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Determination of $msd(L^n)$*

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Abstract. The median stabilization degree (*msd*, for short) of a median algebra measures the largest possible number of steps needed to generate a subalgebra with an arbitrary set of generators. With computer assistance, we found that *msd* of the lattice $\{-1, 0, 1\}^4$ equals 2. This value is of critical importance to determine *msd* of $\{-1, 0, 1\}^n$ for all $n \geq 5$ and to determine *msd* of the free median algebra $\lambda(r)$ for almost all $r \geq 5$.

Keywords: distributive lattice, free median algebra, graphic cube, median operator, median stabilization degree

1. Introduction

In [2], H.-J Bandelt and the author introduced and studied the median stabilization degree (msd) of a median algebra. One of the main open problems is to determine the precise value of *msd* of the lattices $L^n = \{-1, 0, 1\}^n$, $n \geq 4$. These examples are fundamental, since they also yield *msd* of the real space of n dimensions, $\mathbb{R}^n$. All finite median algebras can be embedded in a space of the latter type.

Let us briefly recall the main concepts. By a *median algebra* is meant a set M with an operator $m : M^3 \to M$ satisfying the following three axioms.

(M1) Idempotence: $m(a, a, b) = b$.
(M2) Symmetry: $m(a, b, c) = m(d, e, f)$ for each permutation (d, e, f) of (a, b, c).
(M3) Associativity: $m(a, m(b, c, d), c) = m(m(a, b, c), d, c)$.

A ternary function with these properties is called a *median operator.* It is easy to see that for each $c \in M$, the binary operator $\wedge_c$, defined by $a \wedge_c b = m(a, b, c)$, is a meet operator and c is the least element of the corresponding semilattice.

If ρ is a metric on a set X, such that for each triple $a, b, c \in X$ there is a unique point $m(a, b, c) \in X$ which is geodesically between each two of a, b, c, then the resulting operator m is a median operator. An important example of such a *median metric* is the "sum metric" on the n-dimensional real space, $\mathbb{R}^n$:

$$\rho(x, y) = \sum_{i=1}^{n} |x_i - y_i| \quad \text{(where } x = (x_1, \ldots, x_n);\ y = (y_1, \ldots, y_n)).$$

*This paper is dedicated to the memory of my son Wouter, 1974–1993.

A distributive lattice is a median algebra under the median operator

$$m(a, b, c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a).$$

In case of a totally ordered set, this amounts to taking the middle one of three points. In case of a product of totally ordered sets, this amounts to taking the coordinate-wise median. On $\mathbb{R}^n$, we are lead back to the median, induced by the sum metric.

A subset X of a median algebra M is *median stable* provided $m(X^3) \subseteq X$. Any median algebra can be embedded as a median stable subset of a distributive lattice. If S is a subset of a median algebra M, then its *median stabilization, med*(S), is the smallest median stable set including S. It can be obtained as $\bigcup_{n=0}^{\infty} S_n$, where $S_0 = S$ and (recursively) $S_{n+1} = m(S_n^3)$. The set S_n represents the nth stage of the stabilization process. The *median stabilization degree of* M, *msd*(M), is defined by the following inequalities.

$$msd(M) \leq n \text{ iff } med(S) = S_n \text{ for all subsets } S \text{ of } M \ (n \in \mathbb{N}).$$

Median stabilization is required in several results: cf., e.g., Chepoi [3] (optimal facility location) and van de Vel and Verheul [5] or van de Vel [4, p. 130] (Steiner trees). The amount of computations involved in stabilizing a set is quite sensitive to *msd*.

There is an important class of so-called *free* median algebras, $\lambda(r)$ ($r \in \mathbb{N}$). The algebra $\lambda(r)$ is the median stabilization of an r-point set S, and is free in the sense that any function of S into a median algebra M can be extended (uniquely) to a homomorphism $\lambda(r) \to M$. See [4] for an explicit construction of $\lambda(r)$. Almost by definition, if the r generators of $\lambda(r)$ stabilize in k steps, then *any* r-point set in *any* median algebra stabilizes in at most k steps. It appears that *msd* of $\lambda(r)$ is somewhat larger than this k.

