Journal of Algebraic Combinatorics **8** (1998), 169–191

Combinatorial Statistics on Alternating Permutations

SERGE DULUCQ dulucq@labri.u-bordeaux.fr
LaBRI, Université Bordeaux I, 351 cours de la Libération, 30455 Talence, France

RODICA SIMION simion@math.gwu.edu
Department of Mathematics, The George Washington University, Washington, DC 20052

Received February 7, 1996; Revised April 23, 1997

Abstract. We consider two combinatorial statistics on permutations. One is the genus. The other, $\widehat{\mathrm{des}}$, is defined for alternating permutations, as the sum of the number of descents in the subwords formed by the peaks and the valleys. We investigate the distribution of $\widehat{\mathrm{des}}$ on genus zero permutations and Baxter permutations. Our q-enumerative results relate the $\widehat{\mathrm{des}}$ statistic to lattice path enumeration, the rank generating function and characteristic polynomial of noncrossing partition lattices, and polytopes obtained as face-figures of the associahedron.

Keywords: lattice path, permutation, associahedron, Catalan, Schröder

1. Introduction

We introduce a variation, $\widehat{\mathrm{des}}$, of the descent statistic for permutations, defined for alternating permutations, which we apply to alternating permutations of genus zero and to alternating Baxter permutations. While the original motivation for defining $\widehat{\mathrm{des}}$ was to elucidate the equinumerosity of genus zero alternating permutations and Schröder paths, it turns out that $\widehat{\mathrm{des}}$ enjoys further equidistribution properties related to noncrossing partition lattices and face-figures of the associahedron. Our results involve q-analogues of the Catalan and Schröder numbers, and include extensions of previous work in [5, 9].

A permutation $\alpha \in S_N$ is an *alternating permutation* if $(\alpha(i-1)-\alpha(i))(\alpha(i)-\alpha(i+1)) < 0$ for all $i = 2, 3, \ldots, N-1$. The enumeration of all alternating permutations in S_N is a classical problem which can be viewed in the larger context of permutations with prescribed descent set [7, 22]. For $\alpha \in S_N$, the *descent set* of α is $Des(\alpha) := \{i : 1 \leq i < N, \alpha(i) > \alpha(i+1)\}$, and the *descent statistic* is $des(\alpha) := |Des(\alpha)|$. Thus, the alternating permutations in S_N are the permutations whose descent set is $\{1, 3, 5, \ldots\}$ or $\{2, 4, 6, \ldots\}$, the last descent depending on the parity of N. If α is an alternating permutation, we refer the value $\alpha(i)$ as a *peak* (local maximum) if $i \in Des(\alpha)$, and as a *valley* (local minimum) if $i \notin Des(\alpha)$.

In Section 3 we are concerned with alternating permutations of genus zero. The statistic *genus* which we use here is based on the earlier notion of genus of a pair of permutations appearing in the study of hypermaps (see [8, 10, 23, 24]). Let $z(\rho)$ denote the number of cycles of a permutation ρ. Given $(\sigma, \alpha) \in S_N \times S_N$, the relation

$$z(\sigma) + z(\alpha) + z(\alpha^{-1}\sigma) = N + 2 - 2g$$

defines the *genus of the pair* (σ, α). This suggests considering the permutation statistic $g_\sigma : S_N \to \mathbf{Z}$ obtained by fixing σ and computing the genus of each $\alpha \in S_N$ with respect to σ.

For example, the simplest choice for σ, the identity permutation in S_N, leads to $g_{id}(\alpha) = 1 - z(\alpha)$, essentially the number of cycles, a well studied permutation statistic (see, e.g., [7, 22]).

In order to maintain the topological significance of $g_\sigma(\alpha)$, we will restrict our attention to choices of σ such that the subgroup $\langle \sigma, \alpha \rangle$ is transitive for all α. That is, we will consider only N-cycles σ, and in fact we will normalize our choice for the rest of this paper to $\sigma = (1, 2, \ldots, N)$. Our notion of genus corresponds to the genus of a hypermap with a single vertex.

Definition 1.1 The genus $g(\alpha)$ of a permutation $\alpha \in S_N$ is defined by

$$z(\alpha) + z(\alpha^{-1} \cdot (1, 2, \ldots, N)) = N + 1 - 2g(\alpha).$$

For example, if $\alpha = 2\ 3\ 1\ 5\ 4$ (in cycle notation, $\alpha = (1\ 2\ 3)(4\ 5)$), then $\alpha^{-1}(1\ 2\ 3\ 4\ 5) = (1\ 4)(2)(3)(5)$, hence $g(\alpha) = 0$.

The table in figure 1 shows the number of genus zero alternating permutations beginning with an ascent (*up-down*) and with a descent (*down-up alternating permutations*).

The numerical values in this table are the (big) Schröder numbers: $1, 2, 6, 22, \ldots$, whose generating function is

$$\begin{aligned} Sch(x) := \sum_{n \geq 0} Sch_n x^n &= \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x} \\ &= 1 + 2x + 6x^2 + 22x^3 + \cdots, \end{aligned} \tag{1}$$

and the "small Schröder numbers," $(\frac{1}{2} Sch_n)_{n \geq 1}$. It is this observation, proven in Theorem 3.2, that lies at the origin of this paper.

Schröder numbers have numerous combinatorial interpretations, for example, in terms of incomplete parenthesis systems, certain lattice paths, plane trees with loops allowed, or faces of the associahedron, see [5, 7, 20]. Using the (classical) lattice path interpretation described in the next section, it is natural to consider the number of diagonal steps as a combinatorial statistic and obtain a q-analogue of the Schröder numbers. Is there a statistic

n	0	1	2	3	4	5	6	7	8	9	10	...
$u_n^{(0)}$	1	1	1	2	2	6	6	22	22	90	90	...
$d_n^{(0)}$	0	0	1	1	3	3	11	11	45	45	197	...

Figure 1. The number of up-down and down-up permutations of genus zero.

on genus zero alternating permutations which gives the same q-analogue? In answer to this question (Theorem 3.4), we introduced the statistic $\widehat{\text{des}}$.

Definition 1.2 Given an alternating permutation $\alpha \in S_N$, we say that i, $1 \leq i \leq N-2$, is an alternating descent of α if $\alpha(i) > \alpha(i+2)$, and we define $\widehat{\text{des}}(\alpha)$ to be the number of alternating descents of α. Equivalently, $\widehat{\text{des}}(\alpha) = des(\text{peaks}(\alpha)) + des(\text{valleys}(\alpha))$, where peaks$(\alpha)$ and valleys(α) are the subwords of $\alpha(1)\alpha(2)\ldots\alpha(N)$ formed by the peaks and valleys, respectively.

Although Theorem 3.2 is a special case ($q = 1$) of Theorem 3.4, we prove the former separately for ease of exposition.

It turns out that an alternating permutation of genus zero is necessarily a Baxter permutation (Proposition 4.1). This motivated our investigation of the statistic $\widehat{\text{des}}$ on alternating Baxter permutations. Baxter permutations (whose definition appears in the next section) were investigated in, among other references, [2, 6, 17, 18, 23]. In answer to a question of Mallows, [9] offers a combinatorial proof of the identity

$$C_N C_{N+1} = \sum_{k=0}^{N} \binom{2N}{2k} C_k C_{N-k}, \tag{2}$$

in which $C_N = \frac{1}{N+1}\binom{2N}{N}$ is the Nth Catalan number. The proof in [9] relies on interpreting both sides of (2) as counting alternating Baxter permutations of $\{1, 2, \ldots, 2N+1\}$. Similarly, $(C_N)^2$ is interpreted as the number of alternating Baxter permutations of $\{1, 2, \ldots, 2N\}$.

We show (Theorem 4.2) that this interpretation extends to a q-analogue based on the statistic $\widehat{\text{des}}$ for alternating Baxter permutations and number of cycles for genus zero permutations (equivalently, number of blocks for noncrossing partitions). Our proof takes advantage of the methods developed in [9].

Connections between the rank generating function of the lattice of noncrossing partitions and the enumeration of Schröder lattice paths were described in [5]. Theorem 5.1 relates Schröder path enumeration with the characteristic polynomial of the noncrossing partition lattice. This leads to an alternative derivation (Corollary 5.2) of a certain reciprocity relation [14] between the rank generating function and the characteristic polynomial of successive noncrossing partition lattices. Our proof relies on general facts about the f- and h-vectors of a simplicial complex, exploiting the relation between the lattice of noncrossing partitions, Schröder paths, and the associahedron ([5, 16]). Thus, the rather unusual relation (16) between the rank generating function and the characteristic polynomial for noncrossing partition lattices is explained here as a manifestation of the fact that, for such posets, these two polynomials give the h- and f-vectors of a simplicial complex.

In Section 6 we show that the distribution of the alternating descents statistic $\widehat{\text{des}}$ on alternating Baxter permutations of $\{1, 2, \ldots, N\}$ gives the h-vector of a convex polytope. This can be described as a vertex-figure of the associahedron (Theorem 6.1). The total number of faces of this polytope is either the square of a Schröder number, or the product of two consecutive Schröder numbers, depending on the parity of N. Corollary 6.2 extends this approach and establishes a unified combinatorial interpretation (in terms of h- and f-vectors of face-figures of the associahedron) for multiple products of q-Catalan numbers and of q-Schröder numbers. This generalizes simultaneously results from [5] and [9].

2. Preliminaries and notation

In our discussion, we find it convenient to distinguish several classes of alternating permutations. We denote by U the class of *up-down alternating permutations* (i.e., those beginning with an ascent), and by D the class of *down-up alternating permutations* (i.e., those beginning with a descent). In turn, UD will denote the subset of U consisting of permutations ending with a descent. The classes denoted UU, DU, DD are similarly defined. We convene to view the empty permutation and S_1 in UU. The same notation in lowercase will indicate generating functions, e.g., $ud(x) = \sum_n ud_n x^n$, where $ud_n = |UD \cap S_n|$.

