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A New Distance-Regular Graph Associated to the Mathieu Group M_{10}

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Abstract. We construct a bipartite distance-regular graph with intersection array $\{45, 44, 36, 5; 1, 9, 40, 45\}$ and automorphism group $3^5 : (2 \times M_{10})$ (acting edge-transitively) and discuss its relation to previously known combinatorial structures.

Keywords: distance-regular graph, Mathieu group, spectra of graph

1. Introduction

Let G be the perfect ternary Golay code generated by the rows of the circulant $(-+-+++----+-)_{11}$. Then G is a ternary $[11, 6, 5]$ code. Let Γ be the coset graph of G, that is, the graph with as vertices the 3^5 cosets of G in $\mathbf{F}_3^{11}$, where two cosets are adjacent when their difference contains a vector of weight one. Then Γ is a strongly regular graph with parameters $(v, k, \lambda, \mu) = (243, 22, 1, 2)$, known as the Berlekamp-van Lint-Seidel graph. (See Berlekamp, van Lint and Seidel [1], and Brouwer, Cohen and Neumaier [2], Section 11.3B.)

In [2], p. 360, the question was raised whether the complementary graph of the graph Γ is the halved graph of a bipartite distance-regular graph Δ of diameter 4. In this paper this question is answered affirmatively: the last two authors constructed such a graph Δ. (This also settles the last open case in Riebeek [6], Chapter 7.)

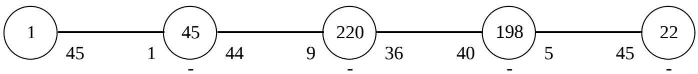
2. Construction

Put $Q := \{1, 3, 4, 5, 9\}$, the set of (nonzero) squares mod 11, and $N := \{2, 6, 7, 8, 10\}$, the nonsquares. Consider in the graph Γ the set D consisting of the following 45 cosets of G (we write u instead of $u + G$):

$$e_j, \ \ -e_0 - e_j \ (j \in N)$$

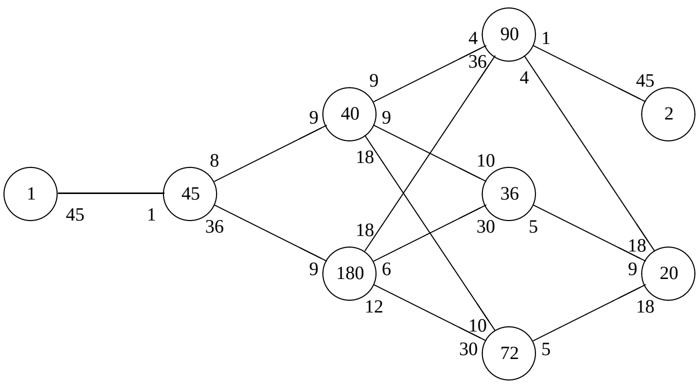
$$e_0 - e_i,\ \ e_i + e_{3i},\ \ \pm(e_i - e_{9i}),\ \ e_i - e_{7i},\ \ -e_i - e_{6i},\ \ -e_i - e_{10i}\ (i \in Q).$$

Then D, as well as each translate of D, is a 45-coclique, and the point-coclique incidence graph Δ on cosets of G and translates of D is distance-regular with intersection array $\{45, 44, 36, 5; 1, 9, 40, 45\}$ and distance distribution diagram



$v = 486.$

All of these properties can be checked easily using GAP [4] and GRAPE [7]. Using these packages and builtin Nauty [5] we find that the automorphism group of Δ has shape $3^5 : (2 \times M_{10})$, and acts edge-transitively with point stabilizer isomorphic to M_{10}. The orbit diagram of the point stabilizer is



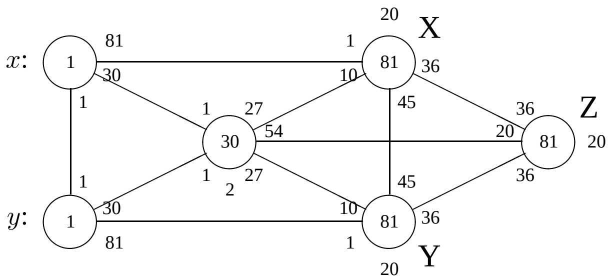
$v = 486.$

3. Structure of the group; related graphs

In order to describe the group of automorphisms more precisely, we have to specify the representation of $2 \times M_{10}$ inside $GL(5, 3)$. The direct factor 2 may be represented by $\pm I$, and then it remains to look at the group $H := 3^5 : M_{10}$, the stabilizer of the bipartition of Δ. This group has a centre of order 3, acting fixed point freely on Δ. The quotient graph is a bipartite graph E of valency 45 on 162 vertices that can be found inside the McLaughlin graph Λ as follows.

Let x, y be two adjacent vertices of Λ. Let X and Y be the sets of vertices of Λ adjacent to x but not to y, and to y but not to x, respectively (see also the figure below). Then $|X| = |Y| = 81$ and E is isomorphic to the graph with vertex set $X \cup Y$, where X and Y are cocliques, and the edges between X and Y are precisely those present in Λ. (Thus, E is not the graph induced by $X \cup Y$; in Λ the sets X and Y induce subgraphs of valency 20. See also Brouwer and Haemers [3], Construction D.)

Λ:



$v = 275.$

A larger graph. Let Z be the set of 81 vertices in Λ nonadjacent to both x and y. The graph induced by Λ on $X \cup Y \cup Z$, after switching with respect to Z, is isomorphic to the Delsarte graph, a strongly regular graph with parameters $(v, k, \lambda, \mu) = (243, 110, 37, 60)$. If we remove from this graph the edges inside X, Y and Z, we obtain a tripartite graph F of valency 90 on 243 vertices such that the subgraph induced on the union of any two of its parts is isomorphic to E. We have $\mathrm{Aut}(F) \simeq 3^5 : (2 \times M_{10})$.

This latter graph has a triple cover Σ, of course again tripartite, such that the subgraph induced on the union of any two of its parts is isomorphic to Δ. We have $\mathrm{Aut}(\Sigma) \simeq 3^6 : (2 \times M_{10})$.

Using [7] this graph Σ can be constructed as follows:

Let $A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}$ and $B := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 2 \end{pmatrix}$.

Let $M := \langle A, B \rangle$ be the matrix group generated by A and B. Then $M \simeq M_{10}$, and M has orbits of sizes 1, 1, 1, 20, 20, 20, 72, 72, 72, 90, 90, 90, 180 on $\mathbf{F}_3^6$. Let $N := \langle A, B, -I \rangle$. Then $N \simeq 2 \times M_{10}$, and N has orbits of sizes 1, 2, 20, 40, 72, 144, 90, 180, 180. The vector (000001) is a representative of the N-orbit O of size 90. The graph Σ is the graph with vertex set $\mathbf{F}_3^6$, where two vertices are adjacent when their difference lies in O. Now the graph Δ is the subgraph of Σ induced on the set of vectors with nonzero last coordinate.

Acknowledgments

Hans Cuypers suggested that $\mathrm{Aut}(\Delta)$ might be related to the edge stabilizer of the McLaughlin graph Λ. The availability of the computer algebra systems GAP [4], GRAPE [7] and Nauty [5] has been very useful. Support of the Dutch Organisation for Scientific Research (NWO) is gratefully acknowledged.

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