In [2], *msd* of L^n has been determined up to one or two units for all n. The first value which is not known is *msd*(L^4); it is predicted to be 2 or 3. In this paper, we discuss an algorithm, based in part on certain results of [2] and which has been implemented in C. The program has been run on the computer system of the Vrije Universiteit Amsterdam with a total runtime (under a time-sharing operating system) of over 31 hours. Our discussion of the algorithm may clarify why it takes so long to produce information on a seemingly simple lattice. The resulting fact that *msd*$(L^4) = 2$ is a key result to compute *msd*(L^n) for all $n \geq 5$ and to compute *msd*$(\lambda(r))$ (with a few exceptions).

2. Preliminary results

A slight modification of an argument in [2] yields the following result on products.

Proposition 2.1 *If $X_1, \ldots, X_{n+1}$ are totally ordered sets, and if msd of a product of any n of them is at most k, then msd$(\prod_{i=1}^{n+1} X_i) \leq k + 1$.*

We next describe some concepts related to convexity and refer to the monograph [4] for further information. A subset C of a median algebra M is *convex* provided

$m(C \times C \times M) \subseteq C$. Note the difference with the definition of a median stable set. Each subset A of M is included in a smallest convex subset $co(A)$, the *convex hull of* A. For an alternative viewpoint on convexity, the *interval* ab joining two points a, b of M is defined as the set of all points $m(a, b, x)$ for $x \in M$. Equivalently,

$$ab = \{x \mid m(a, b, x) = x\}.$$

If m is induced by a metric, then ab is also equal to the set of points which are geodesically between a and b. A set is convex iff $ab \subseteq C$ for each $a, b \in C$.

In Section 1 we already mentioned that each point c of a median algebra M is the least element of a semilattice $(M, \wedge_c)$. The corresponding partial order $\leq_c$ can be described as follows.

$$x \leq_c y \quad \text{iff } x \in yc.$$

This order is known as the *base-point order* based at c.

Proposition 2.2 [2] *Let X be a median algebra and let $S \subseteq X$. Then a point $c \in X$ is generated by S, that is, $c \in med\,(S)$, iff it is generated by the set*

$$\{x \mid x \geq_c s \quad \text{for some} \quad s \in S\}.$$

In either situation, the same number of steps is required.

As a consequence of this result, each minimal set generating c consists of incomparable points in the base-point order of c.

A *median graph* is a connected graph of which the geodesic metric is a median metric. Each finite median algebra can be seen as a median graph, where two distinct points a, b form an edge provided the interval ab equals $\{a, b\}$. In this situation, the median operator induced by the geodesic metric equals the original median operator. The *convex neighborhood* of a vertex in a median graph is the convex hull of the set, consisting of the vertex and all of its neighbors.

Proposition 2.3 [2] *In a finite median graph G, the following assertions are equivalent for $n < \infty$.*

(1) *G has msd at most n.*

(2) *For each vertex the convex neighborhood has msd at most n.*

The previous result has been used to prove that $msd(\mathbb{R}^n)$ and (more generally) *msd* of a product of n non-trivial totally ordered sets, is equal to $msd(L^n)$. A key result in [2] is the determination of *msd* of the graphic n-cube for all n, leading to a slightly unsharp determination of $msd(L^n)$. Some values are presented in Table 1. For $n = 4, 5$, the upper bound is corrected for the result in Proposition 2.1.