For a set X of permutations, $X^{(0)}$ and $X^{(B)}$ will denote the intersection of X with the class of genus zero permutations and with the class of Baxter permutations, respectively. Superscripts for generating functions and cardinalities will be used as above, e.g., $ud^{(0)}(x) = \sum_n ud_n^{(0)} x^n$, where $ud_n^{(0)} = |UD^{(0)} \cap S_n|$. By convention, we set $u_0^{(0)} = uu_0^{(0)} = 1$ (corresponding to the empty permutation) and $u_1^{(0)} = uu_1^{(0)} = 1$. If $\alpha \in S_n$, we write $|\alpha| = n$.

Now we introduce several combinatorial objects which will occur in subsequent sections: noncrossing partitions, Schröder paths, Baxter permutations, and the associahedron.

By a *noncrossing partition* of $[N] := \{1, 2, \ldots, N\}$ we mean a collection $B_1, B_2, \ldots, B_k$ of nonempty, pairwise disjoint subsets of $[N]$ whose union is $[N]$ (that is, a partition of $[N]$), with the property that if $1 \leq a < b < c < d \leq N$ with $a, c \in B_i$ and $b, d \in B_j$, then $i = j$ (that is, no two "blocks" of the partition "cross each other"). Thus, all partitions of $\{1, 2, 3, 4\}$ *except* 1 3 / 2 4 are noncrossing.

If the elements i and j lie in the same block of a partition, we write $i \sim j$. As is customary, we assume that the elements within each block are increasingly ordered and that the blocks are indexed in increasing order of their minimum elements.

The set $\mathrm{NC}(N)$ of noncrossing partitions of $[N]$ is known to form a lattice under the refinement order, whose enumerative and structural properties were investigated, for example, in [12, 13, 15]. We will make reference to the rank of a noncrossing partition, $\mathrm{rank}(\pi) = N - \mathrm{bk}(\pi)$, where $\pi \in \mathrm{NC}(N)$ has $\mathrm{bk}(\pi)$ blocks, and we will use $\mathrm{NC}(N, k)$ to denote the set of noncrossing partitions of $[N]$ having k blocks. If $\pi \in \mathrm{NC}(N)$ we write $|\pi| = N$. It is known (see, e.g., [7, 15]) that $|\mathrm{NC}(N)| = \frac{1}{N+1}\binom{2N}{N}$, the Nth Catalan number, and that the rank generating function of $\mathrm{NC}(N)$ is

$$C_N(q) := \sum_{\pi \in \mathrm{NC}(N)} q^{\mathrm{rank}(\pi)} = \sum_{k=1}^{N} \frac{1}{N} \binom{N}{k} \binom{N}{k-1} q^{k-1}, \tag{3}$$

a q-analogue of the Catalan number whose coefficients are the Narayana numbers ([15]).

Noncrossing partitions will play a role throughout the present paper since it turns out, from [15] and [25], that genus zero permutations can be completely characterized as follows.

Lemma 2.1 *Let $\alpha \in S_N$. Then $g(\alpha) = 0$ if and only if the cycle decomposition of α gives a noncrossing partition of $[N]$, and each cycle of α is increasing.*

Saying that an m-cycle of α is increasing means that its elements are expressible as $a < \alpha(a) < \alpha^2(a) < \cdots < \alpha^{m-1}(a)$. The reader familiar with [15] will recognize that, in the language of Kreweras' paper, a genus zero permutation α is the "trace" of the corresponding noncrossing partition, with respect to the N-cycle σ.

Thus, the number of permutations in S_N whose genus is zero is the Nth Catalan number. A particular encoding of noncrossing partitions which appears in [21] (briefly described, for the reader's convenience, in Section 3) will be used to encode genus zero permutations.

In our q-enumeration results involving the Schröder numbers, we use the following lattice path interpretation (see, e.g., [5]). Let $Sch(N)$ denote the collection of lattice paths in the plane which start at the origin, end at (N, N), are bounded by the horizontal axis and the line $y = x$, and consist of steps of three allowable types: (1, 0) (East), (0, 1) (North), and (1, 1) (diagonal). Then $|Sch(N)| = Sch_N$, the Nth Schröder number, and we refer to such paths as *Schröder paths*. We consider the statistic Diag, number of diagonal steps, on Schröder paths, and we let $Sch_{N,k} = |\{p \in Sch(N) : \mathrm{Diag}(p) = k\}|$. For instance, $Sch_{2,1} = 3$, accounting for the paths DEN, EDN, END, where E, N, D stand for East, North, and diagonal steps. The numbers $Sch_{N,k}$ have the generating function

$$Sch\,\mathrm{D}(x, q) := \sum_{n,k\geq 0} Sch_{N,k} x^N q^k = \frac{1 - qx - \sqrt{1 - (4x + 2qx) + q^2x^2}}{2x}. \tag{4}$$

The statistic $\widehat{\mathrm{des}}$ turns out to have an interesting distribution on alternating Baxter permutations. A permutation $\alpha \in S_N$ is a *Baxter permutation* if it satisfies the following conditions for every $i = 1, 2, \ldots, N-1$: if $\alpha^{-1}(i) < k, m < \alpha^{-1}(i+1)$ and if $\alpha(k) < i$ while $\alpha(m) > i+1$, then $k < m$; similarly, if $\alpha^{-1}(i+1) < k, m < \alpha^{-1}(i)$ and if $\alpha(k) < i$ while $\alpha(m) > i+1$, then $k > m$. Informally, a permutation satisfies the Baxter condition if between the occurrences of consecutive values, i and $i+1$, the smaller values are "near" i and the larger values are "near" $i+1$. Thus, for $N \leq 3$ all permutations are Baxter. In the symmetric group S_4 there are 22 Baxter permutations; the Baxter condition fails for $i = 2$ in 2 4 1 3 and 3 1 4 2. In terms of forbidden subsequences, a permutation $\alpha = \alpha(1)\alpha(2)\ldots\alpha(N) \in S_N$ is Baxter iff it does not contain any 4-term subsequence of the pattern 2 4 1 3 or 3 1 4 2 in which the roles of 2 and 3 are played by consecutive values; that is, there are no indices $1 \leq j < k < l < m \leq N$ such that $\alpha(l) < \alpha(j) < \alpha(m) < \alpha(k)$ and $\alpha(j) + 1 = \alpha(m)$, or $\alpha(k) < \alpha(m) < \alpha(j) < \alpha(l)$ and $\alpha(m) + 1 = \alpha(j)$.

Finally, we summarize a few facts about the associahedron, which will occur in Sections 5 and 6 (see, e.g., [16, 26] for additional information and related developments).

For $n \geq 1$, let Δ_n denote the following simplicial complex: the vertices represent diagonals in a convex $(n+2)$-gon, and the faces are the sets of vertices corresponding to collections of pairwise noncrossing diagonals. Thus, the maximal faces correspond to the triangulations of the polygon and are $(n-2)$-dimensional. In [16], Lee constructs an $(n-1)$-dimensional simplicial polytope, Q_n, called the *associahedron*, with boundary complex Δ_n.

In general, the f-vector of a $(d-1)$-dimensional simplicial complex is $f = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$, where f_i is the number of i-dimensional faces (with $f_{-1} = 1$ accounting for the empty face). The h-vector, $h = (h_0, h_1, \ldots, h_d)$, contains equivalent information and is defined by

$$\sum_{i=0}^{d} f_{i-1}(q-1)^{d-i} = \sum_{i=0}^{d} h_i q^{d-i}. \tag{5}$$

The f- and h-vector of a (simplicial) polytope are those of the (simplicial) complex formed by its boundary faces. Lee [16] shows that the h-vector of the associahedron Q_n is given by the coefficients of $C_n(q)$, that is,

$$\sum_{i=0}^{n-1} h_i(Q_n) q^{n-1-i} = C_n(q). \tag{6}$$

In [5] it is shown that $f_{i-1}(Q_n)$ is the number of Schröder paths with final point (n, n), having $n-1-i$ diagonal steps, and in which the first non-East step is North. In particular, the total number of faces of Δ_n is $\frac{1}{2}Sch_n$.

3. Enumeration and q-enumeration of alternating permutations of genus zero

Our approach to proving the two main results of this section (Theorems 3.2 and 3.4) is as follows. Based on Lemma 2.1, genus zero permutations can be identified with non-crossing partitions. We characterize the descents of a genus zero permutation in terms of a word encoding the associated noncrossing partition. From this characterization, we derive a grammar for the formal language of the words encoding the nonempty genus zero alternating permutations. In turn, the grammar rules lead to a system of equations from which we obtain the generating functions for the classes of up-down and down-up permutations of genus zero, and Theorem 3.2 follows. This approach is then refined to permit q-counting these permutations according to the statistic $\widehat{\text{des}}$. Using a q-grammar (or attribute grammar, see, e.g., [11]) we obtain the q-analogue results of Theorem 3.4.

We begin with a brief description of an encoding (given in [21]) of noncrossing partitions as words over the alphabet $\{B, E, R, L\}$. If $\pi \in \text{NC}(N)$, for $N \geq 1$, then the word associated with π is $w(\pi) = w_1 w_2 \ldots w_{N-1}$ where

$$w_i = \begin{cases} B, & \text{if } i \not\sim i+1 \text{ and } i \text{ is } \textit{not} \text{ the largest element in its block;} \\ E, & \text{if } i \not\sim i+1 \text{ and } i+1 \text{ is } \textit{not} \text{ the smallest element in its block;} \\ L, & \text{if } i \not\sim i+1,\ i \text{ is the largest element in its block, and } i+1 \text{ is the} \\ & \quad \text{smallest element in its block;} \\ R, & \text{if } i \sim i+1. \end{cases}$$

For details on this encoding and applications to deriving structural properties of the lattice of noncrossing partitions, we refer the interested reader to [21]. For our purposes, we also view w as encoding the genus zero permutation whose cycle decomposition gives the noncrossing partition π. Observe that $|\pi| = |w| + 1$ and that the B's and E's are matched in pairs, forming a well-parenthesized system with B's playing the role of left parentheses.

