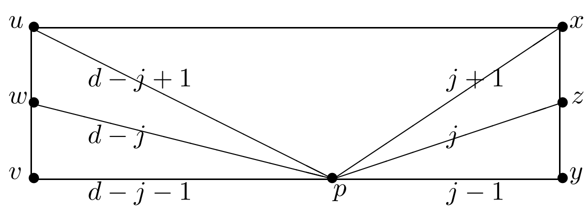
$$b_j - c_{d-j+1} \le b_{j+1} - c_{d-j} \quad \textit{for all} \quad 1 \le j \le s-1.$$

Proof: There exists a 2-brox (u, v, x, y) from Lemma 5. Fix any $w \in C_2(u, v)$ and $z \in C_2(x, y)$. Then we have $\partial(w, y) = d-1$ from the triangle inequalities on (w, u, v, y), and $\partial(x, w) = d$ as otherwise (w, v, x, y) is a $(d, 2)$-box. Similarly, $\partial(z, v) = d-1$ and $\partial(u, z) = d$. We obtain $\partial(w, z) = d$ as otherwise (u, w, x, z) is a box.





Fix any $p \in \Gamma_{j-1}(y) \cap \Gamma_{d-j-1}(v)$ for $1 \le j \le s-1$. Then from the triangle inequalities on (p, y, v, u), (p, y, v, w), (p, y, v, z) and (p, v, y, x), we obtain that $p \in \Gamma_{d-j+1}(u) \cap \Gamma_{d-j}(w) \cap \Gamma_j(z) \cap \Gamma_{j+1}(x)$.



In order to prove the statement, we will show that

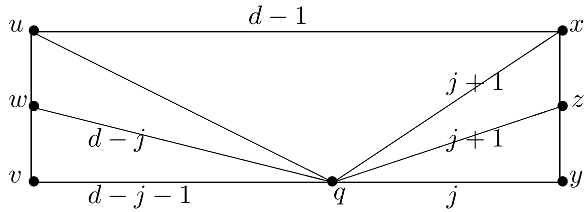
$$B_j(z, p) - B_{j+1}(x, p) \subseteq C_{d-j+1}(u, p) - C_{d-j}(w, p).$$

Take any $q \in B_j(z, p) - B_{j+1}(x, p)$. It is clear that $\partial(y, q) = j$ by the triangle inequalities on (y, z, p, q).

First, we will prove $q \notin C_{d-j}(w, p)$ and $\partial(w, q) = d - j$.

Suppose $q \in C_{d-j}(w, p)$. We have $\partial(x, q) = j + 1$ by the triangle inequalities on (x, w, p, q) and $q \notin B_{j+1}(x, p)$. Since $C_{d-j}(w, p) \subseteq C_{d-j+1}(u, p)$, we obtain (u, w, x, q) is a $(d, j+1)$-box. This contradicts Lemma 3 (1). Hence $q \notin C_{d-j}(w, p)$. Since $q \notin C_j(z, p) = B_{d-j}(w, p)$, we obtain $\partial(w, q) = d - j$.

Then we obtain $\partial(v, q) = d - j - 1$ by the triangle inequalities on (v, w, p, q) and $q \notin C_j(z, p) \subseteq C_{j+1}(x, p) = B_{d-j-1}(v, p)$. Also we have $\partial(x, q) = j + 1$ from the triangle inequalities on (x, v, p, q) and $q \notin B_{j+1}(x, p)$.



Next we will prove $q \in C_{d-j+1}(u,p)$.

From the triangle inequalities on (u,w,y,q), we have $\partial(u,q) \in \{d-j, d-j+1\}$. Suppose $q \notin C_{d-j+1}(u,p)$ to derive a contradiction. Then $\partial(u,q) = d-j+1$. Let $\{y^*\} = \Gamma_d(u) \cap \Gamma_{j-1}(q)$ as $p^{d-j+1}_{d,j-1} = 1$. By the triangle inequalities on (w,u,q,y^*), we get $\partial(w,y^*) = d-1$ and thus let $\{z^*\} = B_{d-1}(w,y^*)$. We obtain $\partial(q,z^*) = j$ and $\partial(v,z^*) = d-1$ by the triangle inequalities on (q,w,y^*,z^*) and on (v,w,q,z^*). Then $\partial(u,z^*) = d$ as otherwise (u,w,z^*,y^*) is a box.

Let $\{x^*\} = B_{d-1}(v,z^*)$. Also we get $\partial(q,x^*) = j+1$, by the triangle inequalities on (q,v,z^*,x^*). Since $\{x, x^*\} \subseteq \Gamma_d(v) \cap \Gamma_{j+1}(q)$ and $p^{d-j-1}_{d,j+1} = 1$, we have $x = x^*$. As $\partial(q,z) = j+1 \neq j = \partial(q,z^*)$, we have $z \neq z^*$. However $\{z, z^*\} \subseteq B_{d-1}(u,x)$. This contradicts $b_{d-1} = 1$. Hence we obtain $q \in C_{d-j+1}(u,p)$. Therefore the lemma is proved. □

Proof of Theorem 1. From Lemma 6 and Lemma 4 (3), we have

$$b_1 - c_d \le b_2 - c_{d-1} \le \cdots \le b_s - c_{t+1} \le 0.$$

On the other hand, from Lemma 4 (1)(2)

$$b_1 - c_d = (k-1) - (k-a_d) = -1 + a_d \ge 1.$$

We have a contradiction. This completes the proof of Theorem 1. ■

Acknowledgments

The authors would like to thank S. Iwamoto, T. Koishi, H. Nakano and S. Sugitani for their valuable comments.

References

1. M. Araya, A. Hiraki and A. Jurišić, “Distance-regular graphs with $b_t = 1$ and antipodal double-covers,” *J. Combin. Th. (B)* **67** (1996), 278–283.
2. M. Araya, A. Hiraki and A. Jurišić, “Distance-regular graphs with $b_2 = 1$ and antipodal covers,” *Europ. J. Combinatorics* **18** (1997), 243–248.
3. E. Bannai and T. Ito, *Algebraic Combinatorics I,* Benjamin-Cummings, California, 1984.
4. A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-Regular Graphs,* Springer-Verlag, Berlin, Heidelberg, 1989.
5. A. Gardiner, “Antipodal covering graphs,” *J. Combin. Th. (B)* **16** (1974), 255–273.