Table 1. Bounds of $msd(\mathbb{R}^n) = msd(L^n)$.

n	Lower	Upper
4	2	3
5	3	4
6	3	5
7, 8, 9	4	6
10–13	5	7
14	6	7
15–19	6	8
20, 21	7	8
22–28	7	9

Another consequence of Proposition 2.3 is, that $msd(L^n)$ is the least number k such that if a subset of L^n generates the origin $\mathbf{0}$, then it does so in at most k steps. Indeed, the convex neighborhood of $\mathbf{0}$ equals the entire L^n, and for each $c \in L^n$ there is a translation of the convex neighborhood of c, mapping c to $\mathbf{0}$.

A set $H \subseteq M$ is a *half-space* provided H and $M \setminus H$ are convex. A median algebra satisfies the *Kakutani separation property:* two disjoint convex sets always extend to complementary half-spaces.

Proposition 2.4 [4] *Let $p \in M$ and $S \subseteq M$. Then $p \in med(S)$ iff each two half-spaces of M containing p have a non-empty intersection with S.*

The half-spaces of L^n are product sets, of which one factor consists of an initial or final segment of $\{-1, 0, 1\}$; all other factors are full. This provides an algorithm of complexity

$$2n(n-1)(\#S)$$

to determine whether a given point of L^n is in the median stabilization of a set S.

Usually, most points are found during the first stages of the stabilizing process and few points persist till the last stage. Hence, the complexity of an algorithm computing $med(S)$ directly should be estimated as $(\#med(S))^4/6$. Even worse, the number of points in $med(S)$ (which can be quite large even for small S) is usually not known in advance. The definition of a free median algebra implies that $\# med(S) \leq \#\lambda(\#S)$. A brief look at a table of values $\#\lambda(r)$ (cf. [4, p. 240]) or at Table 2 (dimension of $\lambda(r)$) may complete these pessimistic remarks. A geometric method for generating median graphs in Boolean algebras has been proposed in [8], but we have no estimation of its complexity.

The last result may help to decrease the number of sets to be investigated. The argument is a simple case analysis.

Proposition 2.5 *Let $S \subseteq L^n$ be a set of incomparable points in the base-point order of the origin. If two members of S are neighbors of the origin $\mathbf{0}$, and if $\#S \geq 3$, then $\mathbf{0} \in med(S)$ and $\mathbf{0}$ is obtained at the first stage of the stabilization process.*

Table 2. Invariants of $\lambda(r)$.

r	$n = dim(\lambda(r))$	$\lceil n \cdot r/(r-1) \rceil$	$msd(\lambda(r))$	r	$n = dim(\lambda(r))$	$msd(\lambda(r))$
3	1	2	1	4	3	2
5	4	5	3	6	10	5
7	15	18	6	8	35	8
9	56	63	9	10	126	11
11	210	231	12–13	12	462	14
13	792	858	15	14	1,716	18
15	3,003	3,218	19	16	6,435	21
17	11,440	12,155	22–23	18	24,310	24
19	43,758	46,189	26	20	92,378	28
21	167,960	176,358	29	22	352,716	31
23	646,646	676,039	32	24	1,352,078	34
25	2,496,144	2,600,150	36	26	5,200,300	37
27	9,657,700	10,029,150	39	28	20,058,300	41
29	37,442,160	38,779,380	42	30	77,558,760	44
31	145,422,675	150,270,097	46	32	300,540,195	47
33	565,722,720	583,401,555	49	34	1,166,803,110	51
35	2,203,961,430	2,268,783,825	52	36	4,537,567,650	54
37	8,597,496,600	8,836,315,950	56	38	17,672,631,900	58
39	33,578,000,610	34,461,632,205	59	40	68,923,264,410	61

3. The algorithm

According to [2], a 4-point set always stabilizes in at most two stages. So we are interested in subsets of L^4 with at least five points. Theorem 2.9(1) of [2] yields that no sets of cardinality >17 need to be considered. This still leaves us with an astronomical number of sets, even when divided by the number of symmetries (384) of L^4. Propositions 2.2, 2.3, and 2.5 reduce the number of critical sets considerably, but the size of this reduction is hard to estimate.