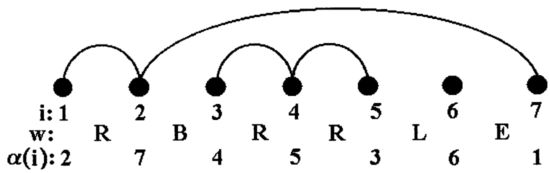

Figure 2. Encoding of a genus zero permutation $\alpha \in S_N$ by a word $w \in \{B, E, R, L\}^{N-1}$.

Figure 2 shows an example: the genus zero permutation $\alpha = 2\ 7\ 4\ 5\ 3\ 6\ 1 \in S_7$, the noncrossing partition determined by its cycle decomposition $\alpha = (1\,2\,7)(3\,4\,5)(6)$, and the word $w = RBRRLE$ which encodes the noncrossing partition and, hence, encodes α.

Next, toward the enumeration of the words w which correspond to alternating permutations of genus zero, we characterize it terms of w the positions $i \in \{1, 2, \ldots, N-1\}$ which are descents for the corresponding permutation.

Lemma 3.1 *Let $\alpha \in S_N$ be a permutation of genus zero encoded by the word $w = w_1 w_2 \ldots w_{N-1} \in \{B, E, R, L\}^*$. Then, for each $1 \le i \le N-1$,*

$$i \notin Des(\alpha) \quad \text{iff } w_i = L, \quad \text{or } w_i \in \{R, E\} \text{ and } w_{i+1} \in \{B, R\}.$$

Equivalently,

$$i \in Des(\alpha) \quad \text{iff } w_i = B, \quad \text{or } w_i \in \{R, E\} \text{ and either } i = N-1 \text{ or else } w_{i+1} \in \{L, E\}.$$

Proof: The proof involves a case-by-case analysis of the conditions on w_i which characterize $i \in Des(\alpha)$.

Suppose $w_i = B$. Using Lemma 2.1, this implies that $\alpha(i) > i + 1$. The noncrossing condition on cycles forces $\alpha(i+1) < \alpha(i)$, so $i \in Des(\alpha)$ whenever $w_i = B$.

If $w_i = R$, then $\alpha(i) = i + 1$, and whether i is a descent depends on w_{i+1}. If $w_{i+1} \in \{R, B\}$, then $\alpha(i+1) \ge i + 2$ and i is an ascent of α. If $w_{i+1} \in \{L, E\}$ or if $i = N - 1$, then $\alpha(i) = i + 1$ is the maximum and $\alpha(i+1)$ is the minimum in their cycle of α. Thus $\alpha(i+1) \le i < \alpha(i)$, so $i \in Des(\alpha)$. Similar arguments, which we omit, apply in the cases $w_i = L$ or E, completing the proof. □

We can now prove the relation between the entries in the table of figure 1 and Schröder numbers.

Theorem 3.2 *The number of up-down and down-up alternating permutations of genus zero satisfies*

$$u_{2n}^{(0)} = u_{2n-1}^{(0)} = Sch_{n-1},$$
$$d_{2n}^{(0)} = d_{2n+1}^{(0)} = \frac{1}{2} Sch_n,$$

for $n \geq 1$, *and* $u_0^{(0)} = 1$, $d_0^{(0)} = d_1^{(0)} = 0$. *Equivalently*,

$$u^{(0)}(x) = 1 + x(1+x)Sch(x^2),$$
$$d^{(0)}(x) = \frac{1}{2}(1+x)[Sch(x^2) - 1].$$

Proof: Let us denote by $\underline{UU}$ the language formed by the words $w \in \{B, E, R, L\}^*$ corresponding to the nonempty permutations of genus zero which alternate starting and ending with an ascent; similarly, we will denote by $\underline{UD}$, $\underline{DU}$, and $\underline{DD}$ the languages for the other three types of alternating permutations of genus zero. Let $\underline{uu}(x) = \sum_{w \in \underline{UU}} x^{|w|}$, and $\underline{ud}(x)$, $\underline{du}(x)$, $\underline{dd}(x)$ be similarly defined as the enumerators according to length of the words in the appropriate language.

Also, $\underline{U} = \underline{UU} + \underline{UD}$ and $\underline{D} = \underline{DU} + \underline{DD}$ denote the languages formed by the words corresponding to the nonempty permutations in $U^{(0)}$ and $D^{(0)}$, respectively.

Using Lemma 3.1, we obtain the following grammar:

$$\underline{UU} \to \epsilon + L + L\ \underline{DU} + R\ \underline{DU}$$
$$\underline{UD} \to L\ \underline{DD} + R\ \underline{DD}$$
$$\underline{DU} \to RL(\epsilon + \underline{DU}) + B(L + L\ \underline{DD}\ L + R\ \underline{DD}\ L)\ E(L + L\ \underline{DU}) + B(\epsilon + \underline{UD})\ E\ \underline{DU}$$
$$\underline{DD} \to R + RL\ \underline{DD} + B(L + L\ \underline{DD}\ L + R\ \underline{DD}\ L)\ E(\epsilon + L\ \underline{DD}) + B(\epsilon + \underline{UD})\ E\ \underline{DD}.$$

The empty word is denoted by ϵ and, of course, B, E, R, L do not commute. Each of the four derivation rules follows from similar arguments. In the interest of brevity, we include only a proof of the third rule (the first two rules are rather obvious, the fourth is similar to the third). Let $\alpha \in DU^{(0)} \cap S_N$ (with N necessarily odd), and $w = w_1 w_2 \ldots w_{N-1} \in \{B, E, R, L\}^*$ be its associated word. By Lemma 3.1, $w_1 \in \{R, B, E\}$, but w_1 cannot be E since the number of B's must be at least equal to the number of E's in every prefix of w. Thus $w_1 \in \{R, B\}$.

Suppose $w_1 = R$. Since α must end with an ascent, we have $N \geq 3$, and Lemma 3.1 implies that w_2 must be E or L. However, w_2 cannot be E because no B precedes it, so $w_2 = L$. This is consistent with α having an ascent in the second position, and by Lemma 3.1 imposes no restriction on w_3. If $N > 3$, then $w_3 w_4 \ldots w_{N-1}$ must be in the language $\underline{DU}$. This gives the first term in the derivation rule for $\underline{DU}$.

Suppose now that $w_1 = B$ and consider the factorization $w = BvEv'$, where BvE is the shortest prefix of w in which the number of B's equals the number of E's. Two cases emerge, depending on the parity of the length of v. If $|v|$ is odd, say, $|v| = 2i - 1$, then $v \in \underline{UU}$ is nonempty, $2i + 1 \in Des(\alpha)$, and $v' \in \underline{UU}$. Moreover, Lemma 3.1 requires $w_2 \in \{L, R, E\}$ so that $2 \notin Des(\alpha)$, but $w_2 = E$ is not possible here by the choice of the factorization $w = BvEv'$. Also, since $2i \notin Des(\alpha)$ and $w_{2i+1} = E$, Lemma 3.1 forces v to end in $w_{2i} = L$. Finally, Lemma 3.1 shows that these requirements are consistent with $w_3 \dots w_{2i-1} \in \underline{DD}$ when $i \geq 2$. Concerning v', Lemma 3.1 implies that it must be either empty or begin with E or L. In fact, v' is nonempty since α ends in an ascent, and it must begin with L in order for $w_1 \dots w_{2i+1} w_{2i+2}$ to contain no more E's than B's. Also, $w_{2i+2} = L$ does not impose additional conditions on v'. This gives the second term of the derivation rule for $\underline{DU}$. If $|v|$ is even, say, $|v| = 2i \geq 0$, then we must have $v = \epsilon$ or $v \in \underline{UD}$, $2i + 2 \notin Des(\alpha)$, and $v' \in \underline{DU}$. Lemma 3.1 applied to $2i + 2 \notin Des(\alpha)$, implies that v' must begin with B or R, which is consistent with $v' \in \underline{DU}$. Hence, the last term of the derivation rule for $\underline{DU}$.

Now, the replacement of ϵ with 1 and of each of B, E, R, L with x, yields a system of equations for $\underline{uu}(x)$, $\underline{ud}(x)$, $\underline{du}(x)$, $\underline{dd}(x)$, the enumerating series (by word-length) for the four languages under consideration. The generating functions (by length) of the permutation classes $UU^{(0)}$, $UD^{(0)}$, $DU^{(0)}$, $DD^{(0)}$ then follow immediately: $uu^{(0)}(x) = 1 + x\,\underline{uu}(x)$, $ud^{(0)}(x) = x\,\underline{ud}(x)$, $du^{(0)}(x) = x\,\underline{du}(x)$ and $dd^{(0)}(x) = x\underline{dd}(x)$. The factors of x account for the fact that $|\alpha| = |w| + 1$ if w encodes the nonempty permutation α, and the additional unit in $uu(x)$ accounts for the empty permutation with no word w associated with it.

Following calculations which we omit, we obtain

$$\begin{aligned} uu^{(0)}(x) &= 1 + x(1 + x Sch\,\mathrm{D}(x^2)), \\ ud^{(0)}(x) &= x(Sch\,\mathrm{D}(x^2) - 1), \\ du^{(0)}(x) &= \frac{x}{2}(Sch\,\mathrm{D}(x^2) - 1), \\ dd^{(0)}(x) &= \frac{1}{2}(Sch\,\mathrm{D}(x^2) - 1). \end{aligned}$$

Hence, $u^{(0)}(x) = uu^{(0)}(x) + ud^{(0)}(x)$ and $d^{(0)}(x) = du^{(0)}(x) + dd^{(0)}(x)$ have the desired expressions, and the values of the coefficients follow. □

Before stating and proving our next result, which refines Theorem 3.2, we illustrate it with an example. The statistic Diag(p), the number of diagonal steps of the path p, leads to a q-analogue of the number Sch_n of Schröder paths with final point (n, n). For $n = 2$ we obtain Diag(EENN) = Diag(ENEN) = 0, Diag(DEN) = Diag(EDN) = Diag(END) = 1, Diag(DD) = 2, hence the q-analogue $2 + 3q + q^2$ of the total number $Sch_2 = 6$ of paths. On the other hand, the statistic $\widehat{\mathrm{des}}(\alpha)$, the number of alternating descents, applied to $U^{(0)} \cap S_5$, the genus zero alternating permutations of {1, 2, 3, 4, 5} which begin with an ascent, gives: $\widehat{\mathrm{des}}(2\ 5\ 3\ 4\ 1) = \widehat{\mathrm{des}}(1\ 5\ 3\ 4\ 2) = 2$, $\widehat{\mathrm{des}}(2\ 3\ 1\ 5\ 4) = \widehat{\mathrm{des}}(2\ 4\ 3\ 5\ 1) = \widehat{\mathrm{des}}(1\ 4\ 3\ 5\ 2) = 1$, $\widehat{\mathrm{des}}(1\ 3\ 2\ 5\ 4) = 0$. This produces the q-analogue $2q^2 + 3q + 1$ of $u_5^{(0)} = 6$. The fact that these two q-analogues are reciprocal polynomials is not an accident. As Theorem 3.4 asserts, Diag and $\widehat{\mathrm{des}}$ have essentially the same distribution. In our proof, Lemma 3.3 plays a role analogous to that of Lemma 3.1 relative to Theorem 3.2. It characterizes the alternating descents of an alternating permutation of genus zero in terms of its word encoding, and is the key to deriving the rules of a q-grammar, from which the relevant generating functions follow.

Lemma 3.3 *Let $\alpha \in S_N$ be an alternating permutation of genus zero encoded by the word $w = w_1 w_2 \dots w_{N-1} \in \{B, E, R, L\}^*$. Then i is an alternating descent of α if and* *only if* $1 \leq i \leq N-2$ *and* $\alpha(i)$ *is a peak and* $w_i w_{i+1} \in \{BL, BR\}$, *or* $\alpha(i)$ *is a valley and* $w_i w_{i+1} \in \{LE, RR, ER\}$.

Proof: Clearly, $i \leq N-2$ is necessary for an alternating descent. Suppose $\alpha(i)$ is a peak. Then $i \in Des(\alpha)$ and $i+1 \notin Des(\alpha)$, and Lemma 3.1 implies that $w_i w_{i+1} \in \{BL, BR, BE, RL, RE, EL, EE\}$. In the first two cases, i and $i+2$ lie in different cycles of α and the noncrossing condition on these cycles forces $\alpha(i) > \alpha(i+2)$, hence i is an alternating descent. By Lemma 3.1, the case $w_i w_{i+1} = BE$ requires in fact that $i \leq N-3$ and $w_i w_{i+1} w_{i+2} \in \{BEB, BER\}$. Consequently, $\alpha(i) = i+2$, $\alpha(i+2) > i+2$, so i is not an alternating descent. Similar arguments show that the remaining cases for $w_i w_{i+1}$ imply that i is not an alternating descent. The characterization of alternating descents among valleys follows similarly, from the consideration of the possibilities $w_i w_{i+1} \in \{LB, LR, LE, RB, RR, EB, ER\}$ and the additional constraints from Lemma 3.1. □

Theorem 3.4 *Let the Schröder paths be enumerated according to their final point and number of diagonal steps*,

$$\begin{aligned} Sch\ D(x, q) &= \sum_N \sum_{p \in Sch_N} x^N q^{Diag(p)} \\ &= 1 + (1+q)x + (2+3q+q^2)x^2 + (5+10q+6q^2+q^3)x^3 + \cdots, \end{aligned}$$

and let the up-down and down-up alternating permutations of genus zero be enumerated according to their length and alternating descents statistic,

$$\begin{aligned} u^{(0)}(x, q) &:= \sum_{\alpha \in U^{(0)}} x^{|\alpha|} q^{\widehat{des}(\alpha)}, \\ d^{(0)}(x, q) &:= \sum_{\alpha \in D^{(0)}} x^{|\alpha|} q^{\widehat{des}(\alpha)}. \end{aligned}$$

Then,

$$\begin{aligned} u^{(0)}(x, q) &= 1 + x(1+x) Sch\ D(x^2 q, q^{-1}) \\ &= 1 + (x + x^2) + (1+q)(x^3 + x^4) + (1+3q+2q^2)(x^5+x^6) + \cdots \end{aligned}$$

and

$$\begin{aligned} d^{(0)}(x, q) &= \frac{1+x}{2 - x^2 + x^2 q}[Sch\ D(x^2 q, q^{-1}) - 1 + x^2 - x^2 q] \\ &= (x^2 + x^3) + (1+q+q^2)(x^4+x^5) \\ &\quad + (1+3q+5q^2+2q^3)(x^6+x^7) + \cdots. \end{aligned}$$

Proof: Using Lemma 3.3 we can refine the grammar rules used in the proof of Theorem 3.2: in the factorization of the words, we introduce the parameter q to record the occurrence of an alternating descent. In doing so, it is useful to distinguish, for each language $\underline{X}$ of words corresponding to the nonempty permutations in the class $X^{(0)}$ weighted by $q^{\widehat{\text{des}}}$, the sublanguage of words beginning with a prescribed letter: $\underline{X}_Y := \{q^{\widehat{\text{des}}(\alpha)} w : \alpha \in X^{(0)} - \{\emptyset\},\ w = w(\alpha),\ w_1 = Y\}$. We obtain

$$
\begin{aligned}
\underline{UU}_L &\to L + L(\underline{DU}_R + \underline{DU}_B)\\
\underline{UU}_R &\to R(q\underline{DU}_R + \underline{DU}_B)\\
\underline{UD}_L &\to L(\underline{DD}_R + \underline{DD}_B)\\
\underline{UD}_R &\to R(q\underline{DD}_R + \underline{DD}_B)\\
\underline{DU}_R &\to RL\ (\epsilon + \underline{DU}_R + \underline{DU}_B)\\
\underline{DU}_B &\to Bq[L + L(\underline{DD}_R + \underline{DD}_B)L + R(q\underline{DD}_R + \underline{DD}_B)L]\\
&\quad \times qE\ (L + L\ (\underline{DU}_R + \underline{DU}_B))\\
&\quad + B(\epsilon + q\underline{UD}_L + q\underline{UD}_R)\ E\ (q\underline{DU}_R + \underline{DU}_B)\\
\underline{DD}_R &\to R + RL(\underline{DD}_R + \underline{DD}_B)\\
\underline{DD}_B &\to Bq[L + L(\underline{DD}_R + \underline{DD}_B)L + R(q\underline{DD}_R + \underline{DD}_B)L]\\
&\quad \times qE\ (\epsilon + L(\underline{DD}_R + \underline{DD}_B))\\
&\quad + B(\epsilon + q\underline{UD}_L + q\underline{UD}_R)\ E\ (q\ \underline{DD}_R + \underline{DD}_B).
\end{aligned}
$$

All the rules are obtained by applying Lemmas 3.1 and 3.3. We include only the treatment of the rules for $\underline{DU}_R$ and $\underline{DU}_B$ which are refinements of the rule for $\underline{DU}$ in the proof of Theorem 3.2.

Let w be the word encoding $\alpha \in DU^{(0)}$. If $w_1 = R$, then $w \in RL(\epsilon + \underline{DU})$, as in the proof of Theorem 3.2, and by Lemma 3.1, $\underline{DU} \to \underline{DU}_R + \underline{DU}_B$. Lemma 3.3 applied to the peak $\alpha(1)$ implies that 1 is not an alternating descent, and applied to the valley $\alpha(2)$ it implies that 2 is not an alternating descent (since $w_1w_2w_3$ cannot be RLE). Thus we get the desired derivation rule for $\underline{DU}_R$. If $w_1 = B$, we consider the factorization $w = BvEv'$ as in the proof of Theorem 3.2. When $|v| = 2i - 1$ for some $i \geq 1$, we know that w_2 must be R or B. In both cases, Lemma 3.3 implies that the peak $\alpha(1)$ gives an alternating descent, hence the first factor of q in the derivation rule for $\underline{DU}_B$. The valley $\alpha(2)$ gives an alternating descent only when $w_2w_3 = RR$, hence the factor of q inside the bracket. We also have $w_{2i}w_{2i+1} = LE$ as in the proof of Theorem 3.2, and Lemma 3.3 shows that the valley $\alpha(2i)$ gives an alternating descent. Hence the factor of q following the bracket. Based on the discussion in the proof of Theorem 3.2, we have $w_{2i+1}w_{2i+2} = EL$ and, by Lemma 3.3, the peak $\alpha(2i+1)$ does not give an alternating descent. This establishes the first term in the derivation rule for $\underline{DU}_B$. The second term refines the last term for $\underline{DU}$ from the proof of Theorem 3.2, corresponding to the case $|v| = 2i \geq 0$. By Lemma 3.3, the peak $\alpha(1)$ gives an alternating descent if $v \neq \epsilon$, while the peak $\alpha(2i+1)$ does not give an alternating descent (since $w_{2i+2} = E$), and the valley $\alpha(2i+2)$ gives an alternating descent only when $w_{2i+3} = R$.