Table 3 is used to represent all vectors of L^4 by a number p, where $0 \leq p \leq 80$. The actual order is irrelevant, except that 0 corresponds with the origin $(0, 0, 0, 0)$. We consider a subset $S \subseteq L^4$, such that

(i) No two members of S are comparable.
(ii) At most one member of S is a neighbor of 0.
(iii) No $q \in S$ is in $med(S \backslash \{q\})$.

We refer to such sets as "admissible". The observations in Section 2 reduce the computation of $msd(L^4)$ to finding out whether an admissible set generates 0 and (if this happens to be the case) to find out whether 0 is introduced before the third stage.

Table 3. Vector codes.

Point	Coordinates				Point	Coordinates				Point	Coordinates			
0	0,	0,	0,	0	27	0,	0,	0,	1	54	0,	0,	0,	−1
1	1,	0,	0,	0	28	1,	0,	0,	1	55	1,	0,	0,	−1
2	−1,	0,	0,	0	29	−1,	0,	0,	1	56	−1,	0,	0,	−1
3	0,	1,	0,	0	30	0,	1,	0,	1	57	0,	1,	0,	−1
4	1,	1,	0,	0	31	1,	1,	0,	1	58	1,	1,	0,	−1
5	−1,	1,	0,	0	32	−1,	1,	0,	1	59	−1,	1,	0,	−1
6	0,	−1,	0,	0	33	0,	−1,	0,	1	60	0,	−1,	0,	−1
7	1,	−1,	0,	0	34	1,	−1,	0,	1	61	1,	−1,	0,	−1
8	−1,	−1,	0,	0	35	−1,	−1,	0,	1	62	−1,	−1,	0,	−1
9	0,	0,	1,	0	36	0,	0,	1,	1	63	0,	0,	1,	−1
10	1,	0,	1,	0	37	1,	0,	1,	1	64	1,	0,	1,	−1
11	−1,	0,	1,	0	38	−1,	0,	1,	1	65	−1,	0,	1,	−1
12	0,	1,	1,	0	39	0,	1,	1,	1	66	0,	1,	1,	−1
13	1,	1,	1,	0	40	1,	1,	1,	1	67	1,	1,	1,	−1
14	−1,	1,	1,	0	41	−1,	1,	1,	1	68	−1,	1,	1,	−1
15	0,	−1,	1,	0	42	0,	−1,	1,	1	69	0,	−1,	1,	−1
16	1,	−1,	1,	0	43	1,	−1,	1,	1	70	1,	−1,	1,	−1
17	−1,	−1,	1,	0	44	−1,	−1,	1,	1	71	−1,	−1,	1,	−1
18	0,	0,	−1,	0	45	0,	0,	−1,	1	72	0,	0,	−1,	−1
19	1,	0,	−1,	0	46	1,	0,	−1,	1	73	1,	0,	−1,	−1
20	−1,	0,	−1,	0	47	−1,	0,	−1,	1	74	−1,	0,	−1,	−1
21	0,	1,	−1,	0	48	0,	1,	−1,	1	75	0,	1,	−1,	−1
22	1,	1,	−1,	0	49	1,	1,	−1,	1	76	1,	1,	−1,	−1
23	−1,	1,	−1,	0	50	−1,	1,	−1,	1	77	−1,	1,	−1,	−1
24	0,	−1,	−1,	0	51	0,	−1,	−1,	1	78	0,	−1,	−1,	−1
25	1,	−1,	−1,	0	52	1,	−1,	−1,	1	79	1,	−1,	−1,	−1
26	−1,	−1,	−1,	0	53	−1,	−1,	−1,	1	80	−1,	−1,	−1,	−1

The algorithm described below aims at enlarging an admissible subset of L^4 by adding one point at the time. When a set "fails" (that is, if it generates 0 in at most two steps), the last introduced point is removed, and the next one is tried. The experimental observation, that specific input sets finish within an hour—and most often within minutes or even seconds—made us realize that a (sharply programmed) algorithm would make a chance.