The solution of the system of equations, obtained with the aid of Maple, gives

$$\begin{aligned}
uu^{(0)}(x,q) &= 1 + x(1 + \underline{uu}_L(x,q) + \underline{uu}_R(x,q)) \\
&= 1 + x(1 + x\mathit{Sch}\,\mathrm{D}(x^2q, q^{-1})), \\
ud^{(0)}(x,q) &= x(\underline{ud}_L(x,q) + \underline{ud}_R(x,q)) \\
&= x(\mathit{Sch}\,\mathrm{D}(x^2q, q^{-1}) - 1), \\
du^{(0)}(x,q) &= x(\underline{du}_R(x,q) + \underline{du}_B(x,q)) \\
&= \frac{x}{2 - x^2 + x^2q}[\mathit{Sch}\,\mathrm{D}(x^2q, q^{-1}) - 1 + x^2 - x^2q], \\
dd^{(0)}(x,q) &= x(\underline{dd}_R(x,q) + \underline{dd}_B(x,q)) \\
&= \frac{1}{2 - x^2 + x^2q}[\mathit{Sch}\,\mathrm{D}(x^2q, q^{-1}) - 1 + x^2 - x^2q],
\end{aligned}$$

with the same notational conventions as in the proof of Theorem 3.2. □

Remark 3.5 The complete solution to the system of eight equations appearing in the proof of Theorem 3.4 exhibits the following relations among the different subclasses of alternating permuations of genus zero:

(i)

$$\begin{aligned}
&\frac{x}{2 - x^2 + qx^2}[\mathit{Sch}\,\mathrm{D}(x^2q, q^{-1}) - 1 + x^2 - qx^2] \\
&\quad = \frac{1}{x}\left(uu_L^{(0)}(x,q) - x^2\right) = ud_L^{(0)}(x,q) = \frac{1}{x^2}\left(du_R^{(0)}(x,q) - x^3\right) = du^{(0)}(x,q) \\
&\quad = \frac{1}{x}\left(dd_R^{(0)}(x,q) - x^2\right) = x\, dd^{(0)}(x,q) \\
&\quad = x^3 + (1 + q + q^2)x^5 + (1 + 3q + 5q^2 + 2q^3)x^7 \\
&\qquad + (1 + 6q + 16q^2 + 16q^3 + 6q^4)x^9 + \cdots,
\end{aligned}$$

(ii)

$$\begin{aligned}
&\frac{x}{2 - x^2 + qx^2}[(1 - x^2 + qx^2)\mathit{Sch}\,\mathrm{D}(x^2q, q^{-1}) - 1] \\
&\quad = \frac{1}{x}uu_R^{(0)}(x,q) = ud_R^{(0)}(x,q) \\
&\quad = qx^3 + (2q + q^2)x^5 + (3q + 5q^2 + 3q^3)x^7 \\
&\qquad + (4q + 14q^2 + 19q^3 + 8q^4)x^9 + \cdots,
\end{aligned}$$

(iii)

$$\begin{aligned}
&x[\mathit{Sch}\,\mathrm{D}(x^2q, q^{-1}) - 1] \\
&\quad = \frac{1}{x}\left(uu^{(0)}(x,q) - 1 - x - x^2\right) = ud^{(0)}(x,q) \\
&\quad = (1 + q)x^3 + (1 + q)(1 + 2q)x^5 + (1 + q)(1 + 5q + 5q^2)x^7 \\
&\qquad + (1 + q)(1 + 9q + 21q^2 + 14q^3)x^9 + \cdots,
\end{aligned}$$

(iv)

$$\begin{aligned}
&\frac{x}{2-x^2+qx^2}[(1-x^2)Sch\,\mathrm{D}(x^2q,q^{-1})-(1+qx^2)] \\
&\quad = du_B^{(0)}(x,q) = x\ dd_B^{(0)}(x,q) \\
&\quad = q(1+q)x^5 + 2q(1+q)^2x^7 + q(1+q)(3+8q+6q^2)x^9 \\
&\qquad + 2q(1+q)^2(2+8q+9q^2)x^{11} \\
&\qquad + q(1+q)(5+40q+115q^2+136q^3+57q^4)x^{13} \\
&\qquad + 2q(1+q)^2(3+32q+118q^2+176q^3+93q^4)x^{15} + \cdots.
\end{aligned}$$

Some of the relations are transparent, e.g., since $\alpha \in UD_L^{(0)}$ is equivalent to $\alpha(1) = 1$ and $\alpha' \in DD^{(0)}$, where $\alpha'(i) := \alpha(i+1) - 1$, we have $ud_L^{(0)}(x,1) = x\ dd^{(0)}(x,1)$; and since the valley $\alpha(1)$ does not give an alternating descent, we have $ud_L^{(0)}(x,q) = x\ dd^{(0)}(x,q)$. Other equalities (e.g., $du_B^{(0)}(x,q) = x\ dd_B^{(0)}(x,q)$ and $du_R^{(0)}(x,q) = x\ dd_R^{(0)}(x,q)$) follow alternatively from the grammar rules and induction.

Still other of these facts can be deduced from the encoding of the permutations, e.g., the divisibility of $ud_R^{(0)}(x,q)$ and $uu_R^{(0)}(x,q)$ by q. Indeed, let $\alpha \in UD_R^{(0)}$. If $w_1 = w_2 = R$, then the valley $\alpha(1)$ gives an alternating descent (hence a factor of q), and if $w_1w_2 = RB$ then it does not. In the latter case, we must have $w_3 \in \{L, R\}$ (in which case the peak $\alpha(2)$ gives an alternating descent), or $w_3w_4 = ER$ (in which case the valley $\alpha(3)$ gives an alternating descent), or yet $w_3w_4 = EB$. The last case leads to $w = RBEBEBE\ldots$, and since α ends with a descent, the repetition of BE terminates with one of the preceding cases which gives an alternating descent.

The factors of $(1+q)$ and $(1+q)^2$ occurring in (iii) and (iv) are not obvious from the grammar rules, and it is a calculus exercise to verify them from the formulae for the generating functions. It would be interesting to explain their presence combinatorially. We also remark that additional observations can be proved through a combination of Lemmas 3.1, 3.3 and induction, e.g.:

$$\text{If } \alpha \in U^{(0)}(2n) \text{ then } \alpha(2n) = 2n, \tag{7}$$

$$\text{If } \alpha \in U^{(0)}(2n+1) \text{ then } \alpha(2n) \in \{2n, 2n+1\}, \tag{8}$$

$$u_{2n}^{(0)} = u_{2n-1}^{(0)} = dconn_{2n}^{(0)} = uconn_{2n+1}^{(0)} = Sch_{n-1}, \tag{9}$$

where $dconn_N^{(0)} = |\{\alpha \in D^{(0)} \cap S_N : 1 \text{ and } N \text{ in the same cycle}\}|$ (thus, $dconn_N^{(0)} = 0$ if N is odd), and $uconn_N^{(0)} = |\{\alpha \in U^{(0)} \cap S_N : 1 \text{ and } N \text{ in the same cycle}\}|$ (thus, $uconn_N^{(0)} = 0$ if N is even), count "connected" alternating permutations of genus zero.

4. Alternating Baxter permutations

We begin this section with the relation between alternating permutations of genus zero and Baxter permutations, which prompted our investigation of the distribution of $\widehat{\mathrm{des}}$ on the class of alternating Baxter permutations.

Proposition 4.1 *If $\alpha \in S_N$ is an alternating permutation of genus zero, then α is a Baxter permutation.*

Proof: Since the Baxter condition is always valid for $i = 1, N-1$, assume $2 \leq i \leq N-2$. By Lemma 2.1, the cycle decomposition of α gives a noncrossing partition and we let $w = w_1 w_2 \ldots w_{N-1} \in \{B, E, R, L\}^*$ be the encoding of α as in Section 3.

Suppose $w_i = B$. Since α is an alternating permutation, Lemma 3.1 implies that $i \in \mathrm{Des}(\alpha)$ and that $w_{i-1} \in \{R, E, L\}$. First suppose $w_{i-1} \in \{R, E\}$. In this case, $\alpha^{-1}(i) < i$ and $\alpha(j) < i$ for all $\alpha^{-1}(i) < j < i$. We also have $\alpha^{-1}(i+1) \geq i+1$, with equality holding if $w_{i+1} \in \{L, E\}$, and the noncrossing condition on cycles implies that $\alpha(k) > i+1$ for all $i+1 < k < \alpha^{-1}(i+1)$, if this interval is nonempty. The Baxter condition also holds for i when $w_{i-1} = L$. This time the noncrossing property of the cycles implies that $\alpha^{-1}(i) > \alpha^{-1}(i+1) \geq i+1$ and that $\alpha(k) > i$ for all $\alpha^{-1}(i+1) < k < \alpha^{-1}(i)$ if this interval is nonempty.

The remaining cases, $w_i = E, R, L$, have similar proofs which we omit. □

It is easy to see that neither of the possible converse statements to Proposition 4.1 is true: the identity permutation is Baxter of genus zero but not alternating; $\alpha = 5\ 6\ 1\ 3\ 2\ 4$ is alternating Baxter but of genus 2.

Our next result extends the combinatorial interpretation of squares and products of two consecutive Catalan numbers from [9] to a q-analogue.

Theorem 4.2 *Consider the q-analogue of the Catalan numbers*

$$C_N(q) := \sum_{\substack{\alpha \in S_N \\ g(\alpha)=0}} q^{z(\alpha)-1} = \sum_{k=1}^{N} \frac{1}{N} \binom{N}{k} \binom{N}{k-1} q^{k-1} \tag{10}$$

and the distribution of the alternating descents statistic on alternating Baxter permutations,

$$u_N^{(B)}(q) := \sum_{\alpha \in U_N^{(B)}} q^{\widehat{des}(\alpha)},$$

$$d_N^{(B)}(q) := \sum_{\alpha \in D_N^{(B)}} q^{\widehat{des}(\alpha)}.$$

Then, for every $n \geq 1$,

$$u_{2n+1}^{(B)}(q) = d_{2n+1}^{(B)}(q) = C_n(q) C_{n+1}(q), \tag{11}$$

$$u_{2n}^{(B)}(q) = d_{2n}^{(B)}(q) = [C_n(q)]^2. \tag{12}$$

Proof: First note that it suffices to prove that $u_{2n+1}^{(B)} = C_n(q) C_{n+1}(q)$ and $u_{2n}^{(B)} = [C_n(q)]^2$. Indeed, it is easy to check that the mapping $S_N \to S_N$ sending a permutation α to β defined by $\beta(i) = N+1-\alpha(i)$, respects the Baxter condition, restricts to a bijection between $D_N^{(B)}$

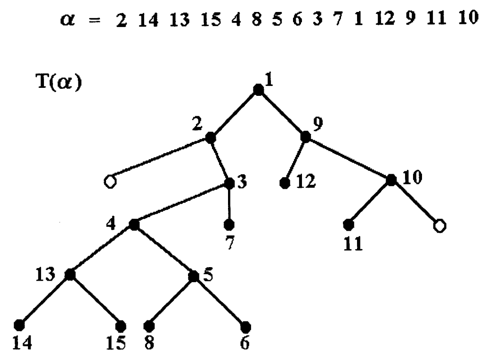

Figure 3. Example of an alternating Baxter permutation α and its associated Baxter tree $T(\alpha)$.

and $U_N^{(B)}$, and complements the value of $\widehat{\text{des}}$. Since the coefficients of the right-hand sides of (11) and (12) are symmetric sequences, we have $u_N^{(B)}(q) = d_N^{(B)}(q)$.

Following [9], there is a bijection between $U_{2n+1}^{(B)}$ and *Baxter trees* with $2n+3$ vertices. These are complete plane binary trees, rooted, and increasingly labeled, generated through an insertion process of the vertices which ensures the Baxter condition for the permutation $\alpha \in S_{2n+1}$ arising from the in-order traversal of the tree (the leftmost and rightmost leaves bear special symbols that are not part of the permutation (figure 3)).

In turn, the Baxter trees are in bijective correspondence with pairs (T', T'') of plane rooted binary trees, having $n+1$ and n vertices, respectively. In this correspondence (see [9]), T' is the plane rooted binary tree formed by the internal vertices of the original Baxter tree and T'' is the plane rooted binary tree obtained after removing the labels of the decreasingly labeled plane rooted binary tree whose in-order traversal gives the subword of α formed by the peaks. It is rather remarkable that the original Baxter tree $T(\alpha)$ can be reconstructed from the *unlabeled* trees (T', T''). In figure 4, the forced labels for T' and T'' are indicated in parenthesis.

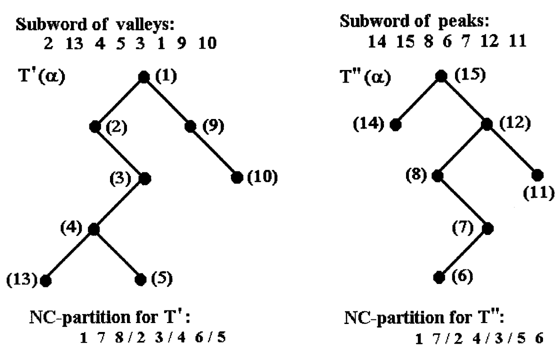

Figure 4. Example of the pair of trees (T', T'') for an alternating Baxter permutation α, and the corresponding noncrossing partitions.

In [9], the insertion process for Baxter trees is encoded by a shuffle of two Dyck words, with the end result that both sides of (2) are interpreted as the count of alternating Baxter permutations of $[2n+1]$. Similarly, alternating Baxter permutations of $[2n]$ are shown to be in bijective correspondence with pairs (T', T'') of plane rooted binary trees, each having n vertices. Thus, for $q = 1$, (11) and (12) are proved in [9].

These facts from [9] and a few simple observations will yield our q-analogues of Theorem 4.2. Clearly, the permutation α obtained from a Baxter tree is alternating because the tree is complete binary. Also, the internal vertices of $T(\alpha)$ give the valleys of α, and left edges in T' correspond to alternating descents among valleys. More precisely, if a is a vertex in T' which has a left child, then a is *not* the first valley, and an alternating descent occurs at the valley preceding it.

On the other hand, a plane rooted binary tree on $n+1$ (unabeled) vertices encodes a noncrossing partition of $[n+1]$, by the following recursive procedure. The root (which corresponds to the minimum element under consideration) and its successive right descendants constitute the block B_1 of the partition. If v is one of these vertices and has a left child, then the noncrossing partition corresponding to the left subtree at v is inserted ("nested") between v and its successor in B_1. (Thus, the tree consisting of a root and two children gives the partition 1 3 / 2.) By construction, the partition is noncrossing and, if the number of vertices that have a left child is s, then the number of blocks of the resulting partition is $s+1$.

Consequently, $C_{n+1}(q)$, which enumerates noncrossing partitions of $[n+1]$ according to the number of blocks diminished by one unit also gives the distribution of descents on the valleys of up-down Baxter permutations of $[2n+1]$.

In T'', right edges correspond with descents between peaks of α, and the correspondence with noncrossing partitions follows similarly (interchanging right and left). Thus, $C_n(q)$ gives the distribution of descents on the peaks of up-down Baxter permutations of $[2n+1]$.

Based on the validity of the bijective correspondence $\alpha \leftrightarrow (T', T'')$ in [9], the proof of (11) is completed. With suitable but minor modifications as in [9], one obtains (12). □

Remark 4.3 In Niven's notation [19], alternating permutations are permutations with monotonicity pattern $+-+\cdots$ or $-+-\cdots$ ("+" for ascents and "−" for descents). Niven proved that the up-down and down-up patterns are the most popular monotonicity patterns over the entire symmetric group. What is the most popular monotonicity pattern over the class of Baxter permutations?

Numerical evidence suggests that the two alternating patterns may still be the answer, but, if this is true in general, it is due to reasons more subtle than in the case of the entire symmetric group. For the symmetric group (see [19]), increasing the number of "sign changes" in a monotonicity pattern leads to an increase in the number of permutations realizing the pattern. However, for Baxter permutations it turns out for instance that the pattern $+++---+$ (with 2 sign changes) is represented by 30 permutations, while the pattern $-+--+++$ (with 3 sign changes) is represented only by 28 permutations.

5. Lattice paths and the associahedron

Just as the q-analogue (3) of the Catalan numbers gives the rank generating function of the noncrossing partition lattice, a q-analogue of the Schröder numbers gives its characteristic polynomial.

Theorem 5.1 *The q-analogue*

$$Sch\ D(n;q) := \sum_{p\,\in\, Sch(n)} q^{Diag(p)} \tag{13}$$

of the Schröder number obtained from counting Schröder paths according to their number of diagonal steps and the characteristic polynomial of the noncrossing partition lattice

$$\chi_{NC(n)}(q) := \sum_{\pi\,\in\, NC(n)} \mu(\hat{0},\pi)q^{n-1-rank(\pi)} \tag{14}$$

are related by

$$(-1)^n Sch\ D(n;q) = \chi_{NC(n+1)}(-q). \tag{15}$$

Proof: It is known that the noncrossing partition lattice is an EL-shellable poset. It admits, for example, the following EL-labeling constructed by Gessel for the lattice of unrestricted set partitions, and which Björner [3] observed works as well for NC: the covering from π to ρ is labeled by the larger of the minima of the two blocks of π which are merged in order to obtain ρ. Therefore, from the general theory of EL-shellable posets (see, e.g., [4]) it follows that if $\pi \in \mathrm{NC}(n+1)$, then $(-1)^{\mathrm{rank}(\pi)}\mu(\hat{0},\pi)$ is equal to the number of maximal $\hat{0}$-π chains in $\mathrm{NC}(n+1)$ whose sequence of labels (starting at $\hat{0}$) is decreasing. In turn, the interval $[\hat{0},\pi]$ is isomorphic to the product of the noncrossing partition lattices $\mathrm{NC}(n_i)$, where n_i are the cardinalities of the blocks of π.

By [13] (Theorem 2.2), there is a bijection between the maximal chains of $\mathrm{NC}(m)$ with prescribed label sequence λ (a permutation of $2, 3, \ldots, m$) and the noncrossing partitions of $\{2, 3, \ldots, m\}$ each of whose blocks constitutes a decreasing subsequence of λ. (For example, in NC(5) there are 7 maximal chains with label sequence $\lambda = 3\,5\,4\,2$, corresponding to the following noncrossing partitions: 2 / 3 / 4 / 5, 2 3 / 4 / 5, 2 4 / 3 / 5, 2 5 / 3 / 4, 2 / 3 / 4 5, 2 4 5 / 3, 2 3 / 4 5.)

In particular, there is a bijection between the decreasingly labeled chains in $\mathrm{NC}(n_j)$ and the noncrossing partitions in $\mathrm{NC}(n_j - 1)$. It is easy to see that this bijection can be extended to a bijection between the decreasingly labeled chains in $[\hat{0},\pi]$ and the product $\prod_j \mathrm{NC}(n_j - 1)$.

Finally, there is a natural bijection between $\mathrm{NC}(m)$ and Schröder paths from (0, 0) to (m, m) with no diagonal steps (we may call such paths "Catalan paths"). Namely, in a Catalan path, view East steps as left parentheses and North steps as right parentheses; number the East steps $1, 2, \ldots, m$ in order, starting from the origin; number each North step with the number of its matching East step; the Catalan path then corresponds biuniquely with the noncrossing partition whose blocks consist of the numbers assigned to contiguous North steps. (For instance, the path EEENENNENN corresponds to the partition 1 5 / 2 4 / 3 $\in$ NC(5).)