3.1. *The basic algorithm*

For convenience, we use *push* $(p, stack)$ and *pop* $(p, stack)$ to describe the operations of pushing the value of p on top of the stack, respectively, of popping the top value from the stack and assigning it to p.

```
Input: a set S and an empty stack of additional points.
p = 0.
while p ≤ 81 do
    begin
    (*) p ← p + 1.
        if p = 81,
        begin
            if the stack is not empty,
            begin
                pop (p, stack),
                S ← S\{p},
                go to (*).
            end
            if the stack is empty, stop: msd equals 2.
        end
        if two points of S ∪ {p} are neighbors of 0, go to (*).
        if p ∈ med(S), or q ∈ med({p} ∪ S/{q}) for some q ∈ S,
    (**) or some member of S is comparable with p in ≤_0, go to
        (*).
        if 0 ∉ med(S ∪ {p}),
        begin
            S ← S ∪ {p},
            push (p, stack),
            go to (*).
        end
        if 0 ∈ med(S ∪ {p}),
        begin
            A_0 ← S ∪ {p}, A_1 ← m(A_0^3), A_2 ← m(A_1^3).
            if 0 ∉ A_2, stop: msd equals 3.
            else
            begin
                pop (p, stack),
                S ← S\{p},
                go to (*).
            end
        end
    end
```

A word of explanation may be necessary concerning (**). It is clear that if $q \leq_0 p$ for some $q \in S$ or if $p \in med(S)$, then p can be skipped. But what if $p \leq_0 q$ or if $q \in med(\{p\} \cup S \backslash \{q\})$ for some $q \in S$? In this situation, one would rather expect the point q to be skipped. However, up to symmetry, the basic algorithm will be called upon for all input sets of a certain size. Hence the set $\{p\} \cup S \backslash \{q\})$ (or a symmetric one) will be considered sooner or later.

The basic algorithm is applied in four steps.

1. Determine all types of 4-point subsets of the "positive" 4-cube of L^4. There are precisely five of them; the following are representative.

$$\{1, 12, 30, 36\}, \{4, 10, 12, 28\}, \{4, 10, 28, 39\}, \{4, 10, 30, 36\}, \{13, 31, 37, 39\}.$$

   (The first set is superfluous, as it generates the origin in two steps.) Then we determine all extensions (up to symmetry) to admissible 5-point subsets. The algorithm is applied to each of them. No superset stabilizes in more than two stages.
2. Assume that admissible 5-point sets contain at most three points in each single cube. To find all possible sets up to symmetry, we first determine all possible types of 3-sets in the "positive" 4-cube of L^4. There are seven types, represented by

$$\{1, 12, 30\}, \{4, 10, 12\}, \{4, 10, 28\}, \{4, 10, 30\}$$
$$\{4, 10, 39\}, \{4, 37, 39\}, \{13, 31, 37\}.$$

   Next we determine all extensions (up to symmetry) to admissible 5-point subsets S. We apply the basic algorithm to each of them, including an additional test just before (**) to eliminate those p for which $S \cup \{p\}$ has more than three points in some cube. No superset stabilizes in more than two stages.
3. Assume that admissible 5-point sets contain at most two points in each single cube. To find all possible sets up to symmetry, we first determine all possible types of 2-point subsets of the "positive" 4-cube of L^4. There are six types of 2-sets to start with.

$$\{1, 12\}, \{1, 39\}, \{4, 10\}, \{4, 36\}, \{4, 37\}, \{13, 31\}.$$

   Then we determine all extensions (up to symmetry) to admissible 5-point subsets. To each of these sets S, we apply the basic algorithm, including an additional test just before (**), eliminating those p for which $S \cup \{p\}$ has more than two points in some cube. Again, no superset stabilizes in more than two stages.
4. Assume that the admissible 5-point sets contain at most one point in each single cube. To find all possible sets up to symmetry, we start with all possible types of singleton subsets of the "positive" 4-cube of L^4, and we determine all extensions S to admissible 5-point subsets. The singletons used were

$$\{1\}, \{4\}, \{13\}, \{40\}.$$

   To each of the sets S, we apply the basic algorithm, including an additional test just before (**), eliminating those p for which $S \cup \{p\}$ has more than one point in some cube. Again, no superset stabilizes in more than two stages.