Using these facts, we will construct a bijection between the Schröder paths from (0, 0) to $(n+1, n+1)$ beginning with a diagonal step and having k diagonal steps and the decreasingly labeled chains in $\bigcup_{\pi \in \mathrm{NC}(n+1,k)} [\hat{0}, \pi]$. A simple translation of the paths (moving (1, 1) to the origin and deleting the initial diagonal step) yields then a bijection between the Schröder paths from (0, 0) to (n, n) and having $k-1$ diagonal steps and the decreasingly labeled chains from $\hat{0}$ to partitions in $\mathrm{NC}(n+1, k)$. Finally, (15) follows from a simple calculation:

$$\begin{aligned}\chi_{\mathrm{NC}(n+1)}(-q) &= (-1)^n \sum_{\pi \in \mathrm{NC}(n+1)} |\mu(\hat{0}, \pi)| q^{bk(\pi)-1} \\ &= (-1)^n \sum_{k=1}^{n+1} q^{k-1} \sum_{\pi \in \mathrm{NC}(n+1,k)} \#\{\text{decreasingly labeled } \hat{0}\text{-}\pi \text{ chains}\} \\ &= (-1)^n Sch\, \mathrm{D}(n; q).\end{aligned}$$

It remains to exhibit a bijection between the Schröder paths from (0, 0) to $(n+1, n+1)$ beginning with a diagonal step and having k diagonal steps total and the decreasingly labeled maximal chains in $\bigcup_{\pi \in \mathrm{NC}(n+1,k)} [\hat{0}, \pi]$. Let p be such a Schröder path. We begin by constructing a partition $\pi = B_1/B_2 \ldots /B_k \in \mathrm{NC}(n+1, k)$. Factor the Schröder path as $p = p'(\mathrm{D}c_k)p''$, where D is the last diagonal step of p and c_k is the longest Catalan path which follows after D (having as its 45-degree barrier the line containing the step D). For example, the path p of figure 5, gives c_5 = ENEENN, formed by the six steps following the diagonal step marked m_5. Let $n_k := 1 + |c_k|/2$, where $|c_k|$ is the number of steps of c_k (necessarily even), be the cardinality of the kth block of π. Replacing the path p with $p'p''$ and repeating the factorization process, we determine the cardinalities of all blocks of π. To

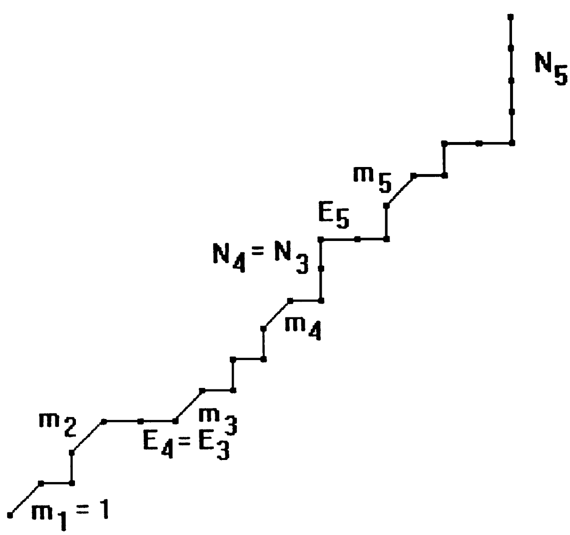

Figure 5. Illustration to the proof of Theorem 5.1.

determine π itself, we apply recursively the following method of establishing the elements of B_k. If $p'' = \emptyset$, then let $B_k = \{n+2-n_k, n+1-n_k, \ldots, n+1\}$. If $p'' \neq \emptyset$, then p'' begins with a North step which matches a unique East step of p' (these are denoted N_k and E_k in figure 5, where they occur for $k = 3, 4, 5$). This East step eventually constitutes, say, the hth horizontal step of a Catalan path c_j arising from the factorization process. Then B_k is "nested" between the $(h-1)$st and the hth element of B_j. For example, the blocks B_3 and B_4 resulting from the Schröder path of figure 5 will lie between the second and third elements of B_2. Due to the noncrossing condition and the indexing of the blocks in increasing order of their minima, we obtain a well-defined noncrossing partition π. The path in figure 5, gives $\pi = 1\ 2\ /\ 3\ 4\ 10\ 15\ 16\ /\ 5\ 6\ 7\ /\ 8\ 9\ /\ 11\ 12\ 13\ 14$.

Having obtained the partition $\pi \in \mathrm{NC}(n+1, k)$ in this manner, each Catalan path c_j corresponds to a unique noncrossing partition of $[n_j - 1]$, which we realize with the elements of $B_j^- := B_j - \{m_j\}$, where m_j is the minimum of B_j. For example, the Catalan path $c_2 = \mathrm{EENEENNN}$ from figure 5 gives the noncrossing partition 4 15 16 / 10 of B_2^-, and $m_2 = 3$. It is easy to verify that the path p can be reconstructed from π and the partitions of B_j^- for $j = 1, \ldots, k$.

In turn, by [13], each of these k noncrossing partitions corresponds bijectively with a decreasingly labeled $\hat{0}$-B_j chain for the appropriate j, with label set B_j^-. For $j = 2$ in our example, this chain is: $\hat{0}$, 3 16 / 4 / 10 / 15, 3 15 16 / 4 / 10, 3 15 16 / 4 10, B_2.

The proof is completed by simply interleaving the k chains in the unique way that gives a decreasingly labeled $\hat{0}$-π chain. □

By combining Theorem 5.1 and results from [5] we recover a reciprocity relation between the rank generating function and characteristic polynomial of noncrossing partition lattices ([14], Lemma 4.5).

Corollary 5.2 *If $\chi_{NC(m)}(q)$ and $R_{NC(m)}(q)$ denote the characteristic polynomial and the rank generating function, respectively, for the noncrossing partition lattice $NC(m)$, then*

$$\chi_{NC(n+1)}(q) = (-1)^n (1-q) R_{NC(n)}(1-q), \tag{16}$$

for all $n \geq 1$.

Proof: For $n \geq 1$, consider the boundary complex Δ_n of the associahedron Q_n and the simplicial complex $\Delta_n * p$, the join of Δ_n with a single vertex simplicial complex. Obviously, $f_{-1}(\Delta_n * p) = 1$ and $f_{i-1}(\Delta_n * p) = f_{i-2}(\Delta_n) + f_{i-1}(\Delta_n)$ for $i \geq 1$. The two terms in this relation can be interpreted in terms of Schröder paths counted by their number of diagonal steps, using Proposition 2.7 of [5]. Namely, the $(i-1)$-dimensional faces of $\Delta_n * p$ which contain p correspond bijectively with the $(i-2)$-dimensional faces of Δ_n, and these are equinumerous with the Schröder paths ending at (n, n) which have $n - i$ diagonal steps and the first non-East step is a North step. On the other hand, the $(i-1)$-dimensional faces of $\Delta_n * p$ which do not contain p correspond bijectively with the $(i-1)$-dimensional faces of Δ_n itself and are equinumerous with the Schröder paths ending at (n, n) which have $n - 1 - i$ diagonal steps and the first non-East step is a North step; in turn, such paths correspond bijectively with the Schröder paths ending at (n, n)

which have $n - i$ diagonal steps and in which the first step or else the first non-East step is diagonal (simply replace the first EN corner with a diagonal step). Consequently,

$$\sum_{i=0}^{n} f_{i-1}(\Delta_n * p)q^{n-i} = Sch\,\mathrm{D}(n; q). \tag{17}$$

Also, the h-vector of $\Delta_n * p$ is that of Δ_n extended by $h_n(\Delta_n * p) = 0$, so,

$$\sum_{i=0}^{n} h_i(\Delta_n * p)q^{n-i} = qC_n(q). \tag{18}$$

Therefore, by the general relation (5) between the f- and h-vector of a simplicial complex we have

$$Sch\,\mathrm{D}(n; q-1) = qC_n(q), \tag{19}$$

for all $n \geq 1$. Combining this with Theorem 5.1, we obtain

$$\begin{aligned}\chi_{\mathrm{NC}(n+1)}(q) &= (-1)^n Sch\,\mathrm{D}(n; -q) = (-1)^n(1-q)C_n(1-q)\\ &= (-1)^n(1-q)\mathrm{R}_{\mathrm{NC}(n)}(1-q).\end{aligned}$$

□

6. Alternating Baxter permutations and polytopes

Let $a_{N,i}^{(B)}$ denote the number of alternating Baxter permutations of $[N]$ with i alternating descents. Note that, by (11) and (12), $a_{N,i}^{(B)}$ is well-defined without specifying whether the permutations begin with an ascent or a descent. Also, (11) and (12) show that the sequence $(a_{N,i}^{(B)})_{i\geq 0}$ is symmetric and unimodal since its generating polynomial is the product of two symmetric and unimodal polynomials, see [1]. These are necessary (but not sufficient) conditions for $(a_{N,i}^{(B)})_{i\geq 0}$ to be the h-vector of a simplicial convex polytope.

Theorem 6.1 *For every $N \geq 1$, there exists an $(N-2)$-dimensional simplicial polytope $Q_N^{(B)}$ whose h-vector is given by the number of alternating Baxter permutations counted according to the $\widehat{\mathrm{des}}$ statistic, that is, for $0 \leq i \leq N-2$,*

$$h_i\big(Q_N^{(B)}\big) = a_{N,i}^{(B)}. \tag{20}$$

Proof: Using (11) and (12), a polytope $Q_N^{(B)}$ as claimed exists if and only if the candidate f-vector given by

$$\begin{aligned}\sum_{i=0}^{N-2} f_{i-1}\big(Q_N^{(B)}\big)q^{N-2-i} &= C_{\lceil\frac{N-1}{2}\rceil}(1+q)C_{\lfloor\frac{N+1}{2}\rfloor}(1+q)\\ &= \frac{1}{(1+q)^2} Sch\,\mathrm{D}\left(\left\lceil\frac{N-1}{2}\right\rceil; q\right) Sch\,\mathrm{D}\left(\left\lfloor\frac{N+1}{2}\right\rfloor; q\right)\end{aligned}$$

is indeed realizable by a simplicial polytope.

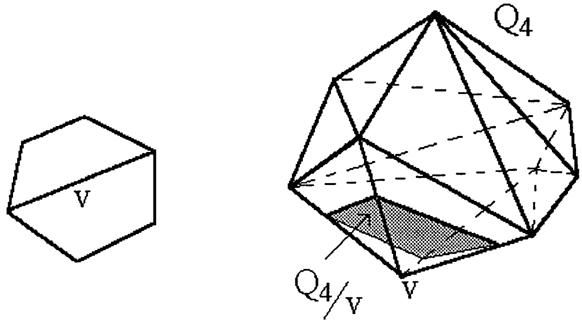

Figure 6. Illustration to Theorem 6.1.

By [5], as discussed in Section 5, the f-vector of the associahedron Q_m is given by

$$\sum_{i=0}^{m-1} f_{i-1}(Q_m)q^{m-1-i} = \frac{1}{1+q} Sch\, \mathrm{D}(m; q). \tag{21}$$

Hence, the desired f-vector for $Q_N^{(B)}$ agrees with that of the simplicial complex $\Delta_N^{(B)} := \Delta_{\lceil \frac{N-1}{2} \rceil} * \Delta_{\lfloor \frac{N+1}{2} \rfloor}$, the join of the boundary complexes of the associahedra $Q_{\lceil \frac{N-1}{2} \rceil}$ and $Q_{\lfloor \frac{N+1}{2} \rfloor}$. We claim that $\Delta_N^{(B)}$ is indeed polytopal. Let v be any of the vertices of Q_N which represents a diagonal dissecting a convex $(N+2)$-gon into a $(\lceil \frac{N-1}{2} \rceil + 2)$-gon P' and a $(\lfloor \frac{N+1}{2} \rfloor + 2)$-gon P''. By the general theory of polytopes (see, e.g., [26]) a vertex figure of Q_N at the vertex v is a polytope whose face lattice is isomorphic to the interval $[v, Q_N]$ of the face lattice of Q_N. More explicitly, if $\mathcal{H}$ is a hyperplane which intersects Q_N and separates v from all other vertices of Q_N, then $Q_N \cap \mathcal{H}$ is an $(N-2)$-dimensional polytope whose j-dimensional faces are in bijective and inclusion-preserving correspondence with the $(j+1)$-dimensional faces of Q_N which contain v. In turn, the latter are precisely the faces with vertex set $\{v\} \cup V' \cup V''$, where V' and V'' are independent dissections of the two polygons P' and P'', respectively. Thus, $\Delta_N^{(B)}$ is polytopal, being relizable as the boundary complex of a vertex figure Q_N/v of the associahedron. □

Figure 6 illustrates the construction of $Q_4^{(B)}$. As another example, $Q_5^{(B)}$ is a convex bipyramid over a pentagon.

Corollary 6.2 *Let $i_1, i_2, \ldots, i_k$ be integers, $i_j \geq 1$ for each $j = 1, \ldots, k$, and let $i_1 + i_2 + \cdots + i_k = N$. For $i \geq 0$, define h_i and f_{i-1} by*

$$\sum_{i=0}^{N-k} h_i q^{N-k-i} := C_{i_1}(q)C_{i_2}(q)\cdots C_{i_k}(q) \tag{22}$$

and

$$\sum_{i=0}^{N-k} f_{i-1} q^{N-k-i} := \frac{1}{(1+q)^k} Sch\, D(i_1; q) Sch\, D(i_2; q) \cdots Sch\, D(i_k; q). \tag{23}$$

Then there exists an $(N-k)$*-dimensional simplicial polytope with* h*-vector* $(h_0, h_1, \ldots, h_{N-k})$ *and* f*-vector* $(f_{-1}, f_0, \ldots, f_{N-k-1})$.

Proof: By the previous discussion, the associahedron satisfies the conclusion when $k = 1$. Let $k \geq 2$ and consider a convex $(N+2)$-gon. Choose $k-1$ noncrossing diagonals which dissect it into k polygons having $i_1+2, i_2+2, \ldots, i_k+2$ vertices, respectively. Denote by $F = F_{i_1,i_2,\ldots,i_k}$ the face of Q_N representing this set of noncrossing diagonals. Then (see [26]), the interval $[F, Q_N]$ in the face lattice of Q_N is isomorphic to the face lattice of a face-figure, Q_N/F, of the polytope Q_N with respect to F. Hence, the j-dimensional faces of Q_N/F are in inclusion-preserving bijection with the $(j+k-1)$-dimensional faces of Q_N which contain F. The desired f- and h-vector for the polytope Q_N/F follow from arguments similar to those in the proof of Theorem 6.1. □

Remark 6.3 In the cases $k = 1, 2$ of Corollary 6.2, the polytopes have h-vectors which enumerate some class of permutations according to a combinatorial statistic: genus zero permutations according to the number of cycles diminished by one unit, and alternating Baxter permutations according to the number of alternating descents, respectively. We are not aware of such interpretations for $k \geq 3$.

Acknowledgments

The second author expresses her appreciation for the hospitality extended to her at LaBRI during her visit, when part of this work was done. She is partially supported by NSF award DMS-9108749 and LaBRI. The authors wish to thank Jean-Guy Penaud and Frank Schmidt for discussions and their interest in this work.

References

1. G. Andrews, "A theorem on reciprocal polynomials with applications to permutations and compositions," *Amer. Math. Monthly* **82** (1975), 830–833.
2. G. Baxter, "On fixed points of the composite of commuting functions," *Proc. Amer. Math. Soc.* **15** (1967), 851–855.
3. A. Björner, "Shellable and Cohen-Macaulay partially ordered sets," *Trans. Amer. Math. Soc.* **260** (1980), 159–183.
4. A. Björner, A. Garsia, and R. Stanley, "An introduction to Cohen-Macaulay partially ordered sets," in *Ordered Sets* (NATO Adv. Study Inst. Ser. C: Math. Phys. Sci.), Ivan Rival (Ed.), Reidel, Dordrecht/Boston, 1982.
5. J. Bonin, L. Shapiro, and R. Simion, "Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths," *J. Stat. Planning and Inference* **34** (1993), 35–55.
6. F. Chung, R. Graham, V. Hoggatt, and M. Kleiman, "The number of Baxter permutations," *J. Combin. Theory Series A* **24** (1978), 382–394.
7. L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.
8. R. Cori, "Un code pour les graphes planaires et ses applications," *Asterisque* **27**, Société Mathématique de France, 1975.
9. R. Cori, S. Dulucq, and X. Viennot, "Shuffle of parenthesis systems and Baxter permutations," *J. Combin. Theory Series A* **43** (1986), 1–22.
10. R. Cori and A. Machi, "Maps, hypermaps and their automorphisms: A survey, I, II, III," *Exposition. Math.* **10** (1992), 403–427, 429–447, 449–467.

11. M. Delest, “Algebraic languages: A bridge between combinatorics and computer science,” in *Algebraic Combinatorics—Formal Power Series and Algebraic Combinatorics/Séries Formelles et Combinatoire Algébrique 1994,* L. Billera, C. Greene, R. Simion, and R. Stanley (Eds.), AMS/DIMACS, **24** (1996), 71–87.
12. P. Edelman, “Multichains, noncrossing partitions and trees,” *Discr. Math.* **40** (1982), 171–179.
13. P. Edelman and R. Simion, “Chains in the lattice of non-crossing partitions,” *Discrete Math.* **126** (1994), 107–119.
14. D. Jackson, “The lattice of noncrossing partitions and the Birkhoff-Lewis equations,” Waterloo Research Report #89-34, 1989, and *European J. Combin.* **15** (1994), 245–250.
15. G. Kreweras, “Sur les partitions noncroisées d’un cycle,” *Discrete Math.* **1** (1972), 333–350.
16. C. Lee, “The associahedron and triangulations of the n-gon,” *European J. Combin.* **10** (1989), 551–560.
17. C.L. Mallows, “Baxter permutations rise again,” *J. Combin. Theory Series A* **27** (1979), 394–396.
18. R. Mullin, “The enumeration of Hamiltonian polygons,” *Pacific J. Math.* **16** (1966), 139–145.
19. I. Niven, “A combinatorial problem of finite sequences,” *Nieuw Arch. Wisk.* **16** (1968), 116–123.
20. D.G. Rogers and L. Shapiro, “Dequeues, trees, and lattice paths,” in *Lecture Notes in Mathematics,* K.L. McAvaney (Ed.), Vol. 884, pp. 293–303, Springer-Verlag, New York, 1981.
21. R. Simion and D. Ullman, “On the structure of the lattice of noncrossing partitions,” *Discrete Math.* **98** (1991), 193–206.
22. R. Stanley, *Enumerative Combinatorics,* Vol. 1, Wadsworth & Brooks/Cole, Monterey, California, 1986.
23. W.T. Tutte, “A census of Hamiltonian polygons,” *Canad. J. Math.* **14** (1962), 402–417.
24. T.R.S. Walsh and A.B. Lehman, “Counting rooted maps by genus, I, II, III,” *J. Combin. Theory* **13** (1972), 122–141, 192–218, and **18** (1975), 259.
25. J.W.T. Youngs, “Minimal imbeddings and the genus of a graph,” *J. Math. Mech.* **12** (1963), 303–315.
26. G. Ziegler, “Lectures on polytopes,” *Graduate Texts in Mathematics # 152,* Springer-Verlag, New York, 1995.