All admissible starting sets S have of course been tested in advance on generating the origin in three steps before letting them grow.

3.2. Comments on the implementation

(1) We originally wanted to run the basic algorithm with all types of 4-point sets as an input list. Note, however, that input sets like $\{13, 31, 37, 39\}$ are highly symmetric. For such sets, the procedure tends to take unnecessarily long, since several choices of a fifth point correspond under an intrinsic symmetry. For this reason, we have specified all possible fifth points as well, eliminating those which correspond under symmetry. (The basic algorithm doesn't test for symmetry because this is rather expensive and hardly effective if the number of points grows larger.)

(2) Another time-saving device is to determine a well-chosen "start value" v for each input set before the basic algorithm is called upon. Initially, $v = 0$ for each set in consideration. If c denotes the maximal number of points allowed in a single cube, then we first determine all types of c-point subsets of the "positive" 4-cube. Next, we consider each point p with $v + 1 \leq p \leq 80$ as a possible next point. If the resulting set is admissible, and if no symmetric set has been introduced yet, then the set is added to the current input list and the start value $v = p$ is assigned to it. If $c + 1 < 5$ then the process is repeated on each of the sets described in the current input list, leading to a new list of enlarged sets, each with a new start value. This is, in fact, a preliminary "breadth-first" search.

(3) In regard to the fact, that a computation of $med(S)$ is rather expensive, our stack keeps more information than just the additional points. In fact, the entire current set is remembered, together with each median and an indication at which stage it was found. When the set is enlarged with a point p, we only have to compute the new medians involving p and to correct for old medians which are now obtained at an earlier stage.

When the basic algorithm is called upon, it reads both a 5-point input set and a start value, and interprets the latter as the initial value of p. Considering that 4-point input sets need up to 1 hour to finish if no precautions are taken, the time profit is spectacular (see Table 4).

Table 4. Runtimes of cases 1–4.

Case	No. of starting sets	Time (s)	No. of sets investigated	Time (s)
1	34	0.1	171,864	2,554
2	362	4.8	3,503,719	53,977
3	721	20.9	3,178,123	52,448
4	742	5.6	406,876	5,113
Total	1,859	31.4	7,260,582	114,092

Table 5. Growth of $msd(Q^n)$ in terms of n.

msd	$n \leq$	*msd*	$n \leq$	*msd*	$n \leq$	*msd*	$n \leq$
0	2	15	711	30	311,073	45	136,216,567
1	3	16	1,066	31	466,609	46	204,324,850
2	4	17	1,599	32	699,913	47	306,487,275
3	6	18	2,398	33	1,049,869	48	459,730,912
4	9	19	3,597	34	1,574,803	49	689,596,368
5	13	20	5,395	35	2,362,204	50	1,034,394,552
6	19	21	8,092	36	3,543,306	51	1,551,591,828
7	28	22	12,138	37	5,314,959	52	2,327,387,742
8	42	23	18,207	38	7,972,438	53	3,491,081,613
9	63	24	27,310	39	11,958,657	54	5,236,622,419
10	94	25	40,965	40	17,937,985	55	7,854,933,628
11	141	26	61,447	41	26,906,977	56	11,782,400,442
12	211	27	92,170	42	40,360,465	57	17,673,600,663
13	316	28	138,255	43	60,540,697	58	26,510,400,994
14	474	29	207,382	44	90,811,045	59	39,765,601,491

In Table 4 the first 'time column' presents the time needed to compute all 5-point input sets. The second 'time column' is the actual runtime of our algorithm. The number of starting sets and the number of sets visited have been communicated by a program counter.

The output of our program leads to the conclusion that $msd(L^4) = 2$. This fact has several consequences.

Corollary 3.1 *For all $n \neq 2, 3$, msd of L^n equals msd of Q^n.*

Proof: The value of $msd(Q^n)$ can be described as follows [2]. Let $q_0 = 2$ and (recursively) $q_{k+1} = \lfloor 3q_k/2 \rfloor$. Then $msd(Q^n) \leq k$ iff $n \leq q_k$; see Table 5. We remind the reader that $msd(L^2) = 1$ and $msd(L^3) = 2$. So, the result is correct for $n \leq 4$. Assume the result to be valid up to but not including $n > 4$. If $n = q_k$ for some k, then (as $n - 1 \neq 2, 3$)

$$msd(L^{n-1}) = msd(Q^{n-1}) = msd(Q^n) - 1,$$

and the result follows from Proposition 2.1. Let $m = q_k < n < q_{k+1}$. By the definition of q_{k+1}, there exist three m-point sets C_j for $j = 1, 2, 3$, each two of which cover the set $\{1, 2, \ldots, n\}$. Let S be any subset of L^n generating the origin $\mathbf{0}$. Consider the projection $\pi_j : L^n \to L^m$ that drops all factors numbered by an index $i \notin C_j$. Then $\pi_j(S)$ generates the origin of L^m and, as $m = q_k \geq 4$, we need at most k stages for this. By imitating this process on the original set S, some point $p_j \in L^n$ can be generated in at most k stages, such that for each $i \in C_j$ the ith coordinate of p_j is 0. We obtain $\mathbf{0} = m(p_1, p_2, p_3)$ from S in at most $k + 1$ stages. □

Note the critical role of the result in dimension 4. It is not difficult to deduce from the previous corollary that a product of n non-degenerate trees has *msd* equal to $msd(Q^n)$ provided $n \neq 1, 2, 3$.

The second corollary involves the "cubical dimension" $dim(M)$ of a finite median algebra M, which is defined as the largest number n such that the graphic n-cube can be embedded in M.

Corollary 3.2 *If $n = dim(\lambda(r))$ then msd of $\lambda(r)$ equals msd of Q^n for even r and for "most" odd r. In exceptional cases, msd may be one unit larger.*

Proof: According to [4], p. 237, the cubical dimension n of $\lambda(r)$ is given by

$$n = \binom{r-1}{\lceil r/2 \rceil}.$$

On the other hand, $\lambda(r)$ can be embedded into $\mathbb{R}^m$ iff

$$\binom{r}{\lceil r/2 \rceil} \leq 2m;$$

cf. [4, Chapter II, Sections 1.22.5, 2.17]. Hence, $\lambda(r)$ can be embedded into the real space of n dimensions if r is even, and it can be embedded into the real space of $\lceil n \cdot r/(r-1) \rceil$ dimensions if r is odd. Assuming that there is no regular connection between the relevant sequences, the probability that the interval $[n, n \cdot r/(r-1)]$ hits some q_k is (roughly) equal to $4/(3 \cdot r)$. □

For $r \leq 40$, the ambiguity about $msd(\lambda(r))$ occurs only if $r = 11, 17$; cf. Table 2. The phenomenon can perhaps be avoided to some extent by noticing that *msd* of $\lambda(r)$ is determined by *msd* of its convex neighborhoods. It is possible that all neighborhoods can be embedded in a real space of dimension less than $n \cdot r/(r-1)$. For $r = 5$ this is provably not the case. For larger r, no information is available yet.

With the exception of the above discussed ambiguity, the original problems on *msd*, raised in [2], have now largely been solved